Billiards, cohomological field theories, and de Jonquières divisors

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Oxford Junior String Seminar

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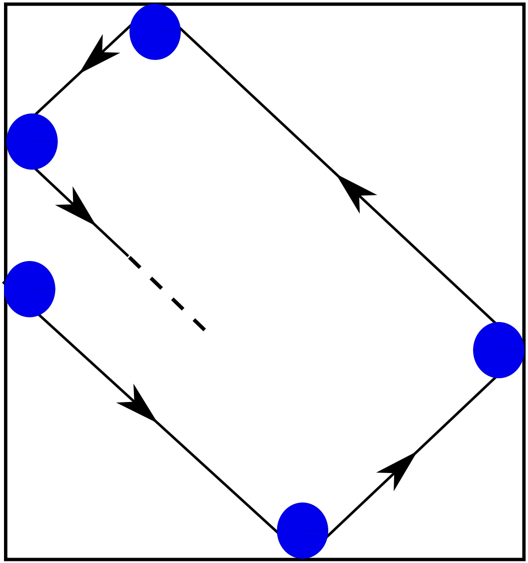
Billiards in convex polygons

- Ideal billiard ball
- Mass concentrated at one point
- No friction, no spin
- Optical rule

Billiards in convex polygons



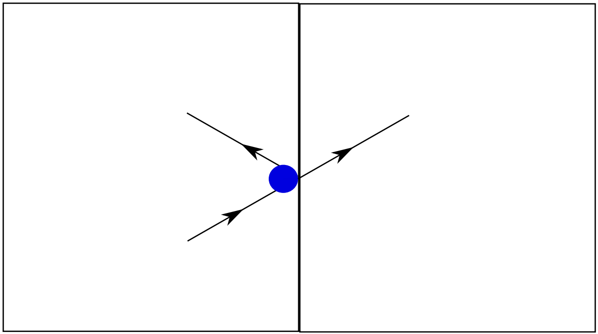
Billiards in rational polygons

Rational polygon: all angles are rational multiples of π

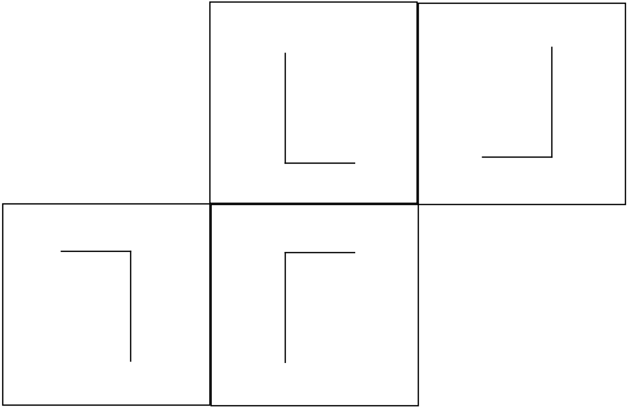
- Many tools available
- Connections with algebraic geometry, Teichmüller theory,...

Motivation: the group generated by the reflections of a rational polygon is finite

Unfolding rational polygons



Unfolding rational polygons



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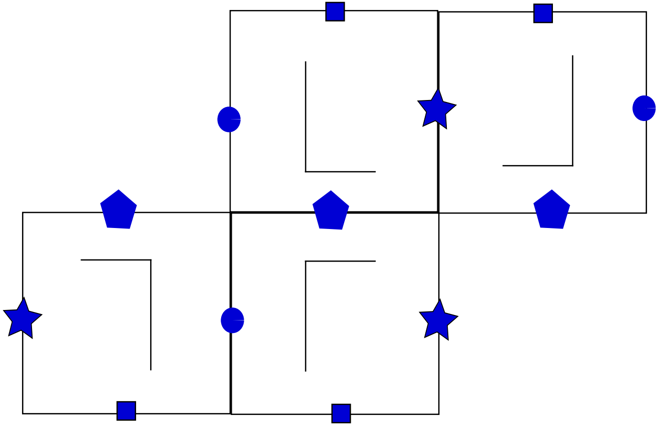

