

Exercise Sheet “Topics in Mathematical Physics”

Sheet Number 4

Due: Monday 17.11.25, 10 a.m., Letterbox 3.13, Ernst-Zemelo-Str. 1.
Please try handing in as pairs

Exercise 9:

(4 Points)

Don't use the Riemann–Lebesgue Lemma to prove the statement.

Let $f \in C_c^0(\mathbb{R})$, meaning continuous and with compact support

Show that $\widehat{f}(k) \rightarrow 0$ as $|k| \rightarrow \infty$.

HINT: For $k \neq 0$, use the change of variable $x \mapsto x + \frac{\pi}{k}$ and add up the two different resulting ways of writing $\widehat{f}(k)$.

Exercise 10:

(4 Points)

Let X_i and P_j be the position and momentum operators, respectively, for $i, j = 1, \dots, d$. Show that

$$[X_i, P_j] = i\hbar \delta_{ij},$$

where δ_{ij} is the Kronecker delta defined as $\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & \text{else.} \end{cases}$

HINT: Simply work on test functions $\psi \in C_0^\infty(\mathbb{R})$ and do not worry about the domains of the above unbounded operators for this exercise.

Exercise 11:

(4 Points)

Let $\psi \in L^2(\mathbb{R})$ with $\|\psi\|_{L^2(\mathbb{R})} = 1$ such that $x\psi \in L^2(\mathbb{R})$, and $x^2\psi \in L^2(\mathbb{R})$. Prove that

$$\langle \psi, X^2 \psi \rangle \geq (\langle \psi, X \psi \rangle)^2$$

where X is the position operator.

HINT: Recall Jensen's inequality: given a probability space $(\Omega, \mathcal{A}, \mu)$, a measurable function $f : \Omega \rightarrow \mathbb{R}$ and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ convex,

$$\varphi\left(\int_{\Omega} f(x) d\mu(x)\right) \leq \int_{\Omega} \varphi(f(x)) d\mu(x).$$