Exercise Sheet
"Topics in Mathematical Physics"

Sheet Number 7

Due: Monday 08.12.25, 10 a.m., Letterbox 3.13, Ernst-Zemelo-Str. 1.
Please try handing in as pairs

Exercise 18: (4 Points)

Let $\phi \in L^2(\mathbb{R}^3)$, $a_{N,\phi} : L^2_a(\mathbb{R}^{3N}) \to L^2_a(\mathbb{R}^{3(N-1)})$ and its adjoint $a^*_{N,\phi} : L^2_a(\mathbb{R}^{3(N-1)}) \to L^2_a(\mathbb{R}^{3N})$ defined as

$$(a_{N,\phi}\psi)(x_1,\dots,x_{N-1}) = \sqrt{N}\int \overline{\phi(x)}\psi(x,x_1,\dots,x_{N-1})\,dx$$

$$(a^*_{N,\phi}\psi)(x_1,\dots,x_N) = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}(-1)^{j+1}\phi(x_j)\psi(x_1,\dots,x_{j-1},x_{j+1},\dots,x_N)$$

Show that

$$a_{N+1,\phi}a^*_{N+1,\phi} + a^*_{N,\phi}a_{N,\phi} = \|\phi\|^2_{L^2(\mathbb{R}^3)}.$$

Exercise 19: (4 Points)

Let γ_N be a fermionic density matrix on $L^2(\mathbb{R}^{3N})$ of the form $\gamma_N = |\psi\rangle\langle\psi|$ where $\psi \in L^2_a(\mathbb{R}^{3N})$. Show that $\forall \psi \in L^2_a(\mathbb{R}^{3N})$,

$$\langle \phi, \gamma_N^{(1)}\phi\rangle_{L^2(\mathbb{R}^3)} = \langle a_{N,\phi}\psi, a_{N,\phi}\psi\rangle_{L^2(\mathbb{R}^{3(N-1)})}$$

where $\gamma_N^{(1)} = N\,\mathrm{tr}_{2,\dots,N}(\gamma_N)$ and $a_{N,\phi}$ as defined in Exercise 18.

Exercise 20: (4 Points)

Let γ_N be a fermionix density matix. Show that the one-body reduced density matrix $\gamma_N^{(1)}$ associated with γ_N satisfies $\gamma_N^{(1)} \leq 1$ in the sense $\langle \phi, \gamma_N^{(1)}\phi\rangle_{L^2(\mathbb{R}^3)} \leq \|\phi\|^2_{L^2(\mathbb{R}^3)}$.

Hint: Show that w.l.o.g. one can restrict the proof to pure states $\gamma_N = |\psi\rangle\langle\psi|$ where $\psi \in L^2_a(\mathbb{R}^{3N})$ with $\|\psi\|_{L^2(\mathbb{R}^{3N})} = 1$. Use the Exercises 18 and 19.