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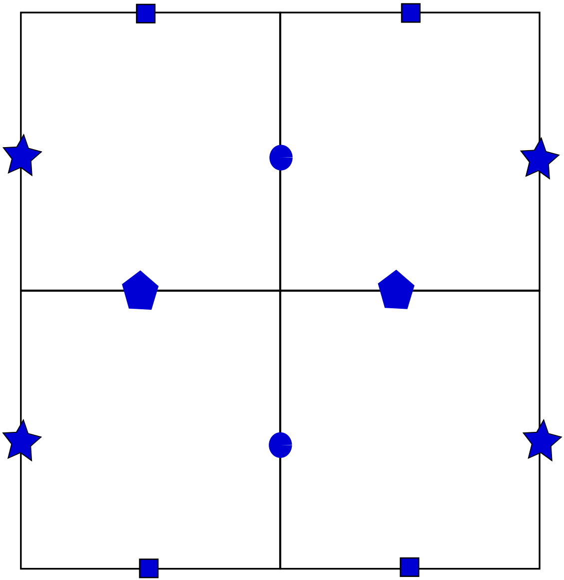

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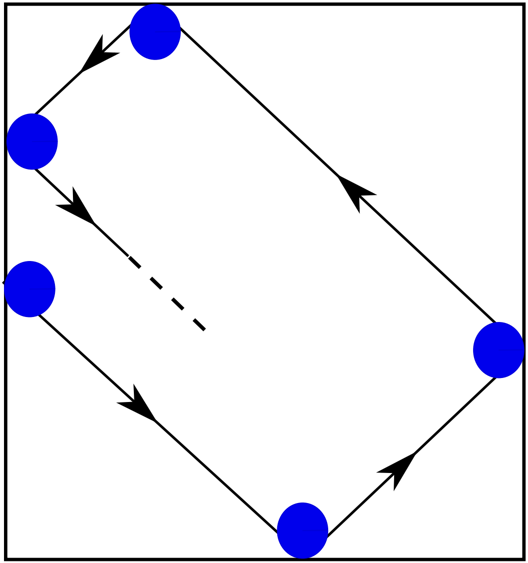

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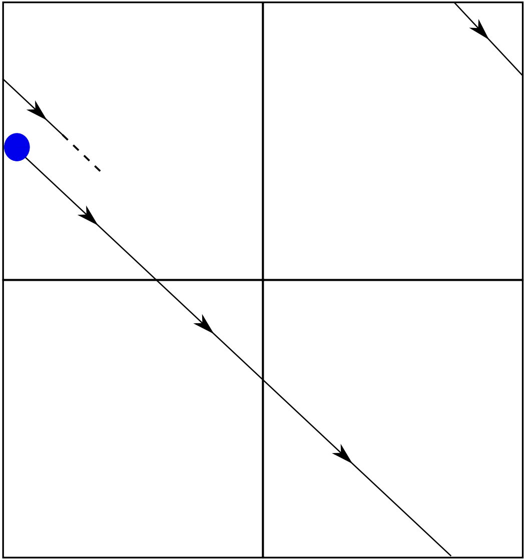


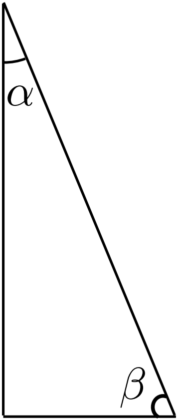
Surfaces from polygons

Glue identified edges of polygon $\Leftrightarrow$ surface with flat metric away from some (conical) singularities

Singularities arise from corners of polygon

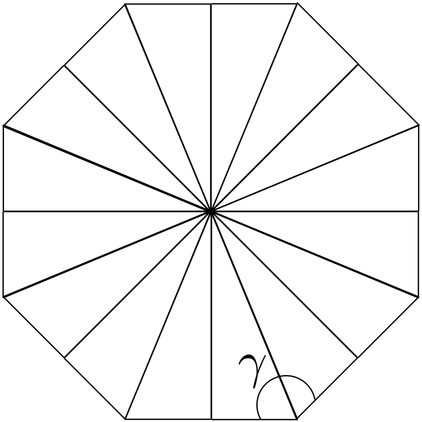
Angle around singularities is integer multiple of 2π

Surfaces from polygons



$$\alpha = \frac{\pi}{8} \text{ and } \beta = \frac{3\pi}{8}$$

Surfaces from polygons



$$\gamma = 2 \times \frac{3\pi}{8} = \frac{3\pi}{4}$$

$$8 \times \frac{3\pi}{4} = 6\pi = 3 \times 2\pi$$

Surfaces from polygons

Gauss-Bonnet type theorem:

