THE CUBIC THREEFOLD AND FRIENDS

1. Background on Threefolds

- Fano, around the 1910s, proved that any smooth quartic threefold is not rational. Later on, in the 1950s, Roth criticised the proof as incomplete. In 1971, Iskovskikh and Manin provide a complete proof.
- In 1972, Artin and Mumford gave an example of another unirational, but not rational Fano threefold. This is a certain double cover X of $\mathbb{P}^3$ ramified over a singular quartic surface. They showed that $H_3(X, \mathbb{Z})$ is a birational invariant and obtained that $H_3(X, \mathbb{Z}) = \mathbb{Z}_2$, from which they concluded that X cannot be rational.
- At the same time, Clemens and Griffiths showed that any smooth cubic threefold V over a field of characteristic zero is unirational, but not rational. In the same year, Murre proved the result in characteristic p.

The idea of Clemens and Griffiths was to consider two auxiliary varieties of a smooth cubic threefold V:

- The *intermediate Jacobian* $J(V)$ - a principally polarised abelian variety playing a role similar to that of the Jacobian for studying divisors on curves.
- the *Fano variety of lines* $F(V)$ - a smooth projective surface parametrising lines on V.

In order to arrive at the result, they represented $J(V)$ as the Albanese variety of $F(V)$, and studied its theta divisors.

2. The Cohomology of the Cubic Threefold

As before, let V be a cubic threefold, i.e. a hypersurface of degree three in $\mathbb{P}^4$. A first idea is to compare the cohomology of V with that of $\mathbb{P}^3$ and hope that we shall thus find a birational invariant that has different values for V and $\mathbb{P}^3$, respectively.

For $\mathbb{P}^3$ we have

$$H^q(\mathbb{P}^3, \Omega^p_{\mathbb{P}^3}) = \begin{cases} \mathbb{C}, & \text{for } p = q \leq 3. \\ 0, & \text{otherwise.} \end{cases}$$

Dolbeault theorem says that $H^{p,q}(X) = H^q(X, \Omega^p_X)$ for a projective variety, and using Serre duality we can write down the relevant half of the Hodge diamond for $\mathbb{P}^3$:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 0 \end{array}$$

To compute the cohomology of V, we first use the Lefschetz theorem to see that

$$H^q(\mathbb{P}^4, \Omega^p_{\mathbb{P}^4}) \simeq H^q(V, \Omega^p_V), \text{ for } p + q \leq 2.$$

Furthermore Serre duality allows us to compute the whole Hodge diamond with the exception of the middle line:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ ? & & ? & & ? & & ? \end{array}$$

One could now hope that $H^{3,0}$ is non-zero, which would prove the irrationality of V. This is because

$$H^{3,0}(V) = H^0(V, \Omega^3_V) = H^0(V, \omega_V)$$

is a birational invariant. Unfortunately $\omega_V = \mathcal{O}_V(-2)$ which implies that $H^{3,0}(V) = 0$. As an aside, if V were a quintic threefold, then it would be a Calabi-Yau manifold meaning that $H^{3,0} = 1$, so this would be enough to prove that it is not rational. Going back to the cubic threefold, we still need to compute the Hodge numbers $h^{1,2} = h^{2,1}$. In particular we want to find $H^2(V, \Omega_V)$.

In order to do this, consider the following exact sequence in $\mathbb{P}^4$:

$$0 \to \mathcal{N}^\vee_{V/\mathbb{P}^4} \to \Omega_{\mathbb{P}^4}|_V \to \Omega_V \to 0, \tag{1}$$

where $\mathcal{N}^\vee_{V/\mathbb{P}^4} = \mathcal{O}_V(-3)$. The idea is to compute $H^2(V, \Omega_V)$ from the long exact sequence associated to (1). From Serre duality and Kodaira vanishing we obtain

$$H^2(V, \mathcal{O}_V(-3)) \simeq H^1(V, \mathcal{O}_V(1))^\vee = 0,$$
$$H^3(V, \mathcal{O}_V(-3)) \simeq H^0(V, \mathcal{O}_V(1))^\vee.$$

