Course Notes for Set Theory and Independence Proofs

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Acknowledgments

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Part I

Working with Objects in Set Theory

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In this chapter, we will discuss the ways in which objects can be manipulated from the perspective of set theory. This includes a precise formulation of the axioms and a survey of what can be done with them on a relatively simple level. We are assuming familiarity with the basics of mathematical logic, including the notions of languages, formulas, sentences, and so on, as well as Gödel's Incompleteness Theorems.

Chapter 1

The Axioms of Zermelo-Fraenkel Set Theory

To start with, we will develop a familiarity with the Zermelo-Fraenkel axioms and the way they can be used to understand relatively simple objects.

1.1 Stating the Axioms and Beginning to Work with Them

Without further ado, we state the axioms.

1. **Extensionality:** $\forall x \forall y(\forall z(z \in x \iff z \in y) \implies x = y)$. (Sets are uniquely defined by their elements.)

2. **Foundation:** $\forall x(\exists y(y \in x) \implies \exists y(y \in x \wedge \neg \exists z(z \in x \wedge z \in y)))$. (Every nonempty set has an $\in$-minimal element.)

3. **Comprehension Scheme:** Let φ be any formula whose free variables are among $\{x, z, w_1, w_2, \ldots, w_n\}$. Then

$$\forall z \forall w_1 \ldots \forall w_n \exists y \forall x(x \in y \iff (x \in z \wedge \varphi)).$$

   (Definable subsets of sets are sets.)

4. **Pairing:** $\forall x, y \exists z(x \in z \wedge y \in z)$. (For any two sets, there is a set with both of those sets as elements.)

5. **Union:** $\forall F \exists U \forall Y, x(x \in Y \wedge Y \in F \iff x \in U)$. (If we have a set which is a family of sets, its union is a set.)

6. **Replacement Scheme:** Let φ be any formula whose free variables are among $\{x, y, A, w_1, \ldots, w_n\}$. Then

$$\forall A \forall w_1 \ldots \forall w_n (\forall x \in A \exists ! y \varphi \implies \exists Y \forall x \in A \exists y \in Y \varphi).$$

(Images of sets under functions are sets.)

We introduce some notation to make the remaining axioms easier to read.

- $x \subseteq y \iff \forall z \in x (z \in y)$.
- $x = \emptyset \iff \neg \exists y (y \in x)$.
- $y = \text{succ}(x) \iff \forall z \in y (z = x \vee z \in x)$.
- $y = v \cap w \iff \forall x (x \in y \iff (x \in v \wedge x \in w))$.
- $\text{singleton}(x) \iff (\exists y \in x \wedge \exists y \forall z \in x (z = y))$.

7. **Infinity:** $\exists x (\emptyset \in x \wedge \forall y \in x (\text{succ}(y) \in x))$. (There exists an infinite set, and in particular, the set of natural numbers is a set.)

8. **Powerset:** $\forall x \exists y (z \subseteq x \iff z \in y)$. (Every set has a power set.)

9. **Choice:** $\forall F (\forall x \in F (x \neq \emptyset) \wedge \forall x, y \in F (x \neq y \implies x \cap y = \emptyset)) \implies \exists C (\forall x \in F (\text{singleton}(C \cap x)))$ (Any nonempty family of sets has a choice function.)

We may want to consider various sub-collections of the axioms.

Definition 1.1.1. We can define subsystems of these axioms.

- ZFC refers to all nine axioms.
- ZF refers to the first eight axioms (excluding choice).
- If X is a sub-collection of these axioms, then $X - P$ is that set minus the powerset axiom.

- And there are many more variations.

Note that we do not explicitly have an axiom asserting the existence of an empty set, but the infinity axiom implies that there is an empty set.

Here we can give a simple example of the usage of these axioms to get something "obvious".

Proposition 1.1.2. ZF *proves that any pair of sets has a union.*

Proof. Let x and y be sets. We want to show that

$$\mathsf{ZF} \vdash \text{“}\exists z \forall u (u \in z \iff (u \in x \vee u \in y))\text{”}.$$

Then by pairing, there is a set w such that $\{x, y\} \subseteq w$. (Notice the actual formulation of the pairing axiom that we are using.) Using the formula $v = x \vee v = y$, we can use the comprehension axiom to ensure that we have a set equal to $\{x, y\}$. Then the union axiom gives us the set $x \cup y = \bigcup\{x, y\}$.

Observe that the formula above uniquely defines the union by the extension axiom. □

1.2 Orders and Their Formalizations

The next task as to develop the notion of orderings. This will help us build towards ordinal and cardinal numbers. Let's clarify the notions that we want.

Definition 1.2.1. A *partial order* is a relation $\leq$ on a set X with the following properties:

1. $\forall a \in X, a \leq a$ (reflexivity),
2. if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry),
3. if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).

We refer to X is the *underlying set* of $\leq$, and we may refer to X itself as a partial order and use the notation $(X, \leq_X)$.

Example 2. Examples of partial orders:

- Let X be the set of closed subsets of $\mathbb{R}$ under the usual Euclidean topology. For $A, B \in X$, we let $A \leq B$ if and only if $A \subseteq B$. We could just as easily let $A \leq B$ if and only if $A \supseteq B$.
- Let $X = \{0,1\}^{<\omega}$, the set of finite sequences of 0's and 1's. Let $s \leq t$ if and only if t end-extends s, i.e. if $\operatorname{dom} s = \{0, \ldots, n\}$ then $s = t \restriction \{0, \ldots, n\}$.

Definition 1.2.3. A *linear order* or *total order* is a partial order $(L, \leq_L)$ such that for all $a, b \in L$, one of the following three hold: $a = b$, $a \leq_L b$, or $a \geq_L b$.

Example 4. $\mathbb{R}$ under its standard ordering, or $\omega + 1 \cong \mathbb{N} \cup \{\infty\}$.

Definition 1.2.5. Let $(L, \leq)$ be a linear ordering. Then $(L, \leq)$ is a *well-ordering* if every nonempty subset $X \subseteq L$ has a $\leq$-least element.

Example 6. Any finite linear ordering is a well-ordering, and so are $\mathbb{N}$ and $\omega + 1$. Both $\mathbb{Q}$ and $\mathbb{R}$ are *not* well-orderings.

Now that we have some definitions in mind, let's show that they can be formalized in terms of ZFC.

Definition 1.2.7. Given sets x and y, the *ordered pair* $\langle x, y\rangle$ is the set $\{\{x\}, \{x, y\}\}$.

Proposition 1.2.8. $\forall x \forall y \forall x' \forall y' (\langle x, y\rangle = \langle x', y'\rangle \iff (x = x' \wedge y = y'))$. *(In other words, ordered pairs define a pair in a specific order.)*

Proof. We will show the forward direction by considering two cases. The other direction is fairly simple.

Suppose $x = y$. Then $\langle x, y\rangle = \{\{x\}\} = \{\{x'\}, \{x', y'\}\}$. By extension we have $\{x\} = \{x'\} = \{x', y'\}$. The first equality gives $x = x'$ and the second gives $x' = y'$. Transitivity of equality gives $x = y = x' = y'$ and so $x = x'$ and $y = y'$.

Suppose $x \neq y$. Then $\langle x, y\rangle = \{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$ (in which the apparent two-element sets are actual two-element sets). Therefore $\{x, y\} = \{x', y'\}$ and $x' \neq y'$ and $\{x'\} = \{x\}$. Therefore $x = x'$ and $y = y'$. $\square$

Proposition 1.2.9. *The Cartesian product of two sets is a set.*

Proof. Given sets A and B, we want to show that $\{\langle x, y\rangle : x \in A \wedge y \in B\}$ is a set.

If $x \in A$ and $y \in B$, then observe that $\langle x, y\rangle = \{\{x\}, \{x, y\}\} \in \mathcal{P}(\mathcal{P}(A \cup B))$ (where $\mathcal{P}$ is indicating the powerset operation). To see this, observe that $\{x\} \in \mathcal{P}(A) \subseteq \mathcal{P}(A \cup B)$ and $\{x, y\} \in \mathcal{P}(A \cup B)$.

Then apply comprehension to obtain

$$C = \{z \in \mathcal{P}(\mathcal{P}(A \cup B)) : \exists x \exists y (x \in A \wedge y \in B \wedge z = \langle x, y\rangle\}$$

and we are done. $\square$

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1.3 Ordinals and Transfinite Induction

Proposition 1.3.1. *The following are equivalent:*

1. *L is a well-ordering, i.e. all subsets have a minimal element.*
2. *L does not contain any infinite descending sequences.*

Proof. *(1) $\implies$ (2)*: Suppose contrapositively that $\langle x_n : n < \omega\rangle$ is an infinite descending sequence through L, i.e. $x_0 > x_1 > x_2 > \ldots$. Then this is a subset of L without a minimal element. *(2) $\implies$ (1)*: Suppose contrapositively that $X \subseteq L$ has no minimal element. Let $x_0 \in X$ be arbitrary, then inductively choose $x_1 < x_0$ and so on. $\square$

But how were we allowed to use induction there? We should to work towards some justification of induction, which is the next task. In order to do this, we first establish some facts about well-ordered sets using our starting definition.

Definition 1.3.2. If L is well-ordered and $x \in L$, then $\{y \in L : y <_L x\}$ is an *initial segment* of L.

Proposition 1.3.3. *No well-ordered set can be isomorphic to an initial segment of itself.*

Proof. First we argue that if $(W, <)$ is well-ordered and $f : W \to W$ is strictly increasing (i.e. $x < y$ implies $f(x) < f(y)$) then $f(x) \geq x$ for all $x \in W$: If $X = \{x \in W : f(x) < x\}$ is nonempty then it has a least element

z. If $w = f(z)$, then $f(w) = f(f(z)) < f(z) = w < z$, which contradicts minimality of z.

Now if $(W, <)$ were isomorphic to an initial segment $\{x : x < u\}$ via f, then $f(u) < u$, which is not possible. □

Proposition 1.3.4. *The only automorphism of a well-ordered set is the identity.*

Proof. Given f, apply Proposition 1.3.3 to both f and f^{-1}. □

Proposition 1.3.5. *If W_1 and W_2 are well-orderings and $f, g : W_1 \to W_2$ are isomorphisms, then $f = g$.*

Proof. If x is the W_1-least element such that $f(x) \neq g(x)$, then you can show that either f or g "misses a spot". □

Proposition 1.3.6. *If W_1 and W_2 are well-ordered, then exactly one of the following three cases will hold:*

1. *$W_1 \cong W_2$,*
2. *W_1 is isomorphic to an initial segment of W_2,*
3. *W_2 is isomorphic to an initial segment of W_1.*

Proof. The previous proposition shows that the cases are mutually exclusive.

If $x \in W_i$ for $i \in \{1, 2\}$ then let $W_i(x)$ denote the initial segment $\{y \in W_i : y <_{W_i} x\}$. Define

$$f = \{(x, y) : W_1(x) \cong W_2(y)\}.$$

We argue that f is a one-to-one function: The fact that it is a function and that fact that it is one-to-one both follow from Proposition 1.3.3.

If $\operatorname{dom} f = W_1$ and $\operatorname{range} f = W_2$ then $W_1 \cong W_2$: Using Proposition 1.3.5, we can argue that the isomorphisms from the $W_1(x)$'s to the $W_2(y)$'s can be unioned up to get an isomorphism from W_1 to W_2.

If $\operatorname{range} f \neq W_2$, then we can argue that W_1 is isomorphic to an initial segment of W_2. Observe that $\operatorname{range} f$ is downwards closed. Hence if $\operatorname{range}(f) \neq W_2$ and y is the least element of $W_2 \setminus \operatorname{range}(f)$ then $\operatorname{range}(f) = W_2(y)$. Then it must be the case that $\operatorname{dom}(f) = W_1$ since otherwise we would have $(x, y) \in f$ where x is the least element of $W_1 \setminus \mathrm{dom}(f)$, but then it would be absurd that this cannot be extended.

If $\mathrm{dom}\, f \neq W_1$, then we can similarly argue that W_2 is isomorphic to an initial segment of W_1. $\square$

Definition 1.3.7. A set X is *transitive* if $\forall y \in X(z \in y \implies z \in X)$.

Example 8. Observe that $\emptyset$ is vacuously transitive. The set $\mathcal{P}(\mathcal{P}(\emptyset))$ is also transitive. However, the set $\{\emptyset, \{\{\emptyset\}\}\}$ is not transitive.

Definition 1.3.9. We say that α is an *ordinal* if it α is transitive and well-ordered by $\in$.

Definition 1.3.10. A *successor* ordinal is an ordinal of the type $\alpha = \beta \cup \{\beta\} := \beta + 1$. A *limit* ordinal α takes the form $\alpha = \bigcup \alpha$.

Example 11. Every natural number can be represented as an ordinal: $0 = \emptyset$, $1 = \{\emptyset\}$, $2 = \{\emptyset, \{\emptyset\}\}$, $3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, etc. We write the set of natural numbers as the limit ordinal $\omega = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}, \dots\}$. $\omega + 1 := \omega \cup \{\omega\}$ is an infinite successor.

Proposition 1.3.12. *The following hold:*

1. $0 := \emptyset$ *is an ordinal.*
2. *If* α *is an ordinal and* $\beta \in \alpha$*, then* β *is an ordinal.*
3. *If* α, β *are ordinals,* $\alpha \neq \beta$*, and* $\alpha \subseteq \beta$*, then* $\alpha \in \beta$*.*
4. *If* α *and* β *are ordinals, then either* $\alpha \subseteq \beta$ *or* $\beta \subseteq \alpha$*.*
5. *If* α *and* β *are ordinals, then exactly one of the following hold:* $\alpha = \beta$*,* $\alpha \in \beta$*, or* $\beta \in \alpha$*.*
6. *If* X *is a set of ordinals, then* $\bigcup X$ *is an ordinal.*
7. *Every ordinal is either a successor or a limit.*

Proof. *(1)* is immediate. *(2)* will actually follow from *(3)* and *(4)*.

(3): Let γ be the $\in$-minimal element of $\beta \setminus \alpha$. If $\delta \in \gamma$ then $\delta \in \alpha$ by minimality. If $\delta \in \alpha$ then $\delta \in \beta$ by $\alpha \subseteq \beta$. Hence $\alpha = \gamma$.

(4): It is immediate that $\alpha \cap \beta$ is an ordinal. Let $\alpha \cap \beta = \gamma$. Then if $\alpha \neq \beta$ and if $\gamma \neq \alpha, \beta$, then it follows that $\gamma \in \alpha, \beta$. This implies that $\gamma \in \gamma$, which is not possible (see homework).

(5): Follows from *(4)* and Proposition 1.3.6.

(6): If $\alpha \in \beta \in \bigcup X$, and $\gamma \in X$ is such that $\beta \in \gamma$, then $\alpha \in \gamma$ so $\alpha \in \bigcup X$. If $Y \subseteq X$ choose some $\alpha \in Y$. Then α is an ordinal, so either $Y \cap \alpha = \emptyset$, in which case α is minimal, or we choose a minimal element of $Y \cap \alpha$. Since we now know that all ordinals are comparable we are done. □

Theorem 1.3.13. *Every well-ordered set is isomorphic to a unique ordinal number.*

Proof. We already have uniqueness.

If W is a well-order and $x \in W$, then let $F(x)$ be the (unique) α (if it exists) such that $\{y \in W : y <_W x\} \cong \alpha$. To show that F is defined for all $x \in W$, suppose for contradiction that x is the $<_W$-least element of W for which W is undefined. Then we can argue that $\{y \in W : y <_W x\} \cong \bigcup_{y <_W x} F(y)$, which is an ordinal, so this is a contradiction.

Now we can argue that $F[W] = \{F(x) : x \in W\}$ is a set by the replacement schema. By the same argument as in the previous paragraph, $F[W]$ is an ordinal α and $F : W \cong \alpha$. □

Now we want to start reasoning about the ordinals as a whole.

Proposition 1.3.14. *There is no set of all ordinals.*

Proof. If X were the set of all ordinals, then it would itself be an ordinal α. But then we would have $\alpha \in \alpha$, and this is not possible (see homework). □

A *class* refers to a collection of sets that is definable with some parameters that is not itself necessarily a set. If a class is not a set, then it is referred to as a *proper class*. By the above proposition, the collection of ordinals is a class which we will denote ON.

There are ways of formalizing classes, such as Bernays-Gödel set theory, but we will not emphasize that here. We want to make sure that whenever we refer to a class, our statements are shorthand for statements made in terms of sets.

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Theorem 1.3.15 (Transfinite Induction)**.** *Suppose C is a subclass of the ordinals such that:*

1. *$0 \in C$ holds,*

2. *for all ordinals α, if $\beta \in C$ for all $\beta < \alpha$, then $\alpha \in C$.*

Then C equals the class of all ordinals.

Proof. This is equivalent to saying if C is a nonempty class of ordinals and $C \neq \mathrm{ON}$, then there is a minimal $\alpha \in \mathrm{ON} \setminus C$.

To see that this is the case, let $\beta \notin C$. Then $\beta \cap C$ is a set by separation. (Remember that our "set operations" on classes are just abbreviations.) If $\beta \cap C = \emptyset$ then β is our witness. Otherwise, by the foundation axiom, there is some minimal $\alpha \in \beta \cap C$, so $\alpha \cap C = \emptyset$. Then α is the witness. □

A *class function* is a class taking the form of a function, i.e. a class of ordered pairs.

Theorem 1.3.16 (Transfinite Recursion)**.** *F be a class function that is defined on all sets. Then there exists a unique class function G that is defined on all ordinals such that $\forall \alpha \in \mathrm{ON}(G(\alpha) = F(G \restriction \alpha)$.*

Proof. If G_1, G_2 are two functions satisfying this description, then we can prove that $\forall \alpha \in \mathrm{ON}, G_1(\alpha) = G_2(\alpha)$ by transfinite induction.

For existence, call g a *δ-approximation* if g is a (set) function with domain δ and

$$\forall \alpha < \delta(g(\alpha) = F(g \restriction \alpha)).$$

If g is a δ-approximation and g' is a δ'-approximation, then $g \restriction (\delta \cap \delta') = g' \restriction (\delta \cap \delta')$: This uses the fact that δ can be compared with respect to the ordering on ordinals. By transfinite induction, we can argue that there is a δ-approximation for each δ. Then let $G(\alpha)$ be the value $g(\alpha)$ where g is a δ approximation for some (equivalently, any) $\delta > \alpha$. □

Example 17 (Schweber)**.** Let $\mathbb{N}^\omega = \mathbb{N} \times \mathbb{N} \times \dots$ be the countable product of the natural numbers with itself where the copies themselves are ordered as in the natural numbers. In other words, there is $\mathbb{N}_0$, $\mathbb{N}_1$, and so on where $\mathbb{N}_m < \mathbb{N}_n$ if $m < n$. Let α be the ordinal that is isomorphic to this ordering.

We will define a function $\alpha \to \omega$ by transfinite induction. For $\beta < \alpha$, if $p : \beta \to X$ is a function, let $F(p)(\beta) = 0$ if β is a limit ordinal and let $F(p)(\gamma) = p(\gamma) + 1$ if $\beta = \gamma + 1$.

Then the function G produced by transfinite inductions "mods out" the copies of $\mathbb{N}$.

We have essentially started doing ordinal arithmetic.

1.4 Ordinal Arithmetic

Recall that *successor operation*: $\beta = \mathrm{succ}(\alpha) = S(\alpha) \iff \forall z(z \in \beta \iff (z = \alpha \vee z \in \alpha))$.

Proposition 1.4.1. *Every ordinal is either a successor ordinal or a limit ordinal.*

Proof. We want to show that for all ordinals α, either there is some β such that $S(\beta) = \alpha$ or else $\alpha = \bigcup\{\beta : \beta \in \alpha\}$. (Note that if $\alpha = S(\beta)$, then $\bigcup\{\beta : \beta \in \alpha\} = \bigcup(\{\beta\} \cup \{\gamma : \gamma \in \beta\}) = \beta \neq \alpha$.)

Suppose that α is not a successor. We can see that $\bigcup\{\beta : \beta \in \alpha\} \subseteq \alpha$, so the task is to show that there are no missing points. For all $\beta \in \alpha$, $S(\beta) \neq \alpha$. This implies that $S(\beta) \in \alpha$, and so $\beta \in S(\beta) \subseteq \bigcup \alpha$. $\square$

According to the definition, 0 counts as a limit ordinal, but it is usually best not to think of it as one.

From now on, if α and β are ordinals, we will (usually) write $\alpha < \beta$ for $\alpha \in \beta$. We will write $\sup_{\xi<\eta} \alpha_\xi$ for $\bigcup_{\xi<\eta} \alpha_\xi$. We are ready to define the rules of ordinal arithmetic.

Remember that we proved that every well-ordered set is isomorphic to an ordinal.

Definition 1.4.2. If X is well-ordered, the *order-type* of X is the unique ordinal α such that $(X, <_X) \cong \alpha$. It is often denoted $\mathrm{ot}(X)$.

Definition 1.4.3. Suppose α and β are ordinals. Let R be

$$\{\langle\langle\xi, 0\rangle, \langle\eta, 0\rangle\rangle : \xi < \eta < \alpha\} \cup \\ \cup \{\langle\langle\xi, 1\rangle, \langle\eta, 1\rangle\rangle : \xi < \eta < \beta\} \cup [(\alpha \times \{0\}) \times (\beta \times \{1\})].$$

Then $\alpha + \beta := \mathrm{ot}(\alpha \times \{0\} \cup \beta \times \{1\}, R)$.

Example 4. $\omega + 1 \neq 1 + \omega$.

Proposition 1.4.5. *Let α, β, γ be ordinals.*

1. $\alpha + 0 = \alpha$.

2. $\alpha + 1 = S(\alpha)$.

3. $\alpha + S(\beta) = S(\alpha + \beta)$.

4. *If β is a limit ordinal, then* $\alpha + \beta = \sup\{\alpha + \xi : \xi < \beta\}$.

5. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

Proof. For *1.*, check that the definition of R above yields some empty sets as components. For *2.*, note that $S(\alpha)$ adds one extra point, which essentially gives *3.* as well.

For *4.*, we can inductively define an order-isomorphism as follows: Remember that $\sup\{\alpha+\xi : \xi < \beta\} = \bigcup_{\xi<\beta} \alpha+\xi$. So we will inductively define f_ξ to have domain $\alpha + \xi$. Define $f_0(\zeta) = \zeta$ for $\zeta < \alpha$. If we have defined f_ξ, let $f_{\xi+1}$ be such that $f_{\xi+1}(\zeta) = \zeta$ for $\zeta \in \operatorname{dom} f_\xi$. Then let $f_{\xi+1}(\xi)$ be the point in $\alpha + \beta$ that corresponds to ξ in the copy of β. If ξ is a limit, let $f_\xi = \bigcup_{\eta<\xi} f_\xi$.

Point *5.* can be proved by induction using the previous points where *3.* gives the base case. □

Definition 1.4.6. Let α and β be ordinals and let R be the lexicographic order on $\beta \times \alpha$, i.e.

$$\langle \xi, \eta \rangle R \langle \xi', \eta' \rangle \iff (\xi < \xi' \vee (\xi = \xi' \wedge \eta < \eta')).$$

Then $\alpha \cdot \beta = \operatorname{ot}(R)$.

Example 7. $2 \cdot \omega = \omega < \omega \cdot 2$.

Proposition 1.4.8. *Let α, β, γ be ordinals.*

1. $\alpha \cdot 0 = 0$.

2. $\alpha \cdot 1 = \alpha$.

3. $\alpha \cdot S(\beta) = \alpha \cdot \beta + \alpha$,

4. *If β is a limit then* $\alpha \cdot \beta = \sup\{\alpha \cdot \xi : \xi < \beta\}$.

5. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.

6. $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.

Proof. Homework. ∎

Definition 1.4.9. Let $\alpha \neq 0$ and β be ordinals.

1. $\alpha^0 = 1$.
2. $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$.
3. If β is a limit then $\alpha^\beta = \sup\{\alpha^\xi : \xi < \beta\}$.

Note: I was indeed wrong about needing transfinite induction to define the ordinals. As suggested in the lecture, we have a solution here: `https://math.stackexchange.com/questions/149158/ordinal-exponentiation-and-transfinite-induction`.

Example 10. $2^\omega = \omega$ in terms of ordinal exponentiation, but not in terms of cardinal exponentiation!

Definition 1.4.11. The natural numbers (in this context) are the set of finite ordinals.

Proposition 1.4.12. *Let α be an ordinal.*

1. *If $\beta < \gamma$ then $\alpha + \beta < \alpha + \gamma$.*
2. *If $\beta < \gamma$ and $\alpha > 0$ then $\alpha \cdot \beta < \alpha \cdot \gamma$.*
3. *If $\alpha < \beta$ then there is a unique ordinal δ such that $\alpha + \delta = \beta$.*
4. *If $\alpha > 0$ and γ is arbitrary, then there exist a unique β and a unique $\rho < \alpha$ such that $\gamma = \alpha \cdot \beta + \rho$.*
5. *If $\beta < \gamma$ and $\alpha > 1$ then $\alpha^\beta < \alpha^\gamma$.*

Proof. The first two points can be proved by induction. Point *3* lets δ be the order type of $\{\gamma \in \beta : \gamma \geq \alpha\}$, and its uniqueness is from the first point.

For *4*, we argue first that there is a greatest ordinal β such that $\alpha \cdot \beta \leq \gamma$. This is because the least ordinal β' such that $\alpha \cdot \beta' > \gamma$ is a successor: if it were a limit, then we would have $\alpha \cdot \epsilon \leq \gamma$ for all $\epsilon < \beta'$ and then our definition would give us $\alpha \cdot \epsilon \leq \gamma$. Once we have defined β, we let ρ be as given from *3*.

For *5* we do an induction similar to *1* and *2*. ∎

Theorem 1.4.13 (Cantor Normal Form)**.** *Every ordinal $\alpha > 0$ can be represented uniquely in the form*

$$\alpha = \omega^{\beta_1} \cdot k_1 + \ldots + \omega^{\beta_n} \cdot k_n,$$

where $n \geq 1$, $\alpha \geq \beta_1 > \ldots > \beta_n$, and $k_1, \ldots, k_n$ are nonzero natural numbers.

Proof. We prove this by induction. If $\alpha = 0$ then there is technically nothing to prove. If $\alpha = 1$ then $\alpha = \omega^0 \cdot 1$.

Suppose that $\alpha > 1$ is an arbitrary ordinal. Therefore there is a maximal β such that $\omega^\beta \leq \alpha$ (as in the argument for *4* in Proposition 1.4.12). By *4* of Proposition 1.4.12 there are unique δ and $\rho < \omega^\beta$ such that $\omega^\beta \cdot \delta + \rho$. By minimality of β, δ must be finite. Then plug in the inductive statement for ρ. $\square$

1.5 The Notion of Cardinality

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Next we will develop the notion of cardinality, and the aspects of it that do not depend on the axiom of choice.

Definition 1.5.1. A *cardinal* is an ordinal κ such that for all $\alpha < \kappa$, there is no surjection $f : \alpha \to \kappa$.

Definition 1.5.2. If (emphasis on if!) A is well-ordered, then the *cardinality* $|A|$ is the least ordinal α such that there is a bijection $f : \alpha \to A$.

We have some structure without necessarily assuming the axiom of choice.

Theorem 1.5.3 (Bernstein-Cantor-Schröder)**.** *If there is an injection $f_1 : A \to B$ and an injection $f_2 : B \to A$ then there is a bijection from A to B.*

Proof. Homework. $\square$

Proposition 1.5.4. *If $|\alpha| \leq \beta \leq \alpha$ then $|\beta| = |\alpha|$.*

Proof. We have an injection $\beta \to \alpha$ since $\beta \subseteq \alpha$. We have an injection $\alpha \to \beta$ via $\alpha \to |\alpha| \to \beta$. $\square$

Definition 1.5.5. Let κ and λ be cardinals.

1. $\kappa + \lambda = |(\kappa \times \{0\}) \cup (\lambda \times \{1\})|$.

2. $\kappa \cdot \lambda = |\kappa \times \lambda|$.

Proposition 1.5.6. *Cardinal addition and multiplication are associative and commutative.*

Definition 1.5.7. Gödel's canonical well-ordering on ON × ON is defined as follows: $(\alpha, \beta) < (\gamma, \delta)$ if and only if

- either $\max\{\alpha, \beta\} < \max\{\gamma, \delta\}$ or
- $\max\{\alpha, \beta\} = \max\{\gamma, \delta\}$ and $\alpha < \gamma$ or
- $\max\{\alpha, \beta\} = \max\{\gamma, \delta\}$, $\alpha = \gamma$ and $\beta < \delta$.

Clearly, this a linear ordering.

Proposition 1.5.8. *Gödel's ordering is a well-ordering.*

Proof. Homework. □

Theorem 1.5.9. *If κ is a cardinal then $\kappa \cdot \kappa = \kappa$.*

Proof. It is clear that $|\kappa \cdot \kappa| \geq \kappa$, so we will prove the other direction by transfinite induction.

Each $\langle \alpha, \beta \rangle \in \kappa \times \kappa$ has $\leq |(\max\{\alpha, \beta\} + 1) \times (\max\{\alpha, \beta\} + 1)| < \kappa$ (which can be checked by cases). Therefore $\text{ot}(\kappa \times \kappa, \triangleleft) \leq \kappa$ where $\triangleleft$ is the Gödel ordering (which can be seen by considering the contrapositive). Therefore $|\kappa \times \kappa| \leq \kappa$. □

Theorem 1.5.10. *If κ and λ are cardinals, then $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$.*

Proof. We have $\kappa, \lambda \leq \kappa + \lambda \leq \kappa \cdot \lambda$. If WLOG $\kappa \leq \lambda$ then $\kappa \cdot \lambda \leq \lambda \cdot \lambda = \lambda$. □

Theorem 1.5.11. *If κ is an infinite cardinal, then $[\kappa]^{<\omega}$ (the set of finite subsets of κ) has cardinality κ.*

Proof. By induction we can use the previous proof to obtain an injection $f : [\kappa]^{<\omega} \to \omega \times \kappa$. And $|\omega \times \kappa| = \kappa$. □

Since the ordinals are ordered, the cardinals are ordered.

Definition 1.5.12. α^+ is the least cardinal larger than α.

Definition 1.5.13. We define $\aleph_\alpha$ be transfinite recursion:

- $\aleph_0 = \omega$,
- $\aleph_{\alpha+1} = \aleph_\alpha^+$,
- if α is a limit then $\aleph_\alpha = \sup_{\beta<\alpha} \aleph_\beta$.

Definition 1.5.14. If α is an ordinal, the *cofinality* of α is the least ordinal β such that there is an increasing unbounded function $f : \beta \to \alpha$. Let $\mathrm{cf}(\alpha)$ denote the cofinality of α.

Definition 1.5.15. A cardinal κ is *singular* if $\mathrm{cf}(\kappa) < \kappa$. Otherwise κ is *regular*.

Example 16. $\aleph_\omega$ is singular.