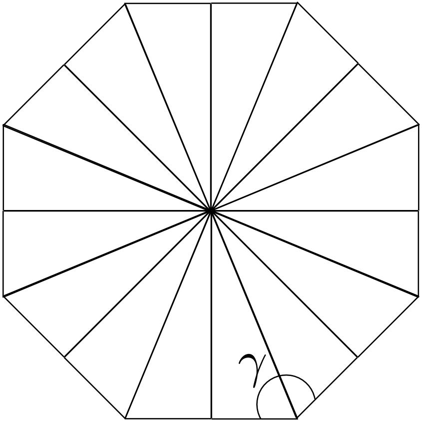
- n singularities with angle $(a_i + 1)2\pi$
- $\sum_{i=1}^{n} a_i = 2g - 2$

Take complex coordinate z on surface

$$\omega := p(z)dz$$

ω vanishes at conical singularities with order a_i

Surfaces from polygons



$$\gamma = 2 \times \frac{3\pi}{8} = \frac{3\pi}{4}$$

$$8 \times \frac{3\pi}{4} = 6\pi = (2+1)2\pi$$

$$2g - 2 = 2 \Rightarrow g = 2$$

Strata of holomorphic differentials

$\mathcal{H}_g(\mu)$

- Riemann surface C of genus g
- n conical singularities p_i with angles $(a_i + 1)2\pi$
- (or with differential ω vanishing at p_i at order a_i)
- $\mu = (a_1, \ldots, a_n)$ partition of $2g - 2$

Strata of holomorphic differentials

Kontsevich and Zorich: $\mathcal{H}_g(\mu)$ has at most three connected components

Lelièvre, Monteil, Weiss: there are at most finitely many points y on the polygon not reachable by a billiard trajectory from an arbitrary point x

To summarise

- Billiards in polygons
- Riemann surface of genus g with n conical singularities
- Riemann surface of genus g with differential vanishing at n points with prescribed order a_i such that

$$\sum a_i = 2g - 2$$

- Strata of differentials $\mathcal{H}_g(\mu)$

Studying $\overline{\mathcal{M}}_{g,n}$

$\mathcal{M}_{g,n}$ = moduli space of Riemann surfaces of genus g with n marked points

$$(C; p_1, \dots, p_n) \in \mathcal{M}_{g,n}$$

- $\dim \mathcal{M}_{g,n} = 3g - 3 + n$
- compactification $\overline{\mathcal{M}}_{g,n}$

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Question

What is the cohomology of $\overline{\mathcal{M}}_{g,n}$?

Studying $\overline{\mathcal{M}}_{g,n}$

$$\mathcal{H}_g(\mu) \subset \mathcal{M}_{g,n}$$

Take closure: $\overline{\mathcal{H}}_g(\mu) \subset \overline{\mathcal{M}}_{g,n}$

Question

What is the fundamental class $[\overline{\mathcal{H}}_g(\mu)]$?

Answer

(potentially) Cohomological field theory!

$\overline{\mathcal{M}}_{g,n}$ and 2-dimensional CFT

CFT

- 2-dimensional QFT invariant under conformal transformations
- defined over compact Riemann surfaces

Stick to holomorphic side

CFT = 2-dimensional QFT covariant w.r.t. holomorphic coordinate changes

$\overline{\mathcal{M}}_{g,n}$ and 2-dimensional CFT

Infinitesimal change of holomorphic coordinate

$$z \mapsto z + \epsilon f(z)$$

Local holomorphic vector field

$$f(z)\frac{d}{dz}$$

$\overline{\mathcal{M}}_{g,n}$ and 2-dimensional CFT

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Local meromorphic vector field

$$f(z)\frac{d}{dz}$$

$$\Downarrow$$

Virasoro algebra:

$$L_n = -z^{n+1}\frac{d}{dz} \Rightarrow [L_n, L_m] = (m-n)L_{m+n}, n \in \mathbb{Z}$$

etc...

