Exercise Sheet 8
18. Juni 2024

Due on July 1 before the exercise session.

Exercise 1 (3 points)**.** Show that for any uncountable $A \subseteq \mathbb{R}$, there is an uncountable $B \subseteq A$ such that all distances between points in B are pairwise distinct. (Hint: Use countably elementary submodels, of course. Think about how some $a \in A \setminus M$ can be used with elementarity.)

Definition 1. If X is a set, a *filter* on X is a set $F \subset P(X)$ such that:

1. $\emptyset \notin F$,
2. $\forall A, B \in F, A \cap B \in F$,
3. $A \in F \wedge A \subseteq B \Rightarrow B \in F$.

A filter U on X is an *ultrafilter* if for all nonempty $A \subseteq X$, either $A \in U$ or $(X \setminus A) \in U$.

Exercise 2. Let τ be the topology on the space of ultrafilters on $\mathbb{N}$ generated by sets of the form $B_X := \{U \text{ an ultrafilter on } \mathbb{N} : X \in U\}$ where $X \subseteq \mathbb{N}$. We let $\beta\mathbb{N}$ denote the space of ultrafilters on $\mathbb{N}$ under this topology.

Prove the following (1 point each):

1. $\beta\mathbb{N}$ is Hausdorff. (What does it mean for two ultrafilters to differ?)
2. Let $f : \mathbb{N} \to \beta\mathbb{N}$ be defined so that $f(n)$ is the principal ultrafilter given by n, i.e. $f(n) = \{X \subseteq \mathbb{N} : n \in X, X \neq \emptyset\}$. Show that $f``\mathbb{N} \subseteq \beta\mathbb{N}$ and that $f``\mathbb{N}$ is dense in $\beta\mathbb{N}$.
3. $\beta\mathbb{N}$ is a compact space. (Argue that it is sufficient to consider an open cover consisting of basic open sets. Use the fact that a collection of sets with the *finite intersection property* generates an ultrafilter, since we are using the axiom of choice.)

Exercise 3 (2 points)**.** Let $\mathbb{P}$ be the partial order consisting of finite partial functions $f : \aleph_0 \rightharpoonup \aleph_1$. We let $f \leq_{\mathbb{P}} g$ if and only if $f \supseteq g$ (this reversal is not a typo!). Show that there is a sequence of open dense sets $\vec{D} = \langle D_\alpha : \alpha < \omega_1 \rangle$ of $\mathbb{P}$ such that there is no filter that is $\vec{D}$-generic. (Hint: The poset $\mathbb{P}$ is "trying" to build something that cannot exist.)