Exercise Sheet
“Topics in Mathematical Physics“

Sheet Number 9

Due: Monday 12.01.26, 10 a.m., Letterbox 3.13, Ernst-Zemelo-Str. 1.
Please try handing in as pairs

Exercise 24: (4 Points)

Let $\{\phi_j\}_{j=1}^N$ orthonormal functions in $L^2(\mathbb{R}^3)$. Let $\phi_j = \phi_j^{e,+} + \phi_j^{e,-}$.
Show that

$$(\sum_{j=1}^N |\phi_j(x)|^2)^{\frac{1}{2}} \le (\sum_{j=1}^N |\phi_j^{e,+}(x)|^2)^{\frac{1}{2}} + (\sum_{j=1}^N |\phi_j^{e,-}(x)|^2)^{\frac{1}{2}}.$$

Remark: The fact that the $\{\phi_j\}_{j=1}^N$ are orthonormal doesn't play any role in the proof.

Exercise 25: (4 Points)

Show that for $a, b > 0$ and $t \ge 0$

$$at - bt^{d/(d+2)} \ge -\frac{2}{d+2}\left[\frac{d}{d+2}\right]^{d/2} a^{-d/2} b^{(d+2)/2}.$$

Exercise 26: (4 Points)

Using the proof of Lieb-Thierring inequality, show that $\exists c > 0$ s.t. $\forall \psi \in L_a^2(\mathbb{R}^{3N})$

$$\langle \psi, \sum_{j=1}^N -\Delta_{x_j} \psi \rangle \ge c \int (\gamma_\psi^{(1)}(x,x))^{5/2}\, dx \tag{1}$$

Hint:

(i) Use the spectral decomposition of $\gamma_\psi^{(1)} = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|$, with $0 \le \lambda_j \le 1$, $\sum \lambda_j = 1$ and $\{\phi_j\}_j$ orthonormal functions in $L^2(\mathbb{R}^3)$.

(ii) Show that (1) is equivalent to

$$\sum_j \int |\nabla \phi_j(x)|^2\, dx \ge c \int (\sum_j \lambda_j |\phi_j(x)|^2)^{5/3} dx. \tag{2}$$

(iii) Use the proof of the Lieb-Thierring inequality to prove (2).