Moreover, we can easily compute $h^0(V, \mathcal{O}_V(1))$ from the exact sequence

$$0 \to \mathcal{I}_V \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_V \to 0, \tag{2}$$

where $\mathcal{I}_V = \mathcal{O}_{\mathbb{P}^4}(-3)$. More precisely, we twist (2) by one and look at its long exact sequence of cohomology

$$0 \to H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(-2)) \to H^0(\mathcal{O}_{\mathbb{P}^4}(1)) \to H^0(\mathcal{O}_V(1)) \to H^1(\mathcal{O}_{\mathbb{P}^4}(-2)) \to \dots.$$

Since the first and last term vanish, we get the isomorphism

$$H^0(\mathcal{O}_{\mathbb{P}^4}(1)) \simeq H^0(\mathcal{O}_V(1)).$$

Therefore we conclude that

$$h^0(\mathcal{O}_V(1)) = h^0(\mathcal{O}_{\mathbb{P}^4}(1)) = \binom{4+1}{1} = 5.$$

Lastly, to obtain some information about the cohomology of $\Omega_{\mathbb{P}^4}|_V$, we use the Euler sequence on $\mathbb{P}^4$ and restrict it to V to get:

$$0 \to \Omega_{\mathbb{P}^4}|_V \to \mathcal{O}_V(-1)^{\oplus 5} \to \mathcal{O}_V \to 0.$$

As before we consider the long exact sequence of cohomology

$$\begin{aligned} \dots \to H^0(\mathcal{O}_V) \to H^2(\Omega_{\mathbb{P}^4}|_V) \to \\ \to \mathcal{O}_V(-1)^{\oplus 5} \to H^2(\mathcal{O}_V) H^3(\Omega_{\mathbb{P}^4}|_V) \to H^3(\mathcal{O}_V(-1))^{\oplus 5} \to \dots. \end{aligned}$$

From the Lefschetz theorem, $h^{0,1} = h^{0,2} = 0$ and therefore the Dolbeault theorem implies

$$H^1(\mathcal{O}_V) \simeq H^2(\mathcal{O}_V) = 0.$$

From Kodaira vanishing we also have that $H^2(\mathcal{O}_V(-1)) = 0$, and from Serre duality

$$H^3(\mathcal{O}_V(-1)) = H^0(\mathcal{O}_V(-1))^\vee = 0.$$

Hence from the long exact sequence above we get that

$$H^2(\Omega_{\mathbb{P}^4}|_V) \simeq H^3(\Omega_{\mathbb{P}^4}|_V) = 0.$$

Therefore from the long exact sequence associated to (1) we conclude that

$$H^2(\Omega_V) \simeq H^3(\mathcal{O}_V(-3)),$$

which finally yields $h^{1,2} = h^0(\mathcal{O}_V(1)) = 5$. Thus the Hodge diamond of V is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 5 & & 5 & & 0. \end{array}$$

However, $h^{1,2}$ is unfortunately not a birational invariant. To see this, consider the example of the blowup $X \to \mathbb{P}^3$ along an elliptic quartic curve. This is a Fano threefold with

$$b_2 = 2, k^3 = 32, \text{ and } h^{1,2} = 1.$$

So even though X is rational (the blowup is a birational morphism), $h^{1,2}$ is nevertheless non-zero, hence it cannot be a birational invariant. We therefore need some additional structure somehow connected to the groups $H^{1,2}(V)$ and $H^{2,1}(V)$.

3. Intermediate Jacobians

We begin by recalling the notion of the *Jacobian of a curve* C:

$$J(C) = H^0(\Omega_C)^\vee / H_1(C, \mathbb{Z}) \simeq H^{1,0}(C)/H^1(C, \mathbb{Z}).$$

Now let X be an n-dimensional variety. We have two possible generalisations of the Jacobian to the case of a higher dimensional variety:

- The Albanese variety
$$Alb(X) = H^0(\Omega_X)^\vee / H_1(X, \mathbb{Z}) \simeq H^{n-1,n}(X)/H^{2n-1}(X, \mathbb{Z}),$$
- The Picard variety
$$Pic^0(X) = H^1(\mathcal{O}_X)/H^1(X, \mathbb{Z}) \simeq H^{0,1}(X)/H^1(X, \mathbb{Z}).$$

The idea now is to generalise this for odd cohomology in order to obtain a torus with a complex structure on it.

Recall also that if X is a compact Kähler manifold, then its cohomology admits a *Hodge structure*, i.e.

