Course Notes for Descriptive Set Theory

Maxwell Levine

Acknowledgments

These are the notes for a short course in descriptive set theory that I taught in 2023 at the University of Freiburg. The sources are Kechris' *Classical Descriptive Set Theory*, Jech's *Set Theory*, Dave Marker's course notes, and Anush Tserunyan's course notes.

Contents

Chapter 1

Introduction to Descriptive Set Theory

April 20, 2023

1.1 The Plan for the Course

What is descriptive set theory, and why study it?

1. Descriptive set theory is where topology meets mathematical logic. It consists of the study of the complexity of sets of real numbers.

2. DST harmonizes with other parts of logic. For example, the *Borel hierarchy* of descriptive set theory resembles the arithmetic hierarchy of computability theory. As an example, the study of *Borel equivalence relations* tackles notions of definability (in a somewhat loose sense) that are of interest in model theory.

3. DST provides connections to the study of basic structures in general mathematics. This includes graphs, groups, and applications in functional analysis.

4. DST is integrally involved with the project of large cardinals and inner models theory. For example, the constency of a model in which all sets of reals are Lebesgue measurable (the canonical example is called the Solovay model) is equiconsistent with the existence of an inaccessible cardinal.

The goals for this course are roughly as follows:

1. Introduce the "bread and butter" notions of descriptive set theory. This includes the basic properties of Polish spaces, Borel sets, and examples of dichotomoy theorems, which typically state that certain objects "are either very simple or very complicated".

2. There are some canonical results that we should definitely get to. This includes Suslin's characterization of Borel sets, the Gale-Stewart Theorem (this has to do with the chess example in the course advertisement), and the Mostowski Absoluteness Theorem. (Time permitting, we can cover the Schoenfield Absoluteness Theorem, which implies that the Riemann Hypothesis cannot be proved independent by forcing.)

3. Ideally, I want to provide enough of a background to make further study possible. For example, if you know about forcing, then I want it to be relatively doable to study the above-mentioned information about Solovay's model.

The main reference for this course is Kechris' textbook *Classical Descriptive Set Theory*, available electronically through the university's library. Those in want of another reference besides Kechris and these notes can look at Marker's notes, available at `http://homepages.math.uic.edu/~marker/math512/dst.pdf`.

Homework sets will be given every week and will consist of one to four questions, sometimes consisting of multiple parts. At the end of the course, a final exam will be given for those students who need a grade. The exam can be a take-home exam if it is generally allowed. When assigning grades, I will account for different levels of preparation. Please consult me via email, and we can also arrange a meeting.

1.2 Polish Spaces

1.2.1 A Motivating Example: Cantor-Bendixson Analysis

Definition 1.1. A set $P \subset \mathbb{R}$ is *perfect* if it is closed and has no isolated points, i.e. for all $x \in P$ and all open sets $U \ni x$, $U \cap (P \setminus \{x\}) \neq \emptyset$.

Proposition 1.2. *If $P \subset \mathbb{R}$ is perfect and non-empty then $|P| = 2^{\aleph_0}$.*

Proof. We already know that $|P| \leq |\mathbb{R}| = 2^{\aleph_0}$. Let $\{0,1\}^{\mathbb{N}}$ denote set of functions f with domain $\mathbb{N}$ and range in $\{0,1\}$. It is well-known from Cantor's famous diagonalization argument that $|\{0,1\}^{\mathbb{N}}| = 2^{\aleph_0}$, so it will be enough to define an injection $F : \{0,1\}^{\mathbb{N}} \to P$.

To define F, we will define a system of closed intervals I_s where s is a finite sequence of 0's and 1's. Specifically, I_s will be defined by induction on $|s|$ with the following requirements:

- $I_s \cap P \neq \emptyset$,
- $I_{s^\frown i} \subset I_s$ for $i \in \{0,1\}$,
- $I_{s^\frown 0} \cap I_{s^\frown 1} = \emptyset$,
- $\operatorname{diam}(I_{s^\frown i}) < 1/|s|$ for $i \in \{0,1\}$ (i.e. the difference between the largest and smallest points in I_s is $1/|s|$).

We describe how we define $I_{s^\frown i}$ given $i \in \{0,1\}$ (we define $I_\emptyset$ similarly). First choose $x \neq y$ such that $x, y \in I_s \cap P$, which is possible because P is perfect, then choose open intervals U_0 and U_1 around x and y respectively such that their closures are disjoint, and then we let $I_{s^\frown i}$ be the closure of U_i for $i \in \{0,1\}$.

Finally, if $f \in \{0,1\}^{\mathbb{N}}$, observe that if $(x_n)_{n\in\mathbb{N}}$ is any sequence such that $x_n \in I_{f\restriction n}$ for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Therefore it converges to some x, and $x \in \bigcap_{n\in\mathbb{N}} I_{f\restriction n} \neq \emptyset$ because $\bigcap_{n\in\mathbb{N}} I_{f\restriction n}$ is closed. Moreover, $\bigcap_{n\in\mathbb{N}} I_{f\restriction n}$ cannot contain more than one element because of the fact that $\lim_{n\to\infty} \operatorname{diam}(I_{f\restriction n}) = 0$. Therefore we let $F(f)$ be the unique element of $\mathbb{R}$ in $\bigcap_{n\in\mathbb{N}} I_{f\restriction n}$. We see that F is injective because if $f \neq g$ for $f, g \in \{0,1\}^{\mathbb{N}}$, then there is some $n \in \mathbb{N}$ such that $f(n) \neq g(n)$ and therefore $I_{f\restriction n} \cap I_{g\restriction n} = \emptyset$. □

The next proof will require us to make use of ordinal induction.

Definition 1.3. An *ordinal* is a set α which is transitive, i.e. $\gamma \in \beta \in \alpha \implies \gamma \in \alpha$, and well-ordered by $\in$. A *successor ordinal* takes the form $\alpha \cup \{\alpha\}$ where α is an ordinal, and it is commonly written $\alpha + 1$. A *limit ordinal* is any ordinal which is not a successor ordinal.

Example 1.4. Some examples of ordinals include the natural numbers, the smallest infinite ordinal ω (which is the same as $\aleph_0$), as well as (when we assume formalization of ordinal arithmetic) $\omega+1$, $\omega+2$, $\omega+\omega$, ω^{ω^ω}, and so on.

Definition 1.5. The ordinal ω (sometimes but rarely written ω_0) is the smallest infinite ordinal, and ω_1 is the smallest uncountable ordinal.

Proposition 1.6. *The real numbers $\mathbb{R}$ have a countable basis in the standard topology, i.e. there are open sets $(U_n)_{n\in\mathbb{N}}$ such that any open $U \subseteq \mathbb{R}$ is a union of some U_n's.*

Proof. A set $U \subseteq \mathbb{R}$ is open if for any $x \in \mathbb{R}$, there are $a, b \in \mathbb{R}$ such that $x \in (a, b) \subseteq U$. Just consider all sets of the form $(q - 1/n, q + 1/n)$ where $q \in \mathbb{Q}$ and $n \in \mathbb{N}$ is positive. □

Theorem 1.7 (Cantor-Bendixson)**.** *If $C \subseteq \mathbb{R}$ is closed, then $C = P \sqcup X$ (i.e. we have a disjoint union) where P is perfect and X is countable.*

Proof. First we give a definition: If $A \subset \mathbb{R}$, then the *Cantor-Bendixson derivative* $\Gamma(A)$ is A without its isolated points, in other words

$$\Gamma(A) = A \setminus \{x \in A : \exists U \text{ open}, U \cap (A \setminus \{x\}) = \emptyset\}.$$

We define a sequence $(A_\alpha)_{\alpha\in\omega_1})$ as follows:

- $A_0 = A$;
- if A_α is defined, then we let $A_{\alpha+1} = \Gamma(A_\alpha)$;
- if A_β is defined for all $\beta \in \alpha$, then we let $A_\alpha = \bigcap_{\beta\in\alpha} A_\beta$.

First observe by induction that all of the A_α's are closed (recall that intersections of closed sets are closed).

We claim that there is some $\delta \in \omega_1$ such that $\Gamma(A_\delta) = A_\delta$. Suppose otherwise. Then for each $\alpha \in \omega_1$, there is some $x \in A_{\alpha+1} \setminus A_\alpha$, and moreover there is some open set U_α from the countable bases from Proposition 1.6 such that $x \in U_\alpha$ and $U_\alpha \cap (A_\alpha \setminus \{x\}) = \emptyset$. But this implies that the U_α's are distinct and therefore that $\{U_\alpha : \alpha \in \omega_1\}$ is uncountable list of elements in a supposedly countable basis, which is a contradiction.

Since $\Gamma(A_\delta) = A_\delta$, it follows by definition that A_δ is perfect. Furthermore, $A \setminus A_\delta$ is countable because each $A_\alpha \setminus A_{\alpha+1}$ is countable by an argument similar to the one in the paragraph above. Specifically, $\Gamma(A) \setminus A$

is always countable: For each element of $x \in \Gamma(A)$, there is a member U of the countable basis given by Proposition 1.6 such that $x \in U$ is the only element in $A \cap U$. Hence $A \setminus A_\delta$ is countable because it is the countable unions of countably many sets. □

Remark 1.8. In fact, the map constructed in Theorem 1.7 is continuous with respect to a topology on $\{0,1\}^{\mathbb{N}}$ (better known as *Cantor space* and more commonly denoted $2^{\mathbb{N}}$) that we will define later.

Corollary 1.9. *All closed subsets of* $\mathbb{R}$ *are finite, countable, or have the same cardinality of* $\mathbb{R}$*.*

In other words, the continuum hypothesis holds for closed subsets of $\mathbb{R}$, even though it is consistent with ZFC that the continuum hypothesis fails in general.

Here are some takeaways:

- The result also a dichotomy: there are no inbetween cardinalities for closed subsets of $\mathbb{R}$.
- This is a result required us to use a notion of a hierarchy that more or less deals with a notion of complexity. This is a fundamental theme in descriptive set theory.
- Two properties of $\mathbb{R}$ stand out in the proof: the fact that has a complete metric, which allows us to define the embedding in Proposition 1.2, and the fact that it has a countable basis, which allows us to show that the process of taking Cantor-Bendixson derivatives in Theorem 1.7 eventually stabilizes.

The last point leads us to the definition of a *Polish space*.

1.2.2 Examples of Polish Spaces

April 27, 2023

Now we are led to a generalization of what the real numbers are. In fact, Cantor-Bendixon analysis can be applied to any *Polish space*.

Definition 1.10. A *topological space* X is a set with an associated topology T. The topology τ is a collection of sets $U \subset X$ that:

- $\emptyset \in \tau$,

- τ is closed under finite intersections,
- τ is closed under arbitrary unions.

