Course Notes for Applications of Set Theory in Algebra and in Topology

Maxwell Levine

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Contents

Chapter 1

Transfinite Induction Arguments for the Real and Complex Numbers

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1.1 The Basics of Linear Orders and Transfinite Induction

1.1.1 Some Examples

Here we will establish some basic terminology.

Definition 1.1. A *partial order* is a relation $\leq$ on a set X with the following properties:

- $\forall a \in X, a \leq a$ (reflexivity),
- if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry),
- if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).

We refer to X is the *underlying set* of $\leq$.

Example 1.2. Examples of partial orders:

- Let X be the set of closed subsets of $\mathbb{R}$ under the usual Euclidean topology. For $A, B \in X$, we let $A \leq B$ if and only if $A \subseteq B$. We could just as easily let $A \leq B$ if and only if $A \supseteq B$.

- Let $X = \{0,1\}^{<\omega}$, the set of finite sequences of 0's and 1's. Let $s \leq t$ if and only if t end-extends s, i.e. if $\operatorname{dom} s = \{0,\ldots,n\}$ then $s = t \restriction \{0,\ldots,n\}$.

Definition 1.3. A *linear order* or *total order* is a relation $\leq$ on a set X that is a partial order and has the property that for all $a, b \in X$, either $a \leq b$ or $b \leq a$.

Example 1.4. Obvious examples: $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ under their natural orderings.

Example 1.5. Constructed examples:

- $\mathbb{N} + 1 \simeq \omega + 1$, i.e. let X be the underlying set where $X = \mathbb{N} \cup \{\infty\}$, we let $n < \infty$ for all $n \in \mathbb{N}$ and we let the ordering be as usual when restricted to $\mathbb{N}$.
- Let $\mathbb{R}$ be the underlying set and let $\leq$ be the usual ordering. Define $\prec$ as follows: If $x \in \mathbb{Q}$ and $y \in \mathbb{R} \setminus \mathbb{Q}$, let $x \prec y$. If $x, y \in \mathbb{Q}$ or $x, y \in \mathbb{R} \setminus \mathbb{Q}$ then let $x \prec y$ if and only if $x \leq y$.
- $\mathbb{Z}^\omega$ i.e. let the underlying set be $X = \bigcup_{n\in\mathbb{N}} \mathbb{Z} \times \{n\}$. Then if $x, y \in X$ and $x, y \in \mathbb{Z} \times \{n\}$ for some $n \in \mathbb{N}$, then let $x \prec y$ if and only if $x <_{\mathbb{Z}} y$. If $x \in \mathbb{Z} \times \{m\}$ and $y \in \mathbb{Z} \times \{n\}$ for $m \neq n$, then let $x \prec y$ if and only if $m < n$, otherwise let $y \prec x$.

Definition 1.6. Two linear orderings L, K are *isomorphic* if there is a bijection $f : L \to K$ such that $x <_L y$ if and only if $f(x) <_K f(y)$.

Do we have $\mathbb{Z} \cong \mathbb{Z}^\omega$? What about $\mathbb{Q} \cong \mathbb{Q}^\omega$?
Remark: We will always be using the axiom of choice here!

Definition 1.7. A linear order L is *dense* if $\forall x, y \in L, \exists z \in L, x <_L z <_L y$. We say that x is an *endpoint* of L if either $\forall y \in L, y \leq_L x$ or $\forall y \in L, x \leq_L y$.

Theorem 1.8. *All countable dense linear orders without endpoints are isomorphic.*

Proof. Let $(A, \leq)$ and $(B, \leq)$ be countable linear orders without endpoints. Let $A = \langle a_n : n \in \mathbb{N}\rangle$ and $B = \langle b_n : n \in \mathbb{N}\rangle$ be one-to-one enumerations. By induction on $n \in \mathbb{N}$, we will define one-to-one sequences $\langle p_n : n \in \mathbb{N}\rangle = A$ and $\langle q_n : n \in \mathbb{N}\rangle = B$ such that the map $f : q_n \mapsto p_n$ is an isomorphism between $(A, \leq)$ and $(B, \leq)$.

Let $p_0 = a_0$ and $q_0 = b_0$. This takes care of the base case. Now suppose that for some $n \in \mathbb{N}$ we have defined p_m and q_m for all $m \leq n$. Then there are two subcases to consider.

If n is even, let p_{n+1} be the element $a_\ell \in A \backslash \{p_0, \ldots, p_n\}$ with the smallest index (i.e. the smallest value of ℓ, chosen for specificity). We distinguish three sub-subcases:

1. For all $m \leq n$, $p_{n+1} < p_m$.
2. For all $m \leq n$, $p_{n+1} > p_m$.
3. Not (1) or (2).

In case *(1)* choose $q_{n+1} \in B$ such that $q_{n+1} < q_m$ for all $m \leq n$. This is possible since $(B, \leq)$ has no endpoints. Similarly, in case *(2)* we can choose $q_{n+1} \in B$ such that $q_{n+1} > q_m$ for all $m \leq n$. In case *(3)* there are $m_0, m_1 \leq m$ such that p_{m_0} is the $\leq$-maximal element of $\{p_m : m \leq n, p_m < p_{n+1}\}$ and p_{m_1} is the $\leq$-minimal element of $\{p_m : m \leq n, p_m > p_{n+1}\}$. Choose $q_{n+1} \in B$ such that $q_{m_0} < q_{n+1} < q_{m_1}$. This is possible since $(B, \leq)$ is a dense linear order.

If n is odd, we choose q_{n+1} to be the $b_\ell \in B \setminus \{q_0, \ldots, q_n\}$ with the smallest index. We choose p_{n+1} exactly as we chose q_{n+1} in the case of even n, with the roles of A and B, a and b, and p and q reversed.

This finishes the definition of the sequences $\langle p_n : n \in \mathbb{N} \rangle$ and $\langle q_n : n \in \mathbb{N} \rangle$. We now that $f : A \to B$ is an isomorphism because we show at each step that it is order-preserving, and we have assured that $\operatorname{dom} f = A$ and $\operatorname{ran} f = B$. □

Where is the "without endpoints" hypothesis used? Where is "countable" used? Can we find a counterexample without "countable" in the hypothesis?

1.1.2 Ordinal Numbers and Transfinite Induction

Definition 1.9. Let $(L, \leq)$ be a linear ordering. Then $(L, \leq)$ is a *well-ordering* if every nonempty subset $X \subseteq L$ has a $\leq$-least element.

Example 1.10. Examples and non-examples:

- Any finite linear ordering.

- Not $\mathbb{Q}$, not $\mathbb{R}$.
- $\mathbb{N}$, $\omega + 1$.

In fact, all examples are essentially ordinals.

Definition 1.11. If L is well-ordered and $x \in L$, then $\{y \in L : y <_L x\}$ is an *initial segment* of L.

Proposition 1.12. *No well-ordered set can be isomorphic to an initial segment of itself.*

Proof. First observe that if $(L, <)$ is well-ordered and $f : L \to L$ is strictly increasing (i.e. $x < y$ implies $f(x) < f(y)$) then $f(x) \geq x$ for all $x \in L$. If $X = \{x \in L : f(x) < x\}$ is nonempty then it has a least element z. If $w = f(z)$, then $f(w) = f(f(z)) < f(z) = w$, which contradicts minimality of z.

Now if $(L, <)$ is isomorphic to an initial segment $\{x : x < u\}$ via f, then $f(u) < u$. □

Proposition 1.13. *If L_1 and L_2 are well-ordered, then exactly one of the following three cases will hold:*

1. *$L_1 \cong L_2$,*
2. *L_1 is isomorphic to an initial segment of L_2,*
3. *L_2 is isomorphic to an initial segment of L_1.*

Proof. Exercise using the previous proposition. □

Definition 1.14. We say that α is an *ordinal* if it is a set such that:

1. It is *transitive*, meaning that if $\beta \in \gamma \in \alpha$, then $\beta \in \alpha$.
2. It is well-ordered by $\in$, i.e. $\in$ is a linear order and every subset of α has a minimal element. ($\in$ and $<$ are usually used interchangably in the context of ordinals.)

A *successor* is an ordinal of the type $\alpha = \beta \cup \{\beta\} := \beta + 1$ and a *limit* ordinal takes the form $\alpha = \bigcup_{\beta \in \alpha} \beta := \sup_{\beta < \alpha} \beta$.

Example 1.15. Every natural number can be represented as an ordinal: $0 = \emptyset,\ 1 = \{\emptyset\},\ 2 = \{\emptyset, \{\emptyset\}\},\ 3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, etc. We write the set of natural numbers as the limit ordinal $\omega = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}, \dots\}$. $\omega + 1 := \omega \cup \{\omega\}$ is an infinite successor.

Ordinals are a generalization of the natural numbers, and the most important usage of ordinal numbers is in the definitions of transfinite induction:

Fact 1.16. *Suppose P is a property of ordinals α such that:*

- *$P(0)$ holds,*
- *if $P(\beta)$ holds for all $\beta < \alpha$, then $P(\alpha)$ holds.*

Then P holds for all ordinals α.

Fact 1.17 (Transfinite Recursion)**.** *Let x be a set $G : V \to V$ be a class function. Then there exists a function $F : On \to V$ such that:*

- *$F(0) = 0$,*
- *$F(\alpha + 1) = G(F(\alpha))$,*
- *$F(\beta) = \cup\{F(\alpha) | \alpha < \beta\}$ for limit β.*

Definition 1.18. A *cardinal* number is an ordinal that does not inject onto a smaller ordinal, i.e. one that is larger than all ordinals preceding it. The α^{th} infinite cardinal is denoted either ω_α or $\aleph_\alpha$. We write $\aleph_0 = \omega_0 = \omega$.

The only infinite cardinals that we will refer to explicitly in this course are ω, i.e. $\aleph_0$, and ω_1, i.e. $\aleph_1$.

Proposition 1.19. *Every vector space V has a basis.*

1.2 Simple Applications of Transfinite Induction

Definition 1.20. The *cofinality* of a limit ordinal δ is the least ordinal γ such that there exists an increasing and unbounded function $f : \gamma \to \delta$. We denote this $\mathrm{cf}(\delta) = \gamma$. We call such a function *cofinal* in δ.

Proposition 1.21. *For all ordinals δ, $\mathrm{cf}(\delta)$ is a cardinal.*

Theorem 1.22. *For all $\alpha < \omega_1$, there is a subset $A \subseteq \mathbb{R}$ such that $(A, <_{\mathbb{R}}) \cong (\alpha, \in)$.*

Proof. Let $f : \mathbb{R} \to (0,1) \cap \mathbb{R}$ be an order-preserving bijection. For example, we can let $f : x \to \tan(\pi x - \pi/2)$.

By induction on $\alpha < \omega_1$, we will construct a set S_α such that $(S_\alpha, <_{\mathbb{R}}) = (\alpha, \in)$.

If $\alpha = 0$ then we can just let $S_0 = \emptyset$ since $(\emptyset, <_{\mathbb{R}}) \cong (\emptyset, \in)$.

Suppose $\alpha = \beta + 1$ and let $S_\beta \subseteq \mathbb{R}$ be such that $g_\beta : (S_\beta, <_{\mathbb{R}}) \cong (\beta, \in)$. Then let $S_{\beta+1} = f[S_\beta] \cup \{1\} := \{f(x) : x \in S_\beta\} \cup \{1\}$. Then we can define an order-isomorphism $g_\alpha : S_\alpha \to \alpha$ be letting $g_\alpha = g_\beta \circ f^{-1}$ on $f[S_\beta]$ and $g_\alpha(1) = \beta$ (the last element of α).

Now suppose α is a limit ordinal. We know that α is countable because $\alpha < \omega_1$. Therefore it has cofinality ω. Let $\langle \beta_n : n < \omega \rangle$ be cofinal in α. Let $S^n = g_{\beta_n}^{-1}[\beta_n \setminus \beta_{n-1}]$. Then let $S_\alpha = \bigcup_{n<\omega}(n + f[S^n])$. Then $(S_\alpha, <_{\mathbb{R}}) \cong (\alpha, \in)$. □

Now we have a primordial example of a diagonalization argument.

Definition 1.23. We say that $A \subseteq \mathbb{R}$ is a *Bernstein set* if for all uncountable closed C, $C \cap A \neq \emptyset$ but $C \not\subseteq A$.

Obviously, an analogous theorem cannot hold for countable sets. Again, we will always be using AC!

Fact 1.24 (Well-Ordering Theorem)**.** *The axiom of choice* AC *is equivalent to the statement that every set can be well-ordered.*

Theorem 1.25. *Every well-ordered set is in bijection with an ordinal.*

Theorem 1.26 (Cantor-Bendixson)**.** *If $C \subseteq \mathbb{R}$ is uncountable and closed, then $|C| = |\mathbb{R}|$.*

Proof. Outlined in the homework. □

Theorem 1.27 (Bernstein). *A Bernstein set exists.*

Proof. We will use transfinite recursion. First observe that there are $2^{\aleph_0}$ There are $2^{\aleph_0}$-many closed subsets of the real numbers $\mathbb{R}$: To see this, observe first that we can equivalently argue that there are $2^{\aleph_0}$-many open subsets of $\mathbb{R}$. Let $\mathcal{B}$ be the set of open intervals of the form (q_0, q_1) where $q_0, q_1 \in \mathbb{Q}$, so we have $\mathcal{B} = \langle b_n : n < \omega \rangle$. Then observe that of $U \subseteq \mathbb{R}$, then $U = \bigcup\{b_n : b_n \subseteq U\}$, so the set of open subsets of $\mathbb{R}$ are in bijection with the set of countable sequences of elements of $\mathcal{B}$.