$\overline{\mathcal{M}}_{g,n}$ and 2-dimensional CFT

Local meromorphic vector field

$$f(z)\frac{d}{dz}$$

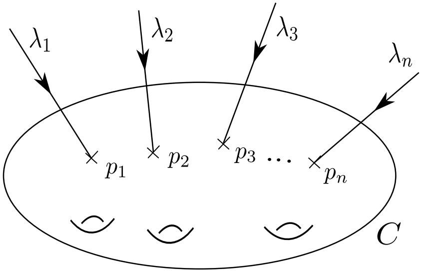
$$\Updownarrow$$

Infinitesimal deformation of complex structure

$$\Updownarrow$$

Infinitesimal deformation of an algebraic curve

$\overline{\mathcal{M}}_{g,n}$ and 2-dimensional CFT

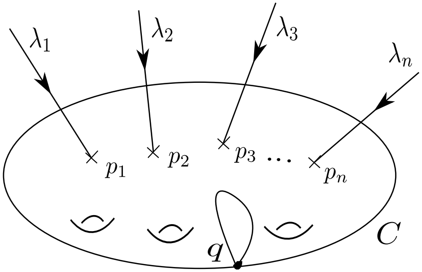


$(C; p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$

$\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ representation labels

$V_{\vec{\lambda}}(C; p_1, \ldots, p_n)$ space of conformal blocks

$\overline{\mathcal{M}}_{g,n}$ and 2-dimensional CFT



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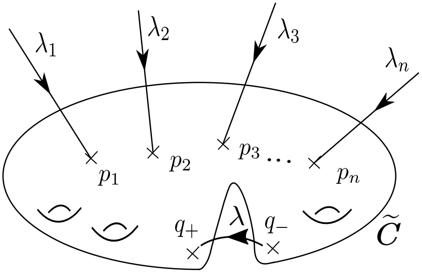

$(\widetilde{C}; p_1, \ldots, p_n, q_+, q_-) \in \overline{\mathcal{M}}_{g,n+2}$

$\vec{\lambda} = (\lambda_1, \ldots, \lambda_n, \lambda, \lambda^\dagger)$ representation labels

$V_{\vec{\lambda}}(\widetilde{C}; p_1, \ldots, p_n, q_+, q_-)$ space of conformal blocks

$\overline{\mathcal{M}}_{g,n}$ and 2-dimensional CFT

Verlinde bundle

$$\mathcal{V}_{\vec{\lambda}} \to \overline{\mathcal{M}}_{g,n}$$

Each fibre is given by space of conformal blocks

$$V_{\vec{\lambda}}(C; p_1, \ldots, p_n) \to (C; p_1, \ldots, p_n)$$

$\overline{\mathcal{M}}_{g,n}$ and CohFT

The characters $ch(\mathcal{V}_{\vec{\lambda}})$ define a CohFT on $\overline{\mathcal{M}}_{g,n}$!

$\overline{\mathcal{M}}_{g,n}$ and CohFT

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A CohFT

- a vector space of fields U
- a non-degenerate pairing η
- a distinguished vector $\mathbf{1} \in U$
- a family of correlators

$$\Omega_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \otimes (U^*)^{\otimes n}$$

satisfying gluing...

$\overline{\mathcal{M}}_{g,n}$ and CohFT

Quantum multiplication $*$ on U

$$\eta(v_1 * v_2, v_3) = \Omega_{0,3}(v_1 \otimes v_2 \otimes v_3) \in \mathbb{Q}$$

$(U, *)$ Frobenius algebra of the CohFT

Teleman: classification of all CohFT with semisimple Frobenius algebra

$\overline{\mathcal{M}}_{g,n}$ and CohFT

$\mathcal{H}_g(\mu) = \{(C; p_1, \ldots, p_n)$ such that differential vanishes at p_i to order $a_i\}$

$[\overline{\mathcal{H}}_g(\mu)] =?$

Maybe $[\overline{\mathcal{H}}_g(\mu)]$ is one of the $\Omega_{g,n}$

$\overline{\mathcal{H}}_g(\mu)$ and CohFT

$[\overline{\mathcal{H}}_g(\mu)]$ is not a CohFT class!