Relative to τ, these sets are considered *open*.

We say that a collection B of subsets of X is a *basis* for a topology τ if τ is the collection of unions of sets of B.

Example 1.11. The basis consisting of open intervals (a, b) for $a, b \in \mathbb{R}$ generates the standard Euclidean topology on $\mathbb{R}$.

Definition 1.12. Given a space X, a *metric* is a function $d : X \times X \to \mathbb{R}$ with the following properties:

1. $\forall a, b \in X$, $d(a, b) = 0$ if and only if $a = b$ (positive definiteness).
2. $\forall a, b \in X$, $d(a, b) = d(b, a)$ (symmetry).
3. $\forall a, b, c \in X$, $d(a, c) \leq d(a, b) + d(b, c)$ (triangle inequality).

Moreover:

- A topological space X is *metrizable* if there is a metric d such that the set of open balls of the form $\{z : d(y, z) < \epsilon\}$ for $y \in X$ and $\epsilon \in \mathbb{R}^+$ form a basis for its topology.
- Given a space X and a metric d, a sequence $(a_n)_{n\in\mathbb{N}}$ is *Cauchy* if $\lim_{m,n\to\infty} d(a_m, a_n) = 0$.
- A metric space is *complete* if all Cauchy sequences converge to a point in the space, i.e. if $(a_n)_{n\in\mathbb{N}}$ is Cauchy then there is some $b \in X$ such that $\lim_{n\to\infty} d(a_n, b) = 0$.

Example 1.13. The set $\mathbb{R}$ under the metrix $d(x, y) := |x - y|$ is completely metrizable, but this is not a complete metrix for the subspace $\mathbb{Q}$.

Definition 1.14. X is a *Polish space* if:

- it is separable (i.e. it has a countable dense set),
- and completely metrizable (i.e. there is a *complete* metric d that generates its topology).

Polish spaces are named for the fact that they were first studied by many Polish mathematicians like Tarski and Sierpinski.

Proposition 1.15. *If X is a metrizable topological space with a countable dense set, then it has a countable basis.*

Proof. Basically the same proof as Proposition 1.6. □

So we do not lose anything by using the slightly stronger requirement of separability.

Remark 1.16. The fact that a given Polish space is metrizable is often indirectly important, and we often do not use the metric.

Example 1.17. We will list some basic examples of Polish spaces.

- $\mathbb{R}$ is a Polish space, and of course it is our motivating example, but $\mathbb{Q}$ is not a Polish space (we have not proved this yet, because even though the usual metric is not complete, it remains to argue that it is impossible to find a complete metric).
- The space of irrationals $\mathbb{R} \setminus \mathbb{Q}$ is a Polish space (will include the argument, possibly as an exercise, later).
- Countable discrete spaces, i.e. spaces X with a metric d such that $d(x, y) = 1$ if and only if $x \neq y$, are Polish spaces, but uncountable discrete spaces are *not* Polish spaces (you can see why this is true from what we have learned so far).
- The set of continuous $\mathbb{R}$-valued functions with domain $[0, 1]$ under the norm $\|f - g\| := \sup_{x\in[0,1]} |f(x) - g(x)|$).

Theorem 1.18 (Baire Category Theorem)**.** *If X is a completely metrizable topological spece and $(D_n)_{n\in\mathbb{N}}$ is a sequence of open dense sets, then $\bigcap_{n\in\mathbb{N}} D_n$ is a dense set.*

Proof. Let d be a complete metric on X. For $z \in X$ and $\epsilon \in \mathbb{R}^+$, we let $B(z, \epsilon)$ denote the *open ball* $\{y \in X : d(x, y) < \epsilon\}$ and let $\bar{B}(z, \epsilon)$ denote the *closed ball* $\{y \in X : d(x, y) \leq \epsilon\}$. Let $y \in X$ and let $U \ni y$ be an arbitrary open set. Our goal is to find an element of $\bigcap_{n\in\mathbb{N}} D_n$ in U.

Define $z_n \in X$ and $r_n \in \mathbb{R}$ by induction as follows: First choose $z_0 \in D_0$ and r_0 such that $B(z_0, r_0) \subset U$ (this uses openness and density of D_0).

Given z_n, choose $z_{n+1} \in D_{n+1} \cap B(z_n, r_n)$, and then let r_{n+1} be small enough such that $\bar{B}(z_{n+1}, r_{n+1}) \subset B(z_n, r_n)$ and $B(z_{n+1}, r_{n+1}) \subset D_{n+1}$ (again using openness and density).

Then $\bigcap_{n\in\mathbb{N}} B(z_n, r_n) = \bigcap_{n\in\mathbb{N}} \bar{B}(z_n, r_n)$ is closed and $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence necessarily converging to some z_*, so $z_* \in \bigcap_{n\in\mathbb{N}} \bar{B}(z_n, r_n) \subseteq \bigcap_{n\in\mathbb{N}} D_n$ and $z_* \in U$. □

The Baire Category Theorem is often stated using the concept of meager sets, and we will get to these later.

Corollary 1.19. $\mathbb{Q}$ *under the standard topology is not a Polish space.*

Proof. The "standard topology" is the subspace topology, under which a set $U \subseteq \mathbb{Q}$ is open if there is some U' open in the standard topology for $\mathbb{R}$ such that $U = U' \cap \mathbb{Q}$. We will show that $\mathbb{Q}$ does not have a complete metric compatible with this topology by showing that it does not satisfy the conclusion of the Baire category theorem. For all $q \in \mathbb{Q}$, let $U_q = \mathbb{Q} \setminus \{q\}$. Then U_q is open in the subspace topology because $\{q\}$ is closed in $\mathbb{R}$. Moreover, U_q is dense because if $x \in \mathbb{Q}$ and V is any open set containing x, there is some rational other than q in V. However, $\bigcap_{q\in\mathbb{Q}} U_q = \emptyset$. □

1.2.3 Universality Properties of Polish Spaces

Definition 1.20. The *Hilbert cube* $\mathbb{H}$ is the space $[0,1]^{\mathbb{N}}$, in other words it consists of sequences $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in [0,1]$ for all $n \in \mathbb{N}$.

Theorem 1.21. *Every Polish space is homeomorphic to a subspace of* $\mathbb{H}$.

Proof. We will prove something a bit more general: every separable metrizable space is homoemorphic to a substance of $\mathbb{H}$.

Let X be a separable metrizable space. Let d be a metric on X duch that $d < 1$ and let $(x_n)_{n\in\mathbb{N}}$ be a dense countable set. Let let $f(x) := (d(x, x_n))_{n\in\mathbb{N}}$.

To see that f is injective: if $d(x,y) = \epsilon$, then choose n such that $d(x, x_n) < \epsilon/2$. Then $d(y, x_n) > \epsilon/2$, so $f(x) \neq f(y)$.

We prove that f is continuous. If $d(x,y) < \epsilon$ then $|d(x, x_i) - d(y, x_i)| < \epsilon$ (otherwise we get a contradiction using the triangle inequality) and therefore $d(f(x), f(y)) \leq \sum_{n\in\mathbb{N}} 1/2^{n+1}\epsilon = \epsilon$ (this is basically a hint for the exercises).

Now we prove that f^{-1} is continuous. Consider $f(x)$ and choose $n \in \mathbb{N}$ such that $d(x, x_n) < \epsilon/3$ (some $\epsilon > 0$). If $d(f(x), f(y)) \leq \frac{1}{3 \cdot 2^{n+1}}\epsilon$ then $|x_n - y| < \epsilon/3$. Hence if $d(f(x), f(y)) \leq \frac{1}{3 \cdot 2^{n+1}}\epsilon$ then $|x - y| < \epsilon$. $\square$

Definition 1.22. The space $\mathbb{N}^{\mathbb{N}}$, in other words the countable product of $\mathbb{N}$ with the discrete topology, is known as *Baire space* and is denoted $\mathcal{N}$.

Remark 1.23. There is a difference between Baire space and *a* Baire space!

Definition 1.24. The space $2^{\mathbb{N}}$ is known as *Cantor space* and is denoted $\mathcal{C}$.

May 4, 2023

Theorem 1.25. *For every compact Polish space X, there is a continuous map from $\mathcal{N}$ onto X.*

Proof. Define a system of closed balls $(C_s)_{s \in \mathcal{N}}$ with the following requirements:

Changes to the version from the lecture!

(1) $C_\emptyset = X$,

(2) $\operatorname{diam}(C_s) = 1/(|s| + 1)$,

(3) $C_s \subset \bigcup_{k \in \mathbb{N}} C_{s^\frown k}$,

(4) If $s \sqsubseteq t$ then $C_t \subseteq C_s$.

We let $F(f)$ be the unique point in $\bigcap_{n \in \mathbb{N}} C_{f \restriction n}$.

First we argue that F is defined on all of $\mathcal{N}$ (the argument is the same as in Proposition 1.2). The set $\bigcap_{n \in \mathbb{N}} C_{f \restriction n}$ is nonempty because any sequence of points from the $C_{f \restriction n}$ will be a Cauchy sequence and the sets are closed. Moreover, this set is a singleton because of (2).

Next we argue that F is surjective. Observe that for all $z \in X$, there is some $f \in \mathcal{N}$ (not necessarily unique) such that $z \in C_{f \restriction n}$ for all $n \in \mathbb{N}$: this f can be defined inductively using (3). Then $z \in \bigcap_{n \in \mathbb{N}} C_{f \restriction n}$ because otherwise there would be an open $U \ni z$ such that $U \cap \left(\bigcap_{n \in \mathbb{N}} C_{f \restriction n}\right) = \emptyset$, which is a contradiction.

Finally, we argue that F is continuous. Let $U \subseteq X$ be open. We want to show that $F^{-1}(U)$ is open. Let $f \in F^{-1}(U)$, so there is some $z \in U$ such that $F(f) = z$. We want to find an open $V \subseteq \mathcal{N}$ such that $f \in V \subseteq F^{-1}(U)$. Choose n large enough that $C_{f \restriction n} \subseteq U$ (using (2)). Then $V := \{x \in \mathcal{N} : (f \restriction n) \sqsubseteq x\}$ is open in $\mathcal{N}$ and $F"V \subseteq C_{f \restriction n} \subseteq U$ by (4), i.e. $N_{f \restriction n} \subseteq F^{-1}(U)$. $\square$

1.3 Trees

Definition 1.26. Let X be any set and let $X^{<\omega}$ denote the set of finite sequences of elements of X. A *tree* on X is a nonempty set $T \subseteq X^{<\omega}$ such that if $s \in T$ and $t \sqsubseteq s$ (meaning that t is an initial segment of s) then $t \in T$.

