Exercise Sheet
Topics in Mathematical Physics

Sheet Number 4

Due: Monday 17.11.25, 10 a.m., Letterbox 3.13, Ernst-Zemelo-Str. 1.
Please try handing in as pairs

Exercise 9: (4 Points)

Don't use the Riemann–Lebesgue Lemma to prove the statement.
Let $f \in C_c^0(\mathbb{R})$, meaning continuous and with compact support
Show that $\widehat{f}(k) \to 0$ as $|k| \to \infty$.

HINT: For $k \neq 0$, use the change of variable $x \mapsto x + \frac{\pi}{k}$ and add up the two different resulting ways of writing $\hat{f}(k)$.

Exercise 10: (4 Points)

Let X_i and P_j be the position and momentum operators, respectively, for $i, j = 1, \dots, d$. Show that

$$[X_i, P_j] = i\hbar\, \delta_{ij},$$

where δ_{ij} is the Kronecker delta defined as $\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & \text{else.} \end{cases}$

HINT: Simply work on test functions $\psi \in C_0^\infty(\mathbb{R})$ and do not worry about the domains of the above unbounded operators for this exercise.

Exercise 11: (4 Points)

Let $\psi \in L^2(\mathbb{R})$ with $\|\psi\|_{L^2(\mathbb{R})} = 1$ such that $x\psi \in L^2(\mathbb{R})$, and $x^2\psi \in L^2(\mathbb{R})$. Prove that

$$\langle \psi, X^2\psi \rangle \geq (\langle \psi, X\psi \rangle)^2$$

where X is the position operator.

HINT: Recall Jensen's inequality: given a probability space $(\Omega, \mathcal{A}, \mu)$, a measurable function $f : \Omega \to \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ convex,

$$\varphi\Big(\int_\Omega f(x)\, d\mu(x)\Big) \leq \int_\Omega \varphi(f(x))\, d\mu(x).$$