Furthermore, we recall the Cantor-Bendixson Theorem that was just mentioned about how all uncountable closed subsets of $\mathbb{R}$ have cardinality equal to $|\mathbb{R}|$.

Now let $\langle C_\alpha : \alpha < 2^{\aleph_0} \rangle$ be a well-ordering of the set of uncountable closed subsets of $\mathbb{R}$. Using transfinite recursion, we will define disjoint sequences $\langle x_\alpha : \alpha < 2^{\aleph_0} \rangle$ and $\langle y_\alpha : \alpha < 2^{\aleph_0} \rangle$ such that for all $\alpha < 2^{\aleph_0}$, $x_\alpha, y_\alpha \in C_\alpha$. Let $x_0, y_0 \in C_0$ be arbitrary. Now suppose we have defined $\langle x_\alpha : \alpha < \beta \rangle$ and $\langle y_\alpha : \alpha < \beta \rangle$. Since $\beta < 2^{\aleph_0}$, $C'_\beta := C_\beta \setminus (\langle x_\alpha : \alpha < \beta \rangle \cup \langle y_\alpha : \alpha < \beta \rangle)$ has cardinality $2^{\aleph_0}$, so we can choose $x_\beta, y_\beta \in C'_\beta$ such that $x_\beta \neq y_\beta$. This finishes the description of the recursion.

Then $A := \langle a_\alpha : \alpha < 2^{\aleph_0} \rangle$ is a Bernstein set. By construction, for all $\beta < 2^{\aleph_0}$, we can $a_\beta \in A \cap C_\beta$ and $b_\beta \in (\mathbb{R} \setminus A) \cap C_\beta$. □

1.3 Sierpinski's Theorem

Theorem 1.28 (Dushnik-Miller). *There exists a dense subset $E \subset \mathbb{R}$ such that the only order-preserving map $f : E \to E$ is the identity.*

Note that the statement is technically slightly stronger than it would be if we only talked about order-isomorphisms!

Lemma 1.29. *Let $g : \mathbb{R} \to \mathbb{R}$ be order-preserving.*

1. *g has countably many discontinuities, i.e. there are countably many $z \in \mathbb{R}$ such that it is not the case that* $\mathrm{glb}\{g(w) : w < z\} = g(z) = \mathrm{lub}\{g(w) : z < w\}$.

2. *g is determined by its values on a countable set H, i.e. if g' is some order-preserving functions $\mathbb{R} \to \mathbb{R}$ such that $g'(z) = g(z)$ for all $z \in H$, then $g' = g$.*

3. *There are $2^{\aleph_0}$-many order-preserving functions $\mathbb{R} \to \mathbb{R}$.*

Proof. 1. If $D \subseteq \mathbb{R}$ is a set of discontinuities, then there is an injection $F : D \to \mathbb{Q}$: For $z \in D$, let $F(z)$ be a rational number between $\text{glb}\{g(w) : w < z\}$ and $\text{lub}\{g(w) : z < w\}$. Therefore there are at most countably many discontinuities.

2. Given g, let D be the (countable by *1.*) set of discontinuities of g. Then $H = D \cup \mathbb{Q}$. Suppose that g' is order-preserving and agrees with g on H. If $z \notin D$, we want to argue that g' is continuous at z. Otherwise, because g' and g are order-preserving, we would have

$$\text{glb}\{g(w) : w < z\} = \text{glb}\{g'(w) : w < z\} < \\ < \text{lub}\{g'(w) : z < w\} = \text{lub}\{g(w) : z < w\}$$

by density of $\mathbb{Q}$, contradicting that $z \notin D$. If $z \notin D$, then $g'(z) = g(z)$ again using density of $\mathbb{Q}$.

3. By *2.* it is enough to "calculate" the value of $|\{f|f : \mathbb{Q} \to \mathbb{R}\}|$ to find an upperbound. This is $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$. To get a lower bound consider $f_r(x) = r \cdot x$ where r is any real number. □

Proposition 1.30. *If $g : \mathbb{R} \to \mathbb{R}$ is order-preserving but not equal to the identity, then there is an open interval $(a, b) \subseteq \mathbb{R}$ such that $g(x) \neq x$ for all $x \in (a, b)$.*

Proof. Otherwise we are saying that for all open intervals (a, b), there is some $x \in (a, b)$ such that $g(x) = x$. Hence the set D on which $g(x) = x$ is dense. Then we can use the order-preservingness of g and density of D to argue that g is equal to the identity. □

Proposition 1.31. *Given an interval $(a, b) \subseteq \mathbb{R}$, there is an order-preserving function $g : \mathbb{R} \to \mathbb{R}$ such that g is the identity outside of (a, b) and such that $g(x) \neq x$ for all $x \in (a, b)$.*

Proof. Let $g(x) = x$ outside of (a, b) and let

$$g(x) = \frac{(x - a)^2}{(b - a)} + a$$

inside of (a, b). One can check that g is continuous, that the derivative of g is positive inside of (a, b), and one can set the equation equal to x and solve to show that $g(x) \neq x$ for $x \in (a, b)$. □

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Proposition 1.32. *If $E \subseteq \mathbb{R}$ is dense in $\mathbb{R}$ and $f : E \to E$ is order-preserving, then there is an order-preserving $g : \mathbb{R} \to \mathbb{R}$ extending f, i.e. such that $g \restriction E = f$.*

Proof of Dushnik-Miller. Let $\langle g_\alpha : \alpha < 2^{\aleph_0} \rangle$ enumerate the order-preserving functions $\mathbb{R} \to \mathbb{R}$ except for the identity map (we can do this by Lemma 1.29). We will define $E = \langle x_\alpha : \alpha < 2^{\aleph_0} \rangle$ by induction on α.

For the base case, let x_0 be any point such that $g_0(x_0) \neq x_0$.

Now for $\alpha \in (0, 2^{\aleph_0})$, we will assume that inductive hypothesis that $x_\beta \neq x_\gamma$ for all $\beta < \gamma < \alpha$ and that $\langle x_\beta : \beta < \alpha \rangle$ is disjoint from $\langle g_\beta(x_\beta) : \beta < \alpha \rangle$.

Suppose now that we have defined $\langle x_\beta : \beta < \alpha \rangle$ for $\alpha < 2^{\aleph_0}$ and that the inductive hypothesis holds. Since g_β is not the identity map, Proposition 1.30 tells as that there are $2^{\aleph_0}$-many real numbers x such that $g(x) \neq x$. We can call this set Z. Then there are $2^{\aleph_0}$ many real numbers $x \in Z$ not equal to any x_β for $\beta < \alpha$, so we call this Z'. Then there are $2^{\aleph_0}$ many $x \in Z'$ such that $g_\beta(x) \neq g_\beta(x_\beta)$ for some $\beta < \alpha$, otherwise we violate order-preservingness. So we call Z'' the set with this property. Now we can just choose $x_\alpha \in Z''$.

Now we argue that $E := \langle x_\alpha : \alpha < 2^{\aleph_0} \rangle$ satisfies the statement of the theorem. By construction we know that there is no order-preserving $g : E \to E$ other than the identity, because if $g = g_\beta : E \to E$ were order-preserving, then $g_\beta(x_\beta) \notin E$. It is dense because if (a, b) is an interval, we can make up an order-preserving function $g_{(a,b)}$ that is the identity outside of (a, b) (by Proposition 1.31), therefore if $g_{(a,b)} = g_\beta$, then we guaranteed that $x_\beta \in (a, b)$ in the construction. □

Theorem 1.33 (Sierpinski)**.** *There is a sequence $\langle L_\alpha : \alpha < 2^{\aleph_0} \rangle$ of linear orders that are mutually non-isomorphic.*

Proof. Outlined in homework, similar to Dushnik-Miller. □

Remark 1.34. We can see that Cantor's Isomorphisms Theorem—the one stating that all countable dense linear orders without endpoints are isomorphic—does not generalize directly, because under CH it is flase.

1.4 Notions of Ramseyness

Definition 1.35. Let $[X]^k$ denote the set of k-sized subsets of a set X. We say that $f : [\mathbb{N}]^k \to m$ is a *coloring* and that a set $X \subseteq \mathbb{N}$ is *homogeneous* (with respect to f) if there is some $i < m$ such that $f(\{x_0, \dots, x_{k-1}\}) = i$ for all $\{x_0, \dots, x_{k-1}\} \in [X]^k$.

Theorem 1.36 (Ramsey)**.** *If $f : [\mathbb{N}]^k \to m$ is a coloring, then there is a set $X \subseteq \mathbb{N}$ that is homogeneous for f.*

Example 1.37. Every sequence $\langle r_n : n \in \mathbb{N}\rangle$ of real numbers has a monotonic subsequence. If $\{m, n\} \in [\mathbb{N}]^2$, let $f(\{m, n\})$ be 0 if $r_m < r_n$, 1 if $r_m = r_n$, and 2 if $r_m > r_n$. If $X \subseteq \mathbb{N}$ is homogeneous for f, then $\langle r_n : n \in \mathbb{N}\rangle$ is monotonic.

Example 1.38. Every infinite undirected graph G has either an infinite complete subgraph or an infinite disconnected subgraph: Let $\langle v_n : n \in \mathbb{N}\rangle$ enumerate the vertex set of G. For each edge $\{v_m, v_n\} \in G$ with $m \neq n$, let $f(\{m, n\}) = 0$ if there is an edge between v_m and v_n and 1 otherwise. Let $X \subseteq \mathbb{N}$ be homogeneous for f. If $f(\{m, n\}) = 0$ for all $\{m, n\} \in [\mathbb{N}]^2$, then $X \subseteq G$ is a complete subgraph and otherwise X is totally disconnected.

Proof of Ramsey's Theorem. We will prove the theorem by induction on k for $f : [X]^k \to m$ and X any countable set. If $k = 1$, then the statement follows from the Pigeonhole Principle: If $f : X \to m$, then there is some $i < m$ and some infinite $Y \subseteq X$ such that $f(n) = i$ for all $n \in Y$.

Now suppose that the theorem is true for k and we want to prove that it is true for $k + 1$.

Start with $f : [\mathbb{N}]^k \to m$. To find a homogenous set for f, we will define an increasing sequence

$$0 = a_0 < a_1 < a_2 < \dots$$

in $\mathbb{N}$ and a $\subseteq$-descending sequence of infinite subsets

$$\mathbb{N} \supseteq X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$$

using the notation: For $a \in \mathbb{N}$, define $f_a : [\mathbb{N} \setminus \{a\}]^{k-1} \to m$ by

$$f_a(\{x_0, \dots, x_{k-2}) = f(\{x_0, \dots, x_{k-2}\} \cup \{a\}).$$

Now, given a_n and X_n, let $X_{n+1} \subseteq X_n \setminus \{a_0, \dots, a_n\}$ be homogeneous for f_{a_n} and let $a_{n+1} = \min X_{n+1}$.

Let c_n be the homogeneous value of f_{a_n} on $[X_{n+1}]^{k-1}$. By the Pigeonhole Principle, there is some $c < m$ such that $\{n \in \mathbb{N} : c_n = c\}$ is infinite. Then let $X = \{a_n : c_n = c\}$. We will argue that X is homogeneous for f.

Suppose that $\{x_0, \dots, x_{k-1}\} \in [X]^k$ and that $x_0 < x_1 < \dots < x_{k-1}$. Let n be such that $x_0 = a_n$. Then $x_1, \dots, x_{k-1} \in X_n$ by construction. Therefore

$$f(\{x_0, \dots, x_{k-1}\}) = f_{a_n}(\{x_1, \dots, x_{k-1}\}) = c_n = c$$

and so we are done. □

Example 1.39. There is a function $F : [2^{\aleph_0}]^2 \to \aleph_0$ that does not even have a homogeneous set of size 3.

Theorem 1.40 (Erdős-Hajnal). *There is a function $f : \mathbb{R}^2 \to \omega$ such that if $f(\mathbf{x}) = f(\mathbf{y})$ then $\|\mathbf{x} - \mathbf{y}\| \notin \mathbb{Q}$, i.e. such that no two points with the same "color" are at a rational distance.*

Proof. For this proof, we will call a function (or "coloring") g on a subset of $\mathbb{R}^2$ "good" if they satisfy the statement of the theorem, i.e. no two points in X^2 with the same color are at a rational distance. The plan is to inductively construct f by defining it on large and larger subsets of $\mathbb{R}^2$. (For this proof let x's and y's denote elements of $\mathbb{R}^2$ and let $\|x\| = |(x_1, x_2)|$ denote the norm $\sqrt{x_1^2 + x_2^2}$.)

Say that $X \subseteq \mathbb{R}^2$ is *closed enough* if $x, y \in X$, $x \neq y$, and $\|x - z\|, \|y - z\| \in \mathbb{Q}$, then $z \in X$.

Claim. *If $X \subseteq \mathbb{R}^2$ is infinite and $W \supseteq X$ is closed enough, then there is a closed enough $X' \supseteq X$ such that $X' \subseteq W$ and $|X| = |X'|$.*

Proof of Claim. Given $x, y \in \mathbb{R}^2$ such that $x \neq y$, let

$$F(x, y) = \bigcup_{q, q' \in \mathbb{Q}} (\{z : \|x - z\| = q\} \cap \{z : \|y - z\| = q'\}).$$

For each x, y, $F(x, y)$ is a countable union of intersections of distinct circles, therefore $F(x, y)$ is countable.