- it satisfies the Hodge decomposition
$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$
- it satisfies the symmetry property
$$H^{p,q}(X) \simeq \overline{H^{q,p}(X)}.$$

Equivalently the cohomology admits a *Hodge filtration*

$$0 \subseteq F^k H^k(X) \subseteq F^{k-1} H^k(X) \subseteq \ldots \subseteq F^0 H^k(X) = H^k(X, \mathbb{C}),$$

where

$$F^r H^k(X) = \bigoplus_{p \geq r} H^{p,k-p}(x),$$

for $0 \leq r \leq k$. For example, if $k = 1$, we have

$$H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X) \text{ and } H^{1,0}(X) \simeq \overline{H^{0,1}(X)}.$$

We have the following isomorphism of real vector spaces

$$H^1(X,\mathbb{R}) \subset H^1(X,\mathbb{C}) \to H^{0,1}(X),$$

where the last arrow is the projection from the Hodge decomposition. Hence the lattice $H^1(X,\mathbb{Z}) \subset H^1(X,\mathbb{R})$ projects onto a lattice in the complex vector space $H^{0,1}(X)$. We therefore get the complex torus

$$Pic^0(X) = H^{0,1}(X)/H^1(X,\mathbb{Z})$$

associated to this decomposition.

For odd degree cohomology groups, we can write

$$H^{2k-1}(X,\mathbb{C}) = F^k H^{2k-1}(X) \oplus \overline{F^k H^{2k-1}(X)}.$$

The the natural map

$$H^{2k-1}(X,\mathbb{R}) \to H^{2k-1}(X,\mathbb{C})/F^k H^{2k-1}(X)$$

is an isomorphism of real vector spaces. Moreover

$$\operatorname{rk} H^{2k-1}(X,\mathbb{Z}) = \dim H^{2k-1}(X,\mathbb{R}),$$

which means that $H^{2k-1}(X,\mathbb{Z})$ gives a full-rank lattice

$$L_k := \operatorname{im}\left(H^{2k-1}(X,\mathbb{Z}) \to H^{2k-1}(X,\mathbb{C})/F^k H^{2k-1}(X)\right)$$

in the complex vector space

$$V_k := H^{2k-1}(X,\mathbb{C})/F^k H^{2k-1}(X).$$

We then define the *k-th intermediate Jacobian* to be the quotient

$$\begin{aligned} J^k(X) &:= V_k/L_k \\ &\simeq H^{2k-1}(X,\mathbb{C})/\left(F^k H^{2k-1}(X) \oplus H^{2k-1}(X,\mathbb{Z})\right). \end{aligned}$$

Note that for $k = 1$,

$$J^1(X) = Pic^0(X),$$

while for $k = 2n-1$

$$J^{2n-1}(X) = Alb(X).$$

Therefore the intermediate Jacobian of the cubic threefold V is given by

$$J(V) = \left(H^{1,2}(V) + H^{0,3}(V)\right)/H^3(V,\mathbb{Z}).$$

In Clemens and Griffiths it is shown that $H^{1,2}(V) + H^{0,3}(V)$ admits a non-degenerate Hermitian form. It corresponds to the cup product mapping on $H^3(V,\mathbb{Z})$ which is non-degenerate by Poincaré duality. Therefore $J(V)$ is a principally polarised abelian variety.

4. Fano Varieties of Lines

In this section we study the *Fano variety of lines* $F(X)$ which parametrises the lines contained in a smooth n-dimensional cubic hypersurface $X \subset \mathbb{P}^{n+1}$. It can be shown that it is always smooth and of dimension $2(n-2)$.

We look at the local structure of $F(X)$: at a point $[l]$ (where $l \subset X$ is a line), the Zariski tangent space to $F(X)$ is $H^0(l, N_{L/X})$. Locally, at the point $[l]$, $F(X)$ is defined by $h^1(l, N_{l/X})$ equations in a smooth scheme of dimension $h^0(l, N_{l/X})$. From Riemann-Roch we have

$$\dim_{[l]} F(X) \geq \chi(l, N_{l/X}) = \deg(N_{l/X}) + \operatorname{rk}(N_{l/X}).$$