The *body* of T, a tree on X, is the set $\{y \in X^\omega : \forall n, x \upharpoonright n \in T\}$ and is denoted $[T]$. (Here X^ω denotes the set of countable sequences of elements of X.

Example 1.27. For example, the tree of all finite binary sequences. Both $2^{\mathbb{N}}$ (Cantor space) and $\mathbb{N}^{\mathbb{N}}$ are bodies of trees.

Theorem 1.28 (König's Theorem). *If $T \subseteq X^{<\omega}$ is an infinite and* finitely branching *(meaning that for all $t \in T$, $\{s \sqsupseteq t : |s| = |t| + 1\}$ is finite), then $[T] \neq \emptyset$.*

Proof. Inductively choose $\langle t_n \rangle n \in \mathbb{N}$ such that $t_n \sqsubseteq t_{n+1}$ for all $n \in \mathbb{N}$ and such that $\{s \in T : s \sqsupseteq t_n\}$ is infinite for all $n \in \mathbb{N}$. Then $(t_n)_{n\in\mathbb{N}} \in [T]$. □

Definition 1.29. Let $T \subseteq X^{<\omega}$ be a tree. If $s \in T$ then we let $T_s := \{t \in T : s \sqsubseteq t \text{ or } t \sqsubseteq s\}$, which is also a tree.

Proposition 1.30. *The topology for Baire space a basis consisting of $N_s := \{x \in \omega^\omega : s \subseteq x\}$ for all $s \in \omega^{<\omega}$. The topology for Cantor space a basis consisting of $N_s := \{x \in 2^\omega : s \subseteq x\}$ for all $s \in 2^{<\omega}$.*

Proposition 1.31. *For Baire space and Cantor space, the N_s's are closed.*

Proof. For Baire space, $\mathcal{N} \setminus N_s = \bigcup_{t \neq s, |t|=|s|} N_t$. The idea for Cantor space is analogous. □

Definition 1.32. A topological space X is *totally disconnected* if it has a basis consisting of clopen sets. It is 0-*dimensional* if it is Hausdorff and has a basis consisting of clopen sets.

Proposition 1.33. *The topologies of both Baire space and Cantor spaces are induced by the metric d such that $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1/2^n$ if n is minimal such that $x(n) \neq y(n)$.*

Example 1.34. Baire space and Cantor space are 0-dimensional.

Definition 1.35. A tree $T \subset X^{<\omega}$ is *pruned* if it has no terminal nodes, i.e. for all $t \in T$, there is some $s \sqsupseteq t$ such that $s \neq t$.

Theorem 1.36. *A set $C \neq \emptyset$ is closed in Baire space (or Cantor space) if and only if $C = [T]$ for some pruned tree $T \subseteq \omega^{<\omega}$ (or $2^{<\omega}$).*

Proof. We will write the proof without disinguishing between Baire space and Cantor space.

Let C be closed and nonempty. Set $T := \{x \upharpoonright n : x \in C, n \in \omega\}$. Clearly T is pruned and $C \subseteq [T]$. On the other hand, if $x \in [T]$, then for all $n \in \omega, \exists y_n \in C$ such that $x \upharpoonright n = y_n \upharpoonright n$. It follows that $(y_n)_{n \in \omega}$ converges to x, and so $x \in C$ by closure.

Conversely, let T be a tree. Then $[T]$ is closed because if $x \notin [T]$ then $\exists n, s := x \upharpoonright n \notin T$, which means that $x \in N_s$ and $N_s \cap [T] = \emptyset$. □

Proposition 1.37. *Baire space is non-compact.*

Proof. Take the open cover $(N_{s_n})_{n \in \mathbb{N}}$ where $s_n = \langle 0, n \rangle$. □

Theorem 1.38. *A nonempty set $K \subseteq \omega^\omega$ is compact if and only if $K = [T]$ for a pruned, finitely branching tree T.*

Proof. Assume that K is compact. Then K is in particular closed, so $K = [T]$ for some pruned $T \subseteq \omega^{<\omega}$ by the previous theorem. Suppose for contradiction that T is not finitely branching. Then there is some $s \in T$ such that $E = \{n \in \omega : s^\frown\langle n \rangle \in T\}$ is infinite. Then $U = \{N_{s^\frown\langle n \rangle} : n \in E\} \cup \{N_t : t \in T, s \perp t\}$ (where $s \perp t$ means that $s \not\sqsubseteq t$ and $t \not\sqsubseteq s$) is an open cover with no finite subcover. May 11, 2023

Now assume that T is pruned and finitely branching. Let $(U_i)_{i \in I}$ be an open cover for $K := [T]$. Suppose for contradiction that it has no finite subcover. Let

$$S = \{s \in T : [T_s] \text{ is } \textit{not} \text{ covered by finitely many } U_i\}.$$

Then $\emptyset \in S$ by assumption. Furthermore, S is finitely branching—because if a node $s \in S$ were infinitely branching, it would have to follow that $s \notin S$. By König's Theorem, there is some $x \in \omega^\omega$ such that $x \in [S] \subseteq [T]$. Then $x \in U_i$ for some $i \in I$. By openness, $N_{x \upharpoonright n} \subseteq U_i$ for some $n \in \omega$, and so $[T_{x \upharpoonright n}] \subseteq U_i$, contradicting that $s \upharpoonright n \in S$. □

Corollary 1.39. *Cantor space is compact.*

So Cantor space and Baire space are meaningfully distinct!

Chapter 2

Borel Sets

2.1 σ-Algebras

Definition 2.1. A collection $\mathcal{A} \subseteq P(S)$ is a *σ-algebra* on S if:

- it is closed under complements,
- it is closed under countable unions (hence also countable intersections),
- both S and $\emptyset$ are in $\mathcal{A}$.

Example 2.2. Lebesgue-measurable subsets of $[0,1]$.

Definition 2.3. Let X be a topological space.

- A set $A \subseteq X$ is *nowhere dense* if whenever $U \subseteq X$ is open and nonempty, there is $V \subseteq U$ also open nonempty such that $A \cap V = \emptyset$.
- A set A is *meager* if it is contained in the countable union of nowhere dense sets.
- A set is *comeager* if its complement is meager.
- A set $B \subseteq X$ has the *Baire property* if there is an open set $U \subseteq X$ such that $B \triangle U = (B \setminus U) \cup (U \setminus B)$ is meager.

Remark 2.4. The Baire Category Theorem states that a completely metrizable space is not meager as a subset of itself.

Example 2.5. $\mathbb{Q}$ is meager is a subspace of $\mathbb{R}$.

Example 2.6. The *Cantor set* is the set constructed by starting with the unit interval $[0,1]$, removing its middle third, then removing the middle third of both $[0,1/3]$ and $[2/3,1]$, and so on. The explicit formula is

$$[0,1] \setminus \bigcup_{n\in\mathbb{N}} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}}\right).$$

The Cantor set has size $2^{\aleph_0}$ because it is perfect (this is similar to earlier proofs), but it is nowhere dense (we are leaving this proof out), so it is meager.

Proposition 2.7. *The subsets of a topological space X with the Baire property form a σ-algebra*

Proof. First we show that the Baire sets are closed under countable unions. Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of Baire sets and let $B = \bigcup_{n\in\mathbb{N}} B_n$. Let $(U_n)_{n\in\mathbb{N}}$ be such that $B_n \triangle U_n$ is meager for all n and let $U = \bigcup_{n\in\mathbb{N}} U_n$. Then $\bigcup_{n\in\mathbb{N}}(B_n \triangle U_n)$ is meager, and so

$$B\triangle U = (B\setminus U)\cup(U\setminus B) = \bigcup_{n\in\mathbb{N}}(B_n\setminus U)\cup\bigcup_{n\in\mathbb{N}}(U_n\setminus B) \subseteq \bigcup_{n\in\mathbb{N}}(B_n\triangle U_n)$$

shows that $B\triangle U$ is meager.

Now we show that the Baire sets are closed under complements, but first we establish some claims.

Claim. *Closed subsets are Baire.*

Proof of Claim. The C be closed and let C° be its interior with $B = C\setminus C^\circ$ being its boundary by definition. Then $B^\circ \subseteq C^\circ$ which implies that $B^\circ = \emptyset$. Also B is closed, and closed plus empty interior implies nowhere density. The claim then follows from $C\triangle(C^\circ) = B$. □

Claim. *If $A, B \subseteq X$ and A is Baire and $A\triangle B$ is meager then B is Baire.*

Proof of Claim. Let U be such that $A\triangle U$ is meager. Then

$$\begin{aligned} B\triangle U &= B\triangle(\emptyset\triangle U) = B\triangle((A\triangle A)\triangle U) = \\ &= (B\triangle A)\triangle(A\triangle U) \subseteq (B\triangle A)\cup(A\triangle U) \end{aligned}$$

where fourth equality works because $\triangle$ is associative. Because $(B\triangle A) \cup (A\triangle U)$ is meager because both sets being unioned are meager, therefore $B\triangle U$ is meager. $\square$

Now we can finish the proof. Let $B\triangle U$ be meager where U is open. Then $(X \setminus B)\triangle(X \setminus U) = B\triangle U$ is meager. The set $X \setminus U$ is closed and therefore Baire by the first claim, and it follows that $X \setminus B$ is Baire by the second claim. $\square$

Proposition 2.8. *Given a topological space X, there is a smallest σ-algebra containing the open sets.*

Proof. Zorn's lemma. $\square$

Definition 2.9. Given a topological space, the set of *Borel sets* is the smallest σ-algebra containing all of the open sets.

Proposition 2.10. *All Borel sets have the Baire property.*

Remark 2.11. The Vitali set does not have the Baire property. In Solovay's model, all sets have the Baire property.

2.2 The Borel Hierarchy

Definition 2.12. Let X be a Polish space. We define the following by induction on $\alpha < \omega_1$.

- $\mathbf{\Sigma}^0_1(X)$ is the collection of open sets in X,
- $\mathbf{\Pi}^0_1(X)$ is the set of closed sets in X (i.e. the set of complements of sets in $\mathbf{\Sigma}^0_1$),
- $\mathbf{\Sigma}^0_\alpha(X)$ is the collection of countable unions of sets in $\bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta$,
- $\mathbf{\Pi}^0_\alpha(X)$ is the collection of countable intersections of sets in $\bigcup_{\beta<\alpha} \mathbf{\Sigma}^0_\beta$(i.e. the set of complements of sets in $\mathbf{\Sigma}^0_\alpha$),
- $\mathbf{\Delta}^0_\alpha(X) = \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X)$.