Define a sequence of sets $\langle X_n : n \in \mathbb{N}\rangle$ as follows: Let $X_0 = X$. Given X_n, let $X_{n+1} = X_n \cup \bigcup\{F(x, y) : x, y \in X, x \neq y\} \subseteq W$. Then $X' := \bigcup_{n \in \mathbb{N}} X_n$ is closed enough. □

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Claim. *If X is closed enough and of infinite uncountable cardinality θ, then $X = \bigcup_{\alpha<\theta} X_\alpha$ where:*

1. *X_0 is countable,*
2. *each X_α is closed enough,*
3. *each X_α has cardinality $|\alpha|$ if α is infinite, hence $|X_\alpha| < \theta$,*
4. *$|X_{\alpha+1}| = |X_\alpha|$ for all $\alpha < \theta$,*
5. *and $\alpha < \beta$ implies $X_\alpha \subset X_\beta$.*

Proof of Claim. Argue by transfinite recursion. Let $\langle r_\alpha : \alpha < \theta\rangle = X$. Let $X^* \subset X$ be any countable set that has r_0 as an element, then the claim gives as a closed enough countable set $X_0 \subseteq X^*$. If $\alpha = \beta+1$, use the claim to find $X_\alpha \supset X_\beta$ closed enough with $|X_\alpha| = |X_\beta|$ where the containment is strict and $r_\beta \in X_\alpha$. If α is a limit then let $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$. By construction we have guaranteed that every $r \in X$ is in X_α for some α. □

We will argue by induction on $|X|$ that a closed enough subset of X has a good coloring. Since $\mathbb{R}$ is closed enough, the above claim shows that this is sufficient.

Let X be closed enough and write $X = \bigcup_{\alpha<\theta} X_\alpha$ as above. We will define a good coloring c_α on X_α by induction in a way such that $c_\alpha \restriction X_\beta = c_\beta$ for $\beta < \alpha$.

For the case X_0, observe that we can define a good coloring c_0 easily using the fact that $|X_0| = \omega$, so the set of pairs of points in X_0 is also countable.

Suppose c_β is defined for $\beta < \alpha$ and α is a limit. Then let $c_\alpha = \bigcup_{\beta<\alpha} c_\beta$.

Now let $\beta = \alpha + 1$ and suppose c_α is defined. Since $|X_{\alpha+1}| = |X_\alpha| < \theta$, there is a good coloring d on $X_{\alpha+1}$ and we can assume that d maps into the evens, i.e. $\{2n : n \in \omega\}$. Define $c_\beta = c_{\alpha+1}$ as follows: If $z \in X_\alpha$, let $c_{\alpha+1}(z) = c_\alpha(z)$. If $z \in X_{\alpha+1} \setminus X_\alpha$, observe that "closed enoughness" of X_α implies that there is at most one $y \in X_\alpha$ such that $\|z - y\| \in \mathbb{Q}$. If $z \in X_{\alpha+1} \setminus X_\alpha$ and there is *no* $y \in X_\alpha$ such that $\|z - y\| \in \mathbb{Q}$, then let $c_{\alpha+1}(z) = d(z)$. If $z \in X_{\alpha+1} \setminus X_\alpha$ and there is *one* $y \in X_\alpha$ then let $c_{\alpha+1}(z) \in \{d(z), d(z) + 1\} \setminus \{c_\alpha(y)\}$.

Then we can argue that $c_{\alpha+1}$ is good. Suppose $\|x - y\| \in \mathbb{Q}$ and for distinct $x, y \in X_{\alpha+1}$. If $y, z \in X_\alpha$, then $c_{\alpha+1}(y) = c_\alpha(y) \neq c_\alpha(z) = c_{\alpha+1}(z)$ using goodness of c_α. If $y, z \in X_{\alpha+1} \setminus X_\alpha$, then $c_{\alpha+1}(y) = d(y) \neq d(z) = c_{\alpha+1}(z)$ using goodness of d. If (WLOG) $y \in X_\alpha$ and $z \in X_{\alpha+1} \setminus X_\alpha$, then $c_{\alpha+1}(z) \neq c_\alpha(y) = c_{\alpha+1}(y)$. □

1.5 Wetzel's Problem

Question 1.41 (Wetzel)**.** *Let $\{f_\alpha : \alpha < \theta\}$ be a family of pairwise distinct analytic functions on the complex numbers such that for each $z \in \mathbb{C}$ the set of values $\{f_\alpha(z)\}$ is at most countable. (Call this statement P.) Does it follow that $\{f_\alpha : \alpha < \theta\}$ is at most countable?*

Theorem 1.42 (Erdős)**.** *If* CH *fails, then every family satisfying P is countable. If* CH *holds, there is an uncountable family satisfying P.*

Fact 1.43. *Let $\langle C_k : k \in \mathbb{N} \rangle$ be a sequence of disks in the complex plane of respective radius k. If f, g are analytic functions that agree on infinitely many points in some C_k, then $f = g$.*

Proof/Sketch/Review. Recall the *Identity Theorem*, stating that any two analytic functions f, g agree on $S \subseteq D$ where S has an accumulation point in D, then $f = g$ on D. Then if D is compact and $V \subset D$ is infinite, then D will contain an accumulation point. Up to scaling we can apply this to the C_k's. □

Proof. First assume that CH fails, i.e. $2^{\aleph_0} > \aleph_1$. We will prove the statement contrapositively. Let $\langle f_\alpha : \alpha < \omega_1 \rangle$ be an uncountable sequence of *distinct* analytic functions.

For $\alpha < \beta < \omega_1$ let

$$S(\alpha, \beta) = \{z \in \mathbb{C} : f_\alpha(z) = f_\beta(z)\}.$$

Then $\forall \alpha < \beta < \omega_1$, $S(\alpha, \beta)$ is countable: By Fact 1.43, we know that for $\alpha < \beta$, f_α and f_β can only agree on finitely many points from each C_k, so only countably many overall.

It follows that $S := \bigcup_{\alpha<\beta<\omega_1} S(\alpha, \beta)$ has cardinality ω_1. Let z_0 be an element of $\mathbb{C} \setminus S$, which exists because $\omega_1 = |S| < |\mathbb{C}| = 2^{\aleph_0}$. By definition, for all $\alpha < \beta < \omega_1$, $f_\alpha(z) \neq f_\alpha(z)$.

For the other direction we also want:

Fact 1.44. *Let* $\sum_{n=0}^{\infty} a_n(z-c)^n$ *and let* r *be equal to* $\liminf_{n<\omega} |a_n|^{-1/n}$, *i.e. the radius of convergence. If* $\|x-c\| < r$ *then the series converges.*

Now assume that CH holds. Let $D = \{p + iq : p, q \in \mathbb{Q}\}$ and observe that D is countable and dense in $\mathbb{C}$. Let $\langle z_\alpha : \alpha < \omega_1 \rangle$ enumerate CH (using that CH holds).

We will construct a family $\langle f_\beta : \beta < \omega_1 \rangle$ of *distinct* analytic functions such that $f_\beta(z_\alpha) \in D$ if $\alpha < \beta$. We can argue that this family satisfies P: If we consider $\{f_\alpha(z) : z \in \mathbb{C}\}$, we have it being equal to the union $\{f_\alpha(z_\beta) : \beta > \alpha\}$, which is countable since it is contained in D, and $\{f_\alpha(z_\beta) : \beta \leq \alpha\}$, which is just countable anyway. Hence, if we succeed in constructing such a family then we are done.

We argue by transfinite induction. Let f_0 be arbitrary. Then suppose that f_β has been defined for $\beta < \gamma$. Re-enumerate $\{f_\beta : \beta < \gamma\}$ as $\{g_n : n \in \mathbb{N}\}$ and $\{z_\alpha : \alpha < \gamma\}$ as $\{w_n : n \in \mathbb{N}\}$. We will construct f_γ such that for all $n \in \mathbb{N}$, (A) $f_\gamma(w_n) \in D$ and (B) $f_\gamma(w_n) \neq g_n(w_n)$. Condition (A) is just what we said we would do in the previous paragraph, and condition (B) makes sure that f_γ is different from the previous f_β's.

Assume that γ is infinite for now. We will choose $\{\epsilon_n : n \in \mathbb{N}\}$ such that

$$f_\gamma(z) = \sum_{n=0}^{\infty} \epsilon_n (z - w_1) \cdots (z - w_n).$$

Observe that the values of ϵ_m for $m \geq n$ do not affect the value of $f_\gamma(w_n)$. We define ϵ_n by induction to ensure that $f_\gamma(w_n) \in D$. Moreover, since D is dense, we can define the ϵ_n's to converge quickly enough so that $f_\gamma(z)$ converges and is an analytic function by Fact 1.44.

The case where γ is finite, and we just construct a polynomial. Now the proof is complete. □

Chapter 2

Elementary Submodels and Topology

2.1 Setting Up

May 29, 2024

2.1.1 Some Motivation

> **Set-Theoretic Topology:** The study of abstract topological spaces with a focus on notions like weakened separation axioms, metrizability, and variations of compactness, questions about which are often independent of ZFC.

Here we will start with some material that is amenable to ZFC proof, albeit with more technical involvement than was dealt with in the previous chapter.

Definition 2.1. Let (X, τ) be a topological space.

- (X, τ) is *first-countable* if every element $z \in X$ has a countable neighborhood basis, i.e. a sequence of open sets $\langle U_n : n \in \mathbb{N} \rangle$ such that if V is any open set containing z, then $z \in U_n \subseteq V$ for some $n \in \mathbb{N}$.
- (X, τ) is *second-countable* if X as a whole has a countable basis.
- (X, τ) is *separable* if it has a countable dense set.

Example 2.2. $\mathbb{R}$ is separable. Second-countable spaces are first-countable. The Sorgenfrey line is first-countable but not second-countable: Given a basis $\mathcal{B}$ we can choose some $U_x \in \mathcal{B}$ with $\inf U_x = x$ for all $x \in \mathbb{R}$, and then all of these must be distinct.

Proposition 2.3. *A separable Hausdorff space has cardinality* $\leq 2^{2^{\aleph_0}}$.

Proof. Let X be a separable Hausdorff space and let Q be a countable dense subset of X. Define a function $f : X \to P(P(Q))$ by setting $f(z) := \{B \subseteq Q : z \in \bar{B}\}$.

We see that f is injective: Since X is Hausdorff, if $y \neq z$ are elements of X, take open sets U and V separating them, meaning $U \cap V = \emptyset$ and $y \in U$ and $z \in V$. Then $(Q \setminus V) \in f(y) \setminus f(z)$ and $(Q \setminus U) \in f(z) \setminus f(y)$.

And we have $|P(P(Q))| = 2^{2^{\aleph_0}}$. □

Example 2.4. Consider the space of ultrafilters $\mathcal{U}$ on $\mathbb{N}$ with the topology generated by sets of the form $B_X = \{U \in \mathcal{U} : X \in U\}$ for each $X \subseteq \mathbb{N}$. This space is compact and has cardinality $2^{2^{\aleph_0}}$. (But this takes work to prove.)

Theorem 2.5 (Arhangel'skii's Theorem)**.** *Every compact, first-countable* Hausdorff *topological space has cardinality at most* $2^{\aleph_0}$.

2.1.2 Reviewing Some Basic Model Theory

We will start by defining models of the form $H(\kappa)$ for a (regular) cardinal κ.

Recall some definitions from model theory, which we will summarize in very loose terms for the sake of haste:

1. A *language* is a set of symbols including constant, function, and relation symbols. In set theory we will only use the language $\mathcal{L} = \{=, \in\}$ (typically the notation for equality is suppressed).

2. Symbols from the language are built up into *terms*, which are build up with variables to create *formulas* using $\neg, \wedge, \vee$ and adding quantifiers $\exists, \forall$. Variables are *free* if they are not included in the scope of quantifiers. A formula with no free variables is called a *sentence* and sentences have truth values.

3. Given a language $\mathcal{L}$, an $\mathcal{L}$-*structure* is a set in which we can interpret truth values of sentences.

4. A *theory* is a set of sentences. It is satisfiable if it has a model M.

5. To structures are *elementary equivalent*, denoted $M \equiv N$, if they satisfy the same sentences. If $M \subseteq N$, then we say M is an *elementary submodel* of N, denoted $M \prec N$, if for all $\bar{a} \in M$ and formulas φ, $M \models \varphi(\bar{a})$ if and only if $N \models \varphi(\bar{a})$.

6. We say that $a \in M$ is *definable* with a parameter $\bar{b}$ if there is a formula $\varphi(v, \bar{b})$ such that $M \models \varphi(a, \bar{b})$ and a is the only element with this property.