Example 17. $\mathrm{cf}(\omega + \omega) = \mathrm{cf}(\omega \cdot \omega) = \omega$.

Proposition 1.5.18. *For all ordinals α,* $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$.

Proof. Composition of functions. □

Proposition 1.5.19. *If α is an ordinal, then* $\mathrm{cf}(\alpha)$ *is a regular cardinal.*

Proof. Let $f : \mathrm{cf}(\alpha) \to \alpha$ be unbounded.

Suppose for contradiction that $\kappa = |\,\mathrm{cf}(\alpha)| < \mathrm{cf}(\alpha)$ (i.e. that $\mathrm{cf}(\alpha)$ is not a cardinal). Let $g : \kappa \to \mathrm{cf}(\alpha)$ be a surjection. Observe that $f \circ g : \kappa \to \alpha$ is unbounded. This is a contradiction because $\mathrm{cf}(\alpha)$ is supposed to be minimal for the existence of such an unbounded function.

Suppose for contradiction that there is an unbounded function $g : \kappa \to \mathrm{cf}(\alpha)$ and then run a similar argument. □

1.6 The Axiom of Choice and Cardinal Arithmetic

We can quickly review some equivalent statements of the axiom of choice.

Definition 1.6.1. Let $(P, <_P)$ be a partially ordered.

1. A subset $X \subseteq P$ is a *chain* if it is totally ordered, i.e. for all $y, z \in X$, either $y <_P z$, $z <_P y$, or $z = y$.

2. An *upper bound* of a chain X is an element $y \in P$ such that for all $z \in X$, $y <_P z$.

3. An element $x \in P$ is *maximal* if for all $y \in P$, if $y \geq_P x$ then $y = x$.

Theorem 1.6.2. *The following are equivalent:*

1. *For every family F of nonempty sets, there is a function $C : F \to \bigcup F$ such that for all $x \in F$, $F(x) \in x$. (The Axiom of Choice,* AC*)*

2. *If P is a partially ordered set such that every chain has an upper bound, then P has a maximal element. (Zorn's Lemma)*

3. *Every set can be well-ordered. (The Well-Ordering Theorem)*

Proof. *3. $\Longrightarrow$ .1* Let $<_F$ be a well-ordering of $\bigcup F$. For all $x \in F$, let $<_F$ be a well-ordering of x. Let $F(x)$ be the $<_x$-least element of x.

1. $\Longrightarrow$ 2. Suppose for contradiction that AC holds and that P is a partially ordered set such that every chain has an upper bound, but that P has no maximal element. Using the choice function we can define a $<_{\mathbb{P}}$-increasing sequence that has no upper bound.

2. $\Longrightarrow$ .3 Let X be a set. Let P consist of all functions f into the ordinals such that $\mathrm{dom}(f) \subseteq X$. Every such chain clearly has an upper bound. Let $g \in P$ be a maximal element. Then it must necessarily have $\mathrm{dom}\, g = X$. $\square$

For the remainder of the section assume AC.

Proposition 1.6.3. *If there is surjection from Y onto X then there is an injection from X to Y.*

Proof. Let $f : Y \to X$ be a surjection. For each $b \in X$ let $a_b \in f^{-1}(a)$ (using the axiom of choice). Then let $g(b) = a_b$. $\square$

Proposition 1.6.4. *Every infinite cardinal is a limit ordinal.*

Proof. $|\alpha + 1| = |\alpha|$. $\square$

Theorem 1.6.5 (Cantor)**.** *If X is a nonempty set then there is no surjection from X onto $\mathcal{P}(X)$.*

Proof. Suppose for contradiction that there is a surjection from X onto $\mathcal{P}(X)$. Let $Y = \{z \in X : z \notin f(z)\}$. Let z be such that $f(z) = Y$. If $z \in Y \iff z \notin f(z) \iff z \notin Y$. $\square$

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Proposition 1.6.6. *A countable union of countably infinite sets is countable.*

Proof. Let $\langle A_n : n < \omega \rangle$ be a sequence of countably infinite sets. Its union will be at least countable.

For each n we *choose* an enumeration $\langle a^n_k : k < \omega \rangle$. Then there is a surjection $f : \omega \times \omega \to \bigcup_{n<\omega} A_n$ given by $(n, k) \mapsto a^n_k$. This shows that the union is at most countable. $\square$

Proposition 1.6.7. *If S is a set then $|\bigcup S| \leq |S| \cdot \sup\{|X| : X \in S\}$.*

Proof. Let $\kappa = |S|$ and let $\lambda = \sup\{|X| : X \subseteq S\}$.

Now we show that $\kappa \cdot \lambda \geq |\bigcup S|$. We have an enumeration $S = \langle X_\alpha : \alpha < \kappa \rangle$. For each $X_\alpha \in S$, we have an enumeration $X_\alpha = \langle a^\alpha_\xi : \xi \leq \lambda_\alpha \rangle$ where $\lambda_\alpha \leq \lambda$. Now we have the mapping $\kappa \times \lambda \to \bigcup S$ given by $(\alpha, \xi) \mapsto a^\alpha_\xi$. $\square$

Proposition 1.6.8. *If α is a successor ordinal, then $\aleph_\alpha$ is regular.*

Proof. Suppose $f : \delta \to \aleph_{\alpha+1}$ where $\delta < \aleph_{\alpha+1}$. Then $f(\xi)$ is an ordinal of cardinality $\leq \aleph_\alpha$. Therefore $|\bigcup \operatorname{im} f| = |\bigcup_{\xi \in \delta} f(\xi)| = |\delta| \cdot \aleph_\alpha = \aleph_\alpha$ by the previous proposition. $\square$

Remark: This is consistently false! The *Axiom of Determinacy* implies that $\aleph_3$ is singular.

1.7 Cardinal Arithmetic with Exponentiation

Definition 1.7.1. If $\kappa \geq \lambda$ are cardinals, then $[\kappa]^\lambda = \{X \subseteq \kappa : |X| = \lambda\}$.

Definition 1.7.2. If A and B are sets, then $A^B = {}^B A = \{f | f : B \to A\}$.

Definition 1.7.3 (Using AC)**.** If κ and λ are cardinals then $\kappa^\lambda = |{}^\lambda \kappa|$ (where the former is indicating the exponential operation and the latter is indicating the cardinality of that particular set of functions).

Proposition 1.7.4. $2^\kappa = |\mathcal{P}(\kappa)|$.

Proof. Use the characteristic functions χ_X where $\chi_X(\alpha) = 1$ if $\alpha \in X$ and $\chi_X(\alpha) = 0$ otherwise. □

Proposition 1.7.5. *If $\kappa \leq \lambda$ then $2^\kappa \leq 2^\lambda$.*

Proof. From $\mathcal{P}(\kappa) \subseteq \mathcal{P}(\lambda)$. □

Proposition 1.7.6. *If κ, λ, μ are cardinals, then $\mu^{\kappa+\lambda} = \mu^\kappa \cdot \mu^\lambda$ and $\mu^{\kappa\cdot\lambda} = (\mu^\kappa)^\lambda$.*

Proof. Check the definitions in terms of functions. For the first assertion, observe that the set of functions from $\kappa \times \{0\} \cup \lambda \times \{1\}$ to μ is in bijection with the disjoint union of the functions from κ to μ and the functions from λ to μ.

You can also argue by cases where, without loss of generality $\lambda \geq \kappa$ and both are assumed to be infinite (since the finite cases are fairly immediate). □

Proposition 1.7.7. *If $2 \leq \kappa \leq \lambda$ and λ is infinite then $\kappa^\lambda = 2^\lambda$.*

Proof. $2^\lambda \leq \kappa^\lambda \leq (2^\kappa)^\lambda = 2^{\kappa\cdot\lambda} = 2^\lambda$. □

Sometimes infinite sums and products are used.

Definition 1.7.8. Given an indexed set of cardinals $\{\kappa_i : i \in I\}$ choose $\{X_i : i \in I\}$ such that $|X_i| = \kappa_i$ for all $i \in I$. We define

$$\sum_{i\in I} \kappa_i = \left|\bigcup_{i\in I} X_i\right|$$

and

$$\prod_{i\in I} \kappa_i = \left|\prod_{i\in I} X_i\right|.$$

Observe that assuming AC, these definitions do not depend on the X_i's.

Proposition 1.7.9. *If λ is infinite and $\kappa_i > 0$ for all $i < \lambda$, then*

$$\sum_{i<\lambda} \kappa_i = \lambda \cdot \sup_{i<\lambda} \kappa_i.$$

Theorem 1.7.10 (König's Theorem)**.** *If $\kappa_i < \lambda_i$ for every $i \in I$ then*

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i.$$

Proof. Choose Y_i's such that $|Y_i| = \lambda_i$. Suppose that X_i's are subsets of $\prod_{i\in I} Y_i$ such that $|X_i| \le \kappa_i$. We want to show that $\bigcup_{i\in I} X_i \ne \prod_{i\in I} Y_i$.

Let $Z_i = \{f(i) : f \in X_i$. Since $|X_i| < |Y_i|$, we have that $Z_i \ne Y_i$. Let f be a function so that $f(i) \notin Z_i$ for all $i \in I$. Then f does not belong to any of the X_i's.

Hence have shown that $\bigcup_{i\in I} X_i \ne \prod_{i\in I} Y_i$. □

Corollary 1.7.11. *If κ is an infinite cardinal then* $\mathrm{cf}(2^\kappa) > \kappa$*.*

Proof. It suffices to show that if $\kappa_i < 2^\kappa$ for all $i < \kappa$, then $\sum_{i<\kappa} \kappa_i < 2^\kappa$. Let $\lambda_i = 2^\kappa$. Then

$$\sum_{i<\kappa} \kappa_i < \prod_{i<\kappa} \lambda_i = (2^\kappa)^\kappa = 2^\kappa.$$

□

Corollary 1.7.12. *If κ is an infinite cardinal, $\kappa^{\mathrm{cf}\,\kappa} > \kappa$.*

Proof. Write $\kappa = \sum_{i<\mathrm{cf}\,\kappa} \kappa_i$ with $\kappa_i < \kappa$. Then

$$\kappa = \sum_{i<\mathrm{cf}\,\kappa} \kappa_i < \prod_{i<\mathrm{cf}\,\kappa} \kappa = \kappa^{\mathrm{cf}\,\kappa}.$$

□

Definition 1.7.13. The *continuum hypothesis*, denoted CH, is the assertion that $2^{\aleph_0} = \aleph_1$. The *generalized continuum hypothesis*, denoted GCH, as the assertion that for all infinite cardinals κ, $2^\kappa = \kappa^+$.

Definition 1.7.14. By transfinite induction, define the *beth function* as follows:

- $\beth_0 = \aleph_0$,
- $\beth_{\alpha+1} = 2^{\beth_\alpha}$,
- $\beth_\alpha = \sup\{\beth_\beta : \beta < \alpha\}$ if α is a limit.

So GCH holds if $\beth_\alpha = \aleph_\alpha$ for all ordinals α.

Proposition 1.7.15. *If* GCH *holds and* κ *and* λ *are infinite cardinals, then the following are true:*

1. *If* $\kappa \leq \lambda$ *then* $\kappa^\lambda = \lambda^+$,
2. *If* $\operatorname{cf} \kappa \leq \lambda < \kappa$ *then* $\kappa^\lambda = \kappa^+$,
3. *If* $\lambda < \operatorname{cf} \kappa$ *then* $\kappa^\lambda = \kappa$.

Proof. *1.* $\kappa^\lambda = 2^\lambda$.

2. We know that $\kappa^\lambda \geq \kappa^{\operatorname{cf} \kappa} \geq \kappa^+$. We also have $\kappa^\lambda \leq \kappa^\kappa \leq (2^\kappa)^\kappa = 2^\kappa = \kappa^+$.

3. Write $\kappa^\lambda = \bigcup_{\alpha<\kappa} \alpha^\lambda$ using the fact that functions from λ into κ are bounded. For $\alpha < \kappa$ we have $|\alpha^\lambda| \leq (2^{|\alpha|})^\lambda = 2^{|\alpha| \cdot \lambda} = (|\alpha| \cdot \lambda)^+ \leq \kappa$ (where the second-to-last relation is using GCH). This then implies that $\kappa^\lambda = \kappa$. □

Chapter 2

Filters, Ideals, and Algebras

2.1 The Basics of Filters and Ideals

Definition 2.1.1. Let X be a set. A *filter* on X is a set $F \subseteq \mathcal{P}(X)$ such that:

1. $\forall A, B \in F, A \cap B \in F$,
2. $\forall A \in F, B \supseteq A, B \in F$,
3. $\emptyset \notin F$.

Some sources omit the requirement that $\emptyset \notin F$ and instead call a filter nontrivial if this holds.

Example 2.

- The trivial filter $\{X\}$.
- Principal filters: Take $z \in X$, then $F = \{Y \subseteq X : z \in Y\}$.
- The Frechet filter, e.g. on ω let $F = \{Y \subseteq \omega : |\omega \setminus Y| < \omega\}$.

Definition 2.1.3. A family G of sets has the *finite intersection property* if for every finite $\{X_1, \ldots, X_n\} \subset G$ has a nonempty intersection, i.e. $X_1 \cap \ldots \cap X_n \neq \emptyset$.

Proposition 2.1.4.

1. *Every filter has the finite intersection property.*
2. *If X is a set and $G \subseteq \mathcal{P}(X)$ has the finite intersection property, then there is a filter F such that $G \subseteq F$.*

Proof. The first assertion follows easily by induction. For the second assertion, let F be the set of all $Y \subseteq X$ such that for some finite $\{X_1, \ldots, X_n\} \subseteq G$, $X_1 \cap \ldots \cap X_n \subseteq Y$. □

Definition 2.1.5. A filter F on a set X is an *ultrafilter* if for all $Y \subseteq X$, either $Y \in F$ or $X \setminus Y \in F$.

Proposition 2.1.6. *The following are equivalent for a set X:*

1. *F is a maximal filter on X, i.e. if $F' \supseteq F$ is a filter then $F' = F$.*

2. *F is an ultrafilter on X.*

Proof. If U is an ultrafilter is is maximal: If we have $F \supsetneq U$ with $Y \in F \setminus U$, then $X \setminus Y \in U$, contradicting that F is a filter.

Suppose F is not an ultrafilter and that $Y, X \setminus Y \notin F$. We will show that there is filter extending F. We can use the finite intesection property to show that either there is a filter containing $F \cup \{Y\}$ or a filter extending $F \cup \{X \setminus Y\}$.

So we want to show that either $F \cup \{Y\}$ has the finite intersection property or else $F \cup \{X \setminus Y\}$. Suppose this is true for neither. Then there is some $A_1, \ldots, A_m \in F$ such that $A_1 \cap \ldots \cap A_m \cap Y = \emptyset$ and $B_1, \ldots, B_n \in F$ such that $B_1 \cap \ldots \cap B_n \cap (X \setminus Y) = \emptyset$. But we know that $A_1 \cap \ldots \cap A_m \cap B_1 \cap \ldots B_n \neq \emptyset$, so this is impossible. □

Theorem 2.1.7. *Every filter can be extended to an ultrafilter.*

Proof. Use Zorn's lemma and the fact that maximal filters are ultrafilters. The main point is to show that if $\langle F_i : i < \theta \rangle$ is a sequence of filters so that $F_i \subseteq F_j$ for $i < j$, then $\bigcup_{i<\theta} F_i$ is a filter (in other words, unions of chains of filters are filters). □

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Definition 2.1.8. A filter F on X is *uniform* if $|Y| = |X|$ for all $Y \in F$.

Example 9. The Frechet filter is uniform, but principal filters on sets of cardinality > 1 are not.

Theorem 2.1.10. *If κ is an infinite cardinal then there are 2^{2^κ}-many uniform filters on κ.*

Proof. We call a family $\mathcal{A}$ of subsets of κ *independent* if for distinct sets $X_1, \dots, X_n, Y_1, \dots, Y_m \in \mathcal{A}$, the intersection

$$X_1 \cap \dots X_n \cap (\kappa \setminus Y_1) \cap \dots \cap (\kappa \setminus Y_m)$$

has cardinality κ.

Claim. *There is an independent family of subsets of κ of cardinality 2^κ.*

Assuming the claim, we can obtain the rest of the theorem: Let $\mathcal{A}$ be the family witnessing the claim. For each $\mathcal{A} \to \{0, 1\}$, consider

$$F_f = \{X : f(X) = 1\} \cup \{\kappa \setminus X : f(X) = 0\} \cup \{X : |\kappa \setminus X| < \kappa\}.$$

We can see that F_f has the finite intersection property: Suppose $X_1, \dots, X_\ell$ are such that $f(X_i) = 1$, $Y_1, \dots, Y_m$ are such that $f(\kappa \setminus Y_i) = 0$, and $Z_1, \dots, Z_n$ are sets that are co-bounded (i.e. they are from the last component of F_f). Then $W_0 := \bigcap_{1 \le i \le \ell} X_i \cap \bigcap_{1 \le i \le m} Y_i$ has cardinality κ by the definition of independence. It is reasonably easy to see (just from De Morgan's Laws) that $W_1 := \bigcap_{1 \le i \le n} Z_i$ also has the property that $|\kappa \setminus W_1| < \kappa$, so there is some $\alpha < \kappa$ such that $(\alpha, \kappa) = \{\beta < \kappa : \alpha \le \beta < \kappa\} \subseteq W_1$. Therefore $W_0 \cap W_1$ has cardinality equal to κ.

We can also see that if $f \neq g$ then $F_f \neq F_g$. Therefore if U_f's are the respective ultrafilters extending the F_f's, then we have 2^{2^κ}-many U_f's because $\{f | f \to \{0, 1\}\}$ has cardinality κ.

Proof of Claim. Let P be the set of all pairs (s, F) where s is a finite subset of κ and F is a finite set of finite subsets of κ. Since $|P| = |[\kappa]^{<\omega}| = \kappa$, it is sufficient to find an independent family of subsets of P of cardinality 2^κ.

For each $u \subseteq \kappa$, let

$$X_u = \{(s, F) \in P : s \cap u \in F\}$$

and let $\mathcal{A} = \{X_u : u \subseteq \kappa\}$. If u and v are distinct subsets of κ, then $X_u \neq X_v$: if (WLOG) $\alpha \in u$ and $\alpha \notin v$, then let $s = \{\alpha\}$, $F = \{s\}$, and $(s, F) \in X_u$. Then $(s, F) \in X_u$ and $(s, F) \notin X_v$.

To say that $\mathcal{A}$ is independent, let $u_1, \dots, u_n, v_1, \dots, v_m$ be distinct subsets of κ. For each $i \le n$ and $j \le m$, let $\alpha_{i,j} \in u_i \Delta v_j$. Now let s be any finite subset of κ such that $s \supseteq \{\alpha_{i,j} : i \le n, j \le m\}$. We have $s \cap u_i \neq s \cap v_j$ for all $i \le n$ and $j \le m$. Thus if we let $F = \{s \cap u_i : i \le n\}$ then $(s, F) \in X_{u_i}$ for $i \le n$ and $(s, F) \notin X_{v_j}$ for $j \le m$.

Hence we have chosen that every such finite set s is an element of

$$X_{u_1} \cap \ldots \cap X_{u_n} \cap (P \setminus X_{v_1}) \cap \ldots \cap (P \setminus X_{v_m}),$$

and since there are κ-many such s, the intersection has cardinality κ. $\square$

Having proven the claim, we are done with the proof. $\square$

2.2 Clubs and Stationary Sets

Definition 2.2.1. A function f whose domain is a subset of the ordinals is *regressive* if $f(\alpha) < \alpha$ for all $\alpha \in \mathrm{dom}(f) \setminus \{0\}$.

Remark 2.2.2. Obviously we have an injective regressive function f with domain ω: Just let $f(n) = n - 1$. But can we get an injective regressive function with domain $\aleph_1$?

Definition 2.2.3. Let κ be an uncountable regular cardinal. A subset $C \subseteq \kappa$ is *club* in κ (or *a* club in κ) if:

1. C is unbounded in κ, i.e. $\forall \beta < \kappa, \exists \alpha \in C$, $\alpha > \beta$;

2. C is *closed*, i.e. if $\langle \alpha_\xi : \xi < \lambda \rangle \subset C$ with $\lambda < \kappa$, then $\sup_{\xi<\lambda} \alpha_\xi \in C$.

The set $\{X \subset \kappa : X$ contains a club$\}$ is called *the club filter on* κ.

Example 4. Consider (1) the set of limit ordinals in κ or perhaps (2) $\kappa \setminus \alpha$ for any $\alpha < \kappa$.

Remark 2.2.5. We can define clubs in limit ordinals that are not cardinals.

Proposition 2.2.6. *The club filter is κ-complete. In other words, if $\langle C_\xi : \xi < \lambda \rangle$ are clubs in κ and $\lambda < \kappa$, then $\bigcap_{\xi<\lambda} C_\xi$ is a club in κ. (In particular, the club filter is a filter.)*

Proof. Closure of $\bigcap_{\xi<\lambda} C_\xi$ is straightforward from the definitions.

For unboundedness, we will first argue that the intersection of any two clubs C and D in κ is unbounded. Fix $\delta < \kappa$. Using the unboundedness of C and D, define by induction sequences $\langle \alpha_n : n < \omega \rangle \subset C$ and $\langle \beta_n : n < \omega \rangle \subset D$ such that $\alpha_0 \geq \delta$ and $\alpha_n < \beta_n < \alpha_{n+1}$ for all $n < \omega$. Then we can see that $\sup_{n<\omega} \alpha_n = \sup_{n<\omega} \beta_n = \gamma$. (This is known as "interleaving.") By closure of C, we know that $\gamma = \sup_{n<\omega} \alpha_n \in C$, and by closure of D, we know that $\sup_{n<\omega} \beta_n = \gamma \in D$, and thus $\gamma \in C \cap D$.

Now let us do the general argument. We will argue that $\bigcap_{\xi<\eta} C_\xi$ is unbounded in κ by induction on $\eta < \kappa$.

- The statement is of course trivial if we are taking only one club, so that gives us the base case.

- Suppose that we are considering

$$\bigcap_{\xi<\eta+1} C_\xi = \left(\bigcap_{\xi<\eta} C_\xi\right) \cap C_{\xi+1}.$$

  The first part is a club by our inductive hypothesis, and the intersection of everything is a club by the same argument we used for two clubs.

- Now suppose we are considering $\bigcap_{\xi<\eta} C_\xi$ where η is a limit ordinal. By induction, $\bigcap_{\xi<\zeta} C_\xi$ is a club for all $\zeta < \eta$. Therefore we can assume without loss of generality that $C_\zeta \subseteq C_\xi$ for all $\xi < \zeta$, i.e. the clubs are "nested." Now define a sequence $\langle \alpha_\xi : \xi < \eta \rangle$ to be an increasing sequence above some fixed $\delta < \kappa$ such that $\alpha_\xi \in C_\xi$ for all $\xi < \eta$. If $\beta = \sup_{\xi<\eta} \alpha_\xi$, then $\beta < \kappa$ by regularity. Because of nestedness, $\alpha_\xi \in C_\zeta$ for all $\zeta \leq \xi$, and so $\beta = \sup_{\zeta\leq\xi<\eta} \alpha_\xi \in C_\zeta$ for all $\zeta < \eta$.

This finishes the proof. $\square$

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Definition 2.2.7. Let κ be an uncountable regular cardinal and let $\langle X_\alpha : \alpha < \kappa \rangle$ be a collection of subsets of κ. Then $\triangle_{\alpha<\kappa} X_\alpha := \{\alpha < \kappa : \alpha \in \bigcap_{\beta<\alpha} X_\beta\}$ is the *diagonal intersection* of this collection. A filter F on κ is *normal* if for all $\langle X_\alpha : \alpha < \kappa \rangle \subset F$, $\triangle_{\alpha<\kappa} X_\alpha \in F$.

Remark 2.2.8. We do not necessarily have $\triangle_{\alpha<\kappa} X_\alpha \subseteq X_\alpha$ for all $\alpha < \kappa$: Consider the example where $X_\alpha = \kappa \setminus \alpha$ for all $\alpha < \kappa$.

Proposition 2.2.9. *If κ is an uncountable regular cardinal and $\langle C_\alpha : \alpha < \kappa \rangle$ is a collection of clubs in κ, then $\triangle_{\alpha<\kappa} C_\alpha$ is a club in κ. (In other words, the club filter is normal.)*

Proof. Notice that the diagonal intersection is the same if we replace each C_α with $\bigcap_{\beta\leq\alpha} C_\beta$. Hence, as in the last proof, we can assume without loss of generality that $C_\beta \subseteq C_\gamma$ for $\gamma \leq \beta$.

Closure: Consider $\langle\gamma_\xi : \xi < \eta\rangle \subset \triangle_{\alpha<\kappa} C_\alpha$ be a strictly increasing sequence where η is a limit ordinal, and let $\sup_{\xi<\eta}\gamma_\xi = \gamma^*$. By the definition of the diagonal intersection, we need to show that $\gamma^* \in \bigcap_{\beta<\gamma^*} C_\beta$.

The definition of diagonal intersections already tells us that $\gamma_\xi \in \bigcap_{\beta<\gamma_\xi} C_\beta$ for all $\xi < \eta$. Using nestedness, this means that $\gamma_\zeta \in C_{\gamma_\xi}$ for all $\zeta \in (\xi,\eta)$, which implies that $\gamma^* = \sup_{\zeta<\eta}\gamma_\zeta = \sup_{\xi\leq\zeta<\eta}\gamma_\zeta \in C_{\gamma_\xi}$ for all $\xi < \eta$. Again using nestedness, we conclude that $\gamma^* \in C_\beta$ for all $\beta < \gamma^*$.

Unboundness: Given $\beta < \kappa$, we will inductively define a sequence $\langle\gamma_n : n < \omega\rangle$ as follows: Let γ_0 be any ordinal in the interval (β,κ). Given γ_n, choose $\gamma_{n+1} \in (\gamma_n,\kappa)$ to be an element of $\bigcap_{\alpha<\gamma_n} C_\alpha$, which we know is a club. Then let $\gamma^* = \sup_{n<\omega}\gamma_n$.

Of course, γ^* is larger than β, so we just need to show that $\gamma^* \in \triangle_{\alpha<\kappa} C_\alpha$, i.e. that $\gamma^* \in C_\alpha$ for all $\alpha < \gamma^*$. Given some particular $\alpha < \gamma^*$, there is some n such that $\alpha < \gamma_n$. Then we see that $\gamma_m \in C_\alpha$ for all $m > n$. As in our previous reasoning, $\gamma^* \in C_\alpha$. □

Definition 2.2.10. Let κ be regular uncountable. We say that $S \subseteq \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for all clubs $C \subset \kappa$.

Example 11. Given a regular uncountable κ, all clubs in κ are stationary. Also, $\{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$ is stationary.

Proposition 2.2.12. *If $S \subset \kappa$ is stationary, then S is unbounded in κ.*

Theorem 2.2.13 (Fodor's Lemma)**.** *Let κ be regular uncountable and let $S \subset \kappa$ be stationary. If f is a regressive function with domain S, then there is a stationary subset $S' \subseteq S$ and some $\gamma < \kappa$ such that for all $\alpha \in S'$, $f(\alpha) = \gamma$.*

Proof. Suppose otherwise. Then for all $\gamma < \kappa$, there is some club C_γ such that for all $\alpha \in C_\gamma \cap S$, $f(\alpha) \neq \gamma$. (We are sort of jumping past a step here.) Now take $C := \triangle_{\gamma<\kappa} C_\gamma$, which we now know is a club. Let $\delta \in C \cap S \neq \emptyset$, and let $f(\delta) = \gamma < \delta$. By the definition of diagonal intersections, $\delta \in \bigcap_{\alpha<\delta} C_\alpha$, meaning that $\delta \in C_\gamma$, but this contradicts the way we defined C_γ. □

Theorem 2.2.14. *If κ is an uncountable regular cardinal, then every stationary subset of $\{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$ is the union of κ-many disjoint stationary sets.*

Proof. Let $S \subseteq E^\kappa_\omega := \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$. For each $\alpha \in S$, let $\langle \delta^\alpha_n : n < \omega \rangle$ be a sequence of ordinals less than α that is cofinal in α.

We argue that there is some $N < \omega$ such that for all $\gamma < \kappa$, $\{\alpha \in S : \gamma \geq \delta^\alpha_N\}$ is stationary. Suppose otherwise. Then for all $n < \omega$, there is some C_n and γ_n such that for all $\alpha \in C_n \cap S$, $\delta^\alpha_n < \gamma_n$. Consider the stationary set $S \cap \bigcap_{n<\omega} C_n$, let $\gamma^* = \sup_{n<\omega} \gamma_n$, and take $\alpha \in S \cap \bigcap_{n<\omega} C_n$ such that $\alpha > \gamma^*$. Then we should be able to choose n such that $\delta^\alpha_n > \gamma^*$ because $\langle \delta^\alpha_n : n < \omega \rangle$ is supposed to be cofinal in α, but this contradicts the choice of C_n and γ_n.

Fix N as in the previous paragraph. Define S_ξ and γ_ξ for $\xi < \kappa$ as follows: Apply Fodor's Lemma with the function $\alpha \mapsto \delta^\alpha_N$ and let $S_0 \subseteq S$ be a stationary set such that this function is constant with value γ_0. Then we continue in this way. Given S_ζ and γ_ζ defined similarly for $\zeta < \xi$, apply $\alpha \mapsto \delta^\alpha_N$ to the set $\{\alpha \in S : \delta^\alpha_N > \sup_{\zeta<\xi} \gamma_\zeta\}$ to get a stationary set S_ξ such that this function is constant with value γ_ξ.

Because κ is regular, we can continue until we have defined $\langle S_\xi : \xi < \kappa \rangle$. These sets are distinct because if $\delta^\alpha_n \neq \delta^\beta_n$ then $\alpha \neq \beta$. (We can "fill in the complement" with S_0.) □

2.3 Boolean Algebras

Definition 2.3.1. A *Boolean algebra* is a set B with at least two elements, 0 and 1, and endowed with two binary operations, $+$ and $\cdot$, is well is a unary operation $-$.

The operations satisfy the following axioms:

1. $u + v = v + u$ and $u \cdot v = v \cdot u$ (commutativity),
2. $u + (v + w) = (u + v) + w$ and $u \cdot (v \cdot w) = (u \cdot v) \cdot w$ (associativity),
3. $u \cdot (v + w) = u \cdot v + u \cdot w$ and $u + (v \cdot w) = (u + v) \cdot (u + w)$ (distributivity),
4. $u \cdot (u + v) = u$ and $u + (u \cdot v) = u$ (absorption),
5. $u + (-u) = 1$ and $u \cdot (-u) = 0$.

Example 2. Let X be any nonempty set. Then there is a Boolean algebra on $\mathcal{P}(X)$ where $0 = \emptyset$, $1 = X$, $+$ is $\cup$, $\cdot$ is $\cap$, and $-$ is $\setminus$.

Proposition 2.3.3. *If B is a Boolean algebra, then for all $x, y \in B$:*

1. $x + x = x$ *and* $x \cdot x = x$,
2. $x + y = y$ *if and only if* $x \cdot y = x$.

