Exercise Sheet 2
April 25, 2024

Due on May 6 *before* the exercise session.

Exercise 1 (4 points, 1 for each part). Let $X \subseteq \mathbb{R}$ and let

$$\Gamma(X) = \{z \in X : z \text{ is not an isolated point in } X\}.$$

(Recall that z is an isolated point of X if there is an open set U such that $z \in U$ and $U \cap X = \{z\}$.) For an ordinal α, we inductively define $\Gamma^\alpha(X)$ as follows:

(i) $\Gamma^0(X) = X$;

(ii) $\Gamma^{\alpha+1}(X) = \Gamma(\Gamma^\alpha(X))$;

(iii) if α is a limit ordinal, $\Gamma^\alpha(X) = \bigcap_{\beta<\alpha} \Gamma^\beta(X)$

Show the following:

1. Show (by transfinite induction) that if C is closed then $\Gamma^\alpha(C)$ is closed for all ordinals α.
2. Show that there is some ordinal $\delta < \omega_1$ such that for all $\alpha > \delta$, $\Gamma^\alpha(C) = \Gamma^\delta(C)$. (Hint: Use the fact that the standard topology on $\mathbb{R}$ has a countable basis.) Let $\Gamma^\delta(C)$ be denoted $\Gamma^\infty(C)$ and show that $\Gamma^\infty(C)$ has no isolated points.
3. Show that $\Gamma^\infty(C)$ has cardinality $2^{\aleph_0}$ if it is nonempty. (Show that there is an injection from $\{f | f : \mathbb{N} \to \{0,1\}\}$ into $\Gamma^\infty(C)$. Note that if U is a neighborhood of $\Gamma^\infty(C)$ then there are $x \neq y$ both in U. This way one can build sequences in C whose limits map to distinct points in C.)
4. Conclude that if $C \subseteq \mathbb{R}$ is closed, then C is the union of an at most countable set and a perfect set. Hence C is either at most countable or has cardinality $2^{\aleph_0}$.

Exercise 2 (2 points). Let S_α be as defined in the limit case of the proof of Theorem 1.22 of the course notes, i.e. $S_\alpha = \bigcup_{n\in\mathbb{N}}(n + f[S_n])$. (Let $S_0 = g^{-1}_{\beta_0}[\beta_0]$ for exactness.) Show that $(S_\alpha, <_\mathbb{R})$ is order-isomorphic to $(\alpha, \in)$.

Exercise 3 (2 points). Prove that the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ has cardinality $2^{\aleph_0}$ and that the set of all functions $f : \mathbb{R} \to \mathbb{R}$ has cardinality $2^{2^{\aleph_0}}$.