Theorem 2.6 (Tarski-Vaught Test). *Let $M \subseteq N$ be $\mathcal{L}$-structures. Suppose it is the case that if $\bar{a} \in M$ and there is $b \in N$ such that $N \models \varphi(b, \bar{a})$, then there is $c \in M$ such that $M \models \varphi(c, \bar{a})$. Then it follows that $M \prec N$.*

Theorem 2.7 (Downward Löwenheim-Skolem). *Let K be a structure and let $A \subset K$. Then there is $M \prec K$ such that $A \subset M$ and $|M| = |A| + \aleph_0$.*

Sketch. We will give the idea for countable A. The idea is to use the Tarski-Vaught test to close everything off. Define a sequence $\langle X_n : n \in \mathbb{N} \rangle$ as folows: Let $A = X_0$. Then given X_n let X_{n+1} include witnesses for all formulas $\exists v \varphi(v, \bar{a})$ where $\bar{a}$ is taken from X_n. □

2.1.3 Elementary Submodels for the Present Context

Definition 2.8.

- A set x is *transitive* if $a \in b \in x$ implies $a \in x$.
- Let x be a set. The *transitive closure* of x, denoted $\mathrm{tc}(x)$, is the smallest transitive set containing x.
- Let θ be a cardinal. Then $H(\theta)$ denotes the set of sets x such that $|\mathrm{tc}(x)| < \theta$.

Example 2.9.

- $\{\aleph_2, \aleph_3\} \notin H(\aleph_1)$.
- We have $\omega \subset H(\omega)$ but $\omega \notin H(\omega)$.

- Interpret the set of rationals $\mathbb{Q}$ as pairs of natural numbers modded out be an equivalence relation. Then $\mathbb{Q} \in H(\aleph_1)$.

Proposition 2.10. *Let θ be an infinite cardinal. Then the following are true:*

1. *$H(\theta)$ is transitive, meaning that if $b \in a \in H(\theta)$ then $b \in H(\theta)$,*
2. *$H(\theta)$ is a set,*
3. *$H(\theta) \cap \mathrm{ON} = \theta$.*

Proof. Exercise. □

Proposition 2.11. *Suppose θ is a regular uncountable cardinal. Then $H(\theta)$ satisfies the* ZFC *axioms besides powerset. More precisely:*

1. *Empty-set: There is a set X containing no elements.*
2. *Extensionality: If X and Y have the same elements then $X = Y$.*
3. *Pairing: For any a and b there exists a set $\{a, b\}$.*
4. *Schema of Separation: If P is a property with parameters $\bar{a}$, then for any set X there exists a set $Y := \{u \in X : P(u, \bar{a})\}$.*
5. *Union: For any set X there is $Y = \bigcup X$.*
6. *Infinity: There exists an infinite set.*
7. *Schema of Replacement: If a class F is a function, then for any set X there exists a set $Y = F(X) = \{F(x) : x \in X\}$.*
8. *Regularity: Every nonempty set has an $\in$-minimal element.*
9. *Axiom of Choice: Any set of sets has a choice function.*

Proof. 1. $|\operatorname{tc}(x)| = 0 < \theta$.

2. We need to say that if $x \neq y$ and $x, y \in H(\theta)$, then there is some $z \in x \Delta y$ such that $z \in H(\theta)$. Since there is definitely some $z \in x \Delta y$, WLOG in x, we know $|\operatorname{tc}(z)| \leq |\operatorname{tc}(x)| < \theta$.

3. $\operatorname{tc}(\{a, b\}) = \{a, b\} \cup \operatorname{tc}(a) \cup \operatorname{tc}(b)$.

4. If $x \subseteq y$ then $\operatorname{tc}(x) \subseteq \operatorname{tc}(y)$.

5. $\mathrm{tc}(\bigcup x) = \bigcup_{y \in x} \mathrm{tc}(y)$.
6. ω is transitive.
7. Exercise.
8. If $y \in x$ then $\mathrm{tc}(y) \subseteq \mathrm{tc}(x)$.
9. Similar to previous items. □

We still get a fragment of the powerset axiom for $H(\theta)$'s though. June 5, 2024

Proposition 2.12. *If $x \in H(\theta)$ and $2^{|x|} < \theta$ then for all $y \subseteq x$, $y \in H(\theta)$.*

Proof. Exercise. □

Proposition 2.13. *Suppose θ is regular and $X \in H(\theta)$. Then $|X| \in H(\theta)$ and there is an enumeration $f : |X| \to X$ such that $f \in H(\theta)$.*

Proof. Since $X \subseteq \mathrm{tc}(X)$, $|\mathrm{tc}(X)| < \theta$ implies that $|X| < \theta$. Also, $|X|$ is transitive. If $f : |X| \to X$ is a function, then it can be argued that $|\mathrm{tc}(f)| \leq |\mathrm{tc}(X)|$. □

Proposition 2.14. *Let $\theta > \omega$ be a regular cardinal and let $M \prec H(\theta)$ be countable with $X \in M$. Then the following are true:*

1. *$M \setminus X \neq \emptyset$,*
2. *If X is countable then $X \subset M$,*
3. *If $X \setminus M \neq \emptyset$ then X is uncountable,*
4. *If $X \subseteq \omega_1$ is uncountable then $X \cap M$ is a cofinal subset of $\sup(M \cap \omega_1)$, i.e. for all $\beta < \sup(M \cap \omega_1)$, there is some $\gamma \in (\beta, \sup(M \cap \omega_1)) \cap (X \cap M)$.*

Proof. *1.* (This more a point where one should pay attention rather than a directly useful fact.) If $M \prec H(\theta)$, then $X \in M$ implies that $X \in H(\theta)$. Since $X \notin X$, $H(\theta) \models \exists y, y \neq X$. By elementarity, there is some $y \in M \setminus X$.

2. First we can argue that X can be expressed as the image of a function. By Proposition 2.13 there is some $f \in H(\theta)$ such that $f : \omega \to X$ is an enumeration. In other words, $H(\theta) \models$ "there is an enumeration of X". By elementarity, $M \models$ "there is an enumeration of X". Let $g \in M$ witness this, so $M \models$ "g is an enumeration of X", so $H(\theta) \models$ "g is an enumeration of X". Since $X \in H(\theta)$, it is true that g is an enumeration of X and hence that $X = g[\omega]$.

Observe that $\omega \subset M$: We have $\emptyset \in M$ by definability of $\emptyset$ and if $n \in M$, then $n + 1 := n \cup \{n\}$ is definable and therefore $n + 1 \in M$. Since $M \models$ "f is a function", we have $f(n) \in M$ for all $n \in \omega$, hence $X \subset M$.

3. This is just the contrapositive of the previous point.

4. Since X is uncountable it is cofinal. Since $X \in H(\theta)$, $H(\theta) \models$ "X is cofinal in ω_1". Suppose that $\gamma < \sup(M \cap \omega_1)$ and let $\beta \in (\gamma, \sup(M \cap \omega_1))$. Then $M \models$ "there is an ordinal above β in X", and this is sufficient. □

Proposition 2.15. *Let $\theta > \omega$ be a regular cardinal and let $M \prec H(\theta)$ be countable. Then $M \cap \omega_1$ is a countable ordinal.*

Proof. Let $\alpha = \sup(M \cap \omega_1)$. Then since M is countable, $\alpha < \omega_1$. We argue that $\alpha = M \cap \omega_1$: Suppose $\beta < \alpha$. Then there is some $\gamma \in (\beta, \alpha)$ such that $\gamma \in M$. Then γ is itself a countable set, so $\gamma \subset M$, hence $\beta \in M$ since $\beta \in \gamma$. □

2.2 Applying the Technique

2.2.1 An Easy Start: the Sunflower Lemma

Definition 2.16. A family F is called a Δ-*system* if there is some d such that for all $a, b \in F$, $a \cap b = d$.

Theorem 2.17 (A case of the Δ-system lemma)**.** *Let F be an $\aleph_1$-sized family of finite sets. Then there is an $\aleph_1$-sized subfamily $F' \subseteq F$ such that F' is a Δ-system.*

Proof. By isomorphism we can assume without loss of generality that $F \subseteq [\aleph_1]^{<\omega}$, i.e. F consists of finite sets of countable ordinals. We know $F \in H(\aleph_2)$, so we can find a countable $M \prec H(\aleph_2)$ such that $F \in M$ by DLS.

Now we identify some key objects. There is some $b \in F \setminus M$ and we let $r = b \cap M$. Since r is a finite sequence, $M \prec H(\theta)$, and $H(\theta)$ satisfies enough of ZFC, we have that $r \in M$. Let β^* be such that $r \subset \beta^*$.

Let $\delta = M \cap \aleph_1$ (use Proposition 2.14).

Then for all $\alpha < \delta$, there is exists $b_\alpha \in M \cap F$ such that (1) $b_\alpha \cap \alpha = r$ and (2) $b_\alpha \setminus \alpha \neq \emptyset$. This is the case because b always witnesses this for $H(\theta)$

and we have $M \prec H(\theta)$. More precisely, if $\alpha \in (\beta^*, \delta)$, then

$$H(\theta) \models \text{“}\exists b_\alpha, b_\alpha \cap \alpha = r \wedge b_\alpha \setminus \alpha \neq \emptyset\text{”}$$

and so

$$H(\theta) \models \text{“}\exists b_\alpha, b_\alpha \cap \alpha = r \wedge b_\alpha \setminus \alpha \neq \emptyset\text{”}.$$

The reason we have this is because $x \in b \cap \alpha \subseteq M \cap b = r$, and $x \in r \subset b \cap \delta$, but anything in $b \cap \delta$ is already in $b \cap \alpha$. Therefore

$$M \models \forall \alpha \in (\beta^*, \delta), \exists b_\alpha, b_\alpha \cap \alpha = r \wedge b_\alpha \setminus \alpha \neq \emptyset\text{”}.$$

So we have that

$$M \models \text{“}\exists \langle c_\alpha : \beta^* \leq \alpha < \omega_1 \rangle, \text{ s.t. } \forall \alpha < \omega_1, c_\alpha \cap \alpha = r, c_\alpha \setminus \alpha \neq \emptyset\text{”},$$

so inside of M we can construct a sequence $\langle c_\alpha : \alpha < \omega_1 \rangle$ (we are dropping β^* in what is sometimes called “abuse of notation”) such that (1) and (2) hold for each c_α. Moreover, we can construct a subsequence $\langle \alpha_\xi : \xi < \omega_1 \rangle$ such that if $\xi < \xi'$ then $c_{\alpha_\xi} \subset \alpha_{\xi'}$. Then if $\xi < \zeta < \delta$ we have

$$c_{\alpha_\xi} \cap c_{\alpha_\zeta} = c_{\alpha_\xi} \cap c_{\alpha_\zeta} \cap \alpha_\zeta = c_{\alpha_\xi} \cap \alpha_\zeta \cap r = r.$$

The first equality comes from the fact that $c_{\alpha_\xi} \subseteq \alpha_\zeta$. The second equality comes from $c_{\alpha_\zeta} \cap \alpha_\zeta = r$. The third equality comes from $r = c_{\alpha_\xi} \cap \alpha_\xi \subseteq c_{\alpha_\xi} \cap \alpha_\zeta$. Hence we are done by elementarity. □

Note: In the lecture I did not have the extra use of α_ζ in the third term, but I thought this way was clearer.
June 12, 2024

2.2.2 Our Main Goal for the Unit: Arhangel'skii's Theorem

Here are some conceptual points that we should not forget about. A *set* is anything that can be proved to be a set by ZFC. A *class* is an object of the form $\{x : \varphi(x)\}$. (Think of statements about classes as statements about formulas.) A *proper class* is a class that is not a set.

Proposition 2.18. *If C is a class, then there is an $\in$-minimal member of C.*

Proof. We can denote C to be the class of x such that $\varphi(x)$ holds (for some φ). Let $S \in C$, so S is in particular a set. Then $S \cap C$ is a set by the separation schema. If $S \cap C = \emptyset$ then we are done. If $S \cap C \neq \emptyset$ then we let $X = T \cap C$ where $T = \text{tc}(S)$. If $z \in X$ is $\in$-minimal element (of which there must be an instance) then z is an $\in$-minimal element of C. □

Definition 2.19. The Von Neumann hierarchy is defined by induction on $\alpha \in \mathrm{ON}$ as follows:

- $V_0 = \emptyset$,
- $V_{\alpha+1} = P(V_\alpha)$ (where the powerset is denoted),
- if α is a limit then $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$.

Proposition 2.20. *For all sets x there is some α such that $x \in V_\alpha$.*

Proof. Let C be the hypothetical class of x such that there is no $\alpha \in \mathrm{ON}$ such that $x \in V_\alpha$. Then let x be an $\in$-minimal element of C. Then for all $z \in x$ there is α_z such that $z \in V_{\alpha_z}$, so there is some $\beta = \sup_{z\in x} \alpha_z$, and we see that $x \in V_{\beta+1}$, a contradiction. □

Proposition 2.21. *If θ is regular and uncountable then $H(\theta) \subseteq V_\theta$, and there is some α such that $H(\theta) \in V_\alpha$.*

Proof. From the axiom schema of separation and the previous proposition. □

We check some notions of elementary submodels next.

Proposition 2.22. *Suppose N is an $\mathcal{L}$-structure and $\langle M_i : i < \kappa\rangle$ is a sequence such that:*

1. *$M_i \prec N$ for all $i < \kappa$,*
2. *for all $i < j < \kappa$, $M_i \prec M_j$.*

Then if $M = \bigcup_{i<\kappa} M_i$, we have $M \prec N$.