Proof. $x + x = x + x \cdot (x + x) = x$ for the first part of the first item.
If $x + y = y$ then $x \cdot y = x \cdot (x + y) = x$. □

Definition 2.3.4. If B is a Boolean algebra, then let $a \leq_B b$ if and only if $a + b = b$.

Proposition 2.3.5. *If B is a Boolean algebra, $\leq_B$ is a partial order. The greatest lower bound of $\{x, y\}$ is $x \cdot y$ and the least upper bound $\{x, y\}$ is $x + y$.*

Proof. Reflexivity is by the previous proposition. For transitivity: Let $x \leq y$ and $y \leq z$. Then

$$x + z = x + (y + z) = (x + y) + z = y + z = z.$$

For antisymmetry: Let $x \leq y$ and $y \leq x$; then $y = x + y = y + x = x$.
To see that, e.g. $x+y$ is the *least* upper bound of x and y, then $z \geq x, y$. Then $(x + y) + z = x + (y + z) = x + z = z$. □

Proposition 2.3.6. *For all x:*

1. $0 \leq x \leq 1$,
2. $x + 0 = x$ *and* $x \cdot 1 = x$,
3. $x \cdot 0 = 0$ *and* $x + 1 = 1$.

Proof. $\mathrm{glb}\{-x, x\} = x \cdot (-x) = 0$, so $0 \leq x$. Hence $0 + x = x$ by definition of $\leq$. Also, $x \cdot 0 = 0$ since $0 + x = x$. You can do the rest with duality. □

Proposition 2.3.7. *The following can be derived from the Boolean algebra axioms:*

1. $u + u = u$,

2. $u \cdot u = u$,

3. $u + 1 = 1$,

4. $-(u + v) = -u \cdot -v$,

5. $-(u \cdot v) = -u + -v$.

Example 8. Let $\mathcal{L}$ be a first-order language and let S be the set of sentences of $\mathcal{L}$. Consider the equivalence relation $\vdash \varphi \iff \psi$. Let B be the set of equivalence classes $[\varphi]$ is a Boolean algebra with: $[\varphi] + [\psi] = [\varphi \vee \psi]$, $[\varphi] \cdot [\psi] = [\varphi \wedge \psi]$, $-[\varphi] = [\neg\varphi]$, $0 = [\varphi \wedge \neg\varphi]$, $1 = [\varphi \vee \neg\varphi]$.

Definition 2.3.9. A *homomorphism* of Boolean algebras will preserve $0, 1$, and the operations. An *isomorphism* (of course) is a bijective homomorphism.

Theorem 2.3.10 (Stone's Representation Theorem)**.** *Every Boolean agebra is isomorphic to an algebra of sets.*

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Proof. Let B be a Boolean algebra. Let $S = \{p : p \text{ is an ultrafilter on } B\}$. For every $u \in B$, let X_u be the set of $p \in S$ such that $u \in p$. Let $\mathcal{S} = \{X_u : u \in B\}$.

Consider the map $\pi : B \to \mathcal{S}$ be given by $\pi(u) = X_u$. We have $\pi(1) = S$ and $\pi(0) = \emptyset$. From the definition of an ultrafilter: $\pi(u \cdot v) = \pi(u) \cap \pi(v)$, $\pi(u + v) = \pi(u) \cup \pi(v)$, $\pi(-u) = S - \pi(u)$. So we have that π is a homomorphism.

We see that π is a surjection by definition. To see that π is injective: If $u \neq v$, then find an ultrafilter p on B containing one element but not the other. $\square$

We will sketch some material briefly with the goal of reaching Theorem 2.3.16.

Definition 2.3.11. If B is a BA then $B^+ = B \setminus \{0\}$.

Definition 2.3.12. Suppose B is a Boolean algebra.

1. Suppose $X \subseteq B$. Then $\sum X = \sup X$ and $\prod X = \inf X$ if these values exist.

2. A Boolean algebra B is *complete* if $\sum X \in B$ and $\prod X \in B$ for all $X \subseteq B$.

3. If B is a Boolean algebra and $A \subseteq B$ is a subalgebra, then A is *dense* in B if for every $u \in B^+ = B \setminus \{0\}$, there is some $0 \neq v \in A$ such that $v \leq u$.

4. If B is a Boolean algebra, then a *completion* C of B is a complete Boolean algebra such that $B \subseteq C$.

Proposition 2.3.13. *If B is a Boolean algebra and $B \subseteq C, D$ where C and D are completions of B, then $C \cong D$.*

Proof. Define a map $\pi : C \to D$ by

$$\pi(c) = \sum_D \{u \in B : u \leq c\}$$

where the supremum is taken within D. Then argue that π is an isomorphism. □

Theorem 2.3.14. *Every Boolean algebra has a completion*

Proof. Let B be a Boolean algebra.

A set $U \subseteq B^+$ is a *cut* if $p \leq q$ and $q \in U$ implies $p \in U$.

For some $p \in B^+$, let $U_p = \{x : x \leq p\}$.

A cut U is *regular* if whenever $p \notin U$, there is some $q \leq p$ such that $U \cap U_q = \emptyset$.

Consider the operations $U \cdot V = U \cap V$ and $U + V = \overline{U \cup V}$, and also $-U = \{p : U_p \cap U = \emptyset\}$.

One can argue that the algebra of regular cuts is complete, and that the assignment $p \mapsto U_p$ shows that B is dense in the algebra of regular cuts. □

Definition 2.3.15. Let B be a Boolean algebra.

- A set $W \subseteq B^+$ is an *antichain* if $u \cdot v = 0$ for all $u, v \in W$ with $u \neq v$.

- If W is an antichain and $\sum W = u$, then W is a *partition* of u. A partition of 1 is called a *partition of B* or a *maximal antichain.*

- B is κ-saturated if there is no partition W of B such that $|W| = \kappa$.

- $\mathrm{sat}(B)$ equals the least κ such that B is κ-saturated.

Theorem 2.3.16. *If B is an infinite complete Boolean algebra, then $\mathrm{sat}(B)$ is a regular uncountable cardinal.*

Proof. Given a Boolean algebra B, we will let cB denote

$$\sup\{|X| : X \text{ is an antichain of } B\}$$

(this is called the *cellularity*). Also, if $x \in B^+$, we let $B \restriction x = \{y \in B : y \leq x\}$. If B is a Boolean algebra, we let $c_B(x) = c(B \restriction x)$. Observe that if $x \in B$ and $c_B(x) > \mu$, then there is some antichain Y of $B \restriction x$ such that $|Y| = \mu$.

Suppose that $\lambda = cB$ is a singular cardinal. Then our goal is to show that there is some partition/antichain of B of cardinality λ.

Let $\lambda = \sup_{\alpha<\kappa} \lambda_\alpha$ where $\kappa = \mathrm{cf}(\lambda)$.

There are three cases to consider.

Case 1: There is some $z \in B$ such that $c(x) = \lambda$ for all $0 < x \leq z$. Since $\kappa < \lambda$, we have an antichain $\{b_\alpha : \alpha < \kappa\}$. For each $\alpha < \kappa$, we have some antichain Z_α in $B \restriction b_\alpha$ of cardinality λ_α. Then if $Z = \bigcup_{\alpha<\kappa} Z_\alpha$, we have an antichain of cardinality λ.

Now assume that the first case does not hold. Let

$$S = \{a \in B^+ : c(a) < \lambda\}$$

and let X be maximal among the antichains of B included in S. This uses Zorn's Lemma: observe that unions of antichains are antichains.

Case 2: Suppose $\sup_{z\in X} c(z) = \lambda$. Inductively choose a sequence of elements $\langle y_\alpha : \alpha < \kappa\rangle$ such that for all $\alpha < \kappa$, $\lambda_\alpha < c(y_\alpha)$. Let Z_α be an antichain of cardinality λ_α in $B \restriction y_\alpha$. Then $Z = \bigcup_{\alpha<\kappa} Z_\alpha$ is an antichain of cardinality λ.

Case 3: Suppose $\sup_{z\in X} c(z) < \lambda$. We will argue that $|X| = \lambda$. Suppose otherwise.

Let $\mu = \sup_{z\in X} c(z)$ and let $\mu' = (\max\{|X|, \mu\})^+$. Then $\mu' < \lambda$ since λ is a limit cardinal. Since $\lambda = cB$ we find an antichain Y of cardinality μ'.

For $z \in X$, let

$$Y_z = \{w \in Y : w \cdot z > 0\}.$$

Since X is maximal, we have that $Y = \bigcup_{z\in X} Y_z$. The set $\{z \cdot y : y \in Y_z\}$ is pairwise disjoint in $B \restriction z$, so $|Y_z| \leq c(z) \leq \mu$. It follows that $|Y| \leq \mu \cdot |X| < \mu'$, which is a contradiction of the choice of Y. □

Definition 2.3.17. Let κ be a regular uncountable cardinal.

- The *nonstationary ideal* on κ is the subset of nonstationary subsets of κ and is often denoted NS_κ.
- The *saturation* of the nonstationary ideal is the saturation of the Boolean algebra $P(\kappa)/\mathrm{NS}_\kappa$, where element are equivalence classes under the relation $X \sim Y$ if and only if $X \Delta Y$ is nonstationary.

Corollary 2.3.18 (of Solovay's Splitting Theorem)**.** *The saturation of the nonstationary ideal on ω_1 is $\geq \omega_2$.*

Of course, this is generalized to κ for the "non-baby" version of Solovay's theorem.

Corollary 2.3.19. *Let κ be a regular uncountable cardinal. Then the saturation of the nonstationary ideal is regular.*

Part II

Working with Models of Set Theory

Chapter 3

Some Examples of Models and Easy Independence Proofs

3.1 The Von Neumann Hierarchy

Definition 3.1.1. The Von Neumann hierarchy is defined by induction on $\alpha \in \mathrm{ON}$ as follows:

- $V_0 = \emptyset$,
- $V_{\alpha+1} = \mathcal{P}(V_\alpha)$,
- if α is a limit then $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$.

Proposition 3.1.2. *For all ordinals α:*

1. *V_α is transitive,*
2. *$\forall \beta < \alpha$, $V_\beta \subset V_\alpha$.*

Proof. Prove *1.* by induction on α. This holds vacuously for $\emptyset$. If we know that V_α is transitive and $y \in x \in V_{\alpha+1} = \mathcal{P}(V_\alpha)$, then $y \in V_\alpha \in \mathcal{P}(V_\alpha) = V_{\alpha+1}$. If α is a limit and $y \in x \in V_\alpha$, then there is some $\beta < \alpha$ so that $x \in V_\beta$, so $y \in V_\beta \subseteq V_\alpha$.

Again, this is vacuous for V_0. If the statement holds for V_α and $\beta < \alpha+1$, then either $\beta < \alpha$ and $V_\beta \subseteq V_\alpha \subseteq \mathcal{P}(V_\alpha)$ or $\beta = \alpha$ we have the same think. The same kind of argument applies to the limit case. $\square$

Definition 3.1.3. If $x \in V$, then $\operatorname{rank}(x)$ if the least α such that $x \in V_{\alpha+1}$.

Proposition 3.1.4. *1. If $x \in y$ then $\operatorname{rank}(x) < \operatorname{rank}(y)$.*

2. $\operatorname{rank}(y) = \sup\{\operatorname{rank}(x) + 1 : x \in y\}$.

Proof. Suppose $\alpha = \operatorname{rank}(y) \leq \operatorname{rank}(x)$. This means that $x \notin V_\alpha$. If it were the case that $x \in y \in \mathcal{P}(V_\alpha)$, then we would have $x \in V_\alpha$.

The second point is clarified if we consider the difference between successor and limit cases. □

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Proposition 3.1.5. *For all $\alpha \in \mathrm{ON}$,*

1. $\alpha \in V$,

2. $\operatorname{rank}(\alpha) = \alpha$,

3. $V_\alpha \cap \mathrm{ON} = \alpha$.

Proof. These are all straightforward with transfinite induction. Observe that $\operatorname{rank}(\emptyset) = 0$. □

Proposition 3.1.6. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \in V_{\omega+\omega}$.

Proof. We have that $\mathbb{N} = \omega \in V_{\alpha+1}$. For the rest, check that we have enough room when ordered-pairs are used. □

3.2 Transitive Closures, Well-Founded Relations and the Mostowski Collapse

Proposition 3.2.1. *If x is a set, there is a transitive set T such that $x \subseteq T$. Moreover, we can show that if T' is transitive and $x \subseteq T'$, then $T \subseteq T'$.*

Proof. Define S_n by induction on $n \in \omega$ as follows: $S_0 = x$ and for $n \in \omega$, $S_{n+1} = \bigcup S_n$. Then let $T = \bigcup_{n<\omega} S_n$. Obviously we have that $T \supseteq S$. To see that T is transitive, suppose that $z \in y \in T$. Then there is some $n \in \omega$ such that $y \in S_n$. Then $z \in S_{n+1} \subseteq T$. To see that T is the smallest transitive set in the sense claimed by the proposition, observe that if A is transitive then $\bigcup A \subseteq A$, so we can argue inductively that if $S_n \subseteq T'$ where T' is as in the statement of the proposition, then $S_{n+1} \subseteq T'$. □

Definition 3.2.2. Let x be a set. The *transitive closure* of x, denoted $\mathrm{tc}(x)$, is the smallest transitive set containing x as a subset.

Proposition 3.2.3.

1. *If x is transitive, then $\mathrm{tc}(x) = x$.*

2. *If $x \subseteq y$ then $\mathrm{tc}(x) \subseteq \mathrm{tc}(y)$.*

3. $\mathrm{tc}(x) = x \cup \{\mathrm{tc}(y) : y \in x\}$.

Proof. *1.* By definition.

2. Use the fact that if $A \subseteq T$ and T is transitive then $\mathrm{tc}(A) \subseteq T$.

3. For the forward direction, it is enough to show that the set on the right hand side is transitive, which is clear.

For the other direction, suppose that $z \in x \cup \{\mathrm{tc}(y) : y \in x\}$. If $z \in x$ then we are done by definition. If there is some $y \in x$ such that $z \in \mathrm{tc}(y)$, then we have $z \in \mathrm{tc}(x)$ because $\mathrm{tc}(y) \subseteq \mathrm{tc}(x)$. □

Proposition 3.2.4. *If C is a class, then there is an $\in$-minimal member of C.*

Proof. We can denote C to be the class of x such that $\varphi(x, \bar{p})$ holds (for some φ). Let $S \in C$, so S is in particular a set. Then $S \cap C$ is a set by the separation schema. If $S \cap C = \emptyset$ then we are done. If $S \cap C \neq \emptyset$ then we let $X = T \cap C$ where $T = \mathrm{tc}(S)$. If $z \in X$ is $\in$-minimal element (of which there must be an instance) then z is an $\in$-minimal element of C. □

Proposition 3.2.5. *For all sets x there is some α such that $x \in V_\alpha$.*

Proof. Let C be the hypothetical class of x such that there is no $\alpha \in \mathrm{ON}$ such that $x \in V_\alpha$. Then let x be an $\in$-minimal element of C. Then for all $z \in x$ there is α_z such that $z \in V_{\alpha_z}$, so there is some $\beta = \sup_{z \in x} \alpha_z$, and we see that $x \in V_{\beta+1}$, a contradiction. □

Theorem 3.2.6 ($\in$-induction)**.** *Let T be a transitive class and let Φ be a property. Assume that:*

1. *$\Phi(\emptyset)$ holds,*

2. *if $x \in T$ and $\Phi(z)$ holds for every $z \in x$, then $\Phi(x)$ holds.*

Then it follows that every $x \in T$ has property Φ.

Proof. Let C be the class of all $x \in T$ that do not have the property Φ. If C is nonempty, then it has an $\in$-minimal element. If it is $\emptyset$ we contradict the first point, otherwise we contradict the second point. $\square$

Theorem 3.2.7 ($\in$-recursion)**.** *Let T be a transitive class and let G be a function defined for all x. Then there is a function F on T such that $F(x) = G(F \restriction x)$ for every $x \in T$. Moreover, F is the unique such function.*

Proof. For every $x \in T$, we let $F(x) = y$ if and only if there exists a function f such that $\mathrm{dom}(f)$ is a transitive subset of T and *(i)* $(\forall z \in \mathrm{dom}\, f) f(z) = G(f \restriction z)$ and *(ii)* $f(x) = y$. As in ordinal induction, we can argue that this definition does not depend on f. $\square$

Definition 3.2.8. A class C is *extensional* if for all $x, y \in C$, if $x \neq y$ then there is some $z \in x \Delta y$ such that $z \in C$.

Theorem 3.2.9 (Mostowski's Collapsing Theorem)**.**

1. *For every extensional class C, there is a transitive class M and an isomorphism π between $(C, \in)$ and $(M, \in)$. Both the class M and the map π are unique.*

2. *If C is an extensional class and $T \subseteq C$ is transitive, then $\pi(x) = x$ for all $x \in T$.*

Proof. We define π by $\in$-induction. Let $\pi_C(x) = \{\pi_C(y) : y \in x \cap C\}$. Let $M := \mathrm{im}\, \pi_C$. To see that M is transitive, suppose that $z \in y \in M$. Then $y = \pi_C(a)$ for some a, and so $z = \pi_C(b)$ for some $b \in a$. Hence $z \in M$.

To see that π_C is one-one, suppose for contradiction that z is of minimal rank such that $z = \pi(x) = \pi(y)$ for some $x \neq y$. It must be the case that x and y are nonempty and that (without loss of generality) there is some u such that $u \in x \setminus y$. Then $\pi(u) \in \pi(x) = \pi(y)$. Therefore there is $v \in y$ such that $\pi(u) = \pi(v)$. But $v \neq u$ so this is a contradiction of minimality of z.

Next we argue that $x \in y$ if and only if $\pi_C(x) \in \pi_C(y)$. First, if $x \in y$ then $\pi_C(x) \in \pi_C(y)$ by definition. If $\pi_C(x) \in \pi_C(y)$ then $\pi_C(x) = \pi_C(z)$ for some $z \in y$. By injectivity, $z = x$.

It can be argued that an isomorphism between transitive classes must be the identity. If π_1, π_2 are isomorphisms from C to M_1 and M_2, then $\pi_2 \circ \pi_1^{-1}$ is an isomorphism and therefore the identity.

Lastly, suppose that C is transitive. Then we can argue by $\in$-induction that $\pi_C(x) = x$ for all $x \in C$: First, $\pi_C(\emptyset) = \emptyset$. Furtherore, $x \subseteq C$ for all $x \in C$, so by induction we have $\pi_C(x) = \{\pi_C(y) : y \in x \cap C = x\} = \{y : y \in x\} = x$. □

Example 10.

- Suppose that X is a set of ordinals. Then the Mostowski collapse of X equals α where $\alpha = \mathrm{ot}(X)$.
- Consider V_α and consider $\gamma > \alpha$. Then the Mostowski collapse of $V_\alpha \cup \{\gamma\} = V_\alpha \cup \{\alpha\}$.

3.3 Hereditary Sets

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Definition 3.3.1. Let κ be an infinite cardinal. Then $H(\kappa)$ denotes the set of sets x such that $|\operatorname{tc}(x)| < \kappa$.

Example 2.

- $\{\aleph_2, \aleph_3\} \notin H(\aleph_1)$.
- We have $\omega \subset H(\omega)$ but $\omega \notin H(\omega)$.
- Interpret the set of rationals $\mathbb{Q}$ as pairs of natural numbers modded out be an equivalence relation. Then $\mathbb{Q} \in H(\aleph_1)$.

Proposition 3.3.3. *Let κ be an infinite cardinal. Then the following are true:*

1. *$H(\kappa)$ is transitive,*
2. *$H(\kappa) \cap \mathrm{ON} = \kappa$.*
3. *If $x \in H(\kappa)$ then $\bigcup x \in H(\kappa)$.*
4. *If $x, y \in H(\kappa)$ then $\{x, y\} \in H(\kappa)$.*
5. *If $x \in H(\kappa)$ and $y \subseteq x$ then $y \in H(\kappa)$.*

Proof. 1. First we argue that $x \in y$ implies that $\operatorname{tc}(x) \subseteq \operatorname{tc}(y)$: It is sufficient to argue that $\operatorname{tc}(y) \supseteq x$ by minimality of $\operatorname{tc}(x)$. Suppose that $z \in x$. Since $z \in x \in y$, it follows that z is an element of any transitive set containing y, so $z \in \operatorname{tc}(y)$.

Note: This was the part where the lecture was sketchy! Hopefully it's clear here.

Use that $x \in y$ implies that $\operatorname{tc}(x) \subseteq \operatorname{tc}(y)$ and hence $|\operatorname{tc}(x)| \leq |\operatorname{tc}(y)|$.

2. Use that $\operatorname{tc}(\alpha) = \alpha$ for ordinals α since they are transitive.

3. We can prove by $\in$-induction that $\operatorname{tc}(\bigcup x) = \bigcup_{y \in x} \operatorname{tc}(y)$. Therefore $|\operatorname{tc}(\bigcup x)| \leq \cdot |x| \cdot \sup_{y \in x} |\operatorname{tc}(y)| \leq |x| \cdot |x| = |x| < \kappa$.

4. $\operatorname{tc}(\{a, b\}) = \{a, b\} \cup \operatorname{tc}(a) \cup \operatorname{tc}(b)$ as in the previous point.

5. Uses that $y \subseteq x$ implies $\operatorname{tc}(y) \subseteq \operatorname{tc}(x)$. □

Proposition 3.3.4. *Suppose κ is a regular uncountable cardinal. Then $H(\kappa)$ satisfies the* ZFC *axioms besides powerset.*

Proof. Extensionality. We need to say that if $x \neq y$ and $x, y \in H(\kappa)$, then there is some $z \in x \Delta y$ such that $z \in H(\kappa)$. Since there is definitely some $z \in x \Delta y$, WLOG in x, we know $|\operatorname{tc}(z)| \leq |\operatorname{tc}(x)| < \kappa$.

Pairing. Done above.

Schema of Comprehension/Separation. Uses that if $x \subseteq y$ then $\operatorname{tc}(x) \subseteq \operatorname{tc}(y)$.

Union. Done above.

Infinity. ω is transitive.

Schema of Replacement. Let $\varphi(v, \bar{w})$ define a function with parameters $\bar{b}$ and domain A.

Regularity/Foundation. If $y \in x$ then $\operatorname{tc}(y) \subseteq \operatorname{tc}(x)$.

Axiom of Choice. Similar to previous items. □

We still get a fragment of the powerset axiom for $H(\kappa)$'s though.

Definition 3.3.5. We say that an uncountable cardinal κ is:

1. *weakly inaccessibe* if $\lambda^+ < \kappa$ for all $\lambda < \kappa$;
2. *strongly inaccessible* or just *inaccessible* if $2^\lambda < \kappa$ for all $\lambda < \kappa$.

Theorem 3.3.6. *Let κ be regular and uncountable. The following are equivalent:*

1. *κ is strongly inaccessible,*
2. *$H(\kappa) = V_\kappa$,*

3. $H(\kappa) \models \mathsf{ZFC}$.

Proof. 1. $\implies$ *2.* We know that $H(\kappa) \subseteq V_\kappa$. The other direction uses induction, the definition of inaccessibility, and the fact that if $2^\lambda < \kappa$ then $|\operatorname{tc}(\mathcal{P}(\lambda))| < \kappa$.

2. $\implies$ *1.* Suppose that $\lambda < \kappa$ and $2^\lambda \geq \kappa$. Then $\mathcal{P}(\lambda) \in V_\kappa \setminus H(\kappa)$.

1. $\implies$ *.3* We only need to check the powerset axiom because we already proved that we have the others: Suppose $X \in H(\kappa)$. Then $|X| \leq |\operatorname{tc}(X)| < \kappa$. Since $2^{|X|} < \kappa$, we have that $\mathcal{P}(X) \in H(\kappa)$.

3. $\implies$ *.1* Again, if the powerset axiom fails, this implies failure of inaccessibility. More precisely, if there is some $X \in H(\kappa)$ such that $\mathcal{P}(X) \notin H(\kappa)$, then this can only be the case if $2^{|X|} \geq \kappa$. $\square$

3.4 Reviewing Some Basic Model Theory

Recall some definitions from model theory, which we will summarize in very loose terms for the sake of haste:

1. A *language* is a set of symbols including constant, function, and relation symbols. In set theory we will only use the language $\mathcal{L} = \{=, \in\}$ (typically the notation for equality is suppressed).

2. Symbols from the language are built up into *terms*, which are build up with variables to create *formulas* using $\neg, \wedge, \vee$ and adding quantifiers $\exists, \forall$. Variables are *free* if they are not included in the scope of quantifiers. A formula with no free variables is called a *sentence* and sentences have truth values.

3. Given a language $\mathcal{L}$, an $\mathcal{L}$-*structure* is a set in which we can interpret truth values of sentences.

4. A *theory* is a set of sentences. It is satisfiable if it has a model M.

5. Two structures are *elementary equivalent*, denoted $M \equiv N$, if they satisfy the same sentences. If $M \subseteq N$, then we say M is an *elementary submodel* of N, denoted $M \prec N$, if for all $\bar{a} \in M$ and formulas φ, $M \models \varphi(\bar{a})$ if and only if $N \models \varphi(\bar{a})$.

6. We say that $a \in M$ is *definable* with a parameter $\bar{b}$ if there is a formula $\varphi(v, \bar{b})$ such that $M \models \varphi(a, \bar{b})$ and a is the only element with this property.

Theorem 3.4.1 (Tarski-Vaught Test)**.** *Let $M \subseteq N$ be $\mathcal{L}$-structures. Suppose it is the case that if $\bar{a} \in M$ and there is $b \in N$ such that $N \models \varphi(b, \bar{a})$, then there is $c \in M$ such that $M \models \varphi(c, \bar{a})$. Then it follows that $M \prec N$.*

Theorem 3.4.2 (Downward Löwenheim-Skolem)**.** *Let K be a structure and let $A \subset K$. Then there is $M \prec K$ such that $A \subset M$ and $|M| = |A| + \aleph_0$.*

3.5 Absoluteness and Reflection

November 19, 2024

Definition 3.5.1. Let $\varphi(x_1, \ldots, x_n)$ be a formula and M a class. Then the *relativization* of φ to M, denoted φ^M, is defined via the following cases:

1. $(x \in y)^M \leftrightarrow x \in y$,
2. $(x = y)^M \leftrightarrow x = y$,
3. $(\varphi \wedge \psi)^M \leftrightarrow \varphi^M \wedge \psi^M$,
4. $(\neg\varphi)^M \leftrightarrow \neg(\varphi^M)$,
5. $(\exists v \varphi(v, \bar{x}))^M \leftrightarrow (\exists v \in M)\varphi(v, \bar{x})$.

Example 2. This can also affect the truth of the statement. Let φ state that there is an uncountable cardinal. Then φ is false in $H(\aleph_1)$.

Definition 3.5.3. Suppose $\varphi(\bar{x})$ is a formula with free variables among $\bar{x}$.

1. Suppose that $M \subseteq N$ are classes. Then φ is *absolute for M and N* if
$$\forall \bar{a} \in M(\varphi^M(\bar{a}) \iff \varphi^N(\bar{a})).$$
2. If M is a class, we say that φ is *absolute* (without explicit reference to another model) if it is absolute for M and V.

Proposition 3.5.4. *A formula Δ_0 if it is generated by the following rules:*

1. $x \in y$ and $x = y$ are Δ_0.

2. *If* φ, ψ *are* Δ_0 *then* $\varphi \wedge \psi$ *and* $\neg\varphi$ *are* Δ_0.

3. *If* φ *is* Δ_0, *then* $\exists v(v \in w \wedge \varphi(v, \bar{u}))$ *is* Δ_0.

Proposition 3.5.5. *Suppose* $M \subseteq N$ *are transitive classes. Then* Δ_0 *formulas are absolute with respect to* M *and* N.

Proof. We induct on formula composition. The only nontrivial case is bounded quantification. First, observe that if $\bar{a}, b \in M$ and $M \models \exists v(v \in b \wedge \varphi(v, \bar{a})$ as witnessed by c, then $c \in N$ and we have $N \models (c \in \wedge \varphi(c, \bar{a})$ by formula induction and therefore $N \models \exists v(v \in b \wedge \varphi(v, \bar{a})$.

Now suppose that $\bar{a}, b \in M$ and $N \models \exists v(v \in b \wedge \varphi(v, \bar{a})$ as witnessed by c. Since $b \in M$, transitivity implies that $c \in M$, and by formula induction we have that $M \models \exists v(v \in b \wedge \varphi(v, \bar{a})$. □

Proposition 3.5.6. *The following formulas are expressible as* Δ_0 *formulas for any model of* $\mathsf{ZF} - \{\text{powerset, foundation, infinity}\}$*: 1.* $x \in y$*, 2.* $x = y$*, 3.* $x \subseteq y$*, 4.* $\{x, y\}$ *(or* $z = \{x, y\}$*), 5.* $\{x\}$*, 6.* $\langle x, y\rangle$*, 7.* $\emptyset$*, 8.* $x \cup y$*, 9.* $x \cap y$*, 10.* $x \setminus y$*, 11.* $S(x) = x \cup \{x\}$*, 12.* x *is transitive, 13.* $\bigcup x$*, 14.* $\bigcap x$ *(where* $\cap\emptyset = 0$*).*

Proof. Since some of these are similar, we will present sufficiently many of the items to give a clear picture.

1,2 are by definition.

3. $\forall z \in x(z \in y)$.

4. $x \in z \wedge y \in z \wedge \forall w \in z(w = x \vee w = y)$.

6. $z = \langle x, y\rangle$ if and only if

$$\exists w \in z(w = \{x\}) \wedge \exists w \in z(w = \{x, y\}) \wedge \forall w \in z(w = \{x\} \vee w = \{x, y\}).$$

7. $z = \emptyset$ iff $\forall w \in z(w \neq w)$.

8. $z = x \cup y$ if and only if $\forall w \in z(w \in x \vee w \in y) \wedge x \subseteq z \wedge y \subseteq z$.

12. x is transitive if and only if $\forall v \in x \forall z \in v(z \in x)$.

13. $y = \bigcup x$ if and only if $\forall v \in x(v \subseteq y) \wedge \forall z \in y \exists v \in x(z \in v)$. □

Lemma 3.5.7. *Suppose* $M \subseteq N$ *and that* $\varphi(\bar{x})$ *and* $G_i(\bar{y})$ $(i = 1, \ldots, n)$ *are absolute for* M, N *(where absoluteness for functions has the natural definition). Then the formula*

$$\varphi(G_1(\bar{y}), \ldots, G_n(\bar{y}))$$

is absolute for M, N.

Proof. If $a \in M$, then with some notational simplifications,

$$(\varphi(G(a)))^M \iff \varphi^M(G^M(a)) \iff \varphi^N(G^N(a)) \iff (\varphi(G(a)))^N.$$

□

Proposition 3.5.8. *The following are absolute for any model of* $\mathsf{ZF} - \{$*powerset, foundation, infinity*$\}$*: 1. z is an ordered pair. 2. $A \times B$. 3. R is a relation. 4.* $\operatorname{dom} R$. *5.* $\operatorname{range} R$. *6. R is a function. 7. $R(x)$. 8. R is a one-one function.*

Proof. Again, we will go through just some of the main points.