Conjecture (Pandharipande, Pixton, Zvonkine): it is related to one

Witten R-spin class

$$W^R_{g,\mu} \in H^{2g-2}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

de Jonquières' divisors

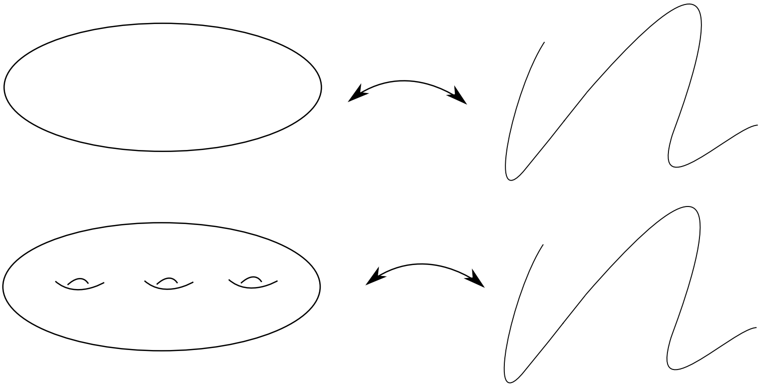
$\mathcal{H}_g(\mu) = \{(C; p_1, \ldots, p_n)$ such that differential vanishes at p_i to order $a_i\}$

$=\{(C; p_1, \ldots, p_n)$ such that Ω^1_C has section that vanishes at p_i to order $a_i\}$

What if $L \neq \Omega^1_C$?

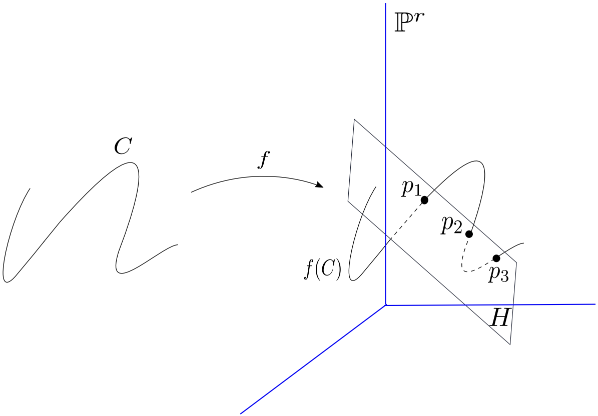
Ernest de Jonquières

Disclaimer



de Jonquières' multitangency conditions

C smooth, genus g
$f : C \to \mathbb{P}^r$ non-degenerate
Degree of $f = \#\{f(C) \cap H\} =: d$



$$d = 3$$

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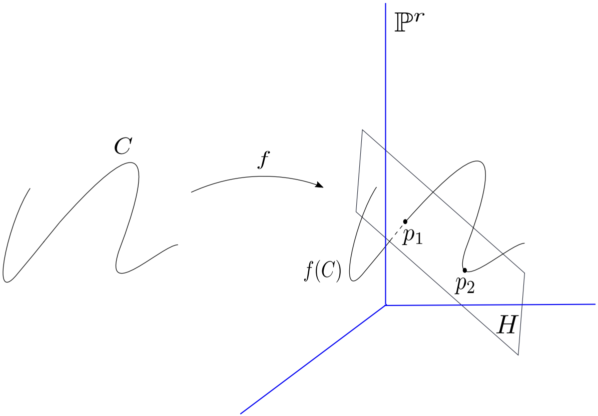

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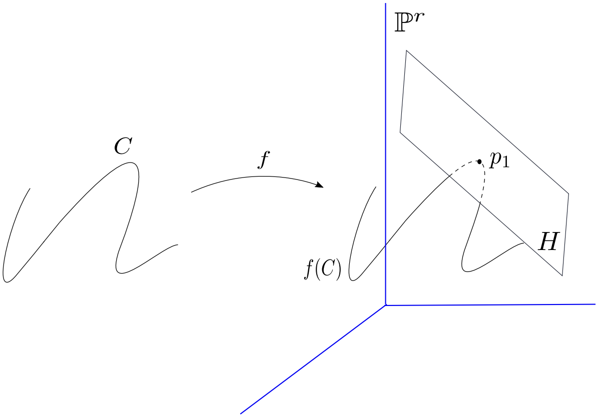