Proof. Apply the Tarski-Vaught test: Suppose $\bar{a} \in M$ and $N \models \exists v \varphi(v, \bar{a})$. Then there is some i such that $\bar{a} \in M_i$. Since $M_i \prec N$, there is some $b \in M_i$ such that $M_i \models \varphi(b, \bar{a})$. We can argue by induction on formula construction. □

Proposition 2.23. *Given a regular uncountable θ and countable $a \in H(\theta)$, there is some $2^{\aleph_0}$-sized $M \prec H(\theta)$ such that $a \subset M$ and M is closed under countable sequences.*

Proof. Define an elementary chain $\langle M_i : i < \omega_1 \rangle$ of submodels of $H(\theta)$ with the inductive hypothesis that $|M_j| \leq 2^{\aleph_0}$ for all $j < i$.

We do this as follows: Let M_0 be obtained by applying the Downward Löwenheim-Skolem Theorem to a, so that we have $|M_0| = \aleph_0$. If $i < \omega_1$ is a limit, then let $M_i := \bigcup_{j<i} M_i$. Then by Proposition 2.22, $M_i \prec H(\theta)$. If $i = j + 1$ is a successor, then let $A = [M_j]^\omega$, i.e. the set of countable sequences of elements of M_j. Note that $|A| \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$. Then apply Downward Löwenheim-Skolem to obtain M_i such that $M_i \supseteq A$. Then $|M_i| = |A| + \aleph_0 \leq 2^{\aleph_0}$ by the statement of the theorem.

By Proposition 2.22, $M := \bigcup_{i<\omega_1} M_i$ is an elementary submodel of $H(\theta)$. We just need to show that it is closed under countable sequences. Suppose that $x \subset M$ is countable. Let $x = \langle y_n : n < \omega \rangle$. Then for all $n < \omega$, there is some $i_n < \omega_1$ such that $y_n \in M_{i_n}$. Let $i^* = \sup_{n<\omega}(i_n + 1)$. Then $i^* < \omega_1$. Then for all n, $y_n \in M_{i^*} \subseteq M$, i.e. $x \subset M$. □

Proposition 2.24. *Let (X, τ) be a first-countable topological space. If $A \subseteq X$ and $z \in \bar{A}$ (i.e. z is in the closure of A) then there is a sequence $(z_n)_{n\in\mathbb{N}}$ that converges to z. Moreover, if (X, τ) is Huasdorff then z is the unique element with this property.*

Proof. We mean that for all open $U \ni z$, there is some m such that $n \geq m$ implies that $z_n \in U$. Let $\langle U_n : n \in \mathbb{N} \rangle$ be a neighborhood basis. Let $z_n \in A \cap \bigcap_{k\leq n} U_n$, which is still open. Then let U be an arbitary open set. If U_m is such that $z \in U_m \subseteq U$ then we have guaranteed that $z_n \in U_m$ for $n \geq m$.

Now assume (X, τ) is also Hausdorff. If $z_A \neq z_B$ and these are separated by U_A and U_B and m is such that $z_n \in U_A$ for all $n \geq m$, then clearly m witnesses that $(z_n)_{n\in\mathbb{N}}$ cannot converge to z_B. □

Finally, recall:

Proposition 2.25. *If (X, τ) is compact then any closed subset of X is compact.*

Proof. Let $C \subseteq X$ be closed. Any open supcover $(U_i)_{i\in I}$ of C can be extended to an open cover $(U_i)_{i\in I} \cup (X \setminus C)$ of the whole space, and a finite subcover of this yields a finite subcover of C. □

Recall our version of Arhangel'skii's Theorem: If (X, τ) is first-countable, compact, and Hausdorff, then $|X| \leq 2^{\aleph_0}$.

Remark 2.26. In fact, Arhangel'skii proved something more general.

Proof of Arhangel'skii's Theorem. Let θ be large enough that $(X, \tau) \in H(\theta)$ and let $M \prec H(\theta)$ be a $2^{\aleph_0}$-sized elementary submodel with $(X, \tau) \in M$.

Observe that if $A \subset X$ and $A \in M$, then $\bar{A} \in M$ (where $\bar{A}$ is the closure of A with respect to τ) since this is definable with the parameters τ and A.

We claim that $X \cap M$ is closed and therefore compact: Suppose that $z \in \overline{X \cap M}$. Then by Proposition 2.24 there is a sequence $\vec{w} := (w_n)_{n \in \mathbb{N}} \subset X \cap M$ converging to z. Since M is closed under countable sequences, we have $\vec{w} \in M$. Then $M \models$ "$\vec{w}$ has a limit", and moreover we have that this is uniquely defined, hence $z \in M$.

To finish the proof, we claim that $X \cap M = X$. For each $z \in X \cap M$ we have a neighborhood base B_z such that $B_z \in M$. (By elementarity, M knows that such a base exists, so there must be one in M.) Since B_z is countable, we have $B_z \subseteq M$. (This is technically distinct from Proposition 2.14 but follows by essentially the same proof, in particular because we did not *need* the model to be only countable for that proof.)

Now suppose there is some $y \in X \setminus M$. Then for each $z \in M$ there is $U_z \in B_z$ and hence $U_z \in M$ such that $y \notin U_z$ (since we choose any $U_z \subset X \setminus \{y\}$). Then we have that $\{U_z : z \in X \cap M\}$ is a cover of the compact space $X \cap M$, hence it has a finite cover $u = \{U_{z_0}, \ldots, U_{z_k}\}$. Since this is finite subset of M we have $u \in M$. This means that

$$M \models \text{“}\forall z \in X, \exists i, z \in U_{z_i}\text{”}.$$

But $H(\theta)$ does not satisfy this formula because of y, which contradicts elementarity. □

2.2.3 A Compactness Theorem

June 19, 2024

Proposition 2.27. *The space ω_1 with the order topology is an example of a topological space X such that X is not second countable but smaller subspaces are.*

For a discussion of possible wrong proofs, see `https://math.stackexchange.com/questions/1878367/how-to-show-that-omega-1-is-not-secound-countable`.

Proof. First we show that (ω_1, τ) is not second countable. Suppose for contradiction that there is a countable base B. We **must** use the fact that

B consists without loss of generality of countable sets—otherwise the most obvious proof will be problematic. However, this is an involved proof. Once we do that we can find some $\alpha \in \omega_1$ outside of all sets in B, thus obtaining a contradiction.

Next we show that all countable subspaces of (ω_1, τ) are second countable. Suppose that $Y \subset \omega_1$ is countable. Then there is some $\alpha < \omega_1$ such that $Y \subseteq \alpha$. Then the set $\{(\beta, \gamma) \cap Y : \beta < \gamma < \omega_1\}$ is a countable base for the subspace topology on Y. □

Remark 2.28. There absolutely are countable topological spaces that are not second countable.

In contrast, we have a sort of compactness theorem:

Theorem 2.29 (Hajnal-Juhász)**.** *If (X, τ) is a topological space such that every subspace $Y \subseteq X$ with $|Y| \leq \aleph_1$ is second countable, then X is second countable.*

First we need:

Definition 2.30. A set M has the *ω-covering property* if for each countable $A \subseteq M$, there is a countable $B \in M$ such that $A \subseteq B$.

(This is not an entirely standard usage of this definition.)

Proposition 2.31. *Let θ be regular and uncountable. Then for all countable $Z \in H(\theta)$, there is an $\aleph_1$-sized $M \prec H(\theta)$ with the ω-covering property such that $Z \subset M$.*

Proof. Build a sequence $\langle M_i : i < \omega_1 \rangle$ of countable elementary submodels of $H(\theta)$ by induction on $i < \omega_1$ as follows:

1. $Z \subseteq M_0$,
2. if i is a limit then $M_i = \bigcup_{j<i} M_j$,
3. for all $i < \omega_1$, $M_i \in M_{i+1}$.

As in arguments that we have seen, we can use Downwards Löwenheim Skolem to define such a sequence. Then let $M = \bigcup_{i<\omega_1} M_i$. Then $M \prec H(\theta)$ by Proposition 2.22.

M has the ω-covering property as in other arguments we have seen: Suppose $A \subseteq M$. If $A = \langle a_n : n \in \mathbb{N} \rangle$, there is for all $n \in \mathbb{N}$ some i_n such that $a_n \in M_i$. Let $i_* = \sup_{n \in \mathbb{N}} i_n$, so $A \subseteq M_i$. Then $M_i \in M_{i+1} \subset M$, so the B from the definition is M_i. □

Proof Theorem 2.29. Let (X, τ) be a topological space and assume that every subspace $Y \subseteq X$ of cardinality $\leq \aleph_1$ is second countable. Let θ be "large enough" and let $M \prec H(\theta)$ be an $\aleph_1$-sized elementary submodel with the ω-covering property such that $(X, \tau) \in M$.

Claim. *The set $\tau \cap M$ is a base for the subspace topology on $X \cap M$.*

Proof of Claim. Suppose that $z \in X \cap M$ and U is an open neighborhood of z. By the assumption of the theorem, $X \cap M \setminus U$ has a countable base and therefore a countable dense subset D. Since M has the ω-covering property we can choose a countable $D' \in M$ so that $D \subseteq D' \subseteq X$. Since $D' \cup \{z\}$ is actually countable and is an element of M, and because $M \prec H(\theta)$, it must be the case that $D' \cup \{z\}$ is second countable in the subspace topology. It follows that there is some $T \in \tau \cap M$ such that $z \in T$ and $T \cap D' \subseteq U$.

Therefore we have

$$X \cap M \setminus U = \overline{D} \subseteq \overline{D' \setminus T} \subseteq X \setminus T.$$

The first equality is true by density of D, the containment that follows is true because we have $T \cap D' \subseteq U$ (hence $D \subseteq D' \setminus T$), and the next containment is true again because of the fact that $T \cap D' \subseteq U$ (which implies that the closure operation will not take any elements of T). It then follows directly from $X \cap M \setminus U \subseteq X \setminus T$ that $M \cap T \subseteq U$. □

Now we can wrap up the proof. It follows from the claim and our assumption that there is a countable subset B of $\tau \cap M$ which is a base for $X \cap M$. We can produce this from any countable base by taking intersections. There is then some $B' \supseteq B$ such that B' by the ω-covering property. Then

$$M \models \text{“}B' \text{ is a countable base for } (X, \tau)\text{”}$$

since for every open $U \in M \cap X$ there is some $V \in B' \cap M$ such that $U \subseteq V$. Therefore it follows from $M \prec H(\theta)$ that

$$H(\theta) \models \text{“there is a countable base for } (X, \tau)\text{”}$$

and it follows that (X, τ) actually has a countable base. □

Chapter 3

Martin's Axiom

3.1 Basic Terminology and Facts

Definition 3.1.

1. $\mathbb{P}$ is a *poset* if it is a partially ordered set with a maximal element $1_{\mathbb{P}}$. We will let $\mathbb{P}$ denote a poset always. Elements $p \in \mathbb{P}$ are called *conditions* and if p, q are conditions such that $q \leq p$, then we say that q is *stronger* than p, meaning that it expresses more information.

2. If $p, q \in \mathbb{P}$, we say that p and q are *compatible* and write $p \| q$ if there is some $r \in \mathbb{P}$ such that $r \leq p, q$. Otherwise we say that p and q are *incompactible* and write $q \perp q$.

3. $\mathbb{P}$ is *non-atomic* if for all $p \in \mathbb{P}$, there exist $q, r \leq p$ such that $q \perp r$. (We will always assume that $\mathbb{P}$ is non-atomic.)

4. $F \subset \mathbb{P}$ is a *filter* if: *(1)* for all $p, q \in F$, there is some $r \in F$ with $r \leq p, q$; and *(2)* for all $p \in F$, if $p \leq q$ then $q \in F$.

5. A subset $D \subseteq P$ is *dense* if for all $p \in \mathbb{P}, \exists q \leq p, q \in D$.

6. If $\mathcal{D}$ is a collection of dense subsets of $\mathbb{P}$, we say that a filter G is $\mathcal{D}$-generic if for all $D \in \mathcal{D}$, $G \cap D \neq \emptyset$.

7. A subset $A \subset \mathbb{P}$ is an *antichain* if for all $p, q \in A$, $p \neq q$ implies $p \perp q$.

8. A poset $\mathbb{P}$ has the *countable chain condition* if all antichains $A \subseteq \mathbb{P}$ are at most countable.

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Definition 3.2. $\mathsf{MA}(\kappa)$ is the statement that if $\mathbb{P}$ has the countable chain condition and $\mathcal{D}$ is a collection of dense subsets of $\mathbb{P}$, then there is a $\mathcal{D}$-generic filter G. *Martin's Axiom*, denoted MA (without reference to a cardinal) is the statement that $\mathsf{MA}(\kappa)$ holds for all $\kappa < 2^{\aleph_0}$.

Proposition 3.3. $\mathsf{MA}(\aleph_0)$ *is true (and does not even need $\mathbb{P}$ to have the countable chain condition).*

Proof. Let $\langle D_n : n \in \mathbb{N}\rangle$ enumerate the family of dense sets in question. Define a $\leq$-decreasing sequence $\langle p_n : n \in \mathbb{N}\rangle$ so that $p_n \in D_n$. Choose $p_0 \in D_0$. Given p_n and D_{n+1}, since D_{n+1} is dense there is some $p_{n+1} \leq p_n$, $p_{n+1} \in D_n$. Then $G := \{q \in \mathbb{P} : \exists n \in \mathbb{N}, p_n \leq q\}$ works. □

Corollary 3.4. CH *implies* MA.

Fact 3.5. *The consistency of* ZFC *implies the consistency of* $\mathsf{ZFC} \wedge \neg\mathsf{CH} \wedge \mathsf{MA}$.