Saying that z is an ordered pair is equivalent to saying that

$$\varphi(z \mapsto \bigcup z, z \mapsto \bigcup z, z)$$

where $\varphi(a, b, c)$ is the formula

$$\exists x \in a \exists y \in b (c = \langle x, y \rangle).$$

The rest of the points clearly build on one another.

For example, R is a relation if and only if $\forall z \in R$, z is an ordered pair. □

Proposition 3.5.9. *Let M and N be classes with $M \subseteq N$. Let $\varphi_1, \ldots, \varphi_n$ be a list of formulas that is closed under subformulas (i.e. if $\varphi_i = \psi_1 \wedge \psi_2$, then $\psi_1 = \varphi_m$ for some m). Then the following are equivalent:*

1. *$\varphi_1, \ldots, \varphi_n$ are absolute for M, N.*
2. *If φ_i is of the form $\exists x \varphi_j(x, \bar{y})$ (with the free variables among $y, \bar{x}$), then*
$$\forall \bar{y} \in M[\exists x \in N \varphi_j^N(x, \bar{y}) \implies \exists x \in M \varphi_j^N(x, \bar{y})].$$

Note that this is basically the Tarski-Vaught test.

Proof. For *1.* $\implies$ *2.*, suppose that $\bar{b} \in M$ and there is $a \in N$ such that $\varphi_j^N(a, \bar{b})$ holds. Then by absoluteness, φ_i^M holds, so there is some $c \in M$ such that $\varphi_j(c, \bar{b})^M$ holds, and so $\varphi_j(c, \bar{b})^N$ holds as well.

Now for the other direction, which is done by induction on formulas. The statement clearly holds for the atomic formulas $x \in y$ and $x = y$, and the connectives $\neg$ and $\wedge$ also can be used in induction easily. The main consideration is the existential quantification, when φ_i is $\exists x, \varphi_j(x, \bar{y})$. If $\bar{b} \in M$, then

$$\varphi_i^M(\bar{y}) \iff \exists a \in M \varphi_j^M(x, \bar{b}) \iff \exists a \in M \varphi_j^N(a, \bar{b}) \iff$$
$$\iff \exists a \in N \varphi_j^N(a, \bar{b}) \iff \varphi_i^N(\bar{b}).$$

□

Theorem 3.5.10. *Suppose Z is a class and suppose that for each α, Z_α is a set such that the following hold:*

1. *$\alpha < \beta \implies Z_\alpha \subseteq Z_\beta$,*
2. *If α is a limit then $Z_\alpha = \bigcup_{\beta<\alpha} Z_\beta$,*
3. *Z is the class of all sets x such that $x \in Z_\alpha$ for some α.*

Then for any formulas $\varphi_1, \dots, \varphi_k$,

$$\forall \alpha \exists \beta > \alpha (\varphi_1, \dots, \varphi_k \text{ are absolute for } Z_\beta, Z).$$

In particular, this applies to V, and hence we are generally working in some V_α when we try to apply an independence proof.

Proof. We will use the second condition in Proposition 3.5.9 in terms of Z and Z_β for a suitable β. Assume that $\varphi_1, \dots, \varphi_n$ is closed under subformulas.

For each $i = 1, \dots, k$, define $F_i : \mathrm{ON} \to \mathrm{ON}$ as follows: If φ_i takes the form $\exists x \varphi_j(x, \bar{y})$, let $G_i(\bar{y}) = 0$ if $\neg \exists x \in Z \varphi_j^Z(x, \bar{y})$, and otherwise let $G_i(\bar{y})$ be the least η such that $\exists x \in Z_\eta \varphi_j^Z(x, \bar{y})$. Then let

$$F_i(\xi) = \sup\{G_i(\bar{y}) : \bar{y} \in Z_\xi\}.$$

If φ_i is not an existential quantification we can let $F_i(\xi) = 0$.

By Proposition 3.5.9, if β is a limit ordinal and for each i, $\forall \xi < \beta (F_i(\xi) < \beta)$, then $\varphi_1, \dots, \varphi_k$ will be absolute for Z_β, Z.

Fixing α, we show that we can define such a $\beta > \alpha$. Let $\beta_0 = \alpha$, and given β_n, let β_{n+1} be the largest of

$$\beta_n + 1, F_1(\beta_n), \dots, F_k(\beta_n).$$

(This is where we use the fact that there are only finitely many formulas to consider.) Let $\beta = \sup\{\beta_n : n < \omega\}$. Observe that $\xi < \xi'$ implies $F_i(\xi) \leq F_i(\xi')$ (using the first hypothesis of the theorem). It follows that β witnesses the theorem. $\square$

Corollary 3.5.11. *No extension of* ZF *is finitely axiomatizable.*

Proof. Suppose that $\varphi_1, \ldots, \varphi_n$ enumerates the axioms of some theory extending ZF. Let β be the least ordinal such that $\bigwedge_{1 \leq i \leq n} \varphi_i$ is true in V_β. Therefore we can construct, in V_β, the "least" α such that the conjunction holds. By absoluteness, $\alpha < \beta$. But this contradicts minimality of β. $\square$

November 21, 2024

Synonyms: $\varphi^M(\bar{a})$ is also phrased as: $\varphi(\bar{a})$ is true in M or M is a model of $\varphi(\bar{a})$. This is also used for a set of sentences: T is true in M and M is a model of T both mean for any $\varphi \in T$, φ^M.

Easy examples of relative consistency.

Lemma 3.5.12. *Let S and T be two sets of sentences in the language of set theory. Suppose that for some non-empty class M we can prove in T that $M \models S$. Then* $\mathrm{Con}(T) \to \mathrm{Con}(S)$.

Proof. From $S \vdash \chi \wedge \neg\chi$ we could prove $T \vdash \chi^M \wedge \neg\chi^M$ $\square$

A more formal version of this lemma is given in Chapter IV Section 8 of Kunen.

Some examples of such T and S are:

Example 13.

1. $\mathrm{Con}(\mathsf{ZF}) \to \mathrm{Con}(\mathsf{ZF} - \mathrm{Inf} + \neg\mathrm{Inf})$.

2. $\mathrm{Con}(\mathsf{ZF}) \to \mathrm{Con}(\text{Extensionality, Comprehension} + \forall x, x = \emptyset)$.

3. $\mathrm{Con}(\mathsf{ZF} - \text{foundation}) \to \mathrm{Con}(\mathsf{ZF})$.

Proof. In ZF we have $M = (V_\omega, \in)$, which models $\mathsf{ZF} - \mathrm{Inf} + \neg\mathrm{Inf}$ and we have $M = (\{0\}, \emptyset)$ which models Extensionality, Comprehension $+ \forall x x = \emptyset$). In ($\mathsf{ZF}$ − foundation) one defines $M = \bigcup\{V_\alpha : \alpha \in \mathrm{ON}\}$ (as in Section 3.1) and shows by induction on α that this is a well-founded model of ZF. $\square$

Examples for absoluteness:

(1) "α is an ordinal" is absolute for transitive models of ZF. We can formalize it: $\forall x \in \alpha, \forall y \in x, y \in \alpha \wedge \in$ is a linear order on α.

In $\mathsf{ZF} -$ foundation, though, we would need a Π_1-clause, i.e., an unrestricted $\forall$-quantification over x, $\forall x(\emptyset \neq x \subseteq \alpha \to \exists y \in x, \forall z \in y, z \notin x)$. This clause of the definition of an ordinal on the basis of $\mathsf{ZF} -$ foundation need not be upwards absolute for transitive models of $\mathsf{ZF} -$ foundation.

(2) "$\alpha = \omega$" is absolute for transitive models of ZF.

We say: α is an Ordinal, and any $x \in \alpha$ is either the empty set or $\exists y \in x, x = y \cup \{y\}$, and there is no $y \in \alpha$ such that $\alpha = y \cup \{y\}$. This is a Δ_0-formula.

Chapter 4

The Consistency of the Continuum Hypothesis and the Axiom of Choice

4.1 Defining Definability

We call a set a "definable" if there is some property P in English such that

$$\forall x(P(x) \leftrightarrow x = a).$$

For example $\mathbb{R}$ is definable. There are at most countably many definitions, and hence there is a least non-definable ordinal. But we just defined it. This paradox shows that "a is definable" is not expressible by a single first order formula.

We formalize a more specific concept. Let A be a transitive set. We would like to find a non-paradoxical formula or set of formulas $\mathrm{Df}(A, n)$ such that

$$\begin{aligned}\mathrm{Df}(A, n) = \Big\{ & X : X \subset A^n : \exists \varphi \in L(\{\in\}), (fr(\varphi) = \{x_1, \ldots, x_n\} \wedge \\ & X = \{\langle x_i : i = 1, \ldots, n\rangle : \varphi^A(x_1, \ldots, x_n)\}) \Big\}.\end{aligned}$$

For an element x of A, we say x is definable in A, if $X = \{x\}$ is definable in A.

We describe the possible φ in a semantic way, using two basic binary relations $\in$ and $=$ and projections, intersections and complementation. For later use, it is important that the definition takes place in ZF, indeed, in $\mathsf{ZF} - P$. We use foundation but we do not use choice.

Definition 4.1.1 ([?, Definition V.1.1])**.** If $n \in \omega$ and $i, j < n$ we let

(a) $\mathrm{Proj}(A, R, n) = \{s \in A^n : \exists t \in R(t \restriction n = s)\}$,

(b) $\mathrm{Diag}_{\in}(A, n, i, j) = \{s \in A^n : s(i) \in s(j)\}$,

(c) $\mathrm{Diag}_{=}(A, n, i, j) = \{s \in A^n : s(i) = s(j)\}$.

(d) By recursion on $k \in \omega$, we define $\mathrm{Df}'(k, A, n)$ for all n simultaneously by:

  (1) $\mathrm{Df}'(0, A, n) = \{\mathrm{Diag}_{\in}(A, n, i, j) : i, j < n\} \cup \{\mathrm{Diag}_{=}(A, n, i, j) : i, j < n\}$,

  (2) $\mathrm{Df}'(k+1, A, n) = \mathrm{Df}'(k, A, n) \cup \{A^n \setminus R : R \in \mathrm{Df}'(k, A, n)\} \cup \{R \cap S : R, S \in \mathrm{Df}'(k, A, n)\} \cup \{\mathrm{Proj}(A, R, n) : R \in \mathrm{Df}'(k, A, n+1)\}$.

(e) $\mathrm{Df}(A, n) = \bigcup\{\mathrm{Df}'(k, A, n) : k \in \omega\}$.

Lemma 4.1.2. *Let $\varphi(x_0, \dots, x_{n-1})$ be any formula whose free variables are among $x_0, \dots, x_{n-1}$ then*

$$\forall A, \{s \in A^n : \varphi^A(s(0), \dots, s(n-1))\} \in \mathrm{Df}(A, n).$$

Proof. By induction on the length of φ simultaneously for all n. If φ is $\exists x_n \psi(x_0, \dots, x_{n-1}, x_n)$ then by induction hypothesis,
$\{t \in A^{n+1} : \psi^A(t(0), \dots, t(n))\} \in \mathrm{Df}'(k, A, n+1)$ for some k. Now

$$\begin{aligned}&\{s \in A^n : \varphi^A(s(0), \dots, s(n-1))\} = \\ &\mathrm{Proj}(A, \{t \in A^{n+1} : \psi^A(t(0), \dots, t(n))\}, n) \in \mathrm{Df}(A, n).\end{aligned}$$

If the quantification is over another free variable of ψ, then we take an equivalent formula by permuting the variables. □

This lemma is actually a scheme of lemmata. The reverse inclusion is not first order: Let φ_i, $i < \omega$, list all the formulars with free variables among $x_0, \dots, x_{n-1}$, then

$$\forall A, \forall Y \in \mathrm{Df}(A, n), \bigvee_{i \in \omega} Y = \{s \in A^n : \varphi_i^A(\bar{s})\}$$

Definition 4.1.3. By recursion on $m \in \omega$ the sequence $\langle \mathrm{En}(m, A, n) : n < \omega \rangle$ is defined by the following clauses:

(a) If $m = 2^i \cdot 3^j$ and $i, j < n$, then $\mathrm{En}(m, A, n) = \mathrm{Diag}_\in(A, n, i, j)$.

(b) If $m = 2^i \cdot 3^j \cdot 5$ and $i, j < n$, then $\mathrm{En}(m, A, n) = \mathrm{Diag}_=(A, n, i, j)$.

(c) If $m = 2^i \cdot 3^j \cdot 5^2$ and $i, j < n$, then $\mathrm{En}(m, A, n) = A^n \setminus \mathrm{En}(i, A, n)$.

(d) If $m = 2^i \cdot 3^j \cdot 5^3$ and $i, j < n$, then $\mathrm{En}(m, A, n) = \mathrm{En}(i, A, n) \cap \mathrm{En}(j, A, n)$.

(e) If $m = 2^i \cdot 3^j \cdot 5^4$ and $i, j < n$, then $\mathrm{En}(m, A, n) = \mathrm{Proj}(A\,\mathrm{En}(i, A, n+1), n)$.

(f) If m is not of the form specified in one of (a) to (e), then $\mathrm{En}(m, A, n) = \emptyset$.

Lemma 4.1.4. *For any n and A,* $\mathrm{Df}(A, n) = \{E(m, A, n) : m \in \omega\}$.

Proof. We show by induction on k,

$$\forall n, \mathrm{Df}'(k, A, n) \subseteq \{\mathrm{En}(m, A, n) : m < \omega\}.$$

For the reverse inclusion we go by induction on m and show $\forall n\, \mathrm{En}(m, A, n) \in \mathrm{Df}(A, n)$. □

Lemma 4.1.5. *The defined function* En *and* Df *are absolute for transitive models of* $\mathsf{ZF} - P$.

Proof. The fact that functions defined recursively (over $(\omega, \in)$) using absolute notions are absolute is applied several times. This fact is like Theorem 3.2.7 (recursion over $(T, \in)$) with the additional premise that T and G are absolute. It is proved like Theorem 3.2.7 with an eye on checking that the induction step extends an absolute function to a larger absolute function. Then the resulting F is absolute. We also use that $\varphi(m, i, j) = m = 2^i 3^j$ for $i, j \in \omega$ is absolute. □

Sometimes we want to define something by recurring over (T, R) where R is some relation on a transitive set T. This works if we know that R is well-founded. An example used frequently later in forcing is $T = V_\alpha \times V_\alpha$ and $\langle \langle x, y \rangle, \langle z, u \rangle \rangle \in R$ iff $x \in z \wedge y \in u$. Is well-foundedness an absolute property?

Lemma 4.1.6. *"(T, R) is well-founded" is an absolute property for transitive models of a sufficiently large finite fragment of* ZF.

Proof. Let $M \subseteq N$ be two transitive models of a bit of ZF (existentiality, foundation, pairing, and some instances of replacement are used in our proof) and $(T, R) \in M$. The property "(T, R) is wellfounded" is formalized as $\forall x \subseteq T(x \neq \emptyset \to \exists y \in x, \forall z \in x, \neg\langle z, y\rangle \in R)$. This is a $\forall$-quantification over a Δ_0-property (such a property is called a Π_1-property), and hence downwards absolute: If it holds in N, then it holds in M.

The property "(T, R) is well-founded" can also be formalized by

$$\exists \alpha \in \mathrm{ON}, \exists f \colon T \to \alpha, \forall x, y \in T, (\langle x, y\rangle \in R \to f(x) \in f(y)).$$

Such an f is called a rank function for (T, R). Now this formalization is existential, also called a Σ_1-formula, and hence upwards absolute: If it holds in M, then it holds in N. □

4.2 Defining Gödel's Constructible Universe

November 26, 2024

Definition 4.2.1. We write

$$\{X \subseteq A : \exists n \in \omega, s \in [A]^n \exists R \in \mathrm{Df}(A, n+1)(X = \{x \in A : s^\frown\langle x\rangle \in R\})\}$$

as $\mathcal{D}$.

The idea is that $\mathcal{D}$ is the set of subsets of A that are definable from finitely many elements from A with respect to the relativization.

Lemma 4.2.2. *Let $\varphi(\bar{v}, x)$ be a formula with free variables among $\bar{v}, x$. Then*

$$\forall A \forall \bar{a} \in A[\{x \in A : \varphi^A(\bar{a}, x)\} \in \mathcal{D}(A)].$$

Technically, this lemma is a schema.

Proof. This is analogous to Lemma 4.1.2. □

Lemma 4.2.3. *For any A,*

1. $\mathcal{D}(A) \subseteq \mathcal{P}(A)$.

2. *If A is transitive, then $A \subseteq \mathcal{D}(A)$.*

3. $\forall X \subseteq A, (|X| < \omega \implies X \in \mathcal{D}(A)$,

4. $(\mathsf{AC})|A| \geq \omega \implies |\mathcal{D}(A)| = |A|$.

Proof. *1.* Observe that $\mathcal{D}(A)$ is defined specifically in terms of subsets of A.

2. If A is transitive, then $A \subseteq \mathcal{D}(A)$. Applying Lemma 4.2.2 in terms of $x \in v$ gives us

$$\forall b \in A[\{x \in A : x \in b\} \in \mathcal{D}(A)],$$

and if A is transitive then this gives us $\forall b \in A(b \in \mathcal{D}(A))$.

3. Let $X = \{b_1, \ldots, b_m\}$. Each of these are defined via some $\varphi_i(x, \bar{a}_i)$. Informally, we use *2.* m-many times to inductively build up a formula defining a set which consists exactly of these elements.

4. Assuming AC, if $|A| \geq \omega$ then $|A^n| = |A|$ for all $n < \omega$. We can argue that $|\operatorname{Df}(A, n+1)| \leq \omega$ since $\operatorname{Df}(A, N) = \{E(m, A, N) : m \in \omega\}$. □

Definition 4.2.4. By induction on $\alpha \in \mathrm{ON}$, we define:

1. $L_0 = \emptyset$,

2. $L_{\alpha+1} = \mathcal{D}(L_\alpha)$,

3. $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$.

Then we informally write $L = \bigcup_{\alpha \in \mathrm{ON}} L_\alpha$.

Proposition 4.2.5. *For all $\alpha \in \mathrm{ON}$:*

1. *L_α is transitive,*

2. $\forall \beta \leq \alpha, L_\beta \subseteq L_\alpha$.

Proof. We argue by induction, almost exactly as with the analogous lemma for V. Assume that both statements hold for $\beta < \alpha$, which is trivial for $\beta = 0$. The only meaningful difference is that we need the second point of Lemma 4.2.3. □

Definition 4.2.6. If $x \in L$, the *L-rank* $\rho(x)$ is the least β such that $x \in L_{\beta+1}$.

Remark: Beware! The L-rank is not the same as the V-rank.

Proposition 4.2.7. *For all $\alpha \in \mathrm{ON}$, $L_\alpha = \{x \in L : \rho(x) < \alpha\}$.*

Proof. This is a tautology. □

Proposition 4.2.8. *For all $\alpha \in \mathrm{ON}$,*

1. $\alpha \in L \wedge \rho(\alpha) = \alpha$,
2. $L_\alpha \cap \mathrm{ON} = \alpha$,
3. $L_\alpha \in L_{\alpha+1}$,
4. $L_\alpha \subseteq V_\alpha$.

Proof. *1.* is implied by *2.*, so we will prove the second one by induction on α. This is trivial if $\alpha = 0$ or if α is a limit, so we suppose $\alpha = \beta + 1$ and $L_\beta \cap \mathrm{ON} = \beta$.

Since $L_\beta \subseteq L_\alpha \subseteq \mathcal{P}(L_\beta)$, we have $\beta \subseteq L_\alpha \cap \mathrm{ON} \subseteq \alpha$, so we are done if we show that $\beta \in L_\alpha$. Recall that there is a Δ_0 formula $\varphi(x)$ such that $\forall x(x$ is an ordinal $\iff \varphi(x))$. Since Δ_0 formulas are absolute for transitive sets,

$$\beta = L_\beta \cap \mathrm{ON} = \{x \in L_\beta : \varphi^{L_\beta}(x)\},$$

and so $\beta \in \mathcal{D}(L_\beta) = L_{\alpha+1}$.

For *3.*, note that $L_\alpha = \{x \in L_\alpha : (x = x)^{L_\alpha}\}$, so $L_\alpha \in \mathcal{D}(L_\alpha)$.

Item *4.* can be proved using a straightforward transfinite induction, essentially using that $\mathcal{D}(A) \subseteq \mathcal{P}(A)$. □

Proposition 4.2.9. *1. Every finite subset of L_α is in $L_{\alpha+1}$.*

2. For all $\alpha \leq \omega$, $L_\alpha = V_\alpha$.

Proof. The first point is using that fact that finite subsets of A are in $\mathcal{D}(A)$.

For the second point, we argue by induction. This is immediate for $L_0 = V_0 = \emptyset$. Then we can see that if A is finite and transitive, then $\mathcal{D}(A) = \mathcal{P}(A)$. This gives us $V_n = L_n$ for $n < \omega$. Then we have $L_\omega = \bigcup_{n<\omega} L_n = \bigcup_{n<\omega} V_n = V_\omega$. □

Lemma 4.2.10. (AC) *For all $\alpha \geq \omega$, $|L_\alpha| = |\alpha|$.*

Proof. Use transfinite induction. Since $\alpha \subseteq L_\alpha$, $|\alpha| \leq |L_\alpha|$.

If α is a limit, then by AC we have $|L_\alpha| = |\alpha| \cdot \sup_{\beta<\alpha} |L_\beta| = |\alpha|$.

If $\alpha = \beta+1$, then $|L_\alpha| = |L_{\beta+1}| = |L_\beta| + |\mathcal{D}(L_\beta)| = |L_\beta| + |L_\beta| = |L_\beta| = |\beta| = |\alpha|$. □

Theorem 4.2.11. $L \models \mathsf{ZF}$.

Proof. *Extensionality:* Works because L is transitive.

Foundation: This will hold in any class.

Comprehension: For each $\psi(x, z, \bar{v})$ with free variables shown, we want to prove that

$$\forall b, \bar{a} \in L(\{x \in b : \psi^L(x, b, \bar{a})\} \in L).$$

Let α be large enough that $b, \bar{a} \in L_\alpha$. Then apply the Reflection Theorem such that ψ is absolute between L and L_β. Then

$$\{x \in b : \psi^L(x, b, \bar{a})\} = \{x \in L_\beta : \psi^{L_\beta}(x, b, \bar{a}) \wedge (x \in b)^{L_\beta}\} \in \mathcal{D}(L_\beta) = L_\alpha.$$

Replacement: We want to show that for each formula $\varphi(x, y, A, \bar{w})$ and each $A, \bar{w} \in L$, if it holds that

$$\forall x \in A \exists! y \in L \varphi^L(x, y, A, \bar{w}),$$

then it follows that

$$\exists Y \in L(\{y : \exists x \in A \varphi^L(x, y, A, \bar{w})\} \subseteq Y).$$

Assuming the hypothesis, let

$$\alpha = \sup\{\rho(y) + 1 : \exists x \in A \varphi^L(x, y, A, \bar{w})\}$$

and then take $Y = L_\alpha$.

Power Set: Let $X \in L$ and let $Y = \mathcal{P}(X) \cap L$. Let α be such that $Y \subseteq L_\alpha$. Then Y is definable over L_α by the Δ_0 formula for the subset relation. Noting that this is absolute, we are done.

Pairing, Union: These are similar to the ones we have already looked at.

Infinity: Holds because $\omega \in L_{\omega+1}$. □

4.3 The Axiom of Choice and the Continuum Hypothesis in L

November 28, 2024

Theorem 4.3.1. $L \models \mathsf{AC}$.

Proof. Recall that the axiom of choice is equivalent to saying that every set can be well-ordered. We will define a global well-ordering for L that is definable within L.

We will write $\lhd_\alpha$ for the ordering on L_α and we will show that it extends nicely.

If α is a limit ordinal and we have defined $\lhd_\beta$ for $\beta < \alpha$ and know

$$\forall \beta < \gamma < \alpha, \forall x \in L_\beta, \forall y \in L_\gamma \setminus L_\beta, x \lhd_\gamma y$$

(this says that $\lhd_\gamma$ is an end extension of $\lhd_\beta$) then we let $x \lhd_\alpha y$ if and only if

$$\rho(x) < \rho(y) \vee (\rho(x) = \rho(y) \wedge x \lhd_{\rho(x)+1} y).$$

Now we take care of the successor step. Suppose we have $\lhd_\alpha$. We will use the lexicographic order $\lhd_\alpha^n$ on $(L_\alpha)^n$ in which

$$s \lhd_\alpha^n t \iff \exists k < n(s \restriction k = t \restriction k \wedge s(k) \lhd_\alpha t(k)).$$

If $X \in L_{\alpha+1} = \mathcal{D}(L_\alpha)$, let n_X be the least n such that

$$\exists s \in (L_\alpha)^n \exists R \in \mathrm{Df}(L_\alpha, n+1)(X = \{x \in L_\alpha : s^\frown \langle x \rangle \in R\}).$$

So n_X checks the arity of the formula defining X. Let s_X be the $\lhd_\alpha^{n_X}$-least $s \in (L_\alpha)^{n_X}$ such that

$$\exists R \in \mathrm{Df}(L_\alpha, n_X + 1)(X = \{x \in L_\alpha : s^\frown \langle x \rangle \in R\}).$$

So s_X is the set of parameters used to define X. Let m_X be the least $m \in \omega$ such that

$$X = \{x \in L_\alpha : {s_X}^\frown \langle x \rangle \in \mathrm{En}(m, L_\alpha, n_X)\}.$$

So m checks the Gödel number of the formula defining X. Then for $X, Y \in L_{\alpha+1}$, we let $X \lhd_{\alpha+1} Y$ if and only if:

(a) $X, Y \in L_\alpha \wedge X \lhd_\alpha Y$ or

(b) $X \in L_\alpha \wedge Y \notin L_\alpha$ or

(c) $X, Y \notin L_\alpha$ and

$$(n_X < n_Y) \vee (n_X = n_Y \wedge s_X \lhd_\alpha^{n_X} s_Y) \vee (n_X = n_Y \wedge s_X = s_Y \wedge m_X < m_Y).$$

These cases are clearly mutually exclusive. Observe that the successor step and limit steps agree with one another.

It can be proved inductively that $\lhd_\alpha$ is a well-order. This follows from observing that case (c) of the successor step gives a well-order.

To formally get a choice function, consider $x \in L$. Then $x \subseteq L_\alpha$ for some $\alpha \in \mathrm{ON}$ and we can see that $\lhd_\alpha$ well-orders x. $\square$

Corollary 4.3.2. $\mathrm{Con}(\mathsf{ZF}) \implies \mathrm{Con}(\mathsf{ZFC})$.

Definition 4.3.3. $\lhd_L$ is the ordering such that takes $\lhd_\alpha$ for $x, y \in L_\alpha$.

Definition 4.3.4.

1. We say that a formula $\varphi(\bar{v})$ is Σ_1 if $\varphi(\bar{v}) \iff \exists w \theta_0(w, \bar{v})$ where θ_0 is Δ_0.
2. We say that a formula $\varphi(\bar{v})$ is Π_1 $\varphi(\bar{v}) \iff \forall w \theta_1(w, \bar{v})$ where θ_1 is Δ_0.
3. We say that a formula $\varphi(\bar{v})$ is Δ_1 if it is both Σ_1 and Π_1. For a background theory T, we say that a formula $\varphi(\bar{v})$ is Δ_1^T if T proves that it is both Σ_1 and Π_1. Our standard background is $T = \mathsf{ZF} - \mathsf{Powerset}$.

Remark 4.3.5. This is part of a broader hierarchy of formulas called the Lévy heirarchy.

Proposition 4.3.6. *If $M \subseteq N$ are transitive models, then Δ_1 formulas are absolute.*

Proof. Firs express t $\varphi(\bar{v})$ as $\exists w \theta_0(w, \bar{v})$ where θ_0 is Δ_0. If $M \models \varphi(\bar{a})$, then there is some $b \in M$ such that $M \models \theta_0(b, \bar{a})$, so $N \models \theta_0(b, \bar{a})$, so $N \models \varphi(\bar{a})$.

Now express $\varphi(\bar{v})$ as $\forall w \theta_1(w, \bar{v})$ where θ_1 is Δ_0. Suppose $N \models \varphi(\bar{a})$ for $\bar{a} \in M$. If b in M then $N \models \theta_1(b, \bar{a})$. Hence we are showing that $M \models \forall v \theta_1(v, \bar{a})$, so $M \models \varphi(\bar{a})$. $\square$

Proposition 4.3.7. *Σ_1 formulas are upwards absolute and Π_1 formulas are downwards absolute.*

Example 8. The formula $\varphi(X,R)$ expressing "R is a well-ordering of X" is Δ_1.

Lemma 4.3.9. *If G is a Δ_1 function on V and F is defined by induction via $F(\alpha) = G(F \restriction \alpha)$, then F is a Δ_1 function.*

Proof. We need to verify that the following expression for $y = F(\alpha)$ has the correct complexity

$$\exists f(f \text{ is a function} \wedge \operatorname{dom}(f) = \alpha \wedge (\forall\beta < \alpha) f(\beta) = G(f \restriction \beta) \wedge y = G(f)).$$

The point is that we can also require $y = G(f)$ to hold for *all* such functions:

$$\forall f((f \text{ is a function} \wedge \operatorname{dom}(f) = \alpha \wedge (\forall\beta < \alpha) f(\beta) = G(f \restriction \beta)) \to y = G(f)).$$

□

Lemma 4.3.10. *$\alpha \mapsto L_\alpha$ is Δ_1 (hence absolute) for transitive models of* $\mathsf{ZF} - \mathsf{Powerset}$.

Proof. The statement regarding $\mathsf{ZF} - \mathsf{Powerset}$ is achieved because we defined L without using the powerset operation.

Because of the previous theorem, we just need to show that the inductive step is Δ_1.

The function $\alpha \mapsto L_\alpha$ is Π_1 because if we look at the definition of $\mathcal{D}(A)$, we see that it looks at all $X \subseteq A$ and applies a requirement.

To see that the that the inductive step is also Σ_1, observe that there is a function W with domain ω the outputs the elements of $\mathcal{D}(A)$ for every formula that potentially gives a definition. □

The definition of $\triangleleft_L$ is Δ_1.

Corollary 4.3.11. *The ordering $\triangleleft_L$ is absolute.*

Definition 4.3.12. The *Axiom of Constructibility* is the statement $V = L$, i.e. $\forall x \exists \alpha, x \in L_\alpha$.