Consider the exact sequence

$$0 \to N_{l/X} \to N_{l/\mathbb{P}^{n+1}} \to N_{l/\mathbb{P}^{n+1}}|_l \to 0,$$

where

$$N_{l/\mathbb{P}^{n+1}} = \mathcal{O}_l(1)^{\oplus n} \text{ and } N_{X/\mathbb{P}^{n+1}}|_l = \mathcal{O}_l(3).$$

We can then easily compute

$$\begin{aligned} \deg N_{l/X} &= \deg \mathcal{O}_l(1)^{\oplus n} - \deg \mathcal{O}_l(3) = n - 3, \\ \operatorname{rk} N_{l/X} &= \operatorname{rk} \mathcal{O}_l(1)^{\oplus n} - \operatorname{rk} \mathcal{O}_l(3) = n - 1. \end{aligned}$$

Therefore $N_{l/X}$ is of form

$$\mathcal{O}_l(a_1) \oplus \ldots \oplus \mathcal{O}_l(a_{n-1}),$$

where $a_1 \leq \ldots \leq a_{n-1} \leq 1$ and $a_1 + \ldots + a_{n-1} = n - 3$. Therefore

$$a_1 = (n-3) - a_2 - \ldots - a_{n-1} \geq -1,$$

from which we conclude that $H^1(l, N_{l/X}) = 0$. Hence $F(X)$ is smooth of dimension

$$\chi(l, N_{l/X}) = (n-3) + (n-1) = 2(n-2).$$

In the case of a cubic threefold V, $F(V)$ is two-dimensional, i.e. we have a two-dimensional family of lines. We can obtain this result in a less sophisticated way: take a general hyperplane H in $\mathbb{P}^4$. Then the intersection

$$S = H \cap V$$

is a smooth cubic surface in $\mathbb{P}^3$ and it therefore contains 27 lines. Consider the following incidence correspondence

$$\Sigma = \{(l, H) \mid l \subset S = H \cap V\} \subset G(1,3) \times \mathbb{P}^4,$$

and the projection maps $p : \Sigma \to G(1,3)$ and $q : \Sigma \to \mathbb{P}^4$. Since the fibre of q is finite, the fibre of p is two-dimensional and both p and q are surjective finite morphisms we conclude that V indeed contains a two-dimensional family of lines.

Finally, by looking at the normal bundles $N_{l/X}$ we can distinguish two types of lines in X:

- if $N_{l/X} \simeq O_l \oplus O_l \oplus O_l(1)^{\oplus(n-3)}$, then l is called a *line of the first type*; these are the lines that go through the general point.
- if $N_{l/X} \simeq O_l(-1) \oplus O_l(1)^{\oplus(n-2)}$, then l is called a *line of the second type*; these lines occur in codimension $(n-2)$ in $F(X)$.

In order to prove the unirationality of the cubic threefold V we make use of the fact that we actually have plenty of lines in V. Pick a line $L_0 \in F(V)$ and define the set

$$W := \{(p, L) \mid p \in L_0 \text{ and } L_0 \text{ is tangent to } V \text{ at } p\}.$$

We have the map $W \to L_0$ given by $(p, L) \mapsto p$ which makes W into a $\mathbb{P}^2$-bundle over L_0. Moreover since the tangent bundle is locally trivial,

$$\bigcup_{p \in L_0} \mathbb{P}(T_p V) \longleftrightarrow L_0 \times \mathbb{P}^2 \longleftrightarrow \mathbb{P}^3 .$$

Therefore W is rational. Now take the other intersection point $q = L \cap V$ with $q \neq p$. This defines a mapping $\varphi : W \to V$ given by $(p, L) \mapsto q$. This map is not defined if $L \in F(V)$, which means it is not defined on a closed set of dimension at most one. Thus φ is a rational map. Moreover φ is also 2-1: for a general point $x \in V$, consider the plane Λ spanned by x and L. Then $\Lambda \cap V = L_0 \cup C$, where C is the residual conic curve. The two lines connecting the two points of intersection of L_0 and C to x are the corresponding tangent lines to V. See picture.

