Exercise Sheet
Topics in Mathematical Physics

Sheet Number 10

Due: Monday 19.01.26, 10 a.m., Letterbox 3.13, Ernst-Zemelo-Str. 1.
Please try handing in as pairs

Exercise 27: (6 Points)

Let μ_1, μ_2 be two charge distributons. Recall that

$$\mathcal{D}(\mu_1, \mu_2) := \frac{1}{2} \int \int \frac{1}{|x-y|} d\mu_1(x) d\mu_2(y).$$

Show that

$$\mathcal{D}(\mu_1, \mu_2) \leq \mathcal{D}(\mu_1, \mu_1)^{\frac{1}{2}} \mathcal{D}(\mu_2, \mu_2)^{\frac{1}{2}}$$

HINT: Starting from $\mathcal{D}(\mu_1, \mu_2)$, use similar steps to the ones in the proof of $\mathcal{D}(\mu, \mu) \geq 0$ and the Cauchy-Schwarz inequality.

Exercise 28: (6 Points)

From Newton's Theorem, we know under the assumption of μ being invariant under rotations, that

$$\phi(x) = \frac{1}{|x|} \int_{|y|<|x|} d\mu(y) + \int_{|y|>|x|} \frac{1}{|y|} d\mu(y).$$

Deduce the following statements:

(i) if μ is s.t. $\operatorname{supp}(\mu) \subset B_R(0)$, then

$$\phi(x) = \frac{\mu(\mathbb{R}^3)}{|x|} \qquad \forall |x| > R$$

(ii) if $\mu(\{y \in \mathbb{R}^3 : |y| < R\}) = 0$, then

$$\phi(x) \leq \frac{\mu(\mathbb{R}^3)}{|x|} \qquad \forall x \in \mathbb{R}^3,$$

where we use the notations:

- μ some Borel measure on $\mathcal{B}(\mathbb{R}^3)$
- $\phi(x) = \int \frac{1}{|x-y|} d\mu(y)$
- $B_R(0) := \{x \in \mathbb{R}^3 : |x| \leq R\}$