Theorem 4.3.13. *$L \models$ "$V = L$".*

Proof. First: $\mathrm{ON} \cap L = \mathrm{ON}$. We have to show: $L \models \forall x \exists \alpha, x \in L_\alpha$. The formula $x \in L_\alpha$ is absolute for transitive models of $\mathsf{ZF} - \mathsf{Powerset}$, and thus in particular for any $x \in L$, $(x \in L_\alpha)^L \leftrightarrow x \in L_\alpha$. □

Theorem 4.3.14. *If M is a transitive model of* $\mathsf{ZF} - \mathsf{Powerset}$ *such that* $\mathrm{ON} \subseteq M$ *then* $L = L^M \subseteq M$. *In other words, L is the smallest such model.*

Proof. Because

$$L^M = \{x \in M : \exists\alpha(x \in L_\alpha)^M\} = \bigcup_{\alpha\in\mathrm{ON}} L_\alpha = L.$$

□

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Lemma 4.3.15 (The Condensation Lemma)**.** *For every limit ordinal $\delta > \omega$, if $M \prec (L_\delta, \in)$ and M is transitive, then $M = L_\gamma$ for some $\gamma \leq \delta$.*

Proof. Let ψ be the conjunction of the following sentences:

1. the axioms needed to prove that the notions of ordinal, rank, and L_α are absolute for transitive models,
2. the assertion that there is no largest ordinal,
3. the assertion that $V = L$.

Then we have that $L_\delta \models \psi$ for limit ordinals δ. If $M \prec L_\delta$, then by the absoluteness of the formula $\alpha \mapsto L_\alpha$, it follows that $M = L_\alpha$. □

Theorem 4.3.16. *If $V = L$, then $\mathcal{P}(\alpha) \subseteq L_{\alpha^+}$.*

Proof. Recall that since we officially know that $L \models \mathsf{AC}$, we have $L_\beta \models$ "$|L_\alpha| = |\alpha|$" for $\omega \leq \alpha < \beta$. In other words, we are saying that in L, $\mathcal{P}(L_\alpha) \subseteq L_{\alpha^+}$.

Fix $X \subseteq \alpha$. By the Downward Löwenheim-Skolem Theorem, there is an elementary submodel $M \prec L_{\alpha^+}$ such that $\{X\} \cup \alpha \subseteq M$ and $|M| = |\{X\} \cup \alpha| = |\alpha|$. Let $N = \pi_M[M]$ (recall the Mostowski collapse). By the Condensation Lemma, there is some $\beta \leq \alpha^+$ such that $N = L_\beta$. We also have $|N| = |M| = |\alpha|$, so it must be the case that $\beta < \alpha^+$. Moreover, because $\alpha \cap M = \alpha$ is transitive, we have that $\pi_M(X) = X$. Therefore $X \in M = L_\beta \subseteq L_{\alpha^+}$, so $X \in L_{\alpha^+}$. □

Corollary 4.3.17. $V = L \implies \mathsf{GCH}$.

Proof. Since $L_\alpha \models \mathsf{AC}$ for limit ordinals α, we apply the previous theorem to show that in L, $|\mathcal{P}(\alpha)| \leq |\bigcup_{\beta<\alpha^+} L_\beta| = |\alpha^+| \cdot \sup_{\beta<\alpha^+} |L_\beta| = |\alpha^+| \cdot \sup_{\beta<\alpha^+} |\beta| = \alpha^+$. □

Corollary 4.3.18. $\mathrm{Con}(\mathsf{ZF}) \implies \mathrm{Con}(\mathsf{ZFC} \wedge \mathsf{GCH})$.

Part III

Forcing

Chapter 5

The Fundamentals of Forcing

5.1 Basic Objects

Definition 5.1.1.

1. A *countable transitive model* M is what it sounds like: a model of some subsystem of ZFC that is countable and transitive.

2. $\mathbb{P}$ is a *poset* if it is a partially ordered set with a maximal element $1_{\mathbb{P}}$. We will let $\mathbb{P}$ denote a poset always. Elements $p \in \mathbb{P}$ are called *conditions* and if p, q are conditions such that $q \leq p$, then we say that q is *stronger* than p, meaning that it expresses more information.

3. If $p, q \in \mathbb{P}$, we say that p and q are *compatible* and write $p \| q$ if there is some $r \in \mathbb{P}$ such that $r \leq p, q$. Otherwise we say that p and q are *incompatible* and write $q \perp q$.

4. $\mathbb{P}$ is *non-atomic* if for all $p \in \mathbb{P}$, there exist $q, r \leq p$ such that $q \perp r$. (We will always assume that $\mathbb{P}$ is non-atomic.)

5. $F \subset \mathbb{P}$ is a *filter* if:

    a) for all $p, q \in F$, there is some $r \in F$ with $r \leq p, q$

    b) for all $p \in F$, if $p \leq q$ then $q \in F$.

6. A subset $D \subseteq P$ is *dense* if for all $p \in \mathbb{P}, \exists q \leq p, q \in D$.

7. A filter $G \subset \mathbb{P}$ is $\mathbb{P}$-*generic over* V if for all dense subsets $D \subset \mathbb{P}$, $G \cap D \neq \emptyset$. If we say that G is "a $\mathbb{P}$-generic" then we mean that it is a $\mathbb{P}$-generic *filter*.

Example 2. Let $\mathrm{Add}(\omega)$ be the poset consisting of finite partial functions $f : \omega \rightharpoonup \{0, 1\}$. If $f \leq_{\mathrm{Add}(\omega)} g$ if and only if $f \supseteq g$.

Proposition 5.1.3. *If M is a ctm and $\mathbb{P}$ is a poset, then there is a filter G that is $\mathbb{P}$-generic over M.*

Proof. Let $\langle D_n : n < \omega \rangle$ enumerate the dense subsets of $\mathbb{P}$ that are elements of M. Inductively build a sequence $\langle p_n : n < \omega \rangle$ as follows: Let $p_0 \in D_0$. If we are given p_n, choose $p_{n+1} \in D_{n+1}$.

Then let $G = \{q \in \mathbb{P} : \exists n, p_n \leq q\}$. Upwards closure (the second point of the definition of filters) of G is basically immediate. If $q_0, q_1 \in G$ let n, m be such that $p_m \leq q_0$ and $q_n \leq q_1$. If, without loss of generality, $m \leq n$, then $p_n \leq q_0, q_1 \in G$. The genericity of G is by construction, since we have explicitly ensured that for each n, $p_n \in G \cap D_n$. ☐

Proposition 5.1.4. *Suppose that M is a transitive model of* ZF-P, *$\mathbb{P} \in M$ is a poset, and $\mathbb{P}$ has the property that*

$$\forall p \in \mathbb{P} \exists q, r \in \mathbb{P}(q \leq p \wedge r \leq p \wedge q \perp r).$$

If G is $\mathbb{P}$-generic over M, then $G \notin M$.

Proof. If $G \in M$, then $D = \mathbb{P} \setminus G \in M$ by absolutness of the set-minus operation. Also, D is dense because if $p \in \mathbb{P}$ and q, r are is in the requirement, then it is not possible for both q and r to be in G. Therefore at least one of these is in D. However, $G \cap D = \emptyset$, so G cannot be generic. ☐

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5.2 The Behavior of Forcing Extensions

Theorem 5.2.1. *(AC) Let Z be any class satisfying the hypotheses of the reflection theorem and let $\varphi_1, \ldots, \varphi_n$ be any list of formulas. Then*

$$\forall X \subseteq Z \exists A[X \subseteq A \subseteq Z \wedge (\varphi_1, \ldots, \varphi_n \text{ are absolute for } A, Z) \\ \wedge |A| \leq \max(\omega, |X|)].$$

Notice that the difference with the previously-stated version of the Reflection Theorem is that we are also insisting the $|A| \leq \max(\omega, |X|)$.

Proof. Take some Z_β so that the φ_i's are absolute between Z and Z_β. Then use AC to fix a well-ordering $\lhd$ on Z_β.

Let H_i be a function such that if φ_i takes the form $\exists v \psi(v, \bar{w})$, then H_i takes that $\bar{a} \in Z_\beta$ and outputs the $\lhd$-least $b \in Z_\beta$ such that $\varphi_i(b, \bar{a})$ holds.

Then we let A be the closure under the H_i's. More precisely, we can define $\langle A_n : n < \omega \rangle$ where $A_0 = X$ and $A_{n+1} = \bigcup\{H_i(\bar{a}) : i \in [1, n], \bar{a} \in A_n\}$. Then $A = \bigcup_{n<\omega} A_n$.

By the Tarski-Vaught test, we have that A is absolute for the φ_i's.

The use of the Axiom of Choice is also to obtain the cardinal bound. □

Proposition 5.2.2. *Suppose $\varphi_1, \ldots, \varphi_n$ is a list of sentences. Then there is a ctm M such that $M \models \bigwedge_{1 \leq i \leq n} \varphi_n$.*

Proof. Use Theorem 5.2.1 and make sure the axiom of extensionality is one of the φ_i's. Then we can apply the Mostowski collapse. □

Remark 5.2.3. We cannot simply use the standard downwards Löwenheim-Skolem and then take the Mostowski collapse. Even we have $N \models$ "*Foundation*" and $M \prec N$, this does not imply that M is actually well-founded, i.e. M may not be "correct" about itself being transitive.

Convention: From now on, when we say that M is a ctm and $M \models \mathsf{ZFC}$, then we mean to say that M satisfies a sufficient fragment of ZFC!

Definition 5.2.4.

1. A set τ is a $\mathbb{P}$-name if τ is a relation and
$$\forall \langle \sigma, p \rangle \in \tau, (\sigma \text{ is a } \mathbb{P}\text{-name } \wedge p \in \mathbb{P}).$$
2. $V^{\mathbb{P}}$ is the class of $\mathbb{P}$-names. If M is a transitive model of ZFC and $\mathbb{P} \in M$, $M^{\mathbb{P}} = V^{\mathbb{P}} \cap M$.
3. If G is $\mathbb{P}$-generic over M and τ is a $\mathbb{P}$-name, then $\operatorname{val}(\tau, G) = \tau_G = \{\operatorname{val}(\sigma, G) : \exists p \in G, \langle \sigma, p \rangle \in \tau\}$.
4. If M is a transitive model of ZFC, $\mathbb{P} \in M$, and $G \subseteq \mathbb{P}$, then $M[G] = \{\tau_G : \tau \in M^{\mathbb{P}}\}$.

Proposition 5.2.5. $\mathrm{val}(\tau, G)$ *is absolute.*

Proposition 5.2.6. *If M is a transitive model of* ZFC*, $\mathbb{P}$ is a poset in M, and G is a non-empty filter on $\mathbb{P}$, then $M[G]$ is transitive.*

Proof. Suppose that $x \in \sigma_G \in M[G]$. Then $x = \tau_G$ for some $\mathbb{P}$-name τ with $\langle \tau, p \rangle \in \sigma$ and $p \in G$. Hence $x = \tau_G \in M[G]$. $\square$

Proposition 5.2.7. *Suppose $\mathbb{P}$ is a poset with $\mathbb{P} \in M$. If M and N are transitive models of* ZFC *with $M \subseteq N$ and G is $\mathbb{P}$-generic over M with $G \in N$, then $M[G] \subseteq N$.*

Proof. If $\tau \in M^{\mathbb{P}}$ then $\tau \in N$. We also have $G \in N$, so $(\tau_G) = (\tau_G)^N \in N$. $\square$

Definition 5.2.8. Suppose $\mathbb{P}$ is a poset and x is a set. The *canonical $\mathbb{P}$-name* $\check{x}$ is defined recursively as $\{\langle \check{y}, 1_{\mathbb{P}} \rangle : y \in x\}$.

Proposition 5.2.9. *If M is a transitive model of* ZFC*, $\mathbb{P}$ is a poset in M, and G is a non-empty filter on $\mathbb{P}$, then*

1. $\forall x \in M, (\check{x} \in M^{\mathbb{P}} \wedge \mathrm{val}(\check{x}, G) = x)$.
2. $M \subseteq M[G]$.

Proof. By induction on the rank, we have that

$$\check{x}_G = \{\check{y}_G : y \in x, 1_{\mathbb{P}} \in G\} = \{y : y \in x\} = y.$$

This gives the first point, and the second one follows from the first. $\square$

Definition 5.2.10. If $\mathbb{P}$ is a poset, let $\Gamma = \{\langle \check{p}, p \rangle : p \in \mathbb{P}\}$.

Proposition 5.2.11. *If M is a transitive model of* ZFC*, $\mathbb{P}$ is a poset in M, and G is a non-empty filter on $\mathbb{P}$, then $G \in M[G]$.*

Proof. $M[G] \ni \Gamma_G = \{\check{p}_G : p \in G\} = \{p : p \in G\} = G$. $\square$

Proposition 5.2.12. *If M is a transitive model of* ZFC*, $\mathbb{P}$ is a poset in M, and G is a non-empty filter on $\mathbb{P}$, then*

1. $\forall \tau \in M^{\mathbb{P}}$, $\mathrm{rank}(\tau_G) \leq \mathrm{rank}(\tau)$.
2. $M[G] \cap \mathrm{ON} = M \cap \mathrm{ON}$.

Proof. The first point follows by $\in$-induction because

$$\operatorname{rank}(\tau_G) = \operatorname{rank}(\{\sigma_G : \exists p \in G, \langle \sigma, p\rangle \in \tau\}) \leq \\ \leq \operatorname{rank}(\{\sigma : \exists p \in G, \langle \sigma, p\rangle \in \tau\}) \leq \operatorname{rank}(\tau).$$

The second point works as follows: We know that $M[G] \cap \mathrm{ON} \subseteq M \cap \mathrm{ON}$ since $\operatorname{rank}(\tau) \in M$ for all $\tau \in M$. Since $M \subseteq M[G]$ we also have $M \cap \mathrm{ON} \subseteq M[G] \cap \mathrm{ON}$. $\square$

Theorem 5.2.13. *If $\mathbb{P}$ is a poset, M is a ctm, and G is $\mathbb{P}$-generic over M, then $M[G]$ satisfies Extensionality, Foundation, and Pairing.*

Proof. Extensionality: This is because $M[G]$ is transitive.

Foundation: This is true since $M[G]$ is formalized as a subclass of V.

Pairing: If $\tau, \sigma \in M^{\mathbb{P}}$ and $\rho = \{\langle \sigma, 1_{\mathbb{P}}\rangle, \langle \tau, 1_{\mathbb{P}}\rangle\}$, then $\rho_G = \{\sigma_G, \tau_G\}$. $\square$

5.3 The Forcing Theorem

December 10, 2024

We look to define a notion $p \Vdash \psi$ where this is understood to mean that p *forces* ψ. We want a definition of forcing that:

1. truth in $M[G]$ (where G is $\mathbb{P}$-generic over M) corresponds to what is forced by p, and

2. the question of what is forced by p can be decided within the model M.

Definition 5.3.1. The *forcing language* (with respect to some M) is the first-order language whose one binary relation is $\in$ and whose constant symbols are elements of $M^{\mathbb{P}}$.

Definition 5.3.2 (Semantic Definition of Forcing)**.** Let $\varphi(\bar{x})$ be a formula with free variable among those mentioned in the tuple $\bar{x}$. Let M be a ctm of ZFC, let $\mathbb{P}$ be a poset with $\mathbb{P} \in M$, $\tau_1, \ldots, \tau \in M^{\mathbb{P}}$, and $p \in \mathbb{P}$. Then $p \Vdash_{\mathbb{P},M} \varphi(\tau_1, \ldots, \tau_n)$ if and only if

$$\forall G[(G \text{ is } \mathbb{P}\text{-generic over } M \wedge p \in G) \implies M[G] \models \varphi((\tau_1)_G, \ldots, (\tau_n)_G)].$$

Again, we really mean that M is a ctm of a relevant finite fragment of ZFC!

Proposition 5.3.3. *Assume that we have some context for $M, \mathbb{P}$.*

1. $(p \Vdash \varphi(\tau_1, \ldots, \tau_n) \wedge q \leq p) \implies q \Vdash \varphi(\tau_1, \ldots, \tau_n)$.
2. $[(p \Vdash \varphi(\tau_1, \ldots, \tau_n) \wedge (p \Vdash \psi(\tau_1, \ldots, \tau_n)] \iff (p \Vdash \varphi(\tau_1, \ldots, \tau_n) \wedge \psi(\tau_1, \ldots, \tau_n))$.

Proof. The first point follows from upwards closure of filters. The second point follows from the definition of truth in a model. □

Definition 5.3.4. If $\mathbb{P}$ is a poset and $p \in \mathbb{P}$, we say that $E \subseteq \mathbb{P}$ is *dense below* p if for all $q \leq p$, there is some $r \in E$ with $r \leq q$.

Proposition 5.3.5. *Assume M is a transitive model of* ZFC*, $\mathbb{P} \in M$, $E \subseteq \mathbb{P}$, and $E \in M$.If $p \in G$ and E is dense below p then $G \cap E \neq \emptyset$.*

Proof. Let
$$D = \{q \in \mathbb{P} : q \perp p \text{ or } (q \leq p \wedge q \in E)\}.$$
Then D is dense: Let $r \in \mathbb{P}$ be arbitrary. If it is not the case that $r \perp p$, then there is some $q' \leq r, p$. Let $q \leq q'$ be such that $q \in E$. Then $q \in D$. Now let $r \in D \cap G$. Since G is a filter, it cannot be the case that $r \perp p$. Therefore $r \in E \cap G$. □

Definition 5.3.6 (Syntactic Definition of Forcing)**.** Fix a poset $\mathbb{P}$ and consider $\varphi(x_1, \ldots, x_n)$ in which free variables are displayed. Let $p \in \mathbb{P}$ and $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$. Then we define $p \Vdash^* \varphi(\tau_1, \ldots, \tau_n)$ by induction on formula composition through the following cases:

1. $p \Vdash^* \tau_1 = \tau_2$ if and only if both of the following are true:

   a) for all $\langle \pi_1, s_1 \rangle \in \tau_1$,
   $$\{q \leq p : q \leq s_1 \implies \exists \langle \pi_2, s_2 \rangle \in \tau_2 (q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}$$
   is dense below p.

   b) for all $\langle \pi_2, s_2 \rangle \in \tau_2$,
   $$\{q \leq p : q \leq s_2 \implies \exists \langle \pi_1, s_1 \rangle \in \tau_1 (q \leq s_1 \wedge q \Vdash^* \pi_1 = \pi_2)\}$$
   is dense below p.

2. $p \Vdash^* \tau_1 \in \tau_2$ if and only if

$$\{q : \exists \langle \pi, s\rangle \in \tau_2 (q \le s \wedge q \Vdash^* \pi = \tau_1)\}$$

is dense below p.

3. $p \Vdash^* \varphi(\tau_1, \dots, \tau_n) \wedge \psi(\tau_1, \dots, \tau_n)$ if and only if $p \Vdash^* \varphi(\tau_n, \dots, \tau_n)$ and $p \Vdash^* \psi(\tau_1, \dots, \tau_n)$.

4. $p \Vdash^* \neg\varphi(\tau_1, \dots, \tau_n)$ if and only if there is no $q \le p$ such that $q \Vdash^* \varphi(\tau_1, \dots, \tau_n)$.

5. $p \Vdash^* \exists x \varphi(x, \tau_1, \dots, \tau_n)$ if and only if

$$\{r : \exists \sigma \in V^{\mathbb{P}}, r \Vdash^* \varphi(\sigma, \tau_1, \dots, \tau_n)\}$$

is dense below p.

Lemma 5.3.7. *For p and $\varphi(\tau_1, \dots, \tau_n)$ as above, the following are equivalent:*

1. $p \Vdash^* \varphi(\tau_1, \dots, \tau_n)$,
2. $\forall r \le p, r \Vdash^* \varphi(\tau_1, \dots, \tau_n)$,
3. $\{r : r \Vdash^* \varphi(\tau_1, \dots, \tau_n)\}$ *is dense below p.*

Proof. Observe the following: If D is dense below p and $r \le p$, then D is dense below r. Moreover, if $\{r : D \text{ is dense below } r\}$ is dense below p, then D is dense below p.

The rest follows by inducting on formula complexity and checking the cases of the syntactic definition of forcing. □

Lemma 5.3.8. *Let $\varphi(\bar{x})$ be a formula with free variables displayed, let M be a transitive model of* ZFC*, $\mathbb{P}$ a poset in M, $\tau_1, \dots, \tau_n \in M^{\mathbb{P}}$, and let G be $\mathbb{P}$-generic over M. Then*

1. *If $p \in G$ and $(p \Vdash^* \varphi(\tau_1, \dots, \tau_n))^M$, then $M[G] \models \varphi((\tau_1)_G, \dots, (\tau_n)_G))$.*
2. *If $M[G] \models \varphi((\tau_1)_G, \dots, (\tau_n)_G))$, then $\exists p \in G, (p \Vdash^* \varphi(\tau_1, \dots, \tau_n))^M$.*

Proof. We will prove both (1) and (2) simultaneously by induction on formula complexity.

First we argue for *1.* and *2.* for atomic formulas. Since these cases are absolute, we ignore the relativizations to M.

For the case of $\tau_1 = \tau_2$, we argue by $\in$-induction on names.

Claim. *(1) holds for $\tau_1 = \tau_2$.*

Suppose $p \in G$ and $p \Vdash^* \tau_1 = \tau_2$. We want to show that $(\tau_1)_G = (\tau_2)_G$. It will be enough to show that $(\tau_1)_G \subseteq (\tau_2)_G$ since the other direction is analogous. Every element of $(\tau_1)_G$ is of the form $(\pi_1)_G$ where $\langle \pi_1, s_1 \rangle \in \tau_1$ for some $s_1 \in G$. Fix $r \in G$ with $r \leq p, s_1$. Then $r \Vdash^* \tau_1 = \tau_2$ (we proved this above). By the proposition about generics intersection dense-below sets, there is some $q \in G$ such that $q \leq r$ and such that $q \leq s_1$ implies

$$\exists \langle \pi_2, s_2 \rangle \in \tau_2 (q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2).$$

We have $q \leq s_1$, so fix $\langle \pi_2, s_2 \rangle$ witnessing the line above. Then $s_2 \in G$ and $(\pi_2)_G \in (\tau_2)_G$. By (1) for $\pi_1 = \pi_2$, $q \Vdash^* \pi_1 = \pi_2$ implies that $(\pi_1)_G = (\pi_2)_G$, and therefore we have $(\pi_1)_G \in (\tau_2)_G$, which was the goal.

Claim. *(2) holds for $\tau_1 = \tau_2$.*

Assume $(\tau_1)_G = (\tau_2)_G$. Let D be the set of $r \in \mathbb{P}$ such that one of the following hold:

(i) $r \Vdash^* \tau_1 = \tau_2$,

(ii) $\exists \langle \pi_1, s_1 \rangle \in \tau_1 (r \leq s_1 \wedge \forall \langle \pi_2, s_2 \rangle \in \tau_2 \forall q \in \mathbb{P}, ((q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2) \implies q \perp r))$,

(iii) $\exists \langle \pi_2, s_2 \rangle \in \tau_2 (r \leq s_2 \wedge \forall \langle \pi_1, s_1 \rangle \in \tau_1 \forall q \in \mathbb{P}, ((q \leq s_1 \wedge q \Vdash^* \pi_1 = \pi_2) \implies q \perp r))$.

We argue that no $r \in G$ can satisfy (ii) or (iii).

Suppose for contradiction that $r \in G$ satisfies (ii) using $\langle \pi_1, s_2 \rangle \in \tau_1$. Then $s_1 \in G$, so $(\pi_1)_G \in (\tau_1)_G = (\tau_2)_G$, so we can fix $\langle \pi_2, s_2 \rangle \in \tau_2$ with $s_2 \in G$ and $(\pi_1)_G = (\pi_2)_G$. Then because we have (2) for $\pi_1 = \pi_2$, fix $q_0 \in G$ with $q_0 \Vdash^* \pi_1 = \pi_2$. Fix $q \in G$ with $q \leq q_0$ and $q \leq s_2$. Since $q \Vdash^* \pi_1 = \pi_2$ we have that $q \perp r$ (following the implication from our assumption), $q \in G$, and $r \in G$, which is a contradiction.

Now we argue that D is dense. Fix $p \in \mathbb{P}$. If we do not have $p \Vdash^* \tau_1 = \tau_2$, then one of the clauses of that definition must fail. Let us assume the first one does, and then we will show that there is some $r \leq p$ such that (ii) holds. If the other clause fails, then we would similarly be able to show that there is some $r \leq p$ such that (iii) holds.

Assume the failure of that first clause, then we apply the definition of "dense below p" to fix $\langle \pi_1, s_1 \rangle \in \tau_1$ and $r \leq p$ such that

$$\forall q \leq r(q \leq s_1 \wedge \forall \langle \pi_2, s_2 \rangle \in \tau_2(\neg(q \Vdash^* \pi_1 = \pi_2 \wedge q \leq s_2))). \tag{5.1}$$

In other words, we are just applying the negation. In particular, we have $r \leq s_1$ of the first piece of 5.1. If $\langle \pi_2, s_2 \rangle \in \tau_2$, $q \leq s_2$, and $q \Vdash^* \pi_1 = \pi_2$, then $q \perp r$, since otherwise we would have a contradiction of 5.1. Therefore we have found $r \leq p$ such that (ii) holds.

Since D is dense, there is some $r \in D \cap G$. By the exclusion of cases (ii) and (iii), we know that $r \Vdash^* \tau_1 = \tau_2$, so we are done.

Now we move on to $\tau_1 \in \tau_2$.

Claim. *(1) holds for $\tau_1 \in \tau_2$.*

Suppose that $p \in G$ and $p \Vdash^* \tau_1 \in \tau_2$. Then

$$D = \{q : \exists \langle \pi, s \rangle \in \tau_2(q \leq s \wedge q \Vdash^* \pi = \tau_1)\}$$

is dense below p by definition. Fix $q \in G \cap D$ and $\langle \pi, s \rangle \in \tau_2$ so that $q \leq s$ and $q \Vdash^* \pi = \tau_1$. Since $s \in G$ and $\langle \pi, s \rangle \in \tau_2$, we have $\pi_G \in (\tau_2)_G$ by definition of $(\tau_2)_G$. Since $q \in G$ and $q \Vdash^* \pi = \tau_1$, it follows that $\pi_G = (\tau_1)_G$ by applying (1) to $\pi = \tau_1$. Therefore $(\tau_1)_G \in (\tau_2)_G$.

Claim. *(2) holds for $\tau_1 \in \tau_2$.*

Suppose that $(\tau_1)_G \in (\tau_2)_G$. By definition of $(\tau_2)_G$, there is a $\langle \pi, s \rangle \in \tau_2$ such that $s \in G$ and $\pi_G = (\tau_1)_G$. By (2) for $\pi = \tau_1$, there is an $r \in G$ such that $r \Vdash^* \pi = \tau_1$. Let $p \in G$ be such that $p \leq s, r$. Then $\forall q \leq p(q \leq s \wedge q \Vdash^* \pi = \tau_1)$ (because of what is forced by r), so $p \Vdash^* \tau_1 \in \tau_2$ by definition. December 12, 2024

Now we argue for the other cases, using the fact that we have obtained the statement of the lemma for the atomic cases. We will ignore the notation for the τ's because it is less necessary than it is for the atomic case. However, we do need to pay attention to the relativizations to M.

Claim. *(1) holds for negation.*

We are assuming (1) and (2) for φ. Assume that $p \in G$ and $(p \Vdash^* \neg\varphi)^M$. We want to show $\neg\varphi^{M[G]}$, i.e. $M[G] \models \neg\varphi$. Otherwise we have $M[G] \models \varphi$, and so by (2) we have some $q \in G$ with $(q \Vdash^* \varphi)^M$. Let $r \in G$ be such that $r \leq p, q$. Then $(r \Vdash^* \varphi)^M$, contradicting the fact that $p \Vdash^* \neg\varphi$.

Claim. *(2) holds for negation.*

First we need:

Proposition 5.3.9. *If φ is a formula in the forcing language and M is a ctm of* ZFC*, then*

$$D = \{p : (p \Vdash^* \varphi)^M \vee (p \Vdash^* \neg\varphi)^M\}$$

is dense in $\mathbb{P}$.

Proof. Induct by formula construction. Fix $r \in \mathbb{P}$. Consider the following cases:

If φ is $\tau_1 \in \tau_2$, then if we do not have $r \Vdash \tau_1 \in \tau_2$, we are saying that a certain set in $\mathbb{P}$ is not dense below r. That means that there is some $p \leq r$ such that there are no members of that certain set below p. Therefore, there is no $q \leq p$ forcing $\tau_1 \in \tau_2$, so by definition $p \Vdash \neg(\tau_1 \in \tau_2)$.

If φ is $\tau_1 = \tau_2$, then $\neg(r \Vdash \varphi)$ implies that one of two sets is not dense below r, so we apply similar reasoning.

If φ is $\psi_1 \wedge \psi_2$ and $r \nVdash \varphi$, then without loss of generality, $r \nVdash \psi_1$. By induction on formula construction, there is some $p \leq r$ such that $p \Vdash \neg\psi_1$, and so $p \Vdash \neg\varphi$.

The remaining cases are analogous. □

Suppose that $M[G] \models \neg\varphi$. Let

$$D = \{p : (p \Vdash^* \varphi)^M \vee (p \Vdash^* \neg\varphi)^M\}.$$

Then $D \in M$ and D is dense. Let $p \in D \cap G$. If $(p \Vdash^* \neg\varphi)^M$, then we are done. If $(p \Vdash^* \varphi)^M$, then we have a contradiction.

Claim. *(1) holds for conjunction.*

Suppose that $p \in G$ and $(p \Vdash^* \varphi \wedge \psi)^M$. Then $(p \Vdash^* \varphi)^M$ and $(p \Vdash^* \psi)^M$, so by induction we have $M[G] \models \varphi$ and $M[G] \models \psi$, hence $M[G] \models \varphi \wedge \psi$.

Claim. *(2) holds for conjunction.*

Suppose $M[G] \models \varphi \wedge \psi$. Then by induction there are $p, q \in G$ such that $(p \Vdash^* \varphi)^M$ and $(q \Vdash^* \psi)^M$. Let $r \in G$ be such that $r \leq p, q$. Then $r \Vdash^* \varphi \wedge \psi$.

Claim. *(1) holds for existential quantification.*

Suppose $p \in G$ and $(p \Vdash^* \exists x \varphi(x))^M$. Then

$$\{r : \exists \sigma \in M^{\mathbb{P}} (r \Vdash^* \varphi(\sigma))^M\}$$

is dense below p and is in M. Fix $r \in G$ and $\sigma \in M^{\mathbb{P}}$ such that $(r \Vdash^* \varphi(\sigma))^M$. By induction, $M[G] \models \varphi(\sigma_G)$, so $M[G] \models \exists x \varphi(x)$.