Remark 2.13. We will often drop the notation for the Polish space X.

May 25, 2023

Example 2.14. $[0,1)$ is $\boldsymbol{\Delta}^0_2$ in $\mathbb{R}$: it is expressible as both a countable union of closed sets and a countable intersection of open sets. However, it is neither $\boldsymbol{\Sigma}^0_1$ nor $\boldsymbol{\Pi}^0_1$ because it is neither closed nor open.

Proposition 2.15. $\bigcup_{\alpha<\omega_1} \boldsymbol{\Sigma}^0_\alpha(X) = \bigcup_{\alpha<\omega_1} \boldsymbol{\Pi}^0_\alpha(X)$ *is a* σ*-algebra and therefore is equal to the collection of Borel sets* $\mathcal{B}(X)$*.*

Proof. It is immediate that $\bigcup_{\alpha<\omega_1} \boldsymbol{\Sigma}^0_\alpha(X) = \bigcup_{\alpha<\omega_1} \boldsymbol{\Pi}^0_\alpha(X)$ is a σ-algebra, so $\supseteq \mathcal{B}(X)$ follows immediately. By definition of the hierarchy, we can argue by induction that we have $\boldsymbol{\Sigma}^0_\alpha(X) \subseteq \mathcal{B}(X)$ for all $\alpha < \omega_1$. □

Example 2.16. $\mathbb{Q}$ is $\boldsymbol{\Sigma}^0_2$ as a subset of $\mathbb{R}$ because it is the union of points, so it is the union of closed sets. It is not open or closed. It also is not $\boldsymbol{\Pi}^0_2$: If we had $\mathbb{Q} = \bigcap_{n\in\mathbb{N}} U_n$ for open U_n's, then it would follow that each U_n is dense because $\mathbb{Q}$ is. But complements of open dense sets are nowhere dense, so this would imply that $\mathbb{R} \setminus \mathbb{Q}$ is meager, which would imply that $\mathbb{R}$ is meager, but this contradicts the Baire Category Theorem.

Example 2.17. Let $A = \{x \in \mathcal{N} : x \text{ is a bijection}\}$. Then A is $\boldsymbol{\Pi}^0_2$.

Let $A_0 = \{x : \forall n \forall m (n \neq m \implies x(n) \neq x(m))\}$. Then

$$A_0 = \bigcap_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}, m\neq n} \{x : x(n) \neq x(m)\}$$

is closed since it is an intersection of closed sets (if $x(n) = x(m)$ then take a stem s below x of length $\max\{m,n\}$ and then N_s will be contained in the complement). Then let $A_1 = \{x : \forall n \exists m, x(m) = n\}$. Then

$$A_1 = \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \{x : x(m) = n\}$$

is $\boldsymbol{\Pi}^0_2$, being the countable intersection of open sets, and $A = A_0 \cap A_1$ is $\boldsymbol{\Pi}^0_2$.

Proposition 2.18. *If* X *is an infinite Polish space, then* $|\mathcal{B}(X)| = 2^{\aleph_0}$*.*

Proof. Since X is infinite metrizable, every infinite subset is $\boldsymbol{\Sigma}^0_2(X)$, so we have $2^{\aleph_0} \leq |\mathcal{B}(X)|$. To get the other direction, first see that $|\boldsymbol{\Sigma}^0_1| \leq 2^{\aleph_0}$ using that because there is a countable basis and every open set is a union of elements in the basic. Then proceed by induction: elements of $\boldsymbol{\Sigma}^0_\alpha$ correspond to countable functions on $\bigcup_{\beta<\alpha} \boldsymbol{\Sigma}^0_\beta$, so we get a bound from the fact that $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$. □

Proposition 2.19. *All $\mathbf{\Sigma}^0_\alpha(X)$'s and $\mathbf{\Pi}^0_\alpha(X)$'s and $\mathbf{\Delta}^0_\alpha(X)$'s are closed under finite unions, finite intersections, and preimages of continuous functions $f : X \to X$.*

Proof. The first two statements follow easily from de Morgan's Laws, so we will prove the third.

The statement for $\mathbf{\Sigma}^0_1(X)$ is just the definition of continuity. Now suppose $B = \bigcup_{n\in\mathbb{N}} B_n$ where $B_n \in \mathbf{\Pi}^0_{\beta_n}$ for some $\beta_n < \alpha$ for all $n \in \mathbb{N}$. Then $f^{-1}(B) = \bigcup_{n\in\mathbb{N}} f^{-1}(B_n)$, so the statement follows by induction. Now we get the same thing for the $\mathbf{\Sigma}^0_\alpha(X)$'s because of the fact that $f^{-1}(X) = f^{-1}(B \cup (X \setminus B)) = f^{-1}(B) \coprod f^{-1}(X \setminus B)$. The statement for the $\mathbf{\Delta}$'s is immediate. □

2.3 Universal and Complete Sets

Proposition 2.20. *$\mathcal{N}$ is homeomorphic to its infinite product $\mathcal{N}^{\mathbb{N}}$.*

Proof. Let $\Gamma : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the canonical pairing function. We define $\Phi : \mathcal{N} \to \mathcal{N}^{\mathbb{N}}$ be setting $\Phi(f)_n(k) = f(\Gamma(n, k))$ (where the subscript indicates the coordinate).

To see that Φ is injective, suppose $f \neq g$ for $f, g \in \mathcal{N}$. Then there is some $m \in \mathbb{N}$ such that $f(m) \neq g(m)$, so if n, k are such that $\Gamma(n, k) = m$, then $\Phi(f)_n(k) \neq \Phi(g)_n(k)$. To see surjectivity, let $\Gamma^{-1}(m) = (\gamma_0(m), \gamma_1(m))$. Then if $(h_n)_{n\in\mathbb{N}} \in \mathcal{N}^{\mathbb{N}}$, let $f(m) = h_{\gamma_0(m)}(\gamma_1(m))$, then $\Phi(f) = (h_n)_{n\in\mathbb{N}}$.

To see continuity: Let $U \subseteq \mathcal{N}^{\mathbb{N}}$ be open and suppose that $\Phi(f) \in I$. Let V be a basic open set such that $\Phi(f) \in V \subseteq U$, so $V = \prod_{n\in\mathbb{N}} V_n$ where $V_n = N_{s_n}$ for finitely many $I \subset \mathbb{N}$ and $V_n = \mathcal{N}$ for the rest. Choose m larger than $\Gamma(n, k)$ for all $n \in I$ and k such that $k \leq |s_n|$. Then for all g such that $g \restriction (m + 1) = f \restriction (m + 1)$, we have $s_n \sqsubseteq \Phi(g)_n \restriction (m + 1)$. It follows that $\Phi'' N_{f\restriction m} \subseteq V$.

The fact that Φ is an open function is similar. One can argue that images of basic open sets are open. □

Definition 2.21. Let X, Y be Polish spaces. If $U \subseteq Y \times X$ and $a \in Y$, then let $U_a = \{b \in X : (a, b) \in U\}$.

We say $U \subset Y \times X$ is *universal-*$\mathbf{\Sigma}^0_\alpha$ if $U \in \mathbf{\Sigma}^0_\alpha(Y \times X)$ and if for all $A \in \mathbf{\Sigma}^0_\alpha(X)$, there is some $a \in A$ such that $A = U_a$. Universal $\mathbf{\Pi}^0_\alpha$ sets are defined similarly.

Proposition 2.22. *$\mathcal{C}$ is homeomorphic to its infinite product $\mathcal{C}^{\mathbb{N}}$.*

Proof. The proof is essentially the same as in the previous proposition. This time we take a pairing function $\Gamma : \mathbb{N} \times \{0,1\} \to \mathbb{N}$ and let $\Phi : \mathcal{C} \to \mathcal{C}^{\mathbb{N}}$ be defined by setting $\Phi(f)_n(k) = f(\Gamma(n,k))$. □

Lemma 2.23. *If X is a separable metric space, then for all $1 \leq \alpha < \omega_1$ there is a $\mathbf{\Sigma}^0_\alpha$-universal set $U_\alpha \subset \mathcal{N} \times X$ and a $\mathbf{\Pi}^0_\alpha$-universal set $V_\alpha \subset \mathcal{N} \times X$.*

Corollary 2.24. *For each $1 \leq \alpha < \omega_1$ there is a set $A \subseteq \mathcal{N}$ that is $\mathbf{\Sigma}^0_\alpha$ but not $\mathbf{\Pi}^0_\alpha$. (In other words, the Borel hierarchy is strict.)*

Proof. Let U be the universal $\mathbf{\Sigma}^0_\alpha$ set and let

$$A = \{x : (x,x) \in U\}.$$

Since $\mathbf{\Sigma}^0_\alpha$ is closed under continuous preimages, A is $\mathbf{\Sigma}^0_\alpha$. If it were also $\mathbf{\Pi}^0_\alpha$, then its complement would be $\mathbf{\Sigma}^0_\alpha$. Then it would be the case that for some $f \in \mathcal{N}$,

$$\{x : (x,x) \notin U\} = \mathcal{N} \setminus A = U_f = \{x \in \mathcal{N} : (f,x) \in U\},$$

which is a contradiction because $f \in A$ if and only if $f \in \mathcal{N} \setminus A$. □

June 15, 2023

Proof of Lemma 2.23. Let $(B_n)_{n\in\mathbb{N}}$ enumerate a countable basis for X. We will give the proof for $\mathbf{\Sigma}$ and $\mathbf{\Pi}$ simultaneously by induction.

Case 1: $\alpha = 1$. First we obtain a $\mathbf{\Sigma}^0_1$-universal set U_1. Let

$$U_1 = \bigcup_{n\in\mathbb{N}} \{(f,x) : x \in B_{f(n)}\}.$$

The fact that U_1 is open will follow from the fact that for all $n \in \mathbb{N}$, $\{(f,x) : x \in B_{f(n)}\}$ is open. Suppose (f_0, x_0) is in this set. Let $s \in \mathbb{N}^{<\mathbb{N}}$ be such that s has length n and $s \upharpoonright (n+1) = f \upharpoonright (n+1)$. Then it follows that $N_s \times B_{f(n)} \subseteq \{(f,x) : x \in B_{f(n)}\}$.

Now if A is open, let f be defined such that $A = \bigcup_{n\in\mathbb{N}} B_{f(n)}$ (there are multiple options). Then we see that $x \in A$ if and only if $(f,x) \in U_1$, so U_1 is in fact $\mathbf{\Sigma}^0_1$-universal.

Next observe that $V_1 := (\mathcal{C} \times X) \setminus U_1$ is Π^0_1-universal: it is closed, and we can show universality by taking complements.