Proposition 3.6. $\mathsf{MA}(2^{\aleph_0})$ *is false.*

Proof. Let $\mathcal{P}(\mathbb{N})$ be enumerated as $\langle A_\alpha : \alpha < 2^{\aleph_0}\rangle$. Let χ_{A_α} be the characteristic function of A_α, i.e., $\chi_A(n) = 1$ if $n \in A$ and $\chi_A(n) = 0$ if $n \in \mathbb{N} \setminus A$. We let $\mathbb{P}$ be the *Cohen forcing*, that is

$$\mathbb{P} = \{p \colon F \to \{0,1\} : F \subseteq \mathbb{N} \text{ is finite}\}, \qquad q \leq p :\Leftrightarrow q \supseteq p.$$

We let for $n \in \mathbb{N}$, $E_n = \{p \in \mathbb{P} : n \in \mathrm{dom}(p)\}$. For $\alpha < 2^{\aleph_0}$ we let

$$D_\alpha = \{p \in \mathbb{P} : \exists n \in \mathrm{dom}(p), p(n) \neq \chi_{A_\alpha}(n)\}$$

We let $\mathcal{D} = \{D_\alpha : \alpha < 2^{\aleph_0}\} \cup \{E_n : n \in \mathbb{N}\}$. It is easy to see that each E_n and each D_α is dense in $\mathbb{P}$. For a contradiction, we assume that G is $\mathcal{D}$-generic. Then $\bigcup G \colon \mathbb{N} \to 2$ Then $\bigcup G \neq \chi_{A_\alpha}$, $\alpha < 2^{\aleph_0}$. But such a G cannot exist.

□

3.2 Simple But Nontrivial Applications

Theorem 3.7. MA *implies that the intersection of fewer than* $2^{\aleph_0}$*-many dense open sets of reals is dense.*

Proof. (Jech 16.23) Let $\kappa < 2^{\aleph_0}$ and let U_α, $\alpha < \kappa$, be dense open sets of reals. For $a \subseteq \mathbb{R}$, we write $\mathrm{cl}(a)$ for its closure. Let I be a bounded open interval. We'll show that $\bigcap_{\alpha<\kappa} U_\alpha \cap I \neq \emptyset$. Let P be the following notion of forcing: Conditions are nonempty open sets p such that $\mathrm{cl}(p) \subseteq I$, with $q \leq p$ if and only if $q \subseteq p$. Since every collection of disjoint open sets is at most countable, P satisfies the countable chain condition. For each $\alpha < \kappa$, let $D_\alpha = \{p \in P : \mathrm{cl}(p) \subseteq U_\alpha\}$; each D_α is dense in P. Let G be a $\mathcal{D}$-generic filter on P where $\mathcal{D} = \{D_\alpha : \alpha < \kappa\}$. Since G is a filter and since $\mathrm{cl}(I)$ is compact, the set $\bigcap\{\mathrm{cl}(p) : p \in G\}$ is nonempty, and is contained (as a subset) in each U_α since $G \cap D_\alpha \neq \emptyset$. □

Remark: By adding the dense sets $E_n = \{p \in P : \mathrm{diam}(p) < \frac{1}{n+1}\}$, $n \in \omega$, we could ensure that $\bigcap\{\mathrm{cl}(p) : p \in G\}$ is a singleton.

Definition 3.8. Let $f, g \colon \mathbb{N} \to \mathbb{N}$. We say "$g$ eventually dominates f" and write $f \leq^* g$, if there is $k \in \omega$ such that for any $n \geq k$, $g(n) \geq f(n)$.

Definition 3.9. *Hechler forcing* is (forcing equivalent to) the following forcing order:

$$\mathbb{P} = \{(s, f) : \exists n,\, s \colon \{0, \dots, n-1\} \to \mathbb{N}, f \colon \mathbb{N} \to \mathbb{N}, s \subseteq f\}.$$

We let $(t, g) \leq (s, f)$ if $t \supseteq s$, $g \geq f$.

$(t, f) \leq (s, f)$ implies, for any $n \in \mathrm{dom}(t) \smallsetminus \mathrm{dom}(s)$, $t(n) \geq f(n)$. Any two conditions (s, f), (s, g) are compatible in Hechler forcing, we just use the pointwise maximum of $f(n), g(n)$ for $n \in \mathbb{N}$ as h and have $(s, h) \leq (s, f), (s, g)$. Since there are countably many different finite sequences s of natural numbers, any antichain A is Hechler forcing is at most countable.

For compatible $(s, f), (t, g) \in \mathbb{P}$, we have a largest/weakest compatibility witness: We let $\mathrm{dom}(r) = \{0, \dots, \max(\mathrm{dom}(s), \mathrm{dom}(t))\}$ and we let $r(n)$ be $s(n)$ on $\mathrm{dom}(s)$ and $t(n)$ on $\mathrm{dom}(t) \setminus \mathrm{dom}(s)$ and $\max(f(n), g(n))$ for the remaining finitely many arguments. We let $h \supseteq r$ be defined as $h(n) = \max(f(n), g(n))$. Then $(r, g) \leq (s, f), (t, g)$ and any other compatibility witness is $\leq (r', g)$ for some variation of r' of r on $\max(\mathrm{dom}(s), \mathrm{dom}(t)) \setminus (\mathrm{dom}(s) \cup \mathrm{dom}(t))$.

Theorem 3.10 (16.24 Jech)**.** *Martin's Axiom implies that every family $\mathcal{G}$ of fewer than $2^{\aleph_0}$ functions from ω to ω is eventually dominated by some $f\colon \omega \to \omega$.*

Proof. Let $\mathcal{G}$ be given. We use Hechler forcing. Let $\mathcal{D} = \{D_g : g \in \mathcal{G}\} \cup \{E_n : n \in \mathbb{N}\}$ with $D_g = \{(s,f) : \forall n \in \mathbb{N} \smallsetminus \mathrm{dom}(s), f(n) \geq g(n)\}$, $E_n = \{(s,f) : n \in \mathrm{dom}(s)\}$. D_f is dense: Let (s,g) be given. We take (s,h) such that $\forall n \in \mathbb{N} \smallsetminus \mathrm{dom}(s), h(n) \geq \max(f(n), g(n))$. Then $(s,h) \leq (s,g)$ and $(s,h) \in D_f$. E_n is dense: Given (s,f) we let $t\colon n+1 \to \mathbb{N}$ be defined by $s \subseteq t$ and for $k \in \mathrm{dom}(t) \smallsetminus \mathrm{dom}(s)$, $t(k) = f(k)$. Then $(t,f) \leq (s,f)$ and $(t,f) \in E_n$. Now let G be $\mathcal{D}$-generic. Then $h = \bigcup\{s : \exists f, (s,f) \in G\}$ eventually dominates any $g \in \mathcal{G}$. To see this, given $g \in \mathcal{G}$, pick $p = (s,f) \in D_g \cap G$. Then for any $n \in \mathbb{N} \smallsetminus \mathrm{dom}(s)$, for any $(t,f') \in G \cap E_n$, we have (t,f') is compatible with (s,f), which means that $t(n) \geq f(n) \geq g(n)$. ☐

This material was improvised in an earlier exercise session, but the LaTeX here is pasted from the notes for the large cardinals course

3.2.1 Clubs and Stationary Sets

Definition 3.11. A function f whose domain is a subset of the ordinals is *regressive* if $f(\alpha) < \alpha$ for all $\alpha \in \mathrm{dom}(f) \setminus \{0\}$.

Remark 3.12. Obviously we have a regressive funtion f with domain ω: Just let $f(n) = n-1$. But can we get a non-constant regressive function with domain $\aleph_1$?

Definition 3.13. The *cofinality* of an ordinal δ is the least ordinal γ such that there exists an unbounded function $f : \gamma \to \delta$. We denote this $\mathrm{cf}(\delta) = \gamma$. We call such a function *cofinal* in δ.

Observation 3.14. *If γ is any ordinal such that $\gamma = \mathrm{cf}(\delta)$, then γ is in fact a regular cardinal.*

Definition 3.15. Let κ be an uncountable regular cardinal. A subset $C \subseteq \kappa$ is *club* in κ (or *a* club in κ) if:

1. C is unbounded in κ, i.e. $\forall \beta < \kappa, \exists \alpha \in C$, $\alpha > \beta$;
2. C is *closed*, i.e. if $\langle \alpha_\xi : \xi < \lambda :\subset\rangle C$ with $\lambda < \kappa$, then $\sup_{\xi<\lambda} \alpha_\xi \in C$.

The set $\{X \subset \kappa : X \text{ contains a club}\}$ is called *the club filter on κ.*

Example 3.16. Consider (1) the set of limit ordinals in κ or perhaps (2) $\kappa \setminus \alpha$ for any $\alpha < \kappa$.

Remark 3.17. We can define clubs in limit ordinals that are not cardinals.

Proposition 3.18. *The club filter is κ-complete. In other words, if $\langle C_\xi : \xi < \lambda \rangle$ are clubs in κ and $\lambda < \kappa$, then $\bigcap_{\xi<\lambda} C_\xi$ is a club in κ. (In particular, the club filter is a filter.)*

Proof. Closure of $\bigcap_{\xi<\lambda} C_\xi$ is straightforward from the definitions.

For unboundedness, we will first argue that the intersection of any two clubs C and D in κ is unbounded. Fix $\delta < \kappa$. Using the unboundedness of C and D, define by induction sequences $\langle \alpha_n : n < \omega \rangle \subset C$ and $\langle \beta_n : n < \omega \rangle \subset D$ such that $\alpha_0 \geq \delta$ and $\alpha_n < \beta_n < \alpha_{n+1}$ for all $n < \omega$. Then we can see that $\sup_{n<\omega} \alpha_n = \sup_{n<\omega} \beta_n = \gamma$. (This is known as "interleaving.") By closure of C, we know that $\gamma = \sup_{n<\omega} \alpha_n \in C$, and by closure of D, we know that $\sup_{n<\omega} \beta_n = \gamma \in D$, and thus $\gamma \in C \cap D$.

Now let us do the general argument. We will argue that $\bigcap_{\xi<\eta} C_\xi$ is unbounded in κ by induction on $\eta < \kappa$.

- The statement is of course trivial if we are taking only one club, so that gives us the base case.

- Suppose that we are considering

$$\bigcap_{\xi<\eta+1} C_\xi = \left(\bigcap_{\xi<\eta} C_\xi \right) \cap C_{\xi+1}.$$

  The first part is a club by our inductive hypothesis, and the intersection of everything is a club by the same argument we used for two clubs.

- Now suppose we are considering $\bigcap_{\xi<\eta} C_\xi$ where η is a limit ordinal. By induction, $\bigcap_{\xi<\zeta} C_\xi$ is a club for all $\zeta < \eta$. Therefore we can assume without loss of generality that $C_\zeta \subseteq C_\xi$ for all $\xi < \zeta$, i.e. the clubs are "nested." Now define a sequence $\langle \alpha_\xi : \xi < \eta \rangle$ to be an increasing sequence above some fixed $\delta < \kappa$ such that $\alpha_\xi \in C_\xi$ for all $\xi < \eta$. If $\beta = \sup_{\xi<\eta} \alpha_\xi$, then $\beta < \kappa$ by regularity. Because of nestedness, $\alpha_\xi \in C_\zeta$ for all $\zeta \leq \xi$, and so $\beta = \sup_{\zeta\leq\xi<\eta} \alpha_\xi \in C_\zeta$ for all $\zeta < \eta$.

This finishes the proof. $\square$

Definition 3.19. Let κ be an uncountable regular cardinal and let $\langle X_\alpha : \alpha < \kappa\rangle$ be a collection of subsets of κ. Then $\triangle_{\alpha<\kappa} X_\alpha := \{\alpha < \kappa : \alpha \in \bigcap_{\beta<\alpha} X_\beta\}$ is the *diagonal intersection* of this collection. A filter F on κ is *normal* if for all $\langle X_\alpha : \alpha < \kappa :\subset\rangle F$, $\triangle_{\alpha<\kappa} X_\alpha \in F$.

Remark 3.20. We do not necessarily have $\triangle_{\alpha<\kappa} X_\alpha \subseteq X_\alpha$ for all $\alpha < \kappa$: Consider the example where $X_\alpha = \kappa \setminus \alpha$ for all $\alpha < \kappa$.

Proposition 3.21. *If κ is an uncountable regular cardinal and $\langle C_\alpha : \alpha < \kappa\rangle$ is a collection of clubs in κ, then $\triangle_{\alpha<\kappa} C_\alpha$ is a club in κ. (In other words, the club filter is normal.)*

Proof. Notice that the diagonal intersection is the same if we replace each C_α with $\bigcap_{\beta\leq\alpha} C_\beta$. Hence, as in the last proof, we can assume without loss of generality that $C_\beta \subseteq C_\gamma$ for $\gamma \leq \beta$.

Closure: Consider $\langle \gamma_\xi : \xi < \eta :\subset\rangle \triangle_{\alpha<\kappa} C_\alpha$ be a strictly increasing sequence where η is a limit ordinal, and let $\sup_{\xi<\eta} \gamma_\xi = \gamma^*$. By the definition of the diagonal intersection, we need to show that $\gamma^* \in \bigcap_{\beta<\gamma^*} C_\beta$.