Claim. *(2) holds for existential quantification.*

Suppose $M[G] \models \exists x \varphi(x)$ and fix $\sigma \in M^{\mathbb{P}}$ such that $M[G] \models \varphi(\sigma_G)$. By (2) and induction, fix $p \in G$ such that $(p \Vdash^* \varphi(\sigma))^M$. Then for all $r \leq p$, $(r \Vdash^* \varphi(\sigma))^M$, so $(p \Vdash^* \exists x \varphi(x))^M$. □

Theorem 5.3.10 (The Forcing Theorem)**.** *Let M be a countable transitive model of* ZFC *and $\mathbb{P}$ a poset in M, and let $\varphi(\bar{x})$ be a formula with all free variables shown, and also let $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$.*

1. *For all $p \in \mathbb{P}$,*

$$p \Vdash \varphi(\tau_1, \ldots, \tau_n) \iff (p \Vdash^* \varphi(\tau_1, \ldots, \tau_n))^M.$$

2. *For all G which are $\mathbb{P}$-generic over M,*

$$M[G] \models \varphi((\tau_1)_G, \ldots, (\tau_n)_G) \iff \exists p \in G (p \Vdash \varphi(\tau_1, \ldots, \tau_n)).$$

Proof. For (1) from right to left: If $p \in G$ and $p \Vdash^* \varphi$, we proved already that $M[G] \models \varphi$.

For (1) from left to right: Suppose that $p \Vdash \varphi(\tau_1, \ldots, \tau_n)$. It is enough to show that

$$D = \{r : (r \Vdash^* \varphi(\tau_1, \ldots, \tau_n))^M\}$$

is dense below p. Otherwise, let $q \leq p$ be such that $\neg \exists r \leq q, r \in D$. Then by the definition of $\Vdash^*$ we have $(q \Vdash^* \neg \varphi(\tau_1, \ldots, \tau_n))^M$, and so we have $q \Vdash \neg \varphi(\tau_1, \ldots, \tau_n)$ (we already did right to left). If G is $\mathbb{P}$-generic over M with $q \in G$, then $M[G] \models \neg \varphi((\tau_1)_G, \ldots, (\tau_n)_G)$, but this contradicts $q \leq p$.

For (2) from right to left: This is immediate from the definition of $\Vdash$.

For (2) from left to right: This follows from (1) and the fact that we proved the same thing about $\Vdash^*$. □

5.4 ZFC in Forcing Extensions

Theorem 5.4.1. *Let M be a ctm of* ZFC*, $\mathbb{P}$ a poset in M, and G $\mathbb{P}$-generic over M. Then $M[G] \models \mathsf{ZFC}$.*

Corollary 5.4.2. *Let M be a ctm of* ZFC*. Then there is a ctm of $N \supset M$ such that $N \models \mathsf{ZFC} + V \neq L$.*

Proof. Observe that $\mathrm{Add}(\omega)$, which we introduced before, it non-atomic. Therefore if we let $\mathbb{P} = \mathrm{Add}(\omega)$ and if G is $\mathbb{P}$-generic over M, then $G \notin M$. Let $N = M[G]$. By absoluteness of L, $L^N = L^M \subseteq M$, therefore $N \models V \neq L$. $\square$

Corollary 5.4.3. $\mathrm{Con}(\mathsf{ZFC}) \implies \mathrm{Con}(\mathsf{ZFC} + V \neq L)$.

Proof of Theorem 5.4.1. Comprehension: We want to show that if $\sigma, \tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$ and $\varphi(x, v, y_1, \ldots, y_n)$ is a formula, then

$$\{a \in \sigma_G : (\varphi(a, \sigma_G, (\tau_1)_G, \ldots, (\tau_n)_G)^{M[G]}\} \in M[G].$$

Let

$$\rho = \{\langle \pi, p\rangle \in \mathrm{dom}(\sigma) \times \mathbb{P} : p \Vdash (\pi \in \sigma \wedge \varphi(\pi, \sigma, \tau_1, \ldots, \tau_n))\}.$$

Then $\rho \in M^{\mathbb{P}}$ because of the fact that forcing is definable.

Now we argue that ρ witnesses comprehension, i.e. $\rho_G = \{a \in \sigma_G : \varphi(a)^{M[G]}\}$. Ignore the τ_i's in what follows. Any element of ρ_G is of the form π_G where $\langle \pi, p\rangle \in \rho$ for some $p \in G$. By definition of ρ, $p \Vdash (\pi \in \sigma \wedge \varphi(\pi))$, so by the forcing theorem, $\pi_G \in \sigma_G$ and $\varphi(\pi_G)^{M[G]}$. Therefore $\rho_G \subseteq \{a \in \sigma_G : \varphi(a)^{M[G]}\}$.

We can next argue for the other direction. Suppose that $a \in \sigma_G$ and $\varphi(a)^{M[G]}$. Then $a = \pi_G$ for some $\pi \in \mathrm{dom}(\sigma)$. Then $(\pi_G \in \sigma_G \wedge \varphi(\pi_G))^{M[G]}$. By the forcing theorem, there is a $p \in G$ such that $p \Vdash (\pi \in \sigma \wedge \varphi(\pi))$, so $\langle \pi, p\rangle \in \rho$, so $a = \pi_G \in \rho_G$.

Replacement: We want to show that for every formula $\varphi(x, v, r, z_1, \ldots, z_n)$ and each $\sigma_G, (\tau_1)_G, \ldots, (\tau_n)_G \in M[G]$, if

$$\forall x \in \sigma_G \exists! y \varphi(x, y, \sigma_G, (\tau_1)_G, \ldots, (\tau_n)_G)^{M[G]},$$

then there is a $\rho \in M^{\mathbb{P}}$ such that

$$\forall x \in \sigma_G \exists y \in \rho_G (\varphi(x, y, \sigma_G, (\tau_1)_G, \ldots, (\tau_n)_G)^{M[G]}.$$

We will again ignore the τ_i's. By the definability of forcing, there is an $S \in M$ be such that $S \subseteq M^{\mathbb{P}}$ and

$$\forall \pi \in \mathrm{dom}(\sigma) \forall p \in \mathbb{P}[\exists \mu \in M^{\mathbb{P}}(p \Vdash \varphi(\pi, \mu)) \implies \exists \mu \in S(p \Vdash \varphi(\pi, \mu))].$$

By reflection in M, we may take $S = V_\alpha^M \cap M^{\mathbb{P}}$ for a suitably-chosen α. Let $\rho = S \times \{1\}$.

We will argue that ρ witnesses replacement. We have $\rho_G = \{\mu_G : \mu \in S\}$. Fix $x \in \sigma_G$. We show that $\exists y \in \rho_G(\varphi(x, y))^{M[G]}$. We have $x = \pi_G$ for some $\pi \in \mathrm{dom}(\sigma)$. By assumption, $(\exists y \varphi(\pi_G, y))^{M[G]}$, so for some $\nu \in M^{\mathbb{P}}$, $\varphi(\pi_G, \nu_G)^{M[G]}$, and so by the forcing theorem there is a $p \in G$ such that $p \Vdash \varphi(\pi, \nu)$. There is then a $\mu \in S$ such that $p \Vdash \varphi(\pi, \mu)$, so we have $\mu_G \in \rho_G$ and $(\varphi(\pi_G, \mu_G))^{M[G]}$. December 17, 2024

Union: By comprehension, it is enough to show that if $a \in M[G]$, then there is some $b \in M[G]$ such that $\bigcup a \subseteq b$. Let $\tau \in M^{\mathbb{P}}$ be such that $a = \tau_G$. Let $\pi = \bigcup \mathrm{dom}\, \tau$. Then $\pi \in M^{\mathbb{P}}$, so $b = \pi_G \in M[G]$. If c is any element of a, then $c = \sigma_G$ for some $\sigma \in \mathrm{dom}\, \tau$. Since $\sigma \subseteq \pi$, $c = \sigma_G \subseteq \pi_G = b$, so $\bigcup a \subseteq b$.

Infinity: Observe that $\omega = (\check{\omega})_G \in M[G]$. (We did not really need the forcing theorem for this.)

Powerset: Give $\sigma_G \in M[G]$. We want to find some $\rho \in M^{\mathbb{P}}$ such that $\forall x \in M[G](x \subseteq \sigma_G \implies x \in \rho_G)$. Let $\rho = S \times \{1\}$ where

$$S = \{\tau \in M^{\mathbb{P}} : \mathrm{dom}(\tau) \subseteq \mathrm{dom}(\sigma)\} = (\mathcal{P}(\mathrm{dom}(\sigma) \times \mathbb{P}))^M.$$

Fix any $\mu \in M^{\mathbb{P}}$ such that $\mu_G \subseteq \sigma_G$. We will show that $\mu_G \in \rho_G$. Let

$$\tau = \{\langle \pi, p \rangle : \pi \in \mathrm{dom}(\sigma) \wedge p \Vdash \pi \in \mu\}.$$

Then $\tau \in S$, so $\tau_G \in \rho_G$, so we are done if we show that $\mu_G = \tau_G$.

To see that $\mu_G \subseteq \tau_G$, note that since $\mu_G \subseteq \sigma_G$, any element of μ_G is of the form π_G for some $\pi \in \mathrm{dom}(\sigma)$. Since $\pi_G \in \mu_G$, there is some $p \in G$ such that $p \Vdash \pi \in \mu$, so $\langle \pi, p \rangle \in \tau$ by the definition of τ, so $\pi_G \in \tau_G$.

To see that $\tau_G \subseteq \mu_G$, note that any element of τ_G is of the form π_G where $\langle \pi, p \rangle \in \tau$ for some $p \in G$. Then $p \Vdash \pi \in \mu$, so $\pi_G \in \mu_G$.

Choice:

Proposition 5.4.4. AC *holds if and only if*

$$\forall x \exists \alpha \in \mathrm{ON} \exists f(f \text{ is a function } \wedge \mathrm{dom}(f) = \alpha \wedge x \subseteq \mathrm{range}(f).$$

Proof. Recall once again that AC holds if and only if every set can be given a well-odering. If x, α and f are as in the statement of the lemma, then we can define a well-ordering of x as follows: Let $g(z) = \min(f^{-1}(z)$ for $z \in x$. Then g is a one-one mapping from x into α. Let $y \prec z$ if and only if $g(y) < g(z)$, so then $\prec$ is a well-ordering of x. $\square$

Also, we can define a $\mathbb{P}$-name for an ordered pair. Let σ, τ be $\mathbb{P}$-names. Then we let $\mathrm{up}(\sigma, \tau) = \{\langle \sigma, 1\rangle, \langle \tau, 1\rangle\}$, which is the $\mathbb{P}$-name for a set of exactly the evaluations of σ and τ as elements. (We used this above to get pairing for $M[G]$.) Then we let $\mathrm{op}(\sigma, \tau) = \mathrm{up}(\mathrm{up}(\sigma, \sigma), \mathrm{up}(\sigma, \tau))$.

Now fix $\sigma_G \in M[G]$. By AC^M, we can choose an enumeration of $\mathrm{dom}(\sigma)$, which we write as $\{\pi_\gamma : \gamma < \alpha\}$. We can choose this enumeration in M. Now let

$$\tau = \{\mathrm{op}(\check{\gamma}, \pi_\gamma) : \gamma < \alpha\} \times \{1\}.$$

Then $\tau \in M$ by definability and $\tau_G = \{\langle \gamma, (\pi_\gamma)_G\rangle : \gamma < \alpha\}$, so τ_G is a function with $\mathrm{dom}(\tau_G) = \alpha$ and $\sigma_G \subseteq \mathrm{range}(\tau_G)$. $\square$

Chapter 6

The Independence of CH and AC

6.1 Dealing with Antichains

Definition 6.1.1. Let $\mathbb{P}$ be a poset.

1. A subset $A \subseteq \mathbb{P}$ is called an *antichain* if for all $p, q \in A$, if $p \neq q$ then $p \perp q$.
2. An antichain $A \subseteq \mathbb{P}$ is called a *maximal antichain* if for all $p \in \mathbb{P}$, there is some $q \in A$ such that $q \| p$.
3. We say that $\mathbb{P}$ has the *countable chain condition*, often denoted *ccc*, if every antichain $A \subseteq \mathbb{P}$ is at most countable.

Example 2. $\mathrm{Add}(\omega)$ has the countable chain condition.

Proposition 6.1.3. *The following are equivalent for a poset* $\mathbb{P}$ *and an antichain* $A \subseteq \mathbb{P}$*:*

1. *A is a maximal antichain.*
2. *For all antichains $B \supseteq A$, $B = A$.*

Proof. *1.* $\Longrightarrow$ *2.*: Suppose that $B \supseteq A$ and suppose for contradiction that $p \in B \setminus A$. Then there is some $q \in A$ such that $q \| p$, contradicting that B is an antichain. *2.* $\Longrightarrow$ *1.*: Let $p \in \mathbb{P}$. Then if $p \perp q$ for all $q \in A$, then $A \cup \{q\}$ is an antichain such that $A \cup \{p\} \supsetneq A$. □

Proposition 6.1.4. *Let M be a ctm of* ZFC *and let $\mathbb{P}$ be a poset with $\mathbb{P} \in M$, and suppose that G is filter on $\mathbb{P}$. Then the following are equivalent:*

1. *G is $\mathbb{P}$-generic over M,*
2. *for all maximal antichains $A \subseteq \mathbb{P}$ with $A \in M$, $G \cap A \neq \emptyset$.*

Proof. We are saying that meeting all dense sets is the same as meeting all maximal antichains.

Suppose that G is $\mathbb{P}$-generic and A is a maximal antichain. Let $D = \{p \in \mathbb{P} : \exists q \in A, q \leq p\}$. We argue that D is dense: Fix $r \in \mathbb{P}$. By maximality of A, there is some $q \in A$ such that $q \| r$, so there is some $p \leq q, r$. Then by definition, $p \in D$.

By assumption we have some $p \in G \cap D$. Let q witness this, so in particular $p \leq q$. Then $q \in G$ since G is a filter, so $G \cap A \neq \emptyset$.

Suppose now that *2.* holds and suppose that D is a dense subset of $\mathbb{P}$ with $D \in M$. Then we define a maximal antichain $A \subseteq D$ as follows: Specifically, use Zorn's Lemma to find an antichain A such that for all antichains A' with $D \supseteq A' \supseteq A$, $A' = A$. Now suppose that B is an antichain such that $B \supseteq A$, and suppose for contradiction that there is some $p \in B \setminus A$. There is some $q \in D$ with $q \leq p$. It cannot be the case that $\{q\} \cup A$ is an antichain, so there is some $r \leq q, s$ where $s \in A$. This implies that $s \| p$, which is a contradiction. It therefore follows that A is a maximal antichain.

Then there is some $p \in G \cap A$ by assumption, so $G \cap D \neq \emptyset$. □

Definition 6.1.5. Let $\mathbb{P}$ be a poset.

1. We say that $\mathbb{P}$ *preserves cardinalities* if whenever G is $\mathbb{P}$-generic over M, then for all $\beta \in M \cap \mathrm{ON}$, $\beta \geq \omega$ and $M \models$ "β is a cardinal", then $M[G] \models$ "β is a cardinal".
2. We say that $\mathbb{P}$ *preserves cofinalities* if whenever G is $\mathbb{P}$-generic over M, and γ is a limit ordinal in M, then $\mathrm{cf}(\gamma)^M = \mathrm{cf}(\gamma)^{M[G]}$.

Proposition 6.1.6. *If $\mathbb{P}$ preserves cofinalities, then $\mathbb{P}$ preserves cardinalities.*

Proof. If κ is regular, then $\mathrm{cf}(\kappa) = \kappa$, so we can see that $\mathbb{P}$ preserves regular cardinals. If λ is a limit cardinal, then $\mathbb{P}$ preserves a cofinal set of cardinals below λ, therefore $\mathbb{P}$ preserves λ. □

Proposition 6.1.7. *If α is a limit and $f : \alpha \to \beta$ is strictly increasing, then* $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$.

Proof. We have that $\mathrm{cf}(\beta) \leq \mathrm{cf}(\alpha)$ by composing a cofinal function from $\mathrm{cf}(\alpha)$ to α with f. To see that $\mathrm{cf}(\alpha) \leq \mathrm{cf}(\beta)$, let $g : \mathrm{cf}(\beta) \to \beta$ be cofinal and let $h(\xi)$ be the least η such that $f(\eta) > g(\xi)$. Then $h : \mathrm{cf}(\beta) \to \alpha$ is a cofinal map. $\square$

Proposition 6.1.8. *Suppose $\mathbb{P} \in M$, a ctm of* ZFC, *and G is $\mathbb{P}$-generic over M. Suppose that for all $\kappa \in M$ such that $M \models$ "κ is regular", we have $M[G] \models$ "κ is regular". Then $\mathbb{P}$ preserves cofinalities.*

Proof. If γ is a limit in M and $M \models \kappa = \mathrm{cf}(\gamma)$, then there is a strictly increasing and cofinal function $f : \kappa \to \gamma$ in M. By assumptiong, $M \models \kappa$ regular implies that $M[G] \models \kappa$ is regular. Since $f \in M[G]$, we can apply the previous proposition to conclude that $M[G] \models \kappa = \mathrm{cf}(\gamma)$. $\square$

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Proposition 6.1.9. *If $\mathbb{P}$ is ccc, then $\mathbb{P}$ preserves cofinalities.*

Proof. We want to argue that if $M \models$ "κ is regular", then $M[G] \models$ "κ is regular". It is enough to show that if $p \Vdash$ "$\dot{f}$ is a function from $\lambda \to \kappa$" where $\kappa < \lambda$ and $\kappa, \lambda \in M$, then there is some $\gamma < \lambda$ such that $p \Vdash$ "$\mathrm{range}(\dot{f}) \subseteq \gamma$".

Suppose $p \Vdash$ "$\dot{f}$ is a function from $\lambda \to \kappa$".

For each $i < \lambda$, let

$$B_i = \{\beta < \kappa : \exists q \leq p, q \Vdash \dot{f}(i) = \beta\}.$$

We claim that B_i is countable: Otherwise we would have an $\aleph_1$-sized set $\{\beta_\xi : \xi < \omega_1\}$ and a sequence $\langle q_\xi : \xi < \omega_1\rangle$ such that for each $\xi < \omega_1$, $q_\xi \leq p$ and $q_\xi \Vdash \dot{f}(i) = \beta_\xi$. But since $\beta_\xi \neq \beta_\eta$ for $\xi \neq \eta$, it follows that $\langle q_\xi : \xi < \omega_1\rangle$ is an uncountable antichain, which is a contradiction.

Now let $B = \bigcup_{i<\lambda} B_i$. By regularity of κ, B is countable.

We have that $p \Vdash \mathrm{range}\, \dot{f} \subseteq B$, so p forces that $\dot{f}$ is not a surjection onto κ. $\square$

6.2 Cohen Forcing

Definition 6.2.1. A family F is called a Δ-*system* if there is some d such that for all $a, b \in F$, $a \cap b = d$. We call the set d the *root* of the system.

Theorem 6.2.2 (A case of the Δ-System Lemma). *Let F be an $\aleph_1$-sized family of distinct finite sets. Then there is an $\aleph_1$-sized subfamily $F' \subseteq F$ such that F' is a Δ-system.*

Proof. Since there are only ω-many possible sizes of elements of F, we can apply the Pigeonhole Principle (or Fodor's Lemma) to find some $F' \subseteq F$ of cardinality $\aleph_1$ and some $n < \omega$ such that for all $a \in F'$, $|a| = n$. Therefore it is enough to argue by induction on $n < \omega$ that if there is a family of $\aleph_1$-many distinct finite sets of size n, then there is a $\aleph_1$-sized subfamily that forms a Δ-system.

The value $n = 0$ is not valid in this context (we cannot have just a bunch of empty sets) and the value $n = 1$ is trivial.

Suppose we have established our statement for n. There are two things that can possibly happen.

Case 1: There is some x such that for $\aleph_1$-many $a \in F'$, $x \in a$.

Let $E = \{a \setminus \{x\} : a \in F', x \in a\}$. Then we apply the inductive hypothesis to find a Δ-system $E' \subseteq E$ of cardinality $\aleph_1$ for which d is a root of E'. Then let $F'' = \{b \cup \{x\} : b \in E'\}$. Then we can see that F'' is a Δ-system with root $d \cup \{x\}$.

Case 2: For all x, there are at most countably-many $a \in F'$ such that $x \in a$.

Then by induction we can construct a sequence $\langle a_i : i < \omega_1 \rangle \subseteq F'$ such that for each $i < j < \omega_1$, $a_i \cap a_j = \emptyset$: Take any a_0. Then if we have a_i for $i < j$, if X_i is the set of elements b of F' such that $a_i \cap b \neq \emptyset$, we take some $a_j \notin \bigcup_{i<j} X_i$. Once we have constructed $\langle a_i : i < \omega_1 \rangle$, we see that it is a Δ-system with root $\emptyset$. $\square$

Definition 6.2.3. If κ is a cardinal, let $\mathrm{Add}(\omega, \kappa)$ consist of functions p such that:

1. $\operatorname{dom} p$ is a finite subset of $\kappa \times \omega$.

2. $\operatorname{range} p \subseteq \{0, 1\}$.

And we have $p \leq q$ if and only if $p \supseteq q$.

Proposition 6.2.4. *For all regular κ,* $\mathrm{Add}(\omega, \kappa)$ *has the countable chain condition.*

Proof. If A^* is an uncountable antichain in $\mathrm{Add}(\omega, \kappa)$, then we can take an $\aleph_1$-sized set $A \subseteq A^*$ to obtain an antichain of cardinality $\aleph_1$. Apply the Δ-System Lemma to the set $\{\operatorname{dom} p : p \in A\}$ to find some $A' \subseteq A$ of cardinality $\aleph_1$ and some d such that for all $p, q \in A$, $\operatorname{dom} p \cap \operatorname{dom} q = d$. We know that for all $p \in a$, $\operatorname{range} p \subseteq \omega$, meaning that there are countably many possibilities for $p \upharpoonright d$, so we apply the Pigeonhole Principle to find A'' sich that for all distinct $p, q \in A$, $\operatorname{dom} p \cap \operatorname{dom} p = d$ *and* $p \upharpoonright d = q \upharpoonright d$.

Since A'' has cardinality $\aleph_1$, we can of course find two $p, q \in A''$ with $p \neq q$. Then we can see that since $\operatorname{dom} p \cap \operatorname{dom} q = d$ and $p \upharpoonright d = q \upharpoonright d$, then p, q are compatible as witnessed by $p \cup q$. $\square$

Theorem 6.2.5. *If* $\mathbb{P} = \mathrm{Add}(\omega, \aleph_2)$*, then* $\Vdash_{\mathbb{P}} \neg CH$.

Proof. We proved that $\mathbb{P}$ is ccc and therefore preserves cardinals and cofinalities. Therefore if M is a ctm of ZFC and G is $\mathbb{P}$-generic over M, then $\aleph_2^M = \aleph_2^{M[G]}$. It is therefore enough to show that $\mathbb{P}$ adds a sequence of $\aleph_2$-many distinct subsets of ω.

Let $\kappa = \aleph_2^M$.

More precisely, we want to argue that $\{p \upharpoonright \{\xi\} \times \omega : p \in G\}$ is a sequence of $\aleph_2$-many distinct subsets of ω. This will work because the fact that G is a filter indicates that $p \upharpoonright \{\times\} \times \omega$ is a function. (Such a set might be called the *generic object* of $\mathbb{P}$.)

We want to show that the following set is dense for all $\xi < \eta < \kappa$:

$$D_{\xi,\eta} = \{p \in \mathbb{P} : \exists n < \omega, p(\xi, n) \neq p(\eta, n)\}$$

Suppose $r \in \mathbb{P}$. Since $|r| < \omega$, choose some N such that there is not ξ with $\langle \xi, N\rangle \in \operatorname{dom} r$. Then let

$$p = r \cup \langle\langle \xi, N\rangle, 0\rangle \cup \langle\langle \eta, N\rangle, 1\rangle.$$

Then we can see that $p \in D_{\xi,\eta}$.

Now let $p \in G \cap D_{\xi,\eta}$. Then this shows that $p \upharpoonright \{\xi\} \times \omega \neq p \upharpoonright \{\eta\} \times \omega$.

(We are eliding slightly over the issue of the countability of M, but notice that these statements are made with respect to $M \cap \mathrm{ON} = M[G] \cap \mathrm{ON}$.) $\square$

Corollary 6.2.6. $\mathrm{Con}(\mathsf{ZFC}) \implies \mathrm{Con}(\mathsf{ZFC} \wedge \neg\mathsf{CH})$.

Definition 6.2.7. We say that $\mathbb{P}$ is *countably closed* if for all $\leq_{\mathbb{P}}$-descending sequences $\langle p_n : n < \omega \rangle$ of conditions in $\mathbb{P}$, there is some $q \in \mathbb{P}$ such that for all $n < \omega$, $q \leq p_n$.

Example 8. Not that $\mathrm{Add}(\omega)$ is *not* countably closed. Suppose that $p_n \in \mathrm{Add}(\omega)$ is such that $\operatorname{dom} p_n = n$ and $p_n(m) = 0$ for all $m \in n$. Then if $n_1 > n_2$, then $p_{n_1} \leq p_{n_2}$. Therefore the sequence $\langle p_n : n < \omega \rangle$ is $\leq_{\mathrm{Add}(\omega)}$-decreasing. If we had $q \leq p_n$ for all $n < \omega$, then we would have $q \supseteq p_n$ for all $n < \omega$, which implies that $|q| \geq \omega$. But elements of $\mathrm{Add}(\omega)$ are finite, so this is impossible.

Lemma 6.2.9. *If $\mathbb{P}$ is countably closed and $p \Vdash$ "$\dot{X} \subseteq \omega$", then there is some $q \leq p$ and some Y such that $q \Vdash$ "$\dot{X} = Y$". In other words, $\mathbb{P}$ does not add any new subsets of ω.*

(We can actually state something more general, but we will deal with that later.)

Proof. Suppose that $p \Vdash$ "$\dot{f} : \omega \to \{0, 1\}$. Build a sequence $\langle p_n : n < \omega \rangle$ be induction on n: Let $p_0 \leq p$ be a condition such that for some ϵ_0, $p_0 \Vdash \dot{f}(n) = \epsilon_0$.

Note that we can find such a p_0: Suppose G is $\mathbb{P}$-generic over M and $p \in G$. Then there is some ϵ_0 such that $M[G] \models$ "$\dot{f}(n) = \check{\epsilon}_0$". Therefore there is some $q \in G$ such that $q \Vdash$ "$\dot{f}(n) = \check{\epsilon}_0$". Then let p_0 be such that $p_0 \leq p, q$.

Given p_n, let $p_{n+1} \leq p_n$ be a condition such that $p_{n+1} \Vdash \dot{f}(n+1) = \epsilon_{n+1}$ for some ϵ_{n+1}. By countable closure, let $q \in \mathbb{P}$ be such that $q \leq p_n$ for all $n < \omega$. Then if $g : n \mapsto \epsilon_n$, then $p \Vdash \dot{f} = g$. $\square$

Definition 6.2.10. Suppose that κ and λ are regular cardinals and $\kappa < \lambda$. The Lévy collapse is denoted by $\mathrm{Col}(\kappa, \lambda)$ and consists of conditions p such that:

1. p is a partial function from κ to λ,
2. $|p| < \kappa$.

We have $p \leq_{\mathrm{Col}(\kappa,\lambda)} q$ if and only if $p \supseteq q$.

Proposition 6.2.11. *If $\lambda > \omega_1$ is a regular cardinal, then the Lévy collapse $\mathrm{Col}(\omega_1, \lambda)$ is countably closed.*

Proof. Suppose that $\langle p_n : n < \omega \rangle$ is a descending sequence of conditions, let $q = \bigcup_{n<\omega} p_n$. Then q is countable because it is a countable union of countable sets. Observe that q is a function: If there were $\alpha, \beta_1 \neq \beta_2$ such that $q(\alpha) = \beta_1 = \beta_2$, then there would be some $n < \omega$ such that $\beta_1, \beta_2 \in \mathrm{range}(p_n)$ and $\alpha \in \mathrm{dom}\, p_n$, and this would contradict the fact that p_n is a function.

Therefore $q \in \mathrm{Col}(\omega_1, \lambda)$, and for all $n < \omega$, we have $q \leq p_n$ because $q \supseteq p_n$. □

Proposition 6.2.12. *If M is a ctm of ZFC with $\lambda \in M$ and G is $\mathrm{Col}(\omega_1, \lambda)$-generic over M, then there is some $f \in M[G]$ which is a surjection from ω_1^M to λ.*

Proof. This is very similar to an exercise in the eighth homework sheet!

We want to show that if $f = \bigcup G$, then $f : \kappa \to \lambda$ is a surjection.

To see that it is a function, suppose for contradiction that there are $\beta_1 \neq \beta_2$ such that $\langle \alpha, \beta_1 \rangle, \langle \alpha, \beta_2 \rangle \in \bigcup G$. Then there are $p_1 \ni \langle \alpha, \beta_1 \rangle$ and $p_2 \ni \langle \alpha, \beta_2 \rangle$, both in G. Let $p \in G$ be such that $p \leq p_1, p_2$. Then $\langle \alpha, \beta_2 \rangle, \langle \alpha, \beta_1 \rangle \in p$, contradicting that p is a function.

Now we want to show surjectivity. We argue for all $\alpha \in \omega_1^M \cap M$, that the set

$$D = \{p \in \mathbb{P} : \alpha \in \mathrm{range}(p)\}$$

is dense in $\mathbb{P}$: Fix $r \in \mathbb{P}$. If $\alpha \in \mathrm{range}(r)$, we are done. Otherwise, since $|\mathrm{dom}\, r| < \omega_1$, there is some $i \in M \cap (\omega_1 \setminus \mathrm{dom}\, r)$. Then let $p = r \cup \langle i, \alpha \rangle$. Then $p \in D$.

By genericity, there is some $p \in G \cap D$, so this proves that α is in the range of $\bigcup G$. □

Theorem 6.2.13. *If $\mathbb{P} = \mathrm{Col}(\omega_1, ((2^{\aleph_0})^+))$, then $\Vdash_{\mathbb{P}} \mathsf{CH}$.*

Proof. Forcing with $\mathbb{P}$ does not add any new subsets of ω, yet it collapses the cardinality of $2^{\aleph_0}$ to be ω_1. □

6.3 Ordinal Definable and Hereditarily Ordinal-Definable Sets

We need to introduce another concept before we get to the independence of the axiom of choice. The formal development is similar to that of L, but we will handle it a bit informally.

Definition 6.3.1.

1. If M is a set, an element $x \in M$ is *definable in* $(M, \in)$ *with parameters in* A if there is a formula $\varphi(v, \bar{w})$ and some $\bar{b} \in A$ such that $M \models \varphi(x, \bar{b})$ and this is unique, i.e. if $M \models \varphi(y, \bar{b})$ then $x = y$.

2. If M is a set, the set OD_M is the set of elements that are definable in $(M, \in)$ with parameters from $M \cap \mathrm{ON}$.

3. $\mathsf{OD} = \bigcup_{\alpha \in \mathrm{ON}} \mathsf{OD}_{V_\alpha}$ (where we are using the usual abuse of notation with the "union").

Lemma 6.3.2. *There is a formula* $\mathrm{Enod}(v)$ *such that* $\mathsf{OD} = \{\mathrm{Enod}(\gamma) : \gamma \in \mathrm{ON}\}$.