Case 2: $\alpha = \beta + 1$. Let U_β be the universal $\mathbf{\Sigma}^0_\beta$ set. Then we let

$$U_\alpha = \bigcup_{n\in\mathbb{N}} \{(f,x) : (\Phi(f)_n, x) \notin U_\beta\}$$

where $\Phi : \mathcal{N} \to \mathcal{N}^\omega$ is continuous and $\Phi(f)_n$ is the n^{th} coordinate of $\Phi(f)$.

First, to show that U_α is $\mathbf{\Sigma}^0_\alpha$, it is enough by definition to show that for all $n \in \mathbb{N}$, $\{(f,x) : (\Phi(f)_n, x) \notin U_\beta\}$ is $\mathbf{\Pi}^0_\beta$. This uses closure under continuous preimages (exercise).

If A is $\mathbf{\Sigma}^0_\alpha$, then $A = \bigcup_{n\in\mathbb{N}} A_n$ where each A_n is $\mathbf{\Pi}^0_\beta$. Choose f_n such that $X \setminus A_n = \{x : (f_n, x) \in U_\beta\}$. Then choose f such that $\Phi(f)_n = f_n$ for all $n \in \mathbb{N}$. Then $A = \{x : (f,x) \in U_\alpha\}$.

Case 3: α is a limit ordinal. Let $(\beta_n)_{n<\omega}$ be a strictly increasing sequence of ordinals such that $\sup_{n<\omega} \beta_n = \alpha$. Let U_{β_n} be the universal $\mathbf{\Sigma}^0_{\beta_n}$ set. Then let

$$U_\alpha = \bigcup_{n\in\mathbb{N}} \{(f,x) : (\Phi(f)_n, x) \notin U_{\beta_n}\}.$$

The rest of the argument is analogous to the successor case. □

Chapter 3

Games of Perfect Information

3.1 Examples of Games

Definition 3.1. Let $A \subseteq \mathcal{N}$. The *Banach-Mazur game on A*, denoted $G^{**}(A)$, is the following game between two players:

- Player I and Player II alternately choose elements of $\omega^{<\omega} \setminus \emptyset$ as in $s_0, s_1, s_2, s_3, \ldots$ where the s_n's denote Player I's moves for even n and Player II's moves for odd n.
- Player I wins the game if the concatenation $s_0{}^\frown s_1{}^\frown s_2{}^\frown s_3{}^\frown \ldots \in A$. Otherwise Player II wins the game.

Definition 3.2. We will define concepts with respect to the Banach-Mazur game for the sake of simplifying notation, but the generalizations will be implicit.

- A *strategy* σ for either Player I or II in the game $G^{**}(A)$ is a function from partial plays of the game to nonempty finite sequences of natural numbers.
- A strategy σ is *compatible* with a play $\vec{s} = \langle s_i : 0 \leq i < n \rangle$ if $\vec{s}$ is in the domain of the strategy.
- Any strategy can be written as a tree consisting of finite sequences *of* finite sequences of natural numbers. A *winning strategy* is a strategy σ for either Player I or II such that if that player uses the strategy,

then the player is guaranteed to win. In other words, a σ is winning for e.g. Player I if every branch $[\sigma]$ is a winning player for Player I, i.e. the intersection of all W_i's in σ is empty.

Theorem 3.3. *Let $A \subseteq \mathcal{N}$.*

1. *Player II has a winning strategy for the Banach-Mazur game if and only if A is meager.*
2. *Player I has a winning strategy for the Banach-Mazur game if and only if there is a basic open set N_s such that $N_s \setminus A$ is meager.*

Proof of (1). First suppose that A is meager. Recall that the closure of a nowhere dense set is nowhere dense, so we are saying that $A \subseteq \bigcup_{n\in\mathbb{N}} C_n$ where C_n is closed and nowhere dense. We describe a strategy σ for Player II as follows: If $s_0, \ldots, s_{2n}$ has been played and s^* is the concatenation of everything played so far, then the nowhere density of C_n implies that there is some t such that $N_{s^\frown t} \cap C_n = \emptyset$, so $\sigma(\langle s_0, \ldots, s_{2n}\rangle) = t$. If $\langle s_n : n \in \mathbb{N}\rangle$ is a play where Player II plays according to σ, and $s^+ = s_0{}^\frown s_1{}^\frown s_2{}^\frown \ldots$, then $s^+ \notin C_n$ for all $n \in \mathbb{N}$, consequently $s^+ \notin A$.

Now suppose that Player II has a winning strategy σ. For each partial play $p = \langle s_0, \ldots, s_{2n-1}\rangle$ where it is Player I's turn, let $p^* = s_0{}^\frown s_1{}^\frown \ldots {}^\frown s_{2n-1}$ and let

$$D_p = \{x \in \mathcal{N} : p^* \subseteq x \implies \exists t \in \omega^{<\omega} \setminus \emptyset, p^{*\frown} t^\frown \sigma(p^\frown \langle t\rangle) \subseteq x\}.$$

Claim. *D_p is open dense.*

Proof of Claim. For openness, suppose $x \in D_p$ as witnessed by t. Let $u = p^{*\frown} t^\frown \sigma(p^\frown\langle t\rangle)$. Then by definition $N_u \subseteq D_p$.

Now for density (and hence nonemptyness), consider some arbitrary $u \in \omega^{<\omega}$. If $p^* \not\subseteq u$ then we vacuously have $u \in D_p$, so suppose $p^* \subseteq u$. Let t be such that $p^{*\frown} u$. Then let x be such that $p^{*\frown} t^\frown \sigma(p^\frown\langle t\rangle) \subseteq x$ (there are many options). Then $x \in N_u \cap D_p$. □

Let P be the set of all partial plays where it is Player I's turn to play. Then P is countable, so consider any $x \in \bigcap_{p\in P} D_p$. Then we can inductively construct a sequence $\langle s_n : n \in \mathbb{N}\rangle$ which is a play of the Banach-Mazur game such that $x = s_0{}^\frown s_1{}^\frown \dots$ and Player II is using the winning strategy σ, so we know that $x \notin A$. Hence $\bigcap_{p\in P} D_p \subseteq \mathcal{N} \setminus A$, so $A \subseteq \bigcup_{p\in P}(\mathcal{N} \setminus D_p)$ showing that A is meager. □

Proof of (2). Suppose that $N_s \setminus A$ is meager. Then Player I uses the same idea that Player II would use for the game $G^{**}(N_s \setminus A)$. More specifically, we would have $N_s \setminus A \subseteq \bigcup_{n\in\mathbb{N}} C_n$ for C_n nowhere dense and Player I would avoid C_n at their n^{th} move.

Now suppose that Player I has a winning strategy σ and $\sigma(\emptyset) = s$. Then Player II has a winning strategy for $G^{**}(A \setminus N_s)$ based on shifting the turns and using Player I's strategy. Therefore $A \setminus N_s$ is meager by the first part. □

Definition 3.4. Given $A \subseteq \mathcal{C}$, the *perfect set game*, denoted $G_2^*(A)$, is the following game between two players:

- Player I plays elements $s_{2n} \in 2^{<\omega}$ and Player II plays elements $k_{2n+1} \in \{0, 1\}$.
- Player II wins the game if the concatenation $s_0{}^\frown\langle k_1\rangle^\frown s_2{}^\frown\langle k_3\rangle^\frown \dots \in A$. Otherwise Player II wins the game.

Theorem 3.5. *Let* $A \subseteq \mathcal{C}$.

1. *Player* II *has a winning strategy in* $G_2^*(A)$ *if and only if* A *is countable.*
2. *Player* I *has a winning strategy in* $G_2^*(A)$ *if and only if* A *contains a perfect subset.*

Proof of (1). Suppose that A is countable with an enumeration $\langle x_n : n \in \mathbb{N}\rangle$. Then the strategy for Player II is to play, at their n^{th} move, the digit k_{2n+1} in such a way that $s_0{}^\frown \dots {}^\frown\langle k_{2n+1}\rangle \not\subseteq x_n$.

Now suppose that Player II has a winning strategy σ. For a partial play $p = \langle s_0, k_1, \dots, s_{2n}, k_{2n+1}\rangle$, let $p^* = s_0 \frown \langle k_1\rangle^\frown \dots {}^\frown s_{2n}{}^\frown k_{2n+1}$ and let

$$D_p = \{x \in \mathcal{C} : p_* \subseteq x \implies \exists t, (p_*{}^\frown t^\frown \sigma(p^\frown\langle t\rangle) \subseteq x\}.$$

As in the argument for the Banach-Mazur game, we have $A \subseteq \bigcup_p(\mathcal{C} \setminus D_p)$.

We are then finished if we can argue that each $\mathcal{C} \setminus D_p$ is a singleton, which we do here. Specifically, we define a unique element x_p inductively.

Let $|p^*| = m$ and $x_p \upharpoonright m = p^*$. Then it must be the case that $x_p(m) = 1 - \sigma(p^\frown\emptyset)$ since we know $x_p \notin D_p$. For $\ell > m$, we can define $x_p(\ell) = 1 - \sigma(p^\frown\langle x_p(m), \ldots, x_p(\ell-1)\rangle)$. □

Proof of (2). Suppose A is perfect and let $T = \{x \upharpoonright n : x \in A, n \in \mathbb{N}\}$. Then the winning strategy for Player I is to play up to splitting points, i.e. if $s_0, k_1, \ldots, k_{2n-1}$ is the play so far then Player I chooses s_{2n} such that if $u = s_0{}^\frown\langle k_1\rangle^\frown \ldots {}^\frown\langle k_{2n-1}\rangle$ then both $u^\frown\langle 0\rangle \in T$ and $u^\frown\langle 1\rangle \in T$. Such an s_{2n} will exist precisely because A has no isolated points.

Now suppose that Player I has a winning strategy σ. Then we can build a perfect tree (i.e. every node has a splitting node above it) using σ. The branches of the tree will be elements of A, and the perfect-ness of the tree witnesses that A is perfect as a subset of $\mathcal{C}$. □

3.2 Open Determinacy

Definition 3.6. Let $A \subseteq \mathcal{N}$. Then game G_A is played between Player I and Player II, who alternate playing natural numbers, i.e. the play is a sequence $\langle s_n : n \in \mathbb{N}\rangle \subseteq A$ where Player I plays s_n for n even and Player II plays s_n for n odd. Player I wins if the sequence is in A and otherwise Player II wins.

Theorem 3.7 (Gale-Stewart)**.** *If $A \subseteq \mathcal{N}$ is open then G_A is determined.*

Proof. We will denote Player I's moves by a_n for $n \in \mathbb{N}$ and we will denote Player II's moves by b_n. Let us assume that Player I does not have a winning strategy, and we will prove that Player II has a winning strategy. The strategy is to always avoid losing positions.

