Exercise Sheet
"Topics in Mathematical Physics"

Sheet Number 12

Due: Monday 02.02.26, 10 a.m., Letterbox 3.13, Ernst-Zemelo-Str. 1.
Please try handing in as pairs

Exercise 31: (4 Points)

For $x_i \in \mathbb{R}^3$, let μ_{x_i} be a non-negative function defined on $\mathbb{R}^3$ such that μ_{x_i} is spherically symmetric with respect to x_i, bounded and such that $\int_{\mathbb{R}^3} \mu_{x_i}(y)\,dy = 1$.
Show for all $\rho \in L^1(\mathbb{R}^3)$ with $\rho \geq 0$,

$$\sum_{i<j}^{N} \frac{1}{|x_i - x_j|} \geq -D(\rho,\rho) + 2\sum_{i=1}^{N} D(\rho,\mu_{x_i}) - \sum_{i=1}^{N} D(\mu_{x_i},\mu_{x_i}).$$

HINT:

(i) Use $D(\rho - \sum_{i=1}^{N}\mu_{x_i}, \rho - \sum_{i=1}^{N}\mu_{x_i}) \geq 0$

(ii) Using that μ_{x_i} is spherically symmetric, apply Newton's Theorem to show that

$$D(\mu_{x_i},\mu_{x_i}) \leq \frac{1}{2}\frac{1}{|x_i - x_j|},$$

where one also uses that $\int_{\mathbb{R}^3} \mu_{x_i}(y)\,dy = 1$

Exercise 32: (6 Points)

Let $\tilde{\mu} \geq 0$ be bounded and such that $\int_{\mathbb{R}^3} \tilde{\mu}(x) = 1$ and $\tilde{\mu}(x) = 0$ for all $|x| > 1$.

Further, let $\mu_z(x) = \rho_\psi(z)\tilde{\mu}(\rho_\psi^{\frac{1}{3}}(z)(x-z))$, where ρ_ψ is the space density associated with $\psi \in L^2_a(\mathbb{R}^{3N})$ defined as

$$\rho_\psi(x) := N\int |\psi(x,x_2,\dots,x_N)|^2\,dx_2\dots dx_N.$$

Show that there exists $C \geq 0$ such that

$$2\int \rho_\psi(y)^{\frac{4}{3}}\,dy = C\iiint \frac{\mu_z(x)\mu_z(y)}{|x-y|}\rho_\psi(z)\,dz\,dy\,dx.$$

HINT: Use the definition of μ_z and perform the change of variables

$$\tilde{x} = \rho_\psi^{\frac{1}{3}}(z)(x-z), \quad \tilde{y} = \rho^{\frac{1}{3}}(z)(y-z).$$

Recall the definition of $D(\tilde{\mu},\tilde{\mu})$ to show that $C \geq 0$.

Exercise 33: (4 Points)

Using Young's Inequality, show that for $\rho : \mathbb{R}^3 \to [0,+\infty)$, there exists $a > 0$ (that can be chosen small), such that

$$\rho(x)^{\frac{4}{3}} \leq a\rho(x)^{\frac{5}{3}} + a^{-1}\rho(x).$$