Sketch of Proof. Use a formula like the ones we used in the "defining definability" section to define the ordinal definable sets. The reflection theorem can be used to show that the formula works. Also, recall the pairing function on the class of ordinals Definition 1.5.7. This idea can be used to obtain a pairing function for the class of finite sequences of ordinals. This can then be used together with the formula for OD to give us Enod. □

Remark 6.3.3. OD is not a model of ZF because OD is not transitive.

Definition 6.3.4.

$$\mathsf{HOD} = \{x \in \mathsf{OD} : \mathrm{tc}(x) \subset \mathsf{OD}\}.$$

Proposition 6.3.5. $\mathrm{ON} \subseteq \mathsf{HOD} \subseteq \mathsf{OD}$ *and* HOD *is transitive.*

Proof. All ordinals are ordinal definable using themselves as parameters. Since ordinals contain only ordinals, it follows that ordinals are also hereditarily ordinal definable. This gives us $\mathrm{ON} \subseteq \mathsf{HOD}$. We also have $\mathsf{HOD} \subseteq \mathsf{OD}$ by definition. If $x \in \mathsf{HOD}$ and $y \in x$, then $\mathrm{tc}(y) \subseteq \mathrm{tc}(x) \subseteq \mathsf{OD}$, so HOD is transitive. □

Proposition 6.3.6. *For any set a, if $a \subseteq \mathsf{HOD}$ and $a \in \mathsf{OD}$ then $a \in \mathsf{HOD}$.*

Proof. This uses that $\mathrm{tc}(a) = a \cup \bigcup_{b\in a} \mathrm{tc}(b)$. □

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Proposition 6.3.7. *For each $\alpha \in \mathrm{ON}$, $(V_\alpha \cap \mathsf{HOD}) \in \mathsf{HOD}$.*

Proof. We immediately have $(V_\alpha \cap \mathsf{HOD}) \subseteq \mathsf{HOD}$. It is therefore sufficient to show $(V_\alpha \cap \mathsf{HOD}) \in \mathsf{OD}$ since this is definable from α. □

Theorem 6.3.8. HOD *is a model of* ZF*.*

Proof. We will go through the list, starting with the easy axioms.

Extension: Works because of transitivity.

Foundation: Works in any class.

Infinity: Works because of ω.

Comprehension: We want to show that for any formula $\psi(v, z, w_1, \ldots, w_n)$,

$$\forall z, w_1, \ldots, w_n \in \mathsf{HOD}(\{v \in z : \psi^{\mathsf{HOD}}(v, z, w_1, \ldots, w_n)\} \in \mathsf{HOD}).$$

Fix α_i's such that $z = \mathrm{Enod}(\alpha_0)$ and $w_i = \mathrm{Enod}(\alpha_i)$. Let

$$a = \{v \in z : \psi^{\mathsf{HOD}}(v, z, w_1, \ldots, w_n)\}.$$

Then a is the unique x satisfying $\varphi(\alpha_0, \alpha_1, \ldots, \alpha_n, x)$ where $\varphi(y_0, y_1, \ldots, y_n, x)$ is saying $y_0, y_1, \ldots, y_n \in \mathrm{ON}$ and

$$x = \{v \in \mathrm{Enod}(y_0) : \psi^{\mathsf{HOD}}(v, \mathrm{Enod}(y_0), \mathrm{Enod}(y_1), \ldots, \mathrm{Enod}(y_n))\}.$$

Therefore $a \in \mathsf{OD}$ and since $a \subseteq z \in \mathsf{HOD}$, we have $a \in \mathsf{HOD}$.

Pairing, Union, Replacement, Power Set: All of these use the same trick of looking for a large enough V_α, so we will just do pairing: Say $x, y \in \mathsf{HOD}$. If $x, y \in V_\beta$, let $\alpha = \beta + \omega$, so $\mathsf{HOD} \cap V_\alpha \in \mathsf{HOD}$. Then apply comprehension to obtain the subset $\{x, y\}$.

Choice: It is sufficient to show that for all $A \in \mathsf{HOD}$, there is a well-ordering $R \in \mathsf{HOD}$ of A. Let $A = \mathrm{Enod}(\alpha)$. Since $A \subseteq \mathsf{OD}$ (by definition), the elements of A are well-ordered by their appearance in Enod. Hence we let

$$R = \{\langle x, y\rangle \in A \times A : \exists \xi(x = \mathrm{Enod}(\xi) \wedge \forall \eta \leq \xi(y \neq \mathrm{Enod}(\eta))\}.$$

□

We need a generalization of HOD, for which we will unfortunately need to black-box the details.

6.4 The Independence of the Axiom of Choice

Black-box 6.4.1. *If A is a set, there is a model $\mathsf{HOD}(A)$ such that:*

1. $A \in \mathsf{HOD}(A)$,
2. $\mathsf{HOD}(A) \models \mathsf{ZF}$,
3. *All elements of $\mathsf{HOD}(A)$ are definable from some formula*

$$\varphi(u, \alpha_1, \ldots, \alpha_n, s, A)$$

where s is a finite sequence of elements of A.

Remark 6.4.2. Jech's textbook contains a broad sketch of $\mathsf{HOD}(A)$. However, there are alternate ways to develop this material. One alternative is the use of *symmetric submodels*, which are covered in the previous course script, Jech's textbook *The Axiom of Choice*, and in Halbeisen's textbook.

Definition 6.4.3. If $\mathbb{P}$ is a poset, we say that π is an *automorphism* of $\mathbb{P}$ if:

1. $\pi : \mathbb{P} \to \mathbb{P}$ is a bijection,
2. for all p, q, if $p \leq_{\mathbb{P}} q$ then $\pi(p) \leq_{\mathbb{P}} \pi(q)$,
3. $\pi(1_{\mathbb{P}}) = 1_{\mathbb{P}}$.

Definition 6.4.4. If $\pi : \mathbb{P} \to \mathbb{P}$ is an isomorphism and τ is a $\mathbb{P}$-name, then we can define by induction

$$\pi_*(\tau) = \{\langle \pi_*(\sigma), \pi(p)\rangle : \langle \sigma, p\rangle \in \tau\}.$$

Proposition 6.4.5. *Let $\varphi(v_1, \ldots, v_n)$ be a formula and let $\mathbb{P}$ be a poset with an automorphism π. Then*

$$p \Vdash \varphi(\dot{x}_1, \ldots, \dot{x}_n) \iff \pi(p) \Vdash \varphi(\pi_*\dot{x}_1, \ldots, \pi_*\dot{x}_n).$$

Proof. Homework. $\square$

Theorem 6.4.6. *There is a model of ZF in which $\mathbb{R}$ cannot be well-ordered.*

Corollary 6.4.7. $\mathrm{Con}(\mathsf{ZF}) \implies \mathrm{Con}(\mathsf{ZF} \wedge \mathsf{AC})$.

Proof of Theorem 6.4.6. Let $\mathbb{P} = \mathrm{Add}(\omega, \omega)$.

Let M be a ctm of ZFC and let G be $\mathbb{P}$-generic over M. For each $i \in \omega$, let

$$a_i = \{n \in \omega : (\exists p \in G) p(i, n) = 1\}$$

and let $A = \{a_i : i \in \omega\}$.

For $i \in \omega$, let $\dot{a}_i$ be a name such that $\mathrm{dom}(\dot{a}_i) = \{\check{n} : n \in \omega\}$ and let

$$\dot{a}_i = \{\langle \check{n}, p \rangle : p(i, n) = 1\}$$

and let

$$\dot{A} = \{\langle \dot{a}_i, 1_{\mathbb{P}} \rangle : i \in \omega\}.$$

Lemma 6.4.8. *If $i \neq j$, then for all $p \in \mathbb{P}$, $p \Vdash \dot{a}_i \neq \dot{a}_j$.*

Proof. This is similar to previous arguments. In particular, show that

$$D = \{p \in \mathbb{P} : \exists n, (i, n), (j, n) \in \mathrm{dom}\, p, p(i, n) \neq p(j, n)\}$$

is dense. □

Therefore, it is enough to show the following:

Lemma 6.4.9. *In $M[G]$, there is no injective function $f : A \to \mathrm{ON}$ that is ordinal-definable over A.*

Proof. First we work in $M[G]$ to find a statement that would have to hold if the lemma were false. If the lemma were false, there would be a one-one function $f : A \to \mathrm{ON}$ that is ordinal-definable over A and a finite sequence $s = \langle x_0, \ldots, x_k \rangle$. This means that every $a \in A$ is ordinal definable from s and A, and moreover there is such an $a \in A$ that is not equal to any of the x_i's.

Now let $\dot{a}$ be a $\mathbb{P}$-name for an element of A that is not equal to any of the $\dot{x}_i$'s, let $\dot{x}_0, \ldots, \dot{x}_k$ be names for elements of A and let $\dot{s}$ be a name for this sequence. Choose ordinals $\alpha_1, \ldots, \alpha_n$. We will show the following:

> If $p_0 \Vdash \varphi(\dot{a}, \check{\alpha}_1, \ldots, \check{\alpha}_n, \dot{s}, \dot{A})$, there is some $\dot{b}$ and $q \leq p_0$ such that $q \Vdash \dot{a} \neq \dot{b}$ and $q \Vdash \varphi(\dot{b}, \check{\alpha}_1, \ldots, \check{\alpha}_n, \dot{s}, \dot{A})$.

There exist $i, i_0, \ldots, i_k$ and $p_1 \leq p_0$ such that

$$p_1 \Vdash \dot{a} = \dot{a}_i \wedge \dot{x}_0 = \dot{a}_{i_0} \wedge \ldots \wedge \dot{x}_k = \dot{a}_{i_k}.$$

There is some $j \in \omega$ such that $j \neq i$ and such that for all $m \in m$, $(j, m) \notin \mathrm{dom}(p_1)$.

Let $\pi_\circ$ be permutation of ω such that:

1. $\pi_\circ(i) = j$,
2. $\pi_\circ(j) = i$,
3. $\pi_\circ(\ell) = \ell$ for all $\ell \neq i, j$.

Now define an automorphism of $\mathbb{P}$ as follows: If $p \in \mathbb{P}$, let πp be such that

$$\mathrm{dom}(\pi p) = \{(\pi_\circ x, m) : (x, m) \in \mathrm{dom}(p)\}$$

and

$$(\pi p)(\pi_\circ x, m) = p(x, m).$$

Then we can check the following:

1. $\pi_*(\dot{a}_i) = \dot{a}_j$,
2. $\pi_*(\dot{a}_j) = \dot{a}_i$,
3. $\pi_*(\dot{a}_\ell) = \dot{a}_\ell$ for $\ell \neq i, j$,
4. $\pi_*(\dot{A}) = \dot{A}$ (because the $\dot{a}_\ell$'s are just permuted).
5. $\pi_*(\dot{s}) = \dot{s}$ (we can assume $\dot{s}$ includes both $\dot{a}_i$ and $\dot{a}_j$ or neither).
6. $\pi_*(\check{\alpha}) = \check{\alpha}$ (because this is the case for all canonical names).

We have

$$p_1 \Vdash \varphi(\dot{a}_i, \check{\alpha}_1, \ldots, \check{\alpha}_n, \dot{s}, \dot{A}),$$

and since $\pi_*\check{\alpha} = \check{\alpha}$, $\pi_*\dot{s} = \dot{s}$, and $\pi_*\dot{A} = \dot{A}$, we have

$$\pi p_1 \Vdash \varphi(\dot{a}_j, \check{\alpha}_1, \ldots, \check{\alpha}_n, \dot{s}, \dot{A})$$

by Proposition 6.4.5. Therefore

$$q \Vdash \varphi(\dot{a}_i, \check{\alpha}_1, \ldots, \check{\alpha}_n, \dot{s}, \dot{A}) \wedge \varphi(\dot{a}_j, \check{\alpha}_1, \ldots, \check{\alpha}_n, \dot{s}, \dot{A})$$

We also have $q \Vdash \dot{a}_i \neq \dot{a}_j$ by Lemma 6.4.8. $\square$

This completes the proof of the theorem. $\square$

6.5 Product Forcing

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Suppose M is a ctm and $\mathbb{P}, \mathbb{Q} \in M$ are partial orders. We want to force with both $\mathbb{P}$ and $\mathbb{Q}$. The easiest way to accomplish this would be to first take a filter G which is $\mathbb{P}$-generic over M, then a filter H which is $\mathbb{Q}$-generic over $M[G]$ and pass to $M[G][H]$. However, the following things are unclear:

1. Do these operations commute, i.e. is H $\mathbb{Q}$-generic over M, G $\mathbb{P}$-generic over $M[H]$ and $M[G][H] = M[H][G]$?

2. Can we view $M[G][H]$ as a forcing extension of the ground model?

Both of these questions are answered by the *Product Lemma*, more specifically, the first question can be answered in the affirmative and in the second question, $M[G][H]$ is a forcing extension of M using the product of $\mathbb{P}$ and $\mathbb{Q}$.

Definition 6.5.1. Let $(\mathbb{P}, \leq_{\mathbb{P}})$ and $(\mathbb{Q}, \leq_{\mathbb{Q}})$ be partial orders. The product order $(\mathbb{P} \times \mathbb{Q}, \leq_{\mathbb{P}\times\mathbb{Q}})$ is given by $(p', q') \leq_{\mathbb{P}\times\mathbb{Q}} (p, q)$ if $p' \leq_{\mathbb{P}} p$ and $q' \leq_{\mathbb{Q}} q$.

And we can already state and prove the product Lemma:

Lemma 6.5.2 (Product Lemma). *Let $\mathbb{P}$ and $\mathbb{Q}$ be two forcing notions in a ctm M. Then the following holds:*

1. *If G is $\mathbb{P}$-generic over M and H is $\mathbb{Q}$-generic over $M[G]$, then $G \times H$ is $\mathbb{P} \times \mathbb{Q}$-generic over M and $M[G \times H] = M[G][H]$.*

2. *If K is $\mathbb{P} \times \mathbb{Q}$-generic over M, let*

$$G := \{p \in \mathbb{P} \mid \exists q (p, q) \in K\}, \quad H := \{q \in \mathbb{Q} \mid \exists p (p, q) \in K\}$$

Then G is $\mathbb{P}$-generic over M, H is $\mathbb{Q}$-generic over $M[G]$ and $M[G][H] = M[G \times H]$.

Proof. First let G be $\mathbb{P}$-generic over M and H $\mathbb{Q}$-generic over $M[G]$. Clearly $G \times H$ is a filter and the only thing required is to show that it is generic. To this end, let $D \subseteq \mathbb{P} \times \mathbb{Q}$ be in M and dense in $\mathbb{P} \times \mathbb{Q}$. Let

$$D_{\mathbb{Q}} := \{q \in \mathbb{Q} \mid \exists p \in G((p, q) \in D)\}$$

$D_{\mathbb{Q}}$ is an element of $M[G]$ and dense in $\mathbb{Q}$: If $q \in \mathbb{Q}$ is arbitrary, it follows from the density of D in $\mathbb{P} \times \mathbb{Q}$ that the set

$$D_{\mathbb{P},q} := \{p \in \mathbb{P} \mid \exists q' \leq q((p, q') \in D)\}$$

is dense in $\mathbb{P}$ (it is also in M). Ergo there is $p \in G \cap D_{\mathbb{P},q}$ and the corresponding q' is below q and in $D_{\mathbb{Q}}$. Lastly, since $G \times H \in M[G][H]$, we know that $M[G \times H] \subseteq M[G][H]$. However, since also G and H are in $M[G \times H]$, $M[G][H] \subseteq M[G \times H]$.

Now let K be $\mathbb{P} \times \mathbb{Q}$-generic over M and define G and H as above. Clearly both G and H are filters and $K \subseteq G \times H$. Moreover, if $(p, q) \in G \times H$, there are $p' \in \mathbb{P}, q' \in \mathbb{Q}$ such that $(p, q') \in K$ and $(p', q) \in K$. As K is a filter, there is $(p'', q'') \in K$ with $(p'', q'') \leq (p, q'), (p', q)$. In particular, $(p'', q'') \leq (p, q)$, so $(p, q) \in K$.

It is easy to see that G is $\mathbb{P}$-generic over M: If $D \in M$ is dense in $\mathbb{P}$, $D \times \mathbb{Q}$ is dense in $\mathbb{P} \times \mathbb{Q}$ so there is $(p, q) \in D \times \mathbb{Q} \cap K$, i.e. $p \in D \cap G$. If $D \in M[G]$ is dense in $\mathbb{Q}$, we can find (by the forcing theorem) a $\mathbb{P}$-name $\dot{D}$ with $\dot{D}_G = D$ and a condition $p \in G$ which forces that $\dot{D}$ is dense in $\check{\mathbb{Q}}$. Let $q \in H$ be arbitrary. We claim that

$$D' := \{(p', q') \in \mathbb{P} \times \mathbb{Q} \mid (p', q') \leq (p, q) \land p' \Vdash q' \in \dot{D}\}$$

is dense in $\mathbb{P} \times \mathbb{Q}$ below (p, q). This would imply that there is some $(p', q') \in D' \cap K$ and then $p' \in G$ would force $q' \in \dot{D}$ so that $q' \in \dot{D}^G \cap H = D \cap H$.

So let $(p'', q'') \leq (p, q)$ be arbitrary. Let G' be an arbitrary $\mathbb{P}$-generic filter containing p''. As $p'' \leq p$, $\dot{D}_{G'}$ is dense in $\mathbb{Q}$ in $M[G']$ (since $p \in G'$ and forces the density of $\dot{D}$), so there exists $q' \leq q''$ in $\dot{D}_{G'}$. By the forcing theorem, there is $p''' \in G'$ which forces $q' \in \dot{D}$ and because G' is a filter, there is $p' \leq p'', p'''$. Then $p' \Vdash q' \in \dot{D}$, so $(p', q') \leq (p'', q'')$ is in D'.

Lastly, we again have $G, H \in M[K]$, so $M[G][H] \subseteq M[K]$, but also $K \in M[G][H]$, so $M[K] \subseteq M[G][H]$. □

6.6 Easton's Theorem

In the first chapter we saw that the powerset function $F \colon \kappa \mapsto |\mathcal{P}(\kappa)|$ restricted to the set of regular cardinals has the following three properties:

1. For any $\kappa \leq \lambda$, $F(\kappa) \leq F(\lambda)$.

2. For any κ, $\mathrm{cf}(F(\kappa)) > \kappa$

Easton's Theorem states that this is precisely what we can prove about the function F:

Theorem 6.6.1 (Easton's Theorem). *Let M be a ctm and suppose that the generalized continuum hypothesis holds in M. Let F be a function (in M) from the regular cardinals into the cardinals with the following properties:*

1. *For any $\kappa \leq \lambda$, $F(\kappa) \leq F(\lambda)$.*

2. *For any κ, $\mathrm{cf}(F(\kappa)) > \kappa$.*

Then there is a generic extension $M[G]$ of M such that M and $M[G]$ have the same cardinals and cofinalities and for every regular κ,

$$M[G] \models |\mathcal{P}(\kappa)| = F(\kappa)$$

For simplicity, we will only prove the theorem in the case where we are working with a set of regular cardinals. To prove Easton's Theorem in full generality, one has to force with a *class* (from the perspective of M) which involves some intricacies: Some class-sized partial orders do not lead to well-defined forcing extensions (consider the poset which adds a Cohen real for every ordinal in M), so one has to restrict to forcing orders where any small part of the forcing extension is determined by a set-sized partial order.

First we have to generalize some results regarding the preservation of cardinals: For this section, we work inside of an arbitrary ctm M. This is because some of our arguments involve taking generic extensions which is not possible over general set-theoretic universes.

Definition 6.6.2. Let $\mathbb{P}$ be a forcing order and κ a cardinal.

1. $\mathbb{P}$ is κ-cc. if $\mathbb{P}$ has no antichains of size κ.

2. $\mathbb{P}$ is κ-Knaster if whenever $A \subseteq \mathbb{P}$ has size κ, there is a subset $B \subseteq A$ with size κ such that any two elements of B are compatible.

3. $\mathbb{P}$ is $<\kappa$-closed (or κ-closed) if for any descending sequence $(p_\alpha)_{\alpha<\mu}$ $(\mu < \kappa)$ there is $p \in \mathbb{P}$ such that $p \leq p_\alpha$ for any $\alpha < \mu$.

4. $\mathbb{P}$ is $<\kappa$-distributive (or κ-distributive) if for any sequence $(D_\alpha)_{\alpha<\mu}$ of open dense subsets of $\mathbb{P}$, the intersection $\bigcap_{\alpha<\mu} D_\alpha$ is open dense.

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Clearly, if $\mathbb{P}$ is κ-Knaster, then $\mathbb{P}$ is κ-cc.. An easy calculation shows the following:

Lemma 6.6.3. *If $\mathbb{P}$ is $<\kappa$-closed, it is $<\kappa$-distributive.*

Proof. Let $(D_\alpha)_{\alpha<\mu}$ be a sequence of open dense subsets of $\mathbb{P}$, where $\mu < \kappa$. For simplicity, assume $D_0 = \mathbb{P}$. We show that $\bigcap_{\alpha<\mu} D_\alpha$ is open dense. It is clearly open. For density, let $p \in \mathbb{P}$ be arbitrary. Inductively define a descending sequence $(p_\alpha)_{\alpha<\mu}$ where $p_\alpha \in D_\alpha$. That this works for successor α is clear, in limit steps λ we first obtain a lower bound p'_λ by $<\kappa$-closure and then take $p_\lambda \leq p'_\lambda$ using the density of D_λ. Let p_μ be a lower bound of $(p_\alpha)_{\alpha<\mu}$. Then by the openness of each D_α, $p_\mu \in D_\alpha$ for any $\alpha \in \mu$ and $p_\mu \leq p$ by transitivity. □

The κ-cc. and $<\kappa$-distributivity are related to the preservation of cardinals. The chain condition implies the preservation of sufficiently large cardinals while distributivity implies the preservation of sufficiently small cardinals:

Similar to Proposition 6.1.9, one shows:

Proposition 6.6.4. *If $\mathbb{P}$ is κ-cc., then $\mathbb{P}$ preserves cofinalities above (and including) κ. This implies that $\mathbb{P}$ preserves cardinalities above (and including) κ.*

And similar to Lemma 6.2.9, we have:

Lemma 6.6.5. *If $\mathbb{P}$ is $<\kappa$-distributive and $p \Vdash \dot{f} \colon \check{\mu} \to V$, where $\mu < \kappa$, then there is $q \leq p$ and g such that $q \Vdash \dot{f} = \check{g}$. In other words, $\mathbb{P}$ does not add any new functions from ordinals $<\kappa$ into the ground model.*

Proof. Assume $p \Vdash \dot{f} \colon \check{\mu} \to \mathrm{ON}$. For any $\alpha < \mu$, let D_α consist of those $q \in \mathbb{P}$ such that for some x, $q \Vdash \dot{f}(\check{\alpha}) = \check{x}$.

Claim. *For any $\alpha < \mu$, D_α is open and dense.*

Proof. For openness: Assume $q \in D_\alpha$ and $r \leq q$. Then $q \Vdash \dot{f}(\check{\alpha}) = \check{x}$ for some x. By Lemma 5.3.7 $r \Vdash \dot{f}(\check{\alpha}) = \check{x}$, so $r \in D_\alpha$.

For density: Assume $q \in \mathbb{P}$ is arbitrary. Let G be any $\mathbb{P}$-generic filter with $q \in G$. In $M[G]$, $\dot{f}^G(\alpha) = x$ where $x \in \mathrm{ON}$. Thus by the forcing theorem and since G is a filter, there is $r \in G$ with $r \leq q$ such that $r \Vdash \dot{f}(\check{\alpha}) = \check{x}$. So $r \in D_\alpha$. $\square$

So by $<\kappa$-distributivity, the intersection $\bigcap_{\alpha<\mu} D_\alpha$ is open dense. In particular there is $q \leq p$ such that $q \in D_\alpha$ for any $\alpha < \mu$. So there exists g such that for any $\alpha < \mu$, $q \Vdash \dot{f}(\check{\alpha}) = \check{g}(\check{\alpha})$. So $q \Vdash \dot{f} = \check{g}$. $\square$

In particular, if $\mathbb{P}$ is $<\kappa$-distributive, $\mathbb{P}$ does not collapse cardinals below (and including) κ: Otherwise there would be a $\mathbb{P}$-generic filter G and a surjection from some $\gamma < \kappa$ onto κ in $M[G]$. However, this surjection would already have been in M, leading to a contradiction.

Definition 6.6.6. Let τ and κ be cardinals. $\mathrm{Add}(\tau, \kappa)$ consists of functions p such that

1. $\mathrm{dom}(p)$ is a $<\tau$-sized subset of $\kappa \times \tau$.

2. $\mathrm{range}(p) \subseteq \{0, 1\}$.

And we have $p \leq q$ if and only if $p \supseteq q$.

In section 7.2, we saw that this poset has the countable chain condition. Something similar is the case here:

Lemma 6.6.7. *Let τ and κ be arbitrary cardinals where τ is regular.*

1. $\mathrm{Add}(\tau, \kappa)$ *is $<\tau$-closed.*

2. *If $\tau^{<\tau} = \tau$,* $\mathrm{Add}(\tau, \kappa)$ *is τ^+-Knaster.*

In particular, assuming $\tau^{<\tau} = \tau$ (which is the case assuming the generalized continuum hypothesis), $\mathrm{Add}(\tau, \kappa)$ does not collapse any cardinals.

To prove the previous lemma, we need another generalization of a result from section 7.2:

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Theorem 6.6.8 (The Δ-System Theorem)**.** *Let κ be an infinite cardinal. Let $\Theta > \kappa$ be regular and assume that for all $\alpha < \Theta$, $|\alpha|^{<\kappa} < \Theta$. Assume that $\mathcal{A}$ is a Θ-sized collection of $<\kappa$-sized sets. Then there is $\mathcal{B} \subseteq \mathcal{A}$ of size Θ and r such that $a \cap b = r$ for any two different $a, b \in \mathcal{B}$.*

Now we can show:

Proof of Lemma 6.6.7. Closure is straightforward: Given a descending sequence $(p_\alpha)_{\alpha<\mu}$ in $\mathrm{Add}(\tau,\kappa)$ where $\mu<\tau$, let $p_\mu := \bigcup_{\alpha<\mu} p_\mu$. Then clearly $p_\mu \in \mathrm{Add}(\tau,\kappa)$ by the regularity of τ and $p_\mu \supseteq p_\alpha$ for any $\alpha<\mu$.

Now assume $2^{<\tau}=\tau$. Assume $W \subseteq \mathrm{Add}(\tau,\kappa)$ has size τ^+. Define $\mathcal{A} := \{\mathrm{dom}(p) \mid p \in W\}$. By assumption $|\alpha|^{<\tau} \leq \tau^{<\tau} = \tau < \tau^+$ for any $\alpha < \tau^+$. Ergo there is $\mathcal{B} \subseteq \mathcal{A}$ with size τ^+ and r such that $a \cap b = r$ for any two different $a, b \in \mathcal{B}$. Let $S \subseteq W$ be such that $\mathcal{B} = \{\mathrm{dom}(p) \mid p \in S\}$. Since $|{}^r 2| < \tau^+$, there is $T \subseteq S$ with size τ^+ and $t \in {}^r 2$ such that $p \restriction r = t$ for any $p \in T$. Then whenever $p, q \in S$, $p \cup q$ is a condition in $\mathrm{Add}(\tau,\kappa)$ because p and q agree on their shared domain. □

The rest of this section is devoted to the proof of Easton's Theorem. So we fix a ctm M in which the generalized continuum hypothesis holds and a function $F \in M$ defined on a set S of regular cardinals such that

1. For any $\kappa \leq \lambda$, $F(\kappa) \leq F(\lambda)$,
2. For any κ, $\mathrm{cf}(F(\kappa)) > \kappa$.

Our intended forcing notion is the *Easton supported* product

$$\mathbb{P} := \prod_{\alpha \in S} \mathrm{Add}(\alpha, F(\alpha))$$

By this we mean the following: Conditions in $\mathbb{P}$ consist of partial functions p on S such that:

1. $\mathrm{dom}(p)$ is an Easton subset of S, i.e. for any regular cardinal δ, $|p \cap \delta| < \delta$.
2. for any $\delta \in \mathrm{dom}(p)$, $p(\delta) \in \mathrm{Add}(\delta, F(\delta))$

We let $p \leq q$ if $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$ and $p(\delta) \leq q(\delta)$ in $\mathrm{Add}(\delta, F(\delta))$ for any $\delta \in \mathrm{dom}(q)$.

For any regular cardinal δ, we let $S^{<\delta} := S \cap \delta$ and $S^{\geq\delta} := \{\mu \in S \mid \mu \geq \delta\}$. We also let

$$\mathbb{P}^{<\delta} := \prod_{\alpha \in S^{<\delta}} \mathrm{Add}(\alpha, F(\alpha)),\ \mathbb{P}^{\geq\delta} := \prod_{\alpha \in S^{\geq\delta}} \mathrm{Add}(\alpha, F(\alpha))$$

again taking Easton Support and we note that for each δ, $\mathbb{P}$ is isomorphic to the product of $\mathbb{P}^{<\delta}$ and $\mathbb{P}^{\geq\delta}$ simply by sending $p \in \mathbb{P}$ to $(p \restriction \delta, p \restriction [\delta, \sup(S)))$.

We are left showing the following facts:

1. $\mathbb{P}$ does not collapse any cardinals.

2. After forcing with $\mathbb{P}$, $|2^\alpha| = F(\alpha)$ for any $\alpha \in S$.

We first show:

Lemma 6.6.9. *Let δ be the successor of a regular cardinal. Then $\mathbb{P}^{<\delta}$ is δ-Knaster and $\mathbb{P}^{\geq\delta}$ is $<\delta$-closed.*

Proof. Let $\delta = \mu^+$ for some regular cardinal μ. By assumption we know that for every $p \in \mathbb{P}^{<\delta}$, $|\operatorname{dom}(p) \cap \mu| < \mu$, but, because S consists of regular cardinals, this also implies that $|\operatorname{dom}(p) \cap \delta| = |\operatorname{dom}(p)| < \mu$.

Let $W \subseteq \mathbb{P}^{<\delta}$ have size δ. For $p \in \mathbb{P}^{<\delta}$, let

$$F(p) := \operatorname{dom}(p) \times \{0\} \cup \bigcup_{\alpha \in \operatorname{dom}(p)} \operatorname{dom}(p(\alpha)) \times \{\alpha\}$$

Let $\mathcal{A} := \{F(p) \mid p \in W\}$. By the definition of $\mathbb{P}^{<\delta}$, $|F(p)| < \mu$, because $F(p)$ is a $<\mu$-sized union of $<\mu$-sized sets. By Theorem 6.6.12, let $\mathcal{B} \subseteq \mathcal{A}$ and r be such that $x \cap y = r$ for different $x, y \in \mathcal{B}$. Let $T \subseteq W$ be such that $F[T] = \mathcal{B}$. For an ordinal γ, let $r^\gamma := \{x \mid (x, \gamma) \in r\}$.