We describe the winning strategy σ for Player I by induction on the length of a play. Suppose that a_0 is an opening move by Player I. Then because Player I does not have a winning strategy, there exists b_0 such that Player I does not have a winning strategy *from the position* $\langle a_0, b_0\rangle$. Therefore let $\sigma(\langle a_0\rangle) = b_0$. Now suppose $\langle a_0, b_0, \ldots, b_{n-1}, a_n\rangle$ is a play where b_{n-1} was chosen such that Player I does not have a winning strategy from the position $\langle a_0, b_0, \ldots, b_{n-1}\rangle$. Then by the same reasoning as in the base case, there must be some b_n such that Player I does not have a winning strategy from the position $\langle a_0, b_0, \ldots, b_{n-1}, a_n, b_n\rangle$, so we let $\sigma(\langle a_0, b_0, \ldots, b_{n-1}, a_n\rangle) = b_n$. This finishes the definition of σ.

Now we argue that σ is a winning strategy. Suppose for contradiction that $x = a_0, b_0, \dots, a_n, b_n, \dots$ is a play of the game according to σ but that the sequence x is in A (i.e. Player I wins). Then there some $s \in \omega^{<\omega}$ such that $x \in N_s \subseteq A$. But if $|s| = 2n$ (without loss of generality) then this means that $a_0, b_0, \dots, a_n, b_n$ is a losing position for Player II, which is a contradiction of the construction. $\square$

Example 3.8. Consider chess. If we temporarily redifine "winning" for Black as either winning or drawing, then chess′ is an open game because all games are decided in finitely many steps. This shows that there as an at least drawing strategy for one of the players in chess.

Fact 3.9 (Martin)**.** *Every Borel game is determined.*

3.3 The Axiom of Determinacy

Definition 3.10. The *axiom of determinacy* is the assertion that for all $A \subseteq \mathcal{N}$, G_A is determined.

Fact 3.11. AD *is consistent from infinitely many Woodin cardinals.*

Corollary 3.12. AD *implies that all sets in* $\mathcal{N}$ *have the Baire property.*

We will need a quick fact that we have not yet established.

Proposition 3.13. *In* $\mathcal{N}$*, no nonempty open set is meager.*

Proof. Let U be an open set and suppose for contradiction that $U \subseteq \bigcup_{n\in\mathbb{N}} X_n$ where the X_n's are nowhere dense. Define a sequence of stems $\langle s_n : n \in \mathbb{N}\rangle$ such that $N_{s_n} \subseteq U$ for all $n \in \mathbb{N}$, such that $n \le m$ implies $s_n \sqsubseteq s_m$, and such that $N_{s_n} \cap X_n = \emptyset$ (using nowhere density). Let x be the union of these stems, which is in the open set U. But $x \notin \bigcup_{n\in\mathbb{N}} X_n$. $\square$

Proof of Corollary 3.12. First we need to argue that the Banach-Mazur game can be coded as a game on real numbers. Let $\langle u_k : k \in \mathbb{N}\rangle$ be an enumeration of finite sequences. Given $A \subseteq \omega^{<\omega}$, let A^* be the set of all sequences $a_0, b_0, \dots, a_n, b_n, \dots$ of naturals such that either there is an n such that $u_{a_0} \subseteq u_{b_0} \subseteq \dots u_{a_n} \not\subseteq u_{b_n}$ or else the sequence of u's is increasing and unions to some element of A.

We claim that Player I has a winning strategy in $G^{**}(A)$ if and only if Player I has a winning strategy in G_{A^*}. (Exercise.)

To finish the argument, we prove the following statement: If $A \subseteq \mathcal{N}$ and

$$N_A = \bigcup \{N_s : s \in \mathbb{N}^{<\mathbb{N}} \text{ and } N_s \setminus A \text{ is meager}\}$$

and $G^{**}(A \setminus N_A)$ is determined, then A has the Baire property.

Let us now prove the statement. If Player I had a winning strategy in $G^{**}(A \setminus N_A)$, then for some stem t, $N_t \setminus (A \setminus N_A)$ would be meager, by what we already know. But then $N_t \setminus A$ would be meager because $N_t \setminus A \subseteq (N_t \setminus (A \setminus N_A))$. Then since $N_t \setminus A$ is meager, it follows by definition of N_A that $N_t \subseteq N_A$, therefore $N_t \setminus (A \setminus N_A) = N_t$ (just computationally speaking). So we just argued that N_t is meager. But nonempty open sets cannot be meager, so this is a contradiction.

So we proved that Player I does not have a winning strategy, and therefore by determinacy it follows that Player II has a winning strategy in $G^{**}(A \setminus N_A)$. Then we know that $A \setminus N_A$ is meager. Also $N_A \setminus A$ is meager because

$$N_A \setminus A \subseteq \bigcup \{N_s \setminus A : s \in \mathbb{N}^{<\mathbb{N}} \text{ and } N_s \setminus A \text{ is meager}\}.$$

so it follows that A has the Baire property: specifically, $A \triangle N_A = (A \setminus N_A) \cup (N_A \setminus A)$ is meager.

Now we have demonstrated that the Banach-Mazur game is indeed determined under AD. □

Corollary 3.14. AD *implies that all sets in* $\mathcal{N}$ *have the perfect set property*

Proof. First we reformulate the perfect set game for $\omega^{<\omega}$, i.e. where the plays consist of finite sequences of naturals alternated with single natural numbers. (Coding can be an exercise.)

The reformulation of the perfect set game works similarly to the Banach-Mazur game, then the result is immediate. □

Theorem 3.15. AD $\implies \neg$AC.

Proof. Assume AC, and we will show that there is an undetermined set. Let $\langle \sigma_\alpha : \alpha < 2^{\aleph_0} \rangle$ and $\langle \tau_\alpha : \alpha < 2^{\aleph_0} \rangle$ enumerate *all* strategies for *all* subsets of $\mathcal{N}$ for Players I and II respectively. This is possible because, in particular, there are $2^{\aleph_0}$-many strategies for any particular game because each can be defined as a tree of heigh $\aleph_0$ with branching of size $2^{\aleph_0}$, and we have $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$.

Now we will inductively define sets $X = \{x_\alpha : \alpha < 2^{\aleph_0}\}$ and $Y = \{y_\alpha : \alpha < 2^{\aleph_0}\}$ such that neither G_X nor G_Y are determined. Suppose that $\{x_\alpha : \alpha < \beta\}$ and $\{y_\alpha : \alpha < \beta\}$ are both defined. Since $\{y_\alpha : \alpha < \beta\}$ has cardinality less than $2^{\aleph_0}$ and $\{\sigma_\alpha * b : b \in \mathcal{N}\}$ has cardinality $2^{\aleph_0}$, we can choose $y_\alpha = \sigma_\alpha * b \notin \{x_\alpha : \alpha < \beta\}$. Similarly, we let $x_\alpha = \tau_\alpha * b \notin \{y_\alpha : \alpha < \beta\}$.

The sets X and Y are disjoint by construction, and that for each α we have some b such that $\sigma_\alpha * b \notin X$ (so σ_α is not a winning strategy for Player I) and some a such that $\tau_\alpha * a \in X$. Therefore G_X is undetermined. □

Corollary 3.16. *There is an undetermined game in Gödel's constructive universe L. (Specifically, $\mathcal{N}^L \cap <_L$ is $\mathbf{\Sigma}^1_2$.)*

Theorem 3.17. AD *implies that every countable subset of $\mathcal{N}$ has a choice function.*

Proof. Let $\langle X_n \rangle n \in \mathbb{N}$ be a sequence of subsets of $\mathcal{N}$. Consider the game where Player I is playing $a_0, a_1, \ldots$ and Player II is playing $b_0, b_1, \ldots$ and Player II wins if and only if $b = (b_0, b_1, \ldots) \in X_{a_0}$. Then Player I does not have a winning strategy because Player II can just start building some $\in X_{a_0}$. Therefore by determinacy Player II has a winning strategy σ, and the choice function with input n is just σ applied to a generic play starting with n. □

Fact 3.18. AD *implies that ω_1 and ω_2 are measurable.*

Chapter 4

Analytic Sets

4.1 The Basics

Remark 4.1. Henri Lebesgue claimed that if $A \subseteq \mathbb{R} \times \mathbb{R}$ is Borel, then its projection, i.e. the image under the function $(x, y) \mapsto x$, would be Borel. This is not necessarily the case!

Definition 4.2. Let X be a Polish space. Then we say that $A \subseteq X$ is *analytic* if there is a Polish space Y, a continuous function $f : Y \to X$, and a Borel set $B \subseteq Y$ such that $A = f(B) = \{f(y) : y \in B\}$. Let $\mathbf{\Sigma}^1_1(X)$ denote the analytic subsets of X.

Note that Borel sets are therefore analytic.

Fact 4.3. *If X is Polish and $B \subseteq X$ is Borel, then there is a topology on X such that X is Polish and B is clopen.*

Proposition 4.4. *If X is a Polish space, then the following are equivalent:*

(1) A is $\mathbf{\Sigma}^1_1(X)$.

(2) Either $A = \emptyset$ or there is a continuous $f : \mathcal{N} \to X$ such that $A = f(\mathcal{N})$.

(3) There is some closed $B \subseteq \mathcal{N} \times X$ such that $A = \pi_X(B)$ where $\pi_X : \mathcal{N} \times X \to X$ is the function $(b, x) \mapsto x$.

(4) There is a Polish space Y and a Borel B such that if $\pi_X : Y \times X \to X$ is the projection then $\pi_X(B) = A$.

Proof. First we show (1) implies (2). If A is analytic, by definition there is some Polish Y, continuous $f : Y \to X$, and Borel $B \subseteq Y$ such that $f(B) = A$. Remember that we have a continuous surjection $g : \mathcal{N} \to Z$ for Polish spaces Z (Theorem 1.25), so we apply to get $g : \mathcal{N} \to B$. Therefore $f \circ g$ gives us (2).

Now suppose (2) is true. Consider the graph $G(f) = \{(b, f(b)) : b \in \mathcal{N}\} \subseteq \mathcal{N} \times X$. Then it is enough to show that $G(f)$ is closed, which works for any Hausdorff space: If $(b, x) \notin G(f)$, then we are saying $f(b) = y \neq x$. Let $U_0, U_1 \subseteq X$ be disjoint open such that $y \in U_0$ and $x \in U_1$, and let V be the preimage of U_0 under f. Then $V \times U_1$ is disjoint from $G(f)$.