The definition of diagonal intersections already tells us that $\gamma_\xi \in \bigcap_{\beta<\gamma_\xi} C_\beta$ for all $\xi < \eta$. Using nestedness, this means that $\gamma_\zeta \in C_{\gamma_\xi}$ for all $\zeta \in (\xi, \eta)$, which implies that $\gamma^* = \sup_{\zeta<\eta} \gamma_\zeta = \sup_{\xi\leq\zeta<\eta} \gamma_\zeta \in C_{\gamma_\xi}$ for all $\xi < \eta$. Again using nestedness, we conclude that $\gamma^* \in C_\beta$ for all $\beta < \gamma^*$.

Unboundness: Given $\beta < \kappa$, we will inductively define a sequence $\langle \gamma_n : n < \omega\rangle$ as follows: Let γ_0 be any ordinal in the interval (β, κ). Given γ_n, choose $\gamma_{n+1} \in (\gamma_n, \kappa)$ to be an element of $\bigcap_{\alpha<\gamma_n} C_\alpha$, which we know is a club. Then let $\gamma^* = \sup_{n<\omega} \gamma_n$.

Of course, γ^* is larger than β, so we just need to show that $\gamma^* \in \triangle_{\alpha<\kappa} C_\alpha$, i.e. that $\gamma^* \in C_\alpha$ for all $\alpha < \gamma^*$. Given some particular $\alpha < \gamma^*$, there is some n such that $\alpha < \gamma_n$. Then we see that $\gamma_m \in C_\alpha$ for all $m > n$. As in our previous reasoning, $\gamma^* \in C_\alpha$. $\square$

Definition 3.22. Let κ be regular uncountable. We say that $S \subseteq \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for all clubs $C \subset \kappa$.

Example 3.23. Given a regular uncountable κ, all clubs in κ are stationary. Also, $\{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$ is stationary.

Observation 3.24. *If $S \subset \kappa$ is stationary, then S is unbounded in κ.*

Theorem 3.25 (Fodor's Lemma)**.** *Let κ be regular uncountable and let $S \subset \kappa$ be stationary. If f is a regressive function with domain S, then there is a stationary subset $S' \subseteq S$ and some $\gamma < \kappa$ such that for all $\alpha \in S'$, $f(\alpha) = \gamma$.*

Proof. Suppose otherwise. Then for all $\gamma < \kappa$, there is some club C_γ such that for all $\alpha \in C_\gamma \cap S$, $f(\alpha) \neq \gamma$. (We are sort of jumping past a step here.) Now take $C := \triangle_{\gamma<\kappa} C_\gamma$, which we now know is a club. Let $\delta \in C \cap S \neq \emptyset$, and let $f(\delta) = \gamma < \delta$. By the definition of diagonal intersections, $\delta \in \bigcap_{\alpha<\delta} C_\alpha$, meaning that $\delta \in C_\gamma$, but this contradicts the way we defined C_γ. $\square$

Corollary 3.26. *There is no non-constant regressive function with domain $\aleph_1$.*

3.3 Normality and Metrizability Problems

July 3, 2024

3.3.1 Dealing with Trees

Definition 3.27. A *tree* is a partially ordered set T with an order $<_T$ such that for every $x \in T$, $\{y \in T : y \leq x\}$ is well-ordered.

- The α^{th} *level* of T is the set $T_\alpha = \{x \in T : \text{ot}\{y \in T : y < x\} = \alpha\}$.
- The *height* of T, denoted $\text{hgt}(T)$, is $\sup\{\alpha : \exists x \in T, \text{ot}\{y \in T : y < x\} = \alpha\}$.

We will assume that trees are *normal*, meaning that

1. All $x \in T$ have immediate successors.
2. If $x, y \in T_\alpha$ for limit α and $\{z : z <_T x\} = \{z : z <_T y\}$ then $x = y$.

Example 3.28. The set T of functions $f : \alpha \to \{0, 1\}$ for $\alpha < \omega_1$ is a tree where $g \leq_T f$ if $g = f \restriction \text{dom} f$. If $\text{dom} f = \alpha$, then f is in the α^{th} level of T. T has height ω_1.

Definition 3.29. A *branch* is a maximal linearly ordered subset of T (specifically, maximal under inclusion). A *cofinal branch* b of T is a branch of order-type height(T), i.e. a function such that for all $\alpha <$ height(T), $b \restriction \alpha \in T$.

Example 3.30. There is a tree with countable levels and countable height but no cofinal branch.

Lemma 3.31 (König's Lemma). *If T is an ω-tree then it has an infinite branch, i.e. there is a set $b \subseteq T$ such that for every n, $|T_n \cap b| = 1$.*

Proof. Construct $b = \{b_n : n < \omega\}$ by induction on n using the inductive hypothesis that for all $m < n$, b_m has infinitely many descendents. Assume without loss of generality that T has a root b_0, i.e. a unique element on level 0. For every $n > 0$, b_n has finitely many immediate descendants $\{x_i : i < k\}$. Let $P_i = \{y \in T : x_i \leq y\}$. At least one P_i must be infinite by the pigeonhole principle, so let $b_{n+1} = x_i$. □

Theorem 3.32 (Aronszajn). *There is a tree of height ω_1 and with countable levels.*

Why can we not just use the full binary tree of height ω_1? Why not "cut off" excess branches?

Proof. We will construct a tree T by defining T_α by induction on $\alpha < \omega_1$. The α^{th} level T_α will consist of sequences of rational numbers of order-type α. In other words, elements of T will take the form $\langle q_\beta : \beta < \alpha \rangle \subset \mathbb{Q}$ where if $\gamma < \beta < \alpha$, then $q_\gamma < q_\beta$. For $t, s \in T$, we will write $t \leq_T s$ if t is an initial segment of s.

Since there are only countably many rational numbers, this tree will not have an unbounded branch. (Having a cofinal branch would be equivalent to having a sequence $\langle q_\beta : \beta < \omega_1 \rangle$ such that for all $\alpha < \omega_1$, $\langle q_\beta : \beta < \alpha \rangle \in T_\alpha$.)

Our inductive hypothesis is the following: For all $\beta < \alpha$, $x \in T_\beta$, and $\sup x < q \in \mathbb{Q}$, there is a sequence y of rationals of order-type α such that $x \leq_T y$ and $\sup y = q$.

Zero Case: First let $T_0 = \emptyset$.

Successor Case: If $\alpha = \beta + 1$, then let $T_\alpha = \{x^\frown\langle q \rangle : x \in T_\beta, q \in \mathbb{Q}, \sup x \leq q\}$. It is fairly immediate to see that if T_β satisfies the inductive hypothesis, then so will T_α.

Limit Case: Suppose α is a limit ordinal.

Claim. *For every $x \in \bigcup_{\beta<\alpha} T_\beta$ and every $q \geq \sup x$, there is a sequence of rationals $y_{x,q}$ of order-type α such that $x \leq_T y_{x,q}$, $\sup y_{x,q} = q$, and for* all *$\beta < \alpha$, $y_{x,q} \restriction \beta \in T_\beta$.*

Assuming the claim is true, we can let $T_\alpha = \{y_{x,q} : x \in \bigcup_{\beta<\alpha} T_\beta, q \geq \sup y_{x,q}\}$ where the $y_{x,q}$'s witness the claim, so T_α is countable.

Proof of Claim. Suppose $q \in \mathbb{Q}$ and $x \in \bigcup_{\beta<\alpha} T_\alpha$ and that the order type of x is $\gamma < \alpha$. Since α is a limit, there is a sequence $\langle \alpha_n : n < \omega \rangle$ such that $\sup_{n<\omega} \alpha_n = \alpha$ and $\alpha_0 > \gamma$. Let q_n be a sequence of rational numbers so that $\lim_{n<\omega} q_n = q$. Then for every n, let y_{n+1} witness the inductive hypothesis for q_{n+1} and y_n, i.e. $y_{n+1} \supset y_n$ and $\sup y_{n+1} = q_{n+1}$. Then $\bigcup_{n<\omega} y_n$ witnesses the claim. □

This finishes the construction. □

3.3.2 Jones' Space

Definition 3.33. Recall that a topological space X is *normal* if two closed sets A, B can be separated by open sets U, V, i.e. $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

We will give an example of a topological space whose properties are more or less reasonable, but whose normality is independent of ZFC.

Definition 3.34 (Jones' Space)**.** Let T be an ω_1-Aronszajn tree, meaning a tree of height ω_1 and countable levels and no cofinal branch. Assume for convenience that there is only one element of T_0 which we denote $\emptyset$.

If $s, t \in T$, let $(s, t) := \{u : s <_T u <_T t\}$ and define $(s, t]$, $[s, t)$, and $[s, t]$ similarly.

Let (T, τ) be the topological space with T as an underlying set and basic open sets of the form (s, t) for $s, t \in T$ and $[\emptyset, t)$ for $t \in T$.

Proposition 3.35. *Jones' space has the following properties:*

1. *T_0 is open.*

2. *If $x \in T_{\alpha+1}$ then $\{x\}$ is open.*

3. *If $x \in T$ then $\{y : y <_T x\}$ is open.*

4. *A (maximal) branch b is closed.*

5. *T is first countable.*

6. *T is a regular space, meaning that closed sets and points can be separated.*

Proof. We will focus on the slightly more substantial bits.

1. If $t \in T_1$ then $[\emptyset, t)$.

2. If s is an immediate predecessor of t and s' is an immediate successor then $\{t\} = (s, s')$. Suppose $t \in T_\alpha$ where α is a limit. let s' be an immediate successor of t. Then $\{(u, s') : u <_T t\}$ is a countable neighborhood base.

4. Let b be a branch and suppose $t \in T \setminus b$. If $t \in T_{\alpha+1}$ for some α then we can use *2.* If $t \in T_\alpha$ for a limit α then there is some $u <_T t$ such that $u \notin b$. Then let t' be an immediate successor of t and take (u, t').

6. Let $t \in T$. If $t \in T_{\alpha+1}$ for some α then this follows from *2.*

7. Let $C \subset T$ be closed and let $t \in T \setminus C$. Let t' be an immediate successor of t. There is some $u <_T t$ such that $C \cap (u, t') = \emptyset$ because C is closed. Then

$$U = \bigcup \{(s_0, s_1) : s_0, s_1 \notin (u, t'), \neg(s_0 \leq_T u \wedge t' \leq_T s_1)\}$$

is an open set containing C and disjoint from (u, t'). □

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Theorem 3.36 (Fleissner)**.** *Assuming* $\mathsf{MA} \wedge \neg\mathsf{CH}$*, T is normal.*

Proof. Assume that T is the tree constructed in Theorem 3.32.

Fix disjoint nonempty closed sets $H, K \subset T$. We describe a poset $\mathbb{P}$ to be equal to the set of functions f such that:

1. f is finite,
2. $\operatorname{dom} f \subseteq H \cup K$,
3. $\forall t \in \operatorname{dom} f$, $f(t)$ is a basic open set containing t *and none of its successors*,
4. $\forall h \in H, k \in K, f(h) \cap f(k) = \emptyset$.

So $\mathbb{P}$ is equal to

$$\{f : |f| < \aleph_0, \operatorname{dom} f \subseteq T, \forall h \in H, k \in K, f(h) \cap f(k) = \emptyset\}.$$

Then $f \leq_{\mathbb{P}} g$ if and only if $\operatorname{dom} g \subseteq \operatorname{dom} f$ and for all $t \in \operatorname{dom} g$, $g(t) \subseteq f(t)$.

Claim. *For $f, g \in \mathbb{P}$, f and g are compatible if and only if for some $h \in H, k \in K$, we have $f(h) \cap g(k) \neq \emptyset$ or $f(k) \cap g(h) \neq \emptyset$.*

Proof. This follows from the fact that f and g are compatible if and only if there is some $j \in \mathbb{P}$ such that $j \subseteq f \cup g$. □

Lemma 3.37. *$\mathbb{P}$ has the countable chain condition.*

Assuming the lemma for now, let us show that $\mathsf{MA} \wedge \neg\mathsf{CH}$ implies the statement of the theorem.

For all $t \in T$, let $D_t = \{f : t \in \operatorname{dom} f\}$.

Claim. *D_t is dense.*

Proof. If $f \in \mathbb{P}$ and $t \in \operatorname{dom} f$ then we are done. If $t \notin f$ then consider the cases knowing that we cannot have $t \in H \cap K$: If $t \in K$, then choose a basic open set U separating t from H. Choose U to avoid $f(k)$ for $k \in \operatorname{dom} f \cap K$. Then let $f' = f \cup \{\langle t, U\rangle\}$. So $f' \in D_t$ and $f' \leq f$. Use the analogous idea if $t \notin K$. □

The set $\vec{D} = \{D_t : t \in T\}$ has cardinality $\aleph_1$, therefore $\mathsf{MA} \wedge \neg\mathsf{CH}$ implies that there is some G that is $\vec{D}$-generic. Then

$$U_H = \bigcup\{V : \exists f \in G, h \in H, V = f(h)\}$$

and

$$U_K = \bigcup\{V : \exists f \in G, k \in K, V = f(k)\}$$

are open sets separating H and K. They are open because they are unions of basic open sets. If we have $t \in U_H \cap U_K$, and we have $f_H \in G$ witnessing $t \in U_H$ and $f_K \in G$ witnessing $t \in U_K$, then we would have some $f' \leq f_H, f_G$ with $f' \in G$, but this contradicts the $f(h) \cap f(k) = \emptyset$ condition from the definition of $\mathbb{P}$.