This argument can be generalised to prove the unirationality of any cubic hypersurface. We give a very rough sketch. Let X be a cubic hypersurface in $\mathbb{P}^n$ and let $\Theta \simeq \mathbb{P}^{l+1}$ be an $(l+1)$-plane. Note that the planes Θ are parametrised by $\mathbb{P}^{n-l-1}$. Consider now the intersection

$$\Theta \cap X = \Gamma \cup X_\Theta,$$

where $\Gamma \simeq \mathbb{P}^l$ and X_Θ is an irreducible quadric hypersurface. But this defines a regular map

$$\pi : X \setminus \Gamma \to \mathbb{P}^{n-l-1}$$

with fibre $\pi^{-1}(\Theta) = X$. Blowing X up along Γ yields a conic bundle

$$\tilde{X} = Bl_\Gamma(X) \to \mathbb{P}^{n-l-1} . \tag{3}$$

We now use the fact that if $E \to B$ is a family of (generically irreducible) quadric hypersurfaces over a rational base B and there exists a rational section $B \to E$ then E is rational to conclude that X is unirational. To do this, apply a base change in (3) to a family H of pointed quadrics. More precisely, let

$$I = \{(\Theta, p) \in \mathbb{P}^{n-l-1} \times \Gamma \mid p \in X_\Theta\}$$

and

$$H = \tilde{X} \times_{\mathbb{P}^{n-l-1}} I.$$

We then have the commutative diagram

$$\begin{array}{ccc} H & \longrightarrow & \tilde{X} \\ \sigma \uparrow \downarrow \rho & & \downarrow \pi \\ I & \longrightarrow & \mathbb{P}^{n-l-1} \end{array}$$

where $\sigma : I \to H$ given by $(\Theta, p) \mapsto (p, \Theta, p)$ is a rational section. One can check that I is rational (as a family of planes) which implies that H is also. Therefore $\tilde{X}$ is dominated by a rational variety, hence $\tilde{X}$ is unirational and so is X.

In fact in [HMP98] it is shown that for any $d \geq 3$ there exists a natural number $N(d)$ such that for any $n \geq N(d)$, any smooth hypersurface of degree d in $\mathbb{P}^n$ is unirational. Moreover the Fano varieties of k-planes all have expected dimension. Interestingly the results also hold for singular hypersurfaces, as long as the singularities occur in high enough codimension.

So far we have discussed only fields of characteristic zero. In the case of an arbitrary field k, Kollár has shown that any smooth cubic hypersurface is unirational if and only if it has a k-point.

5. The Abel-Jacobi Map

Let X be a projective variety of dimension n. Recall our definition of the k-th intermediate Jacobian of X:

$$J^k(X) = H^{2k-1}(X, \mathbb{C}) / \left(F^k H^{2k-1}(X) \oplus H^{2k-1}(X, \mathbb{Z}) \right).$$

After a few manipulations this definition can be equivalently rewritten as

$$J^k(X) = F^{n-k+1} H^{2n-2k+1}(X, \mathbb{C})^\vee / H_{2n-2k+1}(X, \mathbb{Z}).$$

Consider now the cycle class map

$$\begin{aligned} \mathcal{Z}^k(X) &\to H^{2k}(X, \mathbb{Z}) \\ Z &\mapsto [Z], \end{aligned}$$

where $\mathcal{Z}^k$ has as elements the cycles of codimension k in X.

Then the *Abel-Jacobi map* is defined as

$$\begin{aligned}\mathcal{Z}^k_{hom}(X) &\to J^k(X)\\ Z &\mapsto \left(\omega \mapsto \int_{C_Z} \omega\right),\end{aligned}$$

where $\mathcal{Z}^k_{hom}$ are the null-homologous cycles,i.e.

$$\mathcal{Z}^k_{hom} = \ker\left(\mathcal{Z}^k \to H^{2k}(X, \mathbb{Z})\right).$$

In other words Z is a null-homologous cycle if there exists a chain C_Z of dimension $2n - 2k + 1$ such that $\partial C_Z = Z$. Integrating over C_Z gives a functional on forms of degree $2n - 2k + 1$. In fact, it only defines the functional on the whole cohomology $H^{2n-2k+1}(X, \mathbb{C})$ if it vanishes on exact forms. We choose such a piece. One can then show that

$$\left(\omega \mapsto \int_{C_Z} \omega\right) \in F^{n-k+1}H^{2n-2k+1}(X, \mathbb{C})^\vee.$$

Note that there is still an ambiguity left in the definition caused by the choice of cycle C_Z. To get rid of it, we simply take the quotient with respect to $H_{2n-2k+1}$.