(3) $\implies$ (4) $\implies$ (1) follows by definition. □

Definition 4.5. If X is a Polish space and C is the complement of analytic set, we say that C is *coanalytic*. We write $\mathbf{\Pi}^1_1(X)$ for the collection of coanalytic sets.

Note that Borel sets are also coanalytic.

Proposition 4.6. *Both* $\mathbf{\Sigma}^1_1$ *and* $\mathbf{\Pi}^1_1$ *are closed under countable unions and intersections.*

Proof. By using complements, it is enough to get the result for $\mathbf{\Sigma}^1_1$.

Suppose that $\langle A_i : i \in \mathbb{N}\rangle$ is a sequence in $\mathbf{\Sigma}^1_1$. Using item 4.1, let C_i be such that C_i is closed in $\mathcal{N} \times X$ is closed and $\pi_X(C_i) = A_i$. Then $\bigcup_{i\in\mathbb{N}} A_i = \pi_X\left(\bigcup_{i\in\mathbb{N}} C_i\right)$, giving us unions (since $\bigcup_{i\in\mathbb{N}} C_i$ is Borel and we can just use item 4.1).

For intersections, recall the homeomorphism $\Phi : \mathcal{N} \to \mathcal{N}^{\mathbb{N}}$ and consider

$$C = \{(f, x) : \forall i, (\Phi(f)_n i, x) \in C_i\}.$$

Observe that C is closed: If there are $ni \in \mathbb{N}$ such that $(\Phi(f)_i, x) \notin C_i$, then there is some open $U \ni (\Phi(f)_i, x)$ such that $U \cap C_i = \emptyset$. Since $(f, x) \mapsto (\Phi(f)_i, x)$ is continuous, the inverse image of U avoids C. Moreover

$$\bigcap_{i\in\mathbb{N}} A_i = \{x : \forall i, \exists g_i, (g_i, x) \in C_i\} = \pi(C)$$

using bijectivity of Φ, which completes the proof. □

Example 4.7. We will show that, under a certain coding, well orderings are $\mathbf{\Pi}^1_1$.

Let $\mathcal{C}$ be identified with $2^{\mathbb{N}^2}$. Then a set $D \subseteq 2^{\mathbb{N}^2}$ can represent a set of binary relations where $x \in D$ represents some R if $R(i,j)$ holds if and only if $x(i,j) = 1$.

First, we argue that the set LO of linear orders is $\mathbf{\Pi}^0_1$ (i.e. closed) because it is defined by the following formulae

$$\begin{aligned}
&\forall n \forall m (x(n,m) = 0 \vee x(m,n) = 0)\\
&\forall n \forall m (n = m \vee x(n,m) = 1 \vee x(m,n) = 1)\\
&\forall n \forall m \forall k (x(n,m) = x(m,k) = 1 \implies x(n,k) = 1
\end{aligned}$$

where if we remove the quantifiers then we can see that the formulae represent closed sets, and then adding the quantifiers gives an intersection of closed sets.

Now let WO represent the set of well orders. We have

$$\mathsf{WO} = \{x \in \mathsf{LO} : \forall f : \mathbb{N} \to \mathbb{N}, \exists n, x(f(n), f(n+1)) = 0\},$$

which expresses that there are no infinite descending sequences.

We will show that the complement of WO is analytic. So we consider

$$\mathcal{C} \setminus \mathsf{LO} \cup \{x : \exists f : \mathbb{N} \to \mathbb{N}, \forall n, x(f(n), f(n+1)) = 1\}.$$

By Proposition 4.6 we can just consider the expression on the right. It is then enough to see that

$$\{(f,x) : \forall n, x(f(n), f(n+1)) = 1\}$$

is closed in $\mathcal{N} \times X$: since this projects onto

$$\{x : \exists f : \mathbb{N} \to \mathbb{N}, \forall n, x(f(n), f(n+1)) = 1\},$$

the result follows by item 4.1.

Definition 4.8. Let $\{A_s : s \in \omega^{<\omega}\}$ be a family of sets. Then

$$\mathcal{A}\{A_s : s \in \omega^{<\omega}\} := \bigcup_{f \in \mathcal{N}} \bigcap_{n \in \mathbb{N}} A_{f \restriction n}.$$

This is called the *Suslin operation.*

Proposition 4.9. *If $S = \mathcal{A}\{A_s : s \in \omega^{<\omega}\}$ for some family, then there is some family $\{B_s : s \in \omega^{<\omega}\}$ such that $s \sqsubseteq t$ implies $A_t \subseteq A_s$ and $S = \mathcal{A}\{B_s : s \in \omega^{<\omega}\}$.*

Proof. Observe that

$$\mathcal{A}\{A_s : s \in \omega^{<\omega}\} := \bigcup_{f \in \mathcal{N}} \bigcap_{n \in \mathbb{N}} (A_{f\restriction 0} \cap \dots A_{f\restriction n}).$$

□

Theorem 4.10. *If X is Polish, then $A \subseteq X$ is analytic if and only if $A = \mathcal{A}\{C_s : s \in \omega^{<\omega}\}$ for some family of closed sets.*

Proof. First we argue that $A = \mathcal{A}\{C_s : s \in \omega^{<\omega}\}$ is analytic. Let $B_n = \{(f, x) : x \in C_{f\restriction n}\}$ where B_n is Borel. Then we see that $x \in A$ if and only if $\exists f \in \mathcal{N}, (f, x) \in \bigcap_{n \in \mathbb{N}} B_n$.

Let $A \subseteq \mathcal{N}$ be analytic. Then there is some continuous $F : \mathcal{N} \to X$ with $A = F(\mathcal{N})$. For every $f \in \mathcal{N}$, we have

$$\bigcap_{n \in \mathbb{N}} F(N_{f\restriction n}) = \bigcap_{n \in \mathbb{N}} \overline{F(N_{f\restriction n})} = \{F(f)\}.$$

Hence $A = \bigcup_{f \in \mathcal{N}} \bigcap_{n \in \mathbb{N}} \overline{F(N_{f\restriction n})}$. □

4.2 Properties of Analytic Sets

Definition 4.11. If X is Polish and $A \in \mathbf{\Sigma}^1_1(X) \cap \mathbf{\Pi}^1_1(X)$, then we write $X \in \mathbf{\Delta}^1_1(X)$.

Definition 4.12. If $A_1, A_2 \subseteq X$ and $A_1 \cap A_2 = \emptyset$, then we say that A_1 and A_2 are *separated* by a set B if $A_1 \subseteq B \subseteq X \setminus A_2$.

Theorem 4.13. *Any two analytic sets are separated by a Borel set.*

Proof. First we establish:

Claim. *If $A = \bigcup_{n \in \omega} A_n$ and $B = \bigcup_{m \in \omega} B_m$ and each A_n and B_m are separated by a Borel set, then A and B are separated by a Borel set.*

Proof of Claim. For each $n, m \in \omega$ we let $D_{n,m}$ be such that $A_n \subseteq D_{n,m} \subseteq X \setminus B_m$. Then if $D := \bigcup_{n\in\omega} \bigcap_{m\in\omega} D_{n,m}$, we see that D is Borel and $A \subseteq D \subseteq X \setminus B$. □

Now let A and B be two disjoint analytic sets. By item 4.1, we have continuous functions $f : \mathcal{N} \to X$ and $g : \mathcal{N} \to X$ such that $A = f(\mathcal{N})$ and $B = g(\mathcal{N})$. For each $s \in \omega^{<\omega}$, let $A_s = f(N_s)$ and let $B_s = f(N_s)$. The A_s's and B_s's are analytic because the N_s are each isomorphic to $\mathcal{N}$.

Suppose for contradiction that A and B are not separated. We will inductively define $\langle n_i : i \in \mathbb{N}\rangle$ and $\langle m_i : i \in \mathbb{N}\rangle$ such that for every $k \in \mathbb{N}$, the sets $A_{\langle n_0,\ldots,n_k\rangle}$ and $B_{\langle m_0,\ldots,m_k\rangle}$ are not separated. Since $A = \bigcup_{n\in\mathbb{N}} A_{\langle n\rangle}$ and $B = \bigcup_{m\in\mathbb{N}} B_{\langle m\rangle}$, the claim implies that there exist n_0 and m_0 such that $A_{\langle n_0\rangle}$ and $B_{\langle m_0\rangle}$ are not separated. Similarly, there are n_1 and m_1 such that $A_{\langle n_0,n_1\rangle}$ and $B_{\langle m_0,m_1\rangle}$ are not separated, and so on.

Now let $a = \langle n_i : i \in \mathbb{N}\rangle$ and let $b = \langle m_i : i \in \mathbb{N}\rangle$. Since A and B are disjoint, we have $f(a) \neq f(b)$. Let G_a and G_b be disjoint open neighborhoods of a and b respectively. Then by continuity, there is some k such that $A_{a\restriction k} \subseteq G_a$ and $B_{b\restriction k} \subseteq G_b$. But this then means that $A_{a\restriction k}$ and $B_{b\restriction k}$ are separated by Borel set, which is a contradiction of the construction in the above paragraph. □

Theorem 4.14 (Suslin's Separation Theorem)**.** *A set B is Borel if and only if B is $\mathbf{\Delta}^1_1$.*

Proof. item 4.1 tells us that Borel sets are analytic, and since complements of Borel sets are Borel, we know they are coanalytic. If B is analytic and coanalytic, then we have shown that they are separated by a Borel set, but that means that (without loss of generality) the separating set is B! □

Definition 4.15. A function $f : X \to Y$ between topological spaces is *Borel measurable* If for all Borel $B \subseteq Y$, $f^{-1}(B)$ is Borel in X.

Proposition 4.16. *Suppose X and Y are topological spaces and that $f : X \to Y$.*

(i) f is Borel measurable iff $f^{-1}(U)$ is Borel for all open $U \subseteq Y$.

(ii) If Y has a countable basis, then f is Borel measurable implies the graph of f is Borel.

Proof. For *(i)*, suppose the weaker condition holds. Then the result follows from $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$ and $f^{-1}(\bigcup_{n\in\mathbb{N}} A_n) = \bigcup_{n\in\mathbb{N}} f^{-1}(A_n)$.

For *(ii)*, let $\{U_n : n \in \mathbb{N}\}$ be a countable basis for Y. Then the graph of f equals

$$\bigcap_{n\in\mathbb{N}} (\{(x, y) : y \notin U_n\} \cup \{(x, y) : x \in f^{-1}(U_n)\}).$$

If all the $f^{-1}(U_n)$'s are Borel, then the graph is Borel. □

Corollary 4.17. *Suppose X and Y are Polish spaces and $f : X \to Y$. The following are equivalent:*

(i) f is Borel measurable,

(ii) the graph of f is a Borel subset of $X \times Y$,

(iii) the graph of f is an analytic subset of $X \times Y$.

