Exercise Sheet
“Topics in Mathematical Physics“

Sheet Number 6

Due: Monday 01.12.25, 10 a.m., Letterbox 3.13, Ernst-Zemelo-Str. 1.
Please try handing in as pairs

Exercise 15: (6 = 2 + 2 + 2 Points)

Let $H = -\frac{d^2}{dx^2} + \omega^2 x^2$ be the Hamiltonian acting on $H^2(\mathbb{R})$ with $\omega \in \mathbb{R}_{>0}$

(i) Define the ladder operators

$$a_\pm := \frac{1}{\sqrt{2}}\left[\sqrt{\omega}x \mp i\frac{1}{\sqrt{\omega}}\frac{d}{dx}\right]$$

and show that

(a) $[a_-, a_+] = 1$

(b) $H = \omega(2N+1)$, where $N := a_+ a_-$

(c) $[N, a_\pm] = \pm a_\pm$.

(ii) Show that the spectrum of N, $(\sigma(N))$ is contained in the set of natural numbers, i.e. $\sigma(N) \subset \mathbb{N}$.

Hint: Obserse that if $N\psi = n\psi$ for some $\psi \neq 0$, then $Na_\pm\psi = (n \pm 1)\psi$ (therefore the name ”ladder operator”).‘

(iii) Show that $\sigma(N) = \mathbb{N}$.

Hint: It suffices to find $\psi_0 \neq 0$ s.t. $N\psi_0 = 0$. Why?

Exercise 16: (4 Points)

Prove that, for two operators A and B on $\mathcal{H}$

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$$

if $[A, B]$ commutes with A and with B.

Hint: Take the derivative of the function $F(t) = e^{t(A+B)}e^{-tA}e^{-tB}$ with $F(0) = 1$ and deduce the formula.

Exercise 17: (4=2+2 Points)

Consider the kinetic energy of an N-body system $\sum_{j=1}^N -\Delta_{x_j}$ and the two-body interaction $\sum_{i<j}^N K(x_i - x_j)$, $x_i \in \mathbb{R}^3$. Compute their expectations in the state $\psi_N \in L^2(\mathbb{R}^{3N})$ and show that the following identities for the density matrices

$$\gamma_N^{(1)}(x_1; x_1') = \int_{\mathbb{R}^{3(N-1)}} \psi_N(x_1, x_2, \ldots, x_N)\,\overline{\psi_N(x_1', x_2, \ldots, x_N)}\,dx_2 \cdots dx_N.$$

$$\gamma_N^{(2)}(x_1, x_2; x_1', x_2') = \int_{\mathbb{R}^{3(N-2)}} \psi_N(x_1, x_2, x_3, \ldots, x_N)\,\overline{\psi_N(x_1', x_2', x_3, \ldots, x_N)}\,dx_3 \cdots dx_N.$$

holds:

(a) $\langle \psi_N, \sum_{j=1}^N -\Delta_{x_j}\psi_N\rangle = \operatorname{tr}(-\Delta\gamma_N^{(1)})$

(b) $\langle \psi_N, \sum_{i<j} K(x_i - x_j)\psi_N\rangle = \frac{1}{2}\operatorname{tr}(K(x_1 - x_2)\gamma_N^{(2)})$.