Exercise Sheet 4
16. Mai 2024

Due on May 27 before the exercise session. 2 points per exercise.

Here we will prove Sierpinski's theorem, concluding with Exercise 4.

As in the proof of the Dushnik-Miller Theorem $\langle g_\alpha : \alpha < 2^{\aleph_0} \rangle$ enumerate all order-preserving functions $\mathbb{R} \to \mathbb{R}$ other than the identity.

Exercise 1. Construct sets $\langle x_\alpha : \alpha < 2^{\aleph_0} \rangle$ and $\langle y_\alpha : \alpha < 2^{\aleph_0} \rangle$ by transfinite recursion as follows:

(i) Show that there is some x_0 such that $g_0(x_0) \neq x_0$ and there is some y_0 such that $y_0 \neq x_0$, $y_0 \neq g_0(x_0)$, $g_0(y_0) \neq x_0$, and $g_0(y_0) \neq y_0$.

(ii) Suppose that for each $\beta < \alpha$ we have chosen points x_β and y_β such that the three sets

$$\{x_\beta : \beta < \alpha\} \text{ and } \{y_\beta : \beta < \alpha\} \text{ and } (\{g_\beta(x_\beta) : \beta < \alpha\} \cup \{g_\beta(y_\beta) : \beta < \alpha\})$$

are pairwise disjoint.

Show that there is an element x_α such that for all $\beta < \alpha$, $x_\alpha \neq x_\beta$, $x_\alpha \neq y_\beta$, $x_\alpha \neq g_\beta(x_\beta)$, $x_\alpha \neq g_\beta(y_\beta)$, and $g_\alpha(x_\alpha) \neq y_\beta$, and also such that for all $\beta \leq \alpha$, $g_\alpha(x_\alpha) \neq x_\beta$.

(iii) With the setup as in (ii), having chosen x_α, show that there is a y_α such that for all $\beta \leq \alpha$ (notice the non-strict inequality!), $y_\alpha \neq x_\beta$, $y_\alpha \neq g_\beta(x_\beta)$, $g_\alpha(y_\alpha) \neq x_\beta$, $g_\alpha(y_\alpha) \neq y_\beta$, and for all $\beta < \alpha$, $y_\alpha \neq y_\beta$, $y_\alpha \neq g_\beta(y_\beta)$.

Let $E = \langle x_\alpha : \alpha < 2^{\aleph_0} \rangle$ and $F = \langle y_\alpha : \alpha < 2^{\aleph_0} \rangle$.

Exercise 2. Argue that:

(a) $E \cap F = \emptyset$ and $|E| = |F| = 2^{\aleph_0}$,

(b) E and F are both dense subsets of $\mathbb{R}$,

(c) E and F are disjoint from $\{g_\alpha(x_\alpha) : \alpha < 2^{\aleph_0}\} \cup \{g_\alpha(y_\alpha) : \alpha < 2^{\aleph_0}\}$.

(Each of these should be fairly straightforward and analogous to the proof of the Dushnik-Miller Theorem. Specifically, each point can be justified with only a line or two.)

For any subset $T \subseteq F$, let $E_T = E \cup T$.

Exercise 3. Let $S, T \subseteq F$ and suppose that $S \setminus T$ is non-empty. Then there is no order-preserving function $h : E_S \to E_T$. (Hint: Consider $g : \mathbb{R} \to \mathbb{R}$ extending h and think about why and how g is dealt with in the construction.)

Exercise 4. Show that there are $2^{\aleph_0}$-many dense subsets of $\mathbb{R}$, each of cardinality $2^{\aleph_0}$, which are pairwise non-order-isomorphic (i.e. no two of them are order-isomorphic to one another). (Hint: Use the previous exercise. The respective definitions of the sets are quite simple, so be advised not to look for anything technical.)

Hence we have a strong contrast with Cantor's theorem on dense countable linear orders without endpoints!