Proof. The implication *(i)* to *(ii)* is from the proposition we just proved and *(ii)* to *(iii)* is by definition.

Now suppose the graph of f is analytic and A is Borel in Y. Then

$$x \in f^{-1}(A) \iff \exists y(y \in A \wedge f(x) = y) \iff \forall y(f(x) = y \implies y \in A).$$

Hence $f^{-1}(A)$ has a $\mathbf{\Sigma}^1_1$ and a $\mathbf{\Pi}^1_1$ definition, so $f^{-1}(A)$ is Borel. □

Theorem 4.18. *If $A \subseteq X$ Polish and $A \in \mathbf{\Sigma}^1_1(X)$, then either A is countable or A has a perfect subset.*

Proof. Recall that every closed set F in $\mathcal{N}$ takes the form $F = [T] = \{b : \forall n, b \restriction n \in T\}$ where T is a tree in $\omega^{<\omega}$. For each such tree T, let $T_s := T \cap N_s$.

Let A be analytic and let f be a continuous function such that $A = f(\mathcal{N})$. For each tree T we define

$$T' = \{s \in T : f([T_s]) \text{ is uncountable}\}.$$

For each $\alpha < \omega_1$, we define T^α inductively: $T^0 = T$, $T^{\alpha+1} = (T^\alpha)'$, and $T^\alpha = \bigcap_{\beta<\alpha} T^\beta$ if α is a limit ordinal. There is some $\delta < \omega_1$ such that $T^{\delta+1} = T^\delta$. If $T^\delta = \emptyset$, then

$$A = \bigcup_{\beta<\delta} \{f([T^\beta_s]) : s \in T^\beta \setminus T^{\beta+1}\}$$

and therefore A is countable. Therefore if A is uncountable then T^δ is nonempty and for every $s \in T^\delta$, $f([T^\delta_s])$ is uncountable.

We argue that there is a perfect subset of A if A is uncountable. Let $s \in T^\delta$ be arbitrary. Since $f([T^\delta_s])$ has at least two elements, there are $s_{\langle 0\rangle} \sqsupset s$ and $s_{\langle 1\rangle} \sqsupset s$, both in T^α, such that $f([T^\delta_{s_{\langle 0\rangle}}])$ and $f([T^\delta_{s_{\langle 1\rangle}}])$ are disjoint (this is using Hausdorff-ness and continuity). Continue in this way, inducting on the length of all binary sequences to define s_t such that for all t such that $|t| = n$, the $f([T^\delta_{s_t}])$'s are pairwise disjoint. Let $U := \{s : \exists t \in 2^{<\omega}, s \sqsupseteq s_t\}$ of T^δ such that (1) U is perfect (as a tree), (2) every $s \in U$ has two immediate successors in U (hence U is compact), and (3) f is one-one on $[U]$.

Now let P be the image of $[U]$ under the function f. Since $[U]$ is compact and f is continuous, P is compact and therefore closed. Moreover, P has no isolated points since U is perfect and f is continuous. Therefore P is perfect (closed and no isolated points) as a subset of A. □

4.3 Further Reading

Changing Topologies

Reference. Dave Marker's descriptive set theory notes.

Theorem 4.19. *If (X, τ) is a Polish space and $B \subseteq X$ is Borel, then there is a topology $\tau' \supseteq \tau$ on X such that (X, τ') is Polish and B is Borel in the topology τ'.*

Very General Idea of Proof. It is easy to see that if $C \subseteq X$ is closed and X is Polish, then C is Polish in the subspace topology. It is slightly harder to show that the same is true for $U \subseteq X$ open.

Then one verifies the statement that if $F \subseteq X$ is closed where (X, τ) is Polish, then there is $\tau' \supseteq \tau$ such that F is clopen in τ' and (X, τ) and (X, τ') have the same Borel set. This done by taking a natural Polish topology τ' on the disjoint union of F and $X \setminus F$ and then arguing that sets Borel in τ' are Borel in τ.

Then we show that $\Omega := \{B \in B(X) :$ there is a Polish topology on X such that B is clopen $\}$ is a σ-algebra. □

Definition 4.20. If X and Y are topological spaces, then $f : X \to Y$ is a *Borel isomorphism* if it is Borel measurable bijection with a Borel measurable inverse. If there exists such an $f : X \to Y$, we say that X and Y are *Borel isomorphic.*

Theorem 4.21. *If X and Y are Polish spaces then they are Borel isomorphic.*

Solovay's model

Reference. See Kanamori's *The Higher Infinite* or even the original Solovay paper, "A model of set-theory in which every set of reals is Lebesgue measurable".

Definition 4.22 (Solovay's Codes)**.** Let $\langle s_i : i \in \omega \rangle$ be an enumeration of $\omega^{<\omega}$. For each $c \in \omega^\omega$ we let A_c be equal to:

- $\bigcup\{N_{s_i} : c(i+1) = 1\}$ if $c(0) = 0$,
- $\mathcal{N} \setminus \bigcup\{N_{s_i} : c(i+1) = 1\}$ if $c(0) = 1$,
- $\bigcap_{n\in\omega} \bigcup\{N_{s_i} : c(2^n 3^{i+1}) = 1\}$ if $c(0) > 1$.

Remark 4.23. This is not really a full Borel code, but it will be enough for Solovay since G_δ sets approximate Lebesgue measurable sets.

Theorem 4.24. *Assume the consistency of inaccessible cardinal, there is a model of set theory in which the following hold:*

- *All sets of reals are Lebesgue-measurable,*
- *All sets of reals have the perfect set property,*
- *All sets of reals are Baire measurable.*

Also, the inaccessible is not necessary for the Baire property.

Very General Proof Idea. Solovay's model takes the ordinal-definable sets, or instead $L(\mathbb{R})$, in $V[\mathrm{Col}(\omega, <\kappa)]$ where κ is the inaccessible cardinal. The inaccessibility of κ is used to show that countable sets appear in intermediate extensions. □

Theorem 4.25 (Shelah)**.** *If it is consistenct that all sets of reals are Lebesgue measurable, then it is consistent that there is an inaccessible cardinal.*

Reference. See Raisonnier: "A mathematical proof of S. Shelah's theorem on the measure problem and related results".

Ramsey Sets

Reference. Can look at Jech, probably better to look at Kechris.

Definition 4.26. A set $X \subseteq [\omega]^\omega$ is called *Ramsey* if there is $Y \subseteq \omega$ such that either $[Y]^\omega \subseteq X$ or $[Y]^\omega \cap X = \emptyset$.

Theorem 4.27 (Galvin-Prikry, Silver)**.** *Every analytic subset of* $\mathcal{N}$ *is Ramsey.*

Definition 4.28. Given $a \in [\omega]^{<\omega}$ and $A \subseteq \omega$ and $\max a < \min A$, we let $[a, A] = \{X \subseteq \omega : a \subseteq X, X \setminus a \subseteq A\}$. The *Ellentuck topology* on $[\omega]^\omega$ is generated by sets of the form $[a, A]$.

Definition 4.29. We say that $X \subseteq [\omega]^\omega$ is *completely Ramsey* if for all a, A with $[a, A]$ a basic open set, we have some $B \subseteq A$ such that either $[a, B] \subseteq X$ or $[a, B] \cap X = \emptyset$.

Theorem 4.30. *Let* $X \subseteq \omega^\omega$*. Then* X *is completely Ramsey if and only if it has the Baire property in the Ellentuck topology.*

Theorem 4.31 (Mathias)**.** *In Solovay's model (i.e. the* $L(\mathbb{R})$ *part), all sets in* $[\omega]^\omega$ *are completely Ramsey.*

Question. *Is the inaccessible necessary?*

Absoluteness Theorems

Theorem 4.32 (Mostowksi's Absolutness Theorem)**.** *If* P *is a* $\mathbf{\Sigma}^1_1$ *property then* P *is absolute for any model of* ZFC *that can define* P*.*

Theorem 4.33 (Schönfeld's Absoluteness Theorem)**.** $\mathbf{\Pi}^1_2$ *and* $\mathbf{\Sigma}^1_2$ *properties are absolute with* L*.*

Corollary 4.34. *The following two statements cannot be proved independent by forcing:*

- *The Riemann Hypothesis: The zeta function* $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$ *has its zeros precisely at the negative even integers and complex numbers with real part* $\frac{1}{2}$*.*
- $P = NP$*: the assertion that every algorithm checkable in polynomial time converges in polynomial time.*

Determinacy

Reference. See Shehzad Ahmed's masters thesis: `https://andrescaicedo.files.wordpress.com/2008/03/mastersthesis_sharpsandanalyticdeterminacy_shehzadahmed-1.pdf`.

Theorem 4.35 (Martin)**.** *Every Borel game is determined.*

VGPI. We already have open determinacy. Given a Borel game, we define an open game which is winning if and only if the Borel game is winning. □

Definition 4.36. $0^\#$ exists if and only if there is a nontrivial embedding $j : L \to L$.

Corollary 4.37. *"$0^\#$ exists" implies that $V \neq L$.*

Proof. A famous theorem of Kunen states that there is no nontrivial embedding $j : V \to V$. □

Theorem 4.38. *The consistency of "every analytic set is determined" is equal to the consistency of "$0^\#$ exists".*

VGPI. Assuming the existence of $0^\#$, use iterations of the embedding $j : L \to L$ to define a winning strategy. (This is a precursor to the concept of "iteration trees".) □

Borel equivalence relations

Reference. See Hjorth's chapter in *The Handbook of Set Theory.*

Definition 4.39. Let X and Y be topological spaces and let $E \subset X \times X$ and $F \subset Y \times Y$ be equivalence relations. Then we say that E is *Borel reducible* to F, denoted $E \leq_B F$, if there is a Borel function $f : X \to Y$ such that for all $x_1, x_2 \in X$, $x_1 E x_2 \iff f(x_1) F f(x_2)$.

Example 4.40. Let E_0 be the relation on $\mathcal{C}$ such that $x E_0 y$ if and only if x and y eventually agree, i.e. there is some $N \in \omega$ such that $n \geq N$ implies $x(n) = y(n)$. Then $\mathrm{id}_{\mathcal{C}} <_B E_0$. If E_1 is the relation of eventual agreement on infinite sequences of real numbers, then $E_0 < E_1$.

Theorem 4.41 (Silver)**.** *Suppose E is a $\mathbf{\Pi}^1_1$ equivalence relation on $\mathcal{N}$. Then either E has at most $\aleph_0$-many classes or else there is a perfect set $P \subseteq \mathcal{N}$ consisting of E-inequivalent elements.*