To get the countable chain condition, we will need:

Theorem 3.38 (Erdős)**.** *Suppose that $F : [\omega_1]^2 \to 2$, i.e. F partitions the set of pairs of countable ordinals into two sets. Then either (a) there is an unbounded subset $H_1 \subseteq \omega_1$ that $F \restriction [H_1]^2$ is constant with value 0 or else (b) there is a subset H_2 which is order-isomorphic to $\omega + 1$ such that $F \restriction [H_2]^2$ is constant with value 1. (This can be denoted $\omega_1 \to (\omega_1, \omega)^2$.)*

Proof. Let $\{A, B\}$ be a partition of $[\omega_1]^2$.

Case 1: There is an infinite ordinal $\alpha < \omega_1$ such that there is no finite $K_\alpha \subseteq \alpha$ which is maximal such that $[K_\alpha \cup \{\alpha\}]^2 \subseteq B$.

Then build a sequence $\langle K^n : n < \omega \rangle$ of finite subsets of α such $[K_\alpha \cup \{\alpha\}]^2 \subseteq B$. Let $K = \{\alpha\} \cup \bigcup_{n<\omega} K^n$. Then K has an order-type equal to $\omega + 1$.

Case 2: For all infinite $\alpha < \omega_1$, there is a finite $K_\alpha \subseteq \alpha$ which is maximal such that $[K_\alpha \cup \{\alpha\}]^2 \subseteq B$.

Define a function $F : \alpha \to \max K_\alpha$. Since K_α is finite, F is regressive on (ω, ω_1) and therefore by Fodor's Lemma is constant on some stationary $S \subseteq \omega_1$ with value γ. Since $|[\gamma]^{<\omega}| = \omega$, there is a stationary set S' and some K such that for all $\alpha \in S'$, $K_\alpha = K$. (Also we will have $\max K < \alpha$ for all $\alpha \in S'$.) We can then argue that $[S']^2 \subseteq A$: Otherwise, suppose that there is $\{\alpha, \beta\} \in [S']$ such that $\{\alpha, \beta\} \in B$. Then this implies that $[K \cup \{\alpha, \beta\}]^2 \subseteq B$, contradicting maximality of K with respect to β. $\square$

Proof of Lemma 3.37. First some notation: If $t \in T$ and $f \in \mathbb{P}$, let $f^*(t)$ be the $\leq_T$-least element of T in $f(t)$.

Suppose that W is an uncountable subset of $\mathbb{P}$.

Since there are only countably many elements on $\mathbb{N}$ and countably many finite sequences of rationals $\mathbb{Q}$, it follows that there is an uncountable $W' \subseteq W$ and natural numbers n, m such that (1) $\operatorname{dom} f = \{h_0, \ldots, h_{n-1}, k_0, \ldots, k_{m-1}\}$ (where the h's are from H and the k's are from K) and (2) for all $f, g \in W'$, $i < n$, $j < m$, we have $f^*(h_i) = g^*(h_i)$ and $f^*(k_j) = g^*(k_j)$.

Now we argue that it cannot happen simultaneously that $f_1(h_i) \cap f_2(k_j) \neq \emptyset$, $f_1(h_i) \cap f_3(k_j) \neq \emptyset$, and $f_2(h_i) \cap f_3(k_j) \neq \emptyset$ for some f_1, f_2, f_3. Suppose for contradiction that this were the case. For each of these intersections, $f^*(h_i)$ and $f^*(k_j)$ are comparable (with the relevant 1, 2, or 3 plugged in) and therefore have a $\leq_T$-maximum. For e.g. $f_1^*(h_i)$ and $f_2^*(k_j)$ we can call this $s_{1,2}$. So we have $s_{1,2}$, $s_{1,3}$, and $s_{2,3}$. All of these lie on a linearly ordered set below that meeting point of h_i and k_j. Then some two of them must be the same: Many $s_{1,2} = s_{2,3} = s$. It follows then that e.g. $s \in f_2(h_i) \cap f_2(k_j)$ but this contradicts $f_2 \in \mathbb{P}$.

We are now in a position to finish the proof. At this point we have $W' = \langle f_\alpha : \alpha < \omega_1 \rangle$ with the enumeration chosen for our convenience. For $\alpha < \beta < \omega_1$, define $R_{i,j}(\{\alpha, \beta\}) = 1$ if $f_\alpha(h_i) \cap f_\beta(k_j) \neq \emptyset$ and 0 otherwise. Apply the theorem of Erdős (Theorem 3.38) to the partition $R_{1,1}$. Because of the preceding paragraph, there is no set of 3 elements homogeneous for 1, so there is some W'' homogeneous for 0. Than apply Erdős' theorem to the partition $R_{1,2}$ and so on through all $2mn$ partitions to get some $\bar{W}$ of cardinality ω_1 which is homogeneous "in the right way" for each partition. We are saying that for all $\alpha, \beta \in \bar{W}$ and all $i < n, j < m$, we have $f_\alpha(h_i) \cap f_\beta(k_j) = \emptyset$. This means that all conditions in $\bar{W}$ are compatible, so it is not an antichain. □

This finishes the proof of the theorem. □

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Theorem 3.39 (Devlin-Shelah). CH *implies that Jones' space is not normal.*

Fact 3.40 (Devlin-Shelah). *Suppose that* $2^{\aleph_0} < 2^{\aleph_1}$. *Suppose* $F : 2^{<\omega_1} \to 2$. *Then there is a* $g \in 2^{\omega_1}$ *such that for any* $f \in 2^{\omega_1}$, $\{\alpha \in \omega_1 : F(f \upharpoonright \alpha) = g(\alpha)\}$ *is stationary in* ω_1.

Then this statement, which is known as "weak diamond", implies their result.

3.4 Further Reading

Here we will recap some of the subjects that were treated in this course, as well as one fairly well-known subject that we did not cover. The goal is to bring the discussion to the present day and to mention some open problems as a way to show where the discussion is still active. Note that there are some patterns in the types of results considered, even though they touch on different branches of mathematics.

3.4.1 On $\aleph_1$-Dense Sets of Reals

Remark 3.41. Theorem 1.33 tells us that if CH holds, then there are (many) $\aleph_1$-sized subsets of $\mathbb{R}$ that are not order-isomorphic.

Definition 3.42 (Baumgartner). A set $A \subseteq \mathbb{R}$ is $\aleph_1$-*dense* if $|A| = \aleph_1$ and if for all $x, y \in A$, $|\{z \in A : x <_{\mathbb{R}} z <_{\mathbb{R}} y\}| = \aleph_1$ [Bau73].

Theorem 3.43 (Baumgartner, 1971). *The consistency of* ZFC *implies that consistency of* $2^{\aleph_0} = \aleph_2$ *together with the statement that all* $\aleph_1$-*dense sets are order-isomorphic.*

Theorem 3.44 (Abraham and Shelah, 1981)**.** $\mathsf{MA} \wedge \neg\mathsf{CH}$ *does not imply Baumgartner's theorem [AS81].*

Open Question 1 (Baumgartner)**.** *Is it consistent that* $2^{\aleph_0} > \aleph_2$ *and all* $\aleph_2$*-dense sets of reals are order-isomorphic?*

This is an especially famous open problem as far as set theory is concerned.

3.4.2 On Wetzel's Problem

Recall Theorem 1.42.

Definition 3.45. A family $\mathcal{F}$ of holomorphic functions is a *Wetzel family* if for every $z \in \mathbb{C}$, $\{f(z) : f \in \mathcal{F}\}$ has cardinality smaller than $\mathcal{F}$.

Question 3.46 (Erdős)**.** *Does* ZFC *prove that there is a Wetzel family?*

Technically, this is not the language that Erdős used to ask this question. The definitions are used by Schilhan and Weinert in the paper mentioned below.

Theorem 3.47 (Kumar-Shelah, 2017)**.** *Consistently no, where* $2^{\aleph_0} = \aleph_{\omega_1}$ *[KS17].*

Theorem 3.48 (Schilhan-Weinert, 2024)**.** *Consistently yes for all cardinals [SW24].*

Open Question 2 (Schilhan-Weinert)**.** *Does* $\mathsf{MA} \wedge 2^{\aleph_0} = \aleph_2$ *imply the existence of a Wetzel family?*

This is interesting because it is like the question posed by Baumgartner that was answered by Abraham and Shelah.

3.4.3 On Elementary Submodels

See [Cox24].

3.4.4 The Normal Moore Space Problem

See [Nyi01].

Theorem 3.49 (Urysohn)**.** *Every second countable regular Hausdorff space is metrizable.*

Definition 3.50. We say that a topological space X is *developable* if there is a countable collection of open covers $\langle \mathcal{U}_n : n < \omega \rangle$ of X such that for any closed set C and any point $p \notin C$ there exists n such that every neighborhood U of p with $U \in \mathcal{U}_n$ is disjoint from C.

Conjecture 3.51 (The Normal Moore Space Conjecture, circa the 1930's)**.** *Every developable regular Hausdorff space is metrizable.*

Theorem 3.52. CH *implies that the NMSC is false.*

Theorem 3.53 (Fleissner)**.** $\mathsf{MA} \wedge \neg\mathsf{CH}$ *implies that the NMSC is false, and this is because of Jones' space!*

Theorem 3.54 (Nyikos and Kunen, circa 1980)**.** *Assuming the consistency of a strongly compact cardinal, NMSC is consistently true.*

Theorem 3.55 (Fleissner)**.** *The consistency of NMSC implies the consistency of large cardinals.*

Open Question 3 (Tall)**.** *Is it consistent with* $2^{\aleph_0} = \aleph_2$ *that every normal Moore space is metrizable?*

This was posed awhile ago and it seems to be open, but I could be wrong.

3.4.5 The Whitehead Problem

See [Ekl76].

Definition 3.56. If $\pi : B \to A$ is a surjective homomorphism of abelian groups, then π *splits* if there is a homomorphism $\rho : A \to B$ such that $\pi \circ \rho$ is the identity on A. An abelian group A is a *Whitehead group* if for all surjective homomorphisms $\pi : B \to A$, if $\ker \pi \cong \mathbb{Z}$, then π splits.

Proposition 3.57. *Free (abelian) groups are Whitehead groups.*

But are all Whitehead groups free?

Definition 3.58. We say that $\Diamond$ holds if there is a sequence $\langle X_\alpha : \alpha < \omega_1 \rangle$ such that for all $Y \subseteq \omega_1$, $\{\alpha < \omega_1 : Y \cap \alpha = X_\alpha\}$ is stationary in ω_1.

Theorem 3.59 (Jensen)**.** *The consistency of* ZFC *implies the consistency of* ZFC *plus* $\Diamond$*.*

Theorem 3.60 (Shelah, 1970's)**.** *If* $\Diamond$ *holds, then every Whitehead group of cardinality* $\aleph_1$ *is free. If* $\mathsf{MA} \wedge \neg\mathsf{CH}$ *holds, then there is a Whitehead group of cardinality* $\aleph_1$ *that is not free.*

See also [BLHŠ23].

Bibliography

[AS81] Uri Avraham and Saharon Shelah. Martin's axiom does not imply that every two $\aleph_1$-dense sets of reals are isomorphic. *Israel Journal of Mathematics*, 38(1):161–176, 1981.

[AZ99] Martin Aigner and Günter M Ziegler. Proofs from the book. *Berlin. Germany*, 1(2):12, 1999.

[Bau73] James Baumgartner. All $\aleph_1$-dense sets of reals can be isomorphic. *Fundamenta Mathematicae*, 79(2):101–106, 1973.

[BLHŠ23] Jeffrey Bergfalk, Chris Lambie-Hanson, and Jan Šaroch. Whitehead's problem and condensed mathematics. *arXiv preprint arXiv:2312.09122*, 2023.

[Cox24] Sean Cox. Approximation theory and elementary submodels. *arXiv preprint arXiv:2405.19634*, 2024.

[Dow88] Alan Dow. An introduction to applications of elementary submodels to topology. In *Topology Proc*, volume 13, pages 17–72, 1988.

[DS79] Keith J Devlin and Saharon Shelah. A note on the normal moore space conjecture. *Canadian Journal of Mathematics*, 31(2):241–251, 1979.

[Ekl76] Paul C Eklof. Whitehead's problem is undecidable. *The American Mathematical Monthly*, 83(10):775–788, 1976.

[Fle75] William G Fleissner. When is jones' space normal? *Proceedings of the American Mathematical Society*, 50(1):375–378, 1975.

[Jec03] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[KS17] Ashutosh Kumar and Saharon Shelah. On a question about families of entire functions. *Fund. Math*, 239(3):279–288, 2017.

[Mar96] Dave Marker. Model theory and exponentiation. *Notices of the AMS*, 43(7):753–759, 1996.

[Nyi01] Peter J Nyikos. A history of the normal moore space problem. In *Handbook of the History of General Topology*, pages 1179–1212. Springer, 2001.

[Ros82] Joseph G Rosenstein. *Linear orderings*. Academic press, 1982.

[Sou19] Daniel Soukup. Lecture notes on "modern techniques in combinatorial set theory". `https://danieltsoukup.github.io/academic/syllabus_modern_techniques.pdf`, 2019.

[SW24] Jonathan Schilhan and Thilo Weinert. Wetzel families and the continuum. *Journal of the London Mathematical Society*, 109(6):e12918, 2024.