Exercise Sheet
"Topics in Mathematical Physics"

Sheet Number 8

Due: Monday 15.12.25, 10 a.m., Letterbox 3.13, Ernst-Zemelo-Str. 1.
Please try handing in as pairs

Exercise 21: (2+2=4 Points)

Recall that

$$H_{N,M}(\underline{R},\underline{Z}) = \sum_{j=1}^{N}\left[-\Delta_{x_j} - \sum_{i=1}^{M}\frac{Z_i}{|x_j - R_j|}\right] + \sum_{i\le j}\frac{1}{|x_i - x_j|} + \sum_{1\le j<k\le M}\frac{Z_j Z_k}{|R_j - R_k|}.$$

(i) Show that $\forall \psi \in L^2_a(\mathbb{R}^{3N})$, $\langle \psi, H_{N,M}(\underline{R},\underline{Z})\psi\rangle$ fulfills

$$\langle \psi, H_{N,M}(\underline{R},\underline{Z})\psi\rangle = a(\psi) + b(\psi)Z_i,$$

with $a(\psi)$ and $b(\psi)$ depending on ψ, $\underline{R}$ and the Z_j, $j \neq i$.

(ii) Define for fixed $\underline{R}$ and Z_j, $j \neq i$:

$$\begin{aligned} f(Z_i) :&= \inf_{\psi\in L^2_a(\mathbb{R}^{3N}),\|\psi\|_2=1}\langle \psi, H_{N,M}(\underline{R},\underline{Z})\psi\rangle \\ &= \inf_{\psi\in L^2_a(\mathbb{R}^{3N}),\|\psi\|_2=1} a(\psi) + b(\psi)Z_i. \end{aligned}$$

Show that $f(Z_i)$ is a concave function of Z_i.

Exercise 22: (4 Points)

The minimizer of $\mathcal{E}(\psi) = \langle \psi, H_{N,M}(\underline{R},\underline{Z})\psi\rangle$ are bosonic, i.e.

$$\inf_{\psi\in L^2(\mathbb{R}^{3N}),\|\psi\|_2=1}\langle \psi, H_{N,M}(\underline{R},\underline{Z})\psi\rangle = \inf_{\psi\in L^2_s(\mathbb{R}^{3N}),\|\psi\|_2=1}\langle \psi, H_{N,M}(\underline{R},\underline{Z})\psi\rangle$$

Hint: Use Proposition "Restriction to bosonic states has no effect in the search for minimizers".

Exercise 23: (4 Points)

Let $\psi_N \neq 0$ with $\langle \psi_\pi, \psi_\sigma\rangle \geq 0$ for all permutations $\pi, \sigma \in \mathcal{S}_N$. Show that $\psi_N^{(s)} \neq 0$.

Hint: Find a lower bound for the L^2 norm of $\psi_N^{(s)}$ using the expression for the symmetrization of ψ_N.