From de Jonquières' counts to cohomological field theories

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Motivation

Enumerative geometry is an old subject whose goal is to count the number of geometric objects satisfying some given conditions. As his 15th problem, Hilbert asked that a rigorous foundation for the field be established. In this project we verify the validity of a classical enumerative result, namely de Jonquières' formula counting the number of divisors with multiplicities on a curve. Moreover we hope to understand our problem from a different perspective, namely that of cohomological field theories.

Objects of study

Let C be a smooth curve of genus g and L a complete linear series of degree d and dimension r. Fix $n \in \mathbb{N}$. Then $D = p_1 + \cdots + p_n$ is a **de Jonquières divisor of length n** if

$$L = \mathcal{O}(a_1 p_1 + \cdots + a_n p_n)$$

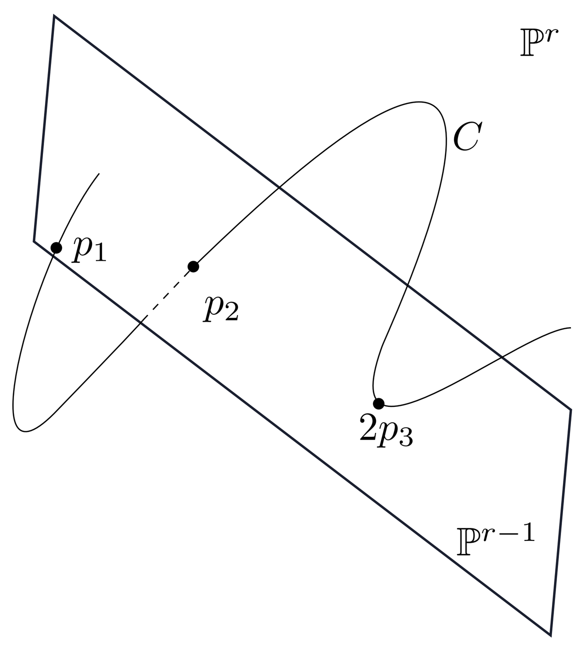
for some $a_i \in \mathbb{N}$ with $\sum a_i = d$. Denote the space of all such divisors by $DJ_n^{r,d}(C, L)$. This is a degeneracy locus in C_n and has expected dimension

$$\exp\dim DJ_n^{r,d}(C, L) = n - d + r.$$

De Jonquières' formula [1] states that, if we expect there to be a finite number of de Jonquières divisors of length n, then this number is given by the coefficient of the monomial $t_1 \cdots t_n$ in

$$(1 + a_1^2 t_1 + \cdots + a_n^2 t_n)^g (1 + a_1 t_1 + \cdots + a_n t_n)^{d-r-g}.$$

Viewing C as embedded in $\mathbb{P}^r$ by L, the formula counts the number of hyperplanes meeting C with prescribed multiplicities a_i at some points p_i.



Hyperplane section of C with de Jonquières divisor $p_1 + p_2 + 2p_3$

Geometric interpretation for the counts:

- If $r = 1$ we recover the number of ramification points of a Hurwitz cover,
- If $r = 2$ we obtain the number of bitangent lines to a plane curve,
- If $L = K_C$ and $a_i = 2$ for all i, we recover the number of odd theta characteristics of C. In this case, if we allow C and the points $p_1, \ldots, p_n$ to vary in $\mathcal{M}_{g,n}$, and we vary the de Jonquières structure with them, we recover the strata of holomorphic differentials, studied for example in [3] and [4].

Goals

First question: What does $DJ_n(C, L)$ look like for a fixed curve C? We must check that it is non-empty, smooth, reduced and of expected dimension $n - d + r$ in order to validate de Jonquières' counts.

Long-term aim: Consider the basic object to be $(C; p_1, \ldots, p_n)$ and allow it to vary in $\mathcal{M}_{g,n}$. Fix also r and d positive integers. We are interested in the space

$\mathcal{DJ}_n^{r,d} = \{(C; p_1, \ldots, p_n)$ such that C has an embedding in $\mathbb{P}^r$ of degree d that admits the de Jonquières divisor $p_1 + \cdots + p_n\} \subset \mathcal{M}_{g,n}$.

We would like to study its closure in $\overline{\mathcal{M}}_{g,n}$ and in particular, we would like to know if it is (related to) a CohFT class in $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.

Results achieved

Expected dimension: A tangent space computation yields the fact that

$$\dim DJ_n^{r,d}(C, L) = n - d + r.$$

Existence and non-existence result: Let C be a general curve and L a general linear series of degree d and dimension r. Then

- if $n - d + r < 0$ then $DJ_n^{r,d}(C, L) = \emptyset$,
- if $n - d + r \geq 0$ then $DJ_n^{r,d}(C, L) \neq \emptyset$.

In the case $r = 1$ the result follows from a deformation theoretic argument à la Kodaira–Spencer. For $r \geq 2$, the proof is based on an induction on degree, genus and dimension of embeddings of certain nodal curves using constructions by Caporaso [2].

Work in progress: The next step is to find a suitable compactification of $\mathcal{DJ}_n^{r,d}$ in $\overline{\mathcal{M}}_{g,n}$.

References

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[3] G. Farkas and R. Pandharipande. "The moduli space of twisted canonical divisors". arXiv:1508.07940 [math.AG].

[4] A. Polishchuk. "Moduli spaces of curves with effective r-spin structures". *Gromov-Witten theory of spin curves and orbifolds*. Vol. 403. Contemporary Mathematics. American Mathematical Society, 2006, pp. 1–20.