# Exercise Sheet
# "Topics in Mathematical Physics"

## Sheet Number 5

**Due:** Monday 24.11.25, 10 a.m., Letterbox 3.13, Ernst-Zemelo-Str. 1.
Please try handing in as pairs

**Exercise 12:** ($3 = \frac{1}{2} + \frac{1}{2} + 1 + 1$ Points)

Consider a vector space $V$ over $\mathbb{C}$ and let $A, B, C$ be bounded linear operators on $V$ and $\alpha \in \mathbb{C}$.

(i) Prove that $[A, B + \alpha C] = [A, B] + \alpha[A, C]$.

(ii) Prove that $[B, A] = -[A, B]$.

(iii) Prove that $[A, BC] = [A, B]C + B[A, C]$.

(iv) Prove that $[A, [B, C]] = [[A, B], C] + [B, [A, C]]$.

**Exercise 13:** (5 Points)

Let $H$ be a Hilbert space and let $A$ and $B$ be linear operators on $H$ such that there exists $\alpha \in \mathbb{C} \setminus \{0\}$ with

$$[A, B] = \alpha \, \mathrm{id}.$$

Prove that $A$ and $B$ cannot both be bounded.

Hint: Assume both are bounded; consider $\|[A, B^n]\|$ and derive a contradiction.

**Exercise 14:** (4 Points)

Recall the definition of $H^2(\mathbb{R})$ as

$$H^2(\mathbb{R}) := \{\psi \in L^2(\mathbb{R}) \mid k^2 \hat{\psi}(k) \in L^2(\mathbb{R})\}.$$

In class we defined the map that to any inital datum $\psi_0 \in L^2(\mathbb{R})$ would associate

$$\psi_t := \hat{U}_0(t)\psi_0,$$

defined via the Hamiltonian $H_0 := -\frac{d^2}{dx^2}$ qith domain $\mathrm{dom}(H_0) = H^2(\mathbb{R})$. Indeed if $U_0(t)\psi_0$ is defined for any $\psi_0 \in \mathcal{S}(\mathbb{R})$ as the unique solution of

$$\begin{cases} i\hbar \, \partial_t\big(U_0(t)\psi_0\big) = H_0 \, U_0(t)\psi_0, \\ U_0(0)\psi_0\big|_{t=0} = \psi_0, \end{cases}$$

then $\tilde{U}_0(t)$ is defined by density on the whole space $L^2(\mathbb{R})$, and coincides with $U_0(t)$ on $\mathcal{S}(\mathbb{R})$. Prove that if $\psi_0 \in D(H_0) = H^2(\mathbb{R})$, then $\psi_t \in D(H_0)$ for all $t \in \mathbb{R}$.