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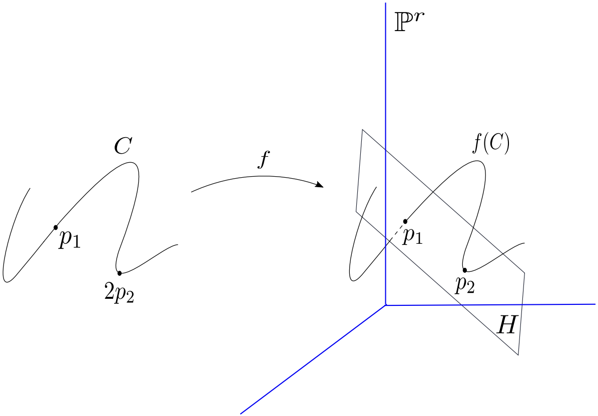

$$f^{-1}\{f(C) \cap H\} = p_1 + 2p_2$$

de Jonquières' multitangency conditions

de Jonquières counts the number of pairs (p_1, p_2) such that there exists a hyperplane $H \subset \mathbb{P}^r$ with

$$f^{-1}\{f(C) \cap H\} = p_1 + 2p_2$$

de Jonquières' multitangency conditions

de Jonquières (and Mattuck, Macdonald) count the n-tuples

$$(p_1, \ldots, p_n)$$

such that there exists a hyperplane $H \subset \mathbb{P}^r$ with

$$f^{-1}\{f(C) \cap H\} = a_1 p_1 + \ldots + a_n p_n$$

where

$$a_1 + \ldots + a_n = d$$

de Jonquières' multitangency conditions

The (virtual) de Jonquières numbers are the coefficients of

$$t_1 \cdot \ldots \cdot t_n$$

in

$$(1 + a_1^2 t_1 + \ldots + a_n^2 t_n)^g (1 + a_1 t_1 + \ldots + a_n t_n)^{d-r-g}$$

Constructing the moduli space

An embedding $f : C \to \mathbb{P}^r$ of degree d is given by

A pair (L, V)

- a line bundle L of degree d on C
- an $(r+1)$-dimensional vector space V of sections of L

Constructing the moduli space

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A pair (L, V)

- a line bundle L of degree d on C
- an $(r+1)$-dimensional vector space V of sections of L

$$\text{Choose } (\sigma_0, \dots, \sigma_r) \text{ basis of } V$$
$$\Downarrow$$
$$f : C \to \mathbb{P}^r$$
$$p \mapsto [\sigma_0(p) : \dots : \sigma_r(p)]$$

Constructing the moduli space

Space of all divisors of degree d on C

$$C_d = \underbrace{C \times \ldots \times C}_{d \text{ times}} / S_d$$

For example

$$p_1 + 2p_2 \in C_3$$

Constructing the moduli space

Space of all divisors of degree d on C

$$C_d = \underbrace{C \times \ldots \times C}_{d \text{ times}} / S_d$$

For example

$$p_1 + 2p_2 \in C_3$$

We define de Jonquières divisors

$$p_1 + \ldots + p_n \in C_n$$

such that

$$f^{-1}\{f(C) \cap H\} = a_1 p_1 + \ldots + a_n p_n$$

Constructing the moduli space

$D = p_1 + \ldots + p_n$ is de Jonquières divisor

$$\Updownarrow$$

there exists a section σ whose zeros are

$$a_1 p_1 + \ldots + a_n p_n$$

To summarise...

- Fix curve C of genus g
- Fix embedding given by (L, V)
- $C_n :=$ space of divisors of degree n
- Defined de Jonquières divisors via multitangency conditions

$$DJ_n = \{D \in C_n \mid D \text{ de Jonquières divisor for } L\}$$

Analysing the moduli space

$$DJ_n = \{D \in C_n \mid D \text{ de Jonquières divisor for } L\}$$

- determinantal subvariety of C_n (degeneracy locus)
- if $DJ_n \neq \emptyset$, then $\dim DJ_n \geq n - d + r$

Analysing the moduli space

Relevant questions

- $n - d + r < 0 \Rightarrow$ non-existence of de Jonquières divisors
- $n - d + r \geq 0 \Rightarrow$ existence of de Jonquières divisors
- $n - d + r = 0 \Rightarrow$ finite number of de Jonquières divisors
- $\dim DJ_n = n - d + r$

Taking a variational perspective

- Allow C to vary in $\mathcal{M}_{g,n}$
- Vary the de Jonquières structure with it

To summarise...

- Billiards in rational polygons
- Obtained Riemann surface with differential vanishing at marked points
- Looked at strata $\mathcal{H}_g(\mu)$ using CohFT
- Generalised to de Jonquières divisors