Claim. *If $p, q \in T$ are different,* $\operatorname{dom}(p) \cap \operatorname{dom}(q) = r(0)$ *and* $\operatorname{dom}(p(\alpha)) = \operatorname{dom}(q(\alpha)) = r(\alpha)$ *for any $\alpha \in r(0)$.*

Proof. If $x \in r(0)$, $(x, 0) \in r$, so $(x, 0) \in F(p) \cap F(q)$. By the definition of F, this implies $x \in \operatorname{dom}(p) \cap \operatorname{dom}(q)$. If $x \in \operatorname{dom}(p) \cap \operatorname{dom}(q)$, $(x, 0) \in F(p) \cap F(q)$, so $(x, 0) \in r$ and $x \in r(0)$.

If $\alpha \in r(0)$, then by the previous proof $\alpha \in \operatorname{dom}(p) \cap \operatorname{dom}(q)$. If $x \in r(\alpha)$, then $(x, \alpha) \in r$ and $(x, \alpha) \in F(p) \cap F(q)$, so $x \in \operatorname{dom}(p(\alpha)) \cap \operatorname{dom}(q(\alpha))$. If $x \in \operatorname{dom}(p) \cap \operatorname{dom}(q)$, then $(x, \alpha) \in F(p) \cap F(q)$, so $(x, \alpha) \in r$ and $x \in r(\alpha)$. $\square$

Now we map $p \in T$ to $G(p) := \langle p(\alpha) \restriction r(\alpha) \rangle_{\alpha \in r(0)}$. There are $\leq \mu$ many possibilities for $p(\alpha) \restriction r(\alpha)$ and so there are $\leq \mu^{|r(0)|} = \mu$ many possibilities for the whole sequence. So there is $S \subseteq T$ with size δ such that

$G(p) = G(q)$ for any $p, q \in S$. Now if $p, q \in S$, $\mathrm{dom}(p) \cap \mathrm{dom}(q) = r(0)$ and for any $\alpha \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$, $p(\alpha)$ and $q(\alpha)$ agree on their common domain. Ergo p and q are compatible.

Luckily, closure is easier: Let $(p_\alpha)_{\alpha<\nu}$ be a descending sequence in $\mathbb{P}^{\geq\delta}$, where $\nu < \delta$. Let p be a function with domain $x := \bigcup_{\alpha<\nu} \mathrm{dom}(p_\alpha)$ defined as follows (we first note that x is an Easton set, because for any regular cardinal κ, either $\kappa < \delta$ and $x \cap \kappa = \emptyset$ or $\kappa \geq \delta$ and $x \cap \kappa$ is a $<\delta$-sized union of $<\kappa$-sized sets and thus has size $<\kappa$). For $\alpha \in x$, let β_α be the minimal ordinal $<\nu$ with $\alpha \in \mathrm{dom}(p_{\beta_\alpha})$ and define $p(\alpha) := \bigcup_{\gamma\in[\beta_\alpha,\nu)} p_\gamma(\alpha)$. Then p is a lower bound of $(p_\alpha)_{\alpha<\nu}$. $\square$

So for any successor cardinal, neither $\mathbb{P}^{<\delta}$ nor $\mathbb{P}^{\geq\delta}$ collapses δ. However, this does not imply that their product does not collapse δ. To show this, we prove the Easton Lemma:

Lemma 6.6.10 (Easton Lemma)**.** *Let $\mathbb{Q}$ and $\mathbb{R}$ be posets and δ a cardinal such that $\mathbb{Q}$ is δ-cc. and $\mathbb{R}$ is $<\delta$-closed. Then:*

1. *$\mathbb{R}$ is $<\delta$-distributive after forcing with $\mathbb{Q}$.*
2. *$\mathbb{Q}$ is δ-cc. after forcing with $\mathbb{R}$.*

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Now we can show our main result of this section:

Theorem 6.6.11. *Let G be $\mathbb{P}$-generic. The following holds:*

1. *$M[G]$ has the same cardinals as M.*
2. *In $M[G]$, $|2^\alpha| = F(\alpha)$ for any $\alpha \in S$.*

Proof. For (1), let δ be a cardinal in M. Assume first that δ is a successor cardinal in M. Let $\pi\colon \mathbb{P} \to \mathbb{P}^{<\delta} \times \mathbb{P}^{\geq\delta}$ be the isomorphism. Then (as on the exercise sheet), $\pi[G]$ is $\mathbb{P}^{<\delta} \times \mathbb{P}^{\geq\delta}$-generic and thus by the Product Lemma 6.5.2 it has the form $H \times I$ where H is $\mathbb{P}^{<\delta}$-generic over M and I is $\mathbb{P}^{\geq\delta}$-generic over $M[H]$. Furthermore, $M[G] = M[H\times I] = M[H][I]$. In $M[H]$, δ remains a cardinal by Proposition 6.6.4 because $\mathbb{P}^{<\delta}$ is δ-Knaster by Lemma 6.6.9. Furthermore, in $M[H]$, $\mathbb{P}^{\geq\delta}$ is $<\delta$-distributive by the Easton Lemma 6.6.13 and Lemma 6.6.9. Ergo δ remains a cardinal in $M[H][I] = M[G]$.

Now assume that δ is a limit cardinal in M. If δ is no longer a cardinal in $M[G]$, $M[G]$ contains a bijection from some $\gamma < \delta$ onto δ. However, since

δ is a limit cardinal in M, $(|\gamma|^+)^M < \delta$ $((|\gamma|)^+$ denotes the minimal ordinal in M which has a larger size than γ in M). Ergo $M[G]$ also contains a bijection from γ onto $(|\gamma|^+)^M$. However, $(|\gamma|^+)^M$ is a successor cardinal in M and no longer a cardinal in $M[G]$, a contradiction.

Now let $\alpha \in S$. Let $\pi\colon \mathbb{P} \to \mathbb{P}^{<\alpha^+} \times \mathbb{P}^{\geq\alpha^+}$ be the isomorphism and let H and I be such that $\pi[G] = H \times I$. Again, $\mathbb{P}^{<\alpha^+}$ is isomorphic to $\mathbb{P}^{<\alpha} \times \mathrm{Add}(\alpha, F(\alpha))$ via some isomorphism σ and thus we can let H_0 and H_1 be such that $\sigma[H] = H_0 \times H_1$.

Claim. *In $V[H]$, $|2^\alpha| = F(\alpha)$.*

Proof. We first show $\geq$. Let $A := \bigcup H_1$, so $A\colon F(\alpha) \times \alpha \to 2$. Then as in the proof of Theorem 6.2.5 the set $(A(\beta, \cdot))_{\beta \in F(\alpha)}$ is a sequence of $F(\alpha)$ many distinct subsets of α.

To show $\leq$, we count names: We first note that $|\mathbb{P}^{<\alpha^+}| = F(\alpha)$: For any $\beta < \alpha^+$ in S, $|\mathrm{Add}(\beta, F(\beta))| = F(\beta) \leq F(\alpha)$ since F is increasing by assumption. Ergo any element of $\mathbb{P}^{<\alpha^+}$ is a sequence of length $\leq \alpha$ with values in sets of size $\leq F(\alpha)$. Ergo $|\mathbb{P}^{<\alpha^+}| \leq F(\alpha)^\alpha = F(\alpha)$ by Proposition 1.7.15 since $\mathrm{cf}(F(\alpha)) > \alpha$ by assumption.

Now we construct a surjection from $F(\alpha)$ onto the set 2^α in $M[G]$. First work in M. Any antichain in $\mathbb{P}^{<\alpha^+}$ has size $\leq \alpha$. So the set of all antichains in $\mathbb{P}^{<\alpha^+}$ has size $F(\alpha)^\alpha = F(\alpha)$ and again, so does the set of all α-sequences of antichains in $\mathbb{P}^{<\alpha^+}$, so let $(\mathcal{A}_\gamma)_{\gamma < F(\alpha)}$ enumerate all such sequences. In $V[H]$, let $\Sigma(\gamma)$ be the following subset of α: For any $\beta < \alpha$, $\beta \in \Sigma(\gamma)$ if and only if $H \cap \mathcal{A}_\gamma(\beta)$ is nonempty.

We will verify that Σ is a bijection. To this end, let $x \subseteq \alpha$ be an element of $M[H]$. Let $x = \tau_G$ for some $\mathbb{P}^{<\alpha^+}$-name τ and let $p \in H$ force $\tau \subseteq \check{\alpha}$. For $\beta < \alpha$, let D_β consist of those $q \leq p$ which force $\check{\beta} \in \tau$ or $\check{\beta} \notin \tau$. Then D_β is open and dense below p. Let $A_\beta \subseteq D_\beta$ be an antichain which is maximal with respect to all antichains contained in D_β. It follows that A_β is a maximal antichain below p. Let B_β consist of all those $q \in A_\beta$ which force $\check{\beta} \in \tau$. Then $(B_\beta)_{\beta<\alpha} = \mathcal{A}_\gamma$ for some $\gamma < F(\alpha)$. We claim that $x = \Sigma(\gamma)$. Let $\beta < \alpha$. Since A_β is a maximal antichain below p, there is exactly one element $q \in A_\beta \cap H$. If $\beta \in x$, q has to force $\check{\beta} \in \tau$, so $q \in B_\beta$ and $\beta \in \Sigma(\gamma)$. If $\beta \notin x$, q has to force $\check{\beta} \in \tau$, so $q \notin B_\beta$ and $\beta \notin \Sigma_\gamma$. $\square$

Now the rest follows easily: $M[G] = M[H][I]$ is an extension of $M[H]$ using a $<\alpha^+$-distributive forcing order. This forcing order does not add any new elements of 2^α and does not collapse any cardinals (since M and

$M[G]$ have the same cardinals, the same holds for any intermediate model), so $|2^\alpha| = F(\alpha)$ in $M[G]$. □

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Theorem 6.6.12. *Let κ be an infinite cardinal. Let $\Theta > \kappa$ be regular and assume that for all $\alpha < \Theta$, $|\alpha|^{<\kappa} < \Theta$. Assume that $\mathcal{A}$ is a Θ-sized collection of $<\kappa$-sized sets. Then there is $\mathcal{B} \subseteq \mathcal{A}$ of size Θ and r such that $a \cap b = r$ for any two different $a, b \in \mathcal{B}$.*

Proof. For simplicity, we assume that $\bigcup \mathcal{A} \subseteq \Theta$. For any $x \in \mathcal{A}$, x is thus a subset of Θ, so it is well-ordered by $\in$ and we can consider its order-type, denoted $\operatorname{ot}(x)$.

Because Θ is regular and greater than κ, there is $\rho < \kappa$ such that $\mathcal{A}_1 = \{x \in \mathcal{A} \mid \operatorname{ot}(x) = \rho\}$ has size Θ. This uses the "Pigeonhole Principle" from before: If $X_\rho = \{x \in \mathcal{A} | \operatorname{ot}(x) = \rho\}$ and all such X_ρ's have cardinality less than Θ, then this contradicts regularity of Θ.

We argue that $\bigcup \mathcal{A}_1$ is unbounded in Θ. Otherwise it would be bounded in some $\alpha < \Theta$. By assumption we have $|\alpha|^{<\kappa} < \Theta$, so we cannot have Θ-many distinct objects that are equal to some element of $\alpha^{<\kappa}$, and this would be a contradiction.

For $x \in \mathcal{A}_1$ and $\xi < \rho$ we let $x(\xi)$ be the ξth element of x (according to the well-order $\in$). Because Θ is regular, there is $\xi < \rho$ such that $\{x(\xi) \mid x \in \mathcal{A}_1\}$ is unbounded in Θ. Let ξ_0 be the minimal ξ with this property. Define

$$\alpha_0 := \sup\{x(\eta) + 1 \mid x \in \mathcal{A}_1 \wedge \eta < \xi_0\}$$

Then $\alpha_0 < \Theta$ by the minimality of ξ_0 and $x(\eta) < \alpha_0$ for all $x \in \mathcal{A}_1$, $\eta < \xi_0$.

By transfinite induction for $\mu < \Theta$ we choose $x_\mu \in \mathcal{A}_1$ such that

$$x_\mu(\xi_0) \geq \max(\alpha_0, \sup\{x_\nu(\eta) \mid \eta < \rho \wedge \nu < \mu\}).$$

Then we let $\mathcal{A}_2 = \{x_\mu \mid \mu < \Theta\}$. We have $|\mathcal{A}_2| = \Theta$ and $x \cap y \subseteq \alpha_0$ for all distinct $x, y \in \mathcal{A}_2$. Because $|\alpha_0|^{<\kappa} < \Theta$ there is $r \subseteq \alpha_0$ such that $x \cap \alpha_0 = r$ for Θ-many $x \in \mathcal{A}_2$. Let $\mathcal{B} := \{x \in \mathcal{A}_2 \mid x \cap \alpha_0 = r\}$. Then for any two different $x, y \in \mathcal{B}$, we have

$$x \cap y = x \cap y \cap \alpha_0 = (x \cap \alpha_0) \cap (y \cap \alpha_0) = r \cap r = r$$

so $\mathcal{B}$ and r are as required. □

Lemma 6.6.13 (Easton Lemma)**.** *Let $\mathbb{P}$ and $\mathbb{Q}$ be posets and δ a cardinal such that $\mathbb{P}$ is δ-cc. and $\mathbb{Q}$ is $<\delta$-closed. Then:*

1. *$\mathbb{Q}$ is $<\delta$-distributive after forcing with $\mathbb{P}$.*
2. *$\mathbb{P}$ is δ-cc. after forcing with $\mathbb{Q}$.*

Proof. First we prove *1.*.

Let G be an arbitrary $\mathbb{P}$-generic filter. In $M[G]$, let $(D_\alpha)_{\alpha<\mu}$ ($\mu < \delta$) be a sequence of open dense subsets. In M, find a sequence $(\dot{D}_\alpha)_{\alpha<\mu}$ with $\dot{D}_\alpha^G = D_\alpha$ and a condition $p \in G$ which forces the preceding statement.

For any $\beta < \mu$, let D'_β be the set of all $q \in \mathbb{Q}$ such that $p \Vdash \check{q} \in \dot{D}_\beta$.

Claim. *For any $\beta < \mu$, D'_β is open dense.*

Proof. Openness is clear: If $p \Vdash \check{q} \in \dot{D}_\beta$ and $q' \leq q$, $p \Vdash \check{q}' \in \dot{D}_\beta$ because p forces $\dot{D}_\beta$ to be open.

For density, fix $q \in \mathbb{Q}$. We will show that there is some $q' \leq q$ in $\dot{D}_\beta$.

By induction on $\alpha < \delta$, we will construct a descending sequence $(q_\alpha)_{\alpha<\delta}$ of elements of $\mathbb{Q}$ below q and a sequence $(p_\alpha)_{\alpha<\delta}$ of pairwise incompatible elements of $\mathbb{P}$ such that for any $\alpha < \delta$, $p_\alpha \Vdash \check{q}_\alpha \in \dot{D}_\beta$. By the δ-cc. of $\mathbb{P}$, the construction will terminate at some $\delta' < \delta$. Assume both sequences have been defined for $\alpha < \gamma$. By the $<\delta$-closure of $\mathbb{Q}$, let q'_γ be a lower bound of $(q_\alpha)_{\alpha<\delta}$. If $\{p_\alpha \mid \alpha < \gamma\}$ is a maximal antichain below p, we are finished. Otherwise, choose some $q_\gamma \leq q'_\gamma$ and some p_γ incompatible with all of $\{p_\alpha \mid \alpha < \gamma\}$ such that $p_\gamma \Vdash q_\gamma \in \dot{D}_\beta$. (To see that we can do this, choose some p'_γ incompatible with all elements of $\{p_\alpha \mid \alpha < \gamma\}$. let G' be a $\mathbb{P}$-generic containing p'_γ. In $M[G']$, there is some $q_\gamma \leq q'_\gamma$ that is in $(\dot{D}_\beta)_{G'}$. Choose $p_\gamma \leq p'_\gamma$ forcing this.)

Assume the process terminates at $\delta' < \delta$. By $<\delta$-closure, take a lower bound q^* for $(q_\alpha)_{\alpha<\delta'}$. We argue that $p \Vdash \check{q}^* \in \dot{D}_\beta$. If $p' \leq p$, then there is some p_α with $\alpha < \delta'$ such that $p' \| p_\alpha$. Choose $p'' \leq p', p_\alpha$. Then $p'' \Vdash q_\alpha \in \dot{D}_\beta$, so it follows that $p'' \Vdash q^* \in \dot{D}_\beta$. □

In M, $\mathbb{Q}$ is in particular $<\delta$-distributive since it is $<\delta$-closed, so $D := \bigcap_{\alpha<\mu} D'_\alpha$ is open dense. Let $r \in D$ be below q. Then $p \Vdash r \in \dot{D}_\beta$ and we are done.

Now we prove part *2.*, which is easier. Let H be an arbitrary $\mathbb{Q}$-generic filter. Assume that in $M[H]$, there is a function f on δ such that $f(\alpha) \perp f(\beta)$ whenever $\alpha \neq \beta$. Let $\dot{f}$ be an $\mathbb{Q}$-name such that $\dot{f}^H = f$ and let r

force the preceding statement. By induction on $\alpha < \delta$, define a descending sequence $(p_\alpha)_{\alpha<\delta}$ such that for any $\alpha < \delta$ there is x_α such that $p_\alpha \Vdash \dot{f}(\check{\alpha}) = \check{x}_\alpha$.

Claim. $\{x_\alpha \mid \alpha < \delta\}$ *is an antichain.*

Proof. Let $\alpha \neq \beta$, assume WLoG that $\alpha < \beta$. Then $p_\beta \leq p_\alpha$, so $p_\beta \Vdash \dot{f}(\check{\alpha}) = \check{x}_\alpha \wedge \dot{f}(\check{\beta}) = \check{x}_\beta$. In particular, $p_\beta \Vdash \check{x}_\alpha \perp \check{x}_\beta$, so $x_\alpha \perp x_\beta$ because that statement is sufficiently absolute. $\square$

This is a contradiction of the fact that we assumed that $\mathbb{P}$ is δ-cc. $\square$

Chapter 7

Iterated Forcing and Martin's Axiom

7.1 Two-Step Iterations

January 23, 2025

Let us begin using the convention of referring to V of the ground model.

Definition 7.1.1. Let $\mathbb{P}$ be a poset and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a poset.

1. $\mathbb{P} * \dot{\mathbb{Q}} = \{(p, \dot{q}) : p \in \mathbb{P}, 1_{\mathbb{P}} \Vdash \dot{q} \in \dot{\mathbb{Q}}\}$.
2. $(p_1, \dot{q}_1) \leq_{\mathbb{P} * \dot{\mathbb{Q}}} (p_2, \dot{q}_2)$ if and only if $p_1 \leq_{\mathbb{P}} p_2$ and $p_1 \Vdash \dot{q}_1 \leq \dot{q}_2$.

Remark 7.1.2. Technically this definition gives a proper class for $\mathbb{P} * \dot{\mathbb{Q}}$, but we can always consider $(p, \dot{q})$ for $\dot{q}$ in a "sufficiently large" $H(\lambda)$. We can also consider $(p, \dot{q})$ such that $\dot{q} \in \operatorname{dom} \dot{\mathbb{Q}}$, but this will become more complex when we work with longer iterations.

Theorem 7.1.3. *Let $\mathbb{P}$ be a poset and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a poset.*

1. *Let G be $\mathbb{P}$-generic over V, let $\mathbb{Q} = \dot{\mathbb{Q}}_G$, and let H be $\mathbb{Q}$-generic over $V[G]$. Then*
$$G * H = \{(p, \dot{q}) : p \in G, \dot{q}_G \in H\}$$
is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V.
2. *Suppose that K is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V. Then*
$$G := \{p \in \mathbb{P} : \exists \dot{q}, (p, \dot{q}) \in K\}$$

is $\mathbb{P}$-generic over V and

$$H := \{\dot{q}_G : \exists p, (p, \dot{q}) \in K\}$$

is $\dot{\mathbb{Q}}_G$-generic over $V[G]$.

Also, $K = G * H$.

Proof. First we prove *1.*

Let us check that $G * H$ is a filter: For compatibilities, suppose we have $(p_0, \dot{q}_0), (p_1, \dot{q}_1) \in G * H$. Then there is some $p_2' \in G$ such that $p_2' \leq p_0, p_1$. Since H is a filter, there is some $\dot{q}_2$ such that $(\dot{q}_2)_G \leq (\dot{q}_0)_G, (\dot{q}_1)_G$. By the Forcing Theorem there is some $p_2 \leq p_2'$ such that $p_2 \Vdash \dot{q}_2 \leq \dot{q}_0, \dot{q}_1$. Therefore $(p_2, \dot{q}_2) \leq (p_i, \dot{q}_i)$ for $i = 0, 1$. For upwards closure, suppose that $(p_0, \dot{q}_0) \in G * H$ and $(p_1, \dot{q}_1) \geq (p_0, \dot{q}_0)$. Then by definition of the ordering, $p_0 \Vdash \dot{q}_0 \leq \dot{q}_1$, therefore $(\dot{q}_0)_G \leq (\dot{q}_1)_G$, and therefore $(\dot{q}_1)_G \in H$. We also have $p_1 \in G$, so $(p_1, \dot{q}_1) \in G * H$.

Now we need to argue for genericity. Let $D \in V$ be dense in $\mathbb{P} * \dot{\mathbb{Q}}$. Let

$$D_1 = \{\dot{q}_G : \exists p \in G, (p, \dot{q}) \in D\}.$$

We argue that in $V[G]$, D_1 is dense in $\mathbb{Q}$.

Before we do this, we fix $(\dot{q}_0)_G \in \mathbb{Q}$ and define

$$D_2 = \{p \in \mathbb{P} : \exists \dot{q}_1 (p \Vdash \dot{q}_1 \leq \dot{q}_0, (p, \dot{q}_1) \in D)\}.$$

We argue that D_2 is dense: If $r \in \mathbb{P}$, find some $(p, \dot{q}_1) \leq (r, \dot{q}_0)$ such that $(p, \dot{q}_1) \in D$. Then by definition we have $p \Vdash \dot{q}_1 \leq \dot{q}_0$, so $p \in D_2$.

Now we can argue that D_1 is dense: Let $p \in G \cap D_2$. Suppose that this is witnessed by $\dot{q}_1$. Then $(\dot{q}_1)_G \leq (\dot{q}_0)_G$.

Finally, let $\dot{q}_G \in D_1 \cap H$ and let $p \in G$ witness this. Then $(p, \dot{q}) \in D \cap G * H$.

Now we prove *2.* The fact that G and H are filters has a proof similar to the one for *1*, so we will skip it and focus on genericity.

To prove that G is generic: Let $D \in V$ be dense in $\mathbb{P}$. Then $D_1 = \{(p, \dot{q}) : p \in D\}$ is dense in $\mathbb{P} * \dot{\mathbb{Q}}$, so there is some $(p, \dot{q}) \in K \cap D_1$, so $p \in G \cap D$.

Now we prove that H is generic while working in $V[G]$. Let $\dot{D}$ be a $\mathbb{P}$-name for a dense subset of $\dot{\mathbb{Q}}$ and that this is forced by $\bar{p} \in G$. Then the set $\{(p, \dot{q}) : p \Vdash \dot{q} \in \dot{D}\}$ is dense in $\mathbb{P} * \dot{\mathbb{Q}}$ below $(\bar{p}, 1)$. Therefore it follows that $D \cap H \neq \emptyset$.

The proof of $K = G * H$ is rote verification. $\square$

Lemma 7.1.4. *If $\mathbb{P}$ is $< \kappa$-closed and $\Vdash$ "$\dot{\mathbb{Q}}$ is $< \kappa$-closed" then $\mathbb{P} * \dot{\mathbb{Q}}$ is $< \kappa$-closed.*

Proof. Let $\langle (p_\alpha, \dot{q}_\alpha) : \alpha < \delta \rangle$ be a $\leq_{\mathbb{P}*\dot{\mathbb{Q}}}$-decreasing sequence of conditions for some $\delta < \kappa$. Using the fact that $\mathbb{P}$ is $< \kappa$-closed, there is some $p_* \leq p_\alpha$ for all $\alpha < \delta$.

If G is $\mathbb{P}$-generic over some M with $p_* \in G$, work in $M[G]$ and consider $\langle (\dot{q}_\alpha)_G : \alpha < \delta \rangle$. The poset $\mathbb{Q} = \dot{\mathbb{Q}}_G$ is $< \delta$-closed, so there is some q_* such that $q_* \leq (\dot{q}_\alpha)_G$ for all $\alpha < \delta$. Therefore there is some $\mathbb{P}$-name $\dot{q}_*$ and some $\bar{p} \leq p$ such that

$$\bar{p} \Vdash \text{“}\dot{q}_* \text{ is a lower bound of } \langle \dot{q}_\alpha : \alpha < \delta \rangle\text{”}.$$

Therefore $(\bar{p}, \dot{q}_*)$ is a lower bound of $\langle (p_\alpha, \dot{q}_\alpha) : \alpha < \delta \rangle$. □

Lemma 7.1.5. *Suppose that κ is regular, $\mathbb{P}$ is κ-cc, and $\dot{X}$ is a $\mathbb{P}$-name such that for some $p \in \mathbb{P}$, $p \Vdash \dot{X} \subseteq \kappa \wedge |\dot{X}| < \kappa$. Then there is some $\beta < \kappa$ such that $p \Vdash \dot{X} \subseteq \beta$.*

Proof. Let

$$B = \{\alpha < \kappa : \exists q \leq p, q \Vdash \sup(\dot{X}) = \alpha\}.$$

For each $\alpha \in B$, choose some p_α witnessing it. Then the set $\{p_\alpha : \alpha \in B\}$ is an antichain. Therefore $|B| < \kappa$ be the κ-cc of $\mathbb{P}$. Let $\beta \in (\alpha, \kappa)$. Since $\mathbb{P}$ forces that κ is regular (from section 6.1), it follows that $p \Vdash \dot{X} \subseteq \beta$. □

Lemma 7.1.6. *If κ is regular and $\mathbb{P}$ has the κ-chain condition and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a poset with the κ-chain condition, it follows that $\mathbb{P} * \dot{\mathbb{Q}}$ has the κ-chain condition.*

Proof. Suppose $\langle (p_\alpha, \dot{q}_\alpha) : \alpha < \kappa \rangle$ is a sequence of incompatible conditions in $\mathbb{P} * \dot{\mathbb{Q}}$.

Let $\dot{Z}$ be the $\mathbb{P}$-name

$$\dot{Z} = \{(\check{\alpha}, p_\alpha) : \alpha < \kappa\}.$$

Let G be $\mathbb{P}$-generic over some M. We argue that if $\alpha, \beta \in \dot{Z}_G$, then $(\dot{q}_\alpha)_G, (\dot{q}_\beta)_G$ are pairwise incompatbile. Since $\alpha, \beta \in \dot{Z}$, it follows that $p_\alpha, p_\beta \in G$. Therefore choose some $r \leq p_\alpha, p_\beta$ and some $\dot{q}_*$ such that $r \Vdash \dot{q}_* \leq \dot{q}_\alpha, \dot{q}_\beta$. It is then implied that $(r, \dot{q}_*) \leq (p_\alpha, \dot{q}_\alpha), (p_\beta, \dot{q}_\beta)$.

Since $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is κ-cc", it follows that $1_{\mathbb{P}} \Vdash |\dot{X}| < \kappa$.

It follows from Lemma 7.1.5 and $\mathbb{P}$ being κ-cc that there is some β such that $\Vdash_{\mathbb{P}} \dot{X} \subseteq \beta$. But $p_\beta \Vdash \beta \in \dot{X}$, which is a contradiction. □

7.2 Iterations with Finite Support

January 28, 2025

Definition 7.2.1. Let $\alpha \geq 1$. We inductively define a forcing notion $\mathbb{P}_\alpha$ as an *iteration of length* α if it is a set of α-sequences with the following properties:

1. If $\alpha = 1$, then for some forcing notion $\mathbb{Q}_0$,

   a) $\mathbb{P}_1$ is the set of all 1-sequences $\langle p(0)\rangle$ where $p(0) \in \mathbb{Q}_0$,

   b) $\langle p(0)\rangle \leq_1 \langle q(0)\rangle$ if and only if $p(0) \leq_{\mathbb{Q}_0} q(0)$.

2. If $\alpha = \beta + 1$, then $\mathbb{P}_\beta = \mathbb{P}_\alpha \restriction \beta = \{p \restriction \beta : p \in \mathbb{P}_\alpha\}$ is an iteration of length β, and there is a $\mathbb{P}_\beta$-name for a forcing notion $\dot{\mathbb{Q}}_\beta$ such that,

   a) $p \in \mathbb{P}_\alpha$ if and only if $p \restriction \beta \in \mathbb{P}_\beta$ and $\Vdash_{\mathbb{P}_\beta} p(\beta) \in \dot{\mathbb{Q}}_\beta$,

   b) $p \leq_{\mathbb{P}_\alpha} q$ if and only if $p \restriction \beta \leq_{\mathbb{P}_\beta} q \restriction \beta$ and $p \restriction \beta \Vdash_{\mathbb{P}_\beta} p(\beta) \leq q(\beta)$.

3. If α is a limit ordinal, then for every $\beta < \alpha$, $\mathbb{P}_\beta = \mathbb{P}_\alpha \restriction \beta = \{p \restriction \beta : p \in \mathbb{P}_\alpha\}$ is an iteration of length β and

   a) if $p \in \mathbb{P}_\alpha$ then $p \restriction \beta \in \mathbb{P}_\beta$ for all $\beta < \alpha$,

   b) $p \leq_{\mathbb{P}_\alpha} q$ if and only if $\forall \beta < \alpha, p \restriction \beta \leq_{\mathbb{P}_\beta} q \restriction \beta$.

Here we say that $\mathbb{P}_\alpha$ is defined from $\langle \dot{\mathbb{Q}}_\beta : \beta < \alpha\rangle$.

Note that this is not a complete description of what happens when α is a limit!

Definition 7.2.2. Let $\mathbb{P}_\gamma$ be an iteration of length γ.

1. $\mathbb{P}_\gamma$ is an *iteration with finite support* if for all limit ordinals $\alpha \leq \gamma$, $p \in \mathbb{P}_\alpha$ if and only if $\forall \beta < \alpha, p \restriction \beta \in \mathbb{P}_\beta$ for all but finitely many $\beta < \alpha$, $\Vdash_{\mathbb{P}_\beta} p(\beta) = 1$.

2. $\mathbb{P}_\gamma$ is an *iteration with* $\leq \kappa$*-support* if for all limit ordinals $\alpha \leq \gamma$, $p \in \mathbb{P}_\alpha$ if and only if $\forall \beta < \alpha, p \restriction \beta \in \mathbb{P}_\beta$ for all but $\leq \kappa$ many $\beta < \alpha$, $\Vdash_{\mathbb{P}_\beta} p(\beta) = 1$.

3. If $p \in \mathbb{P}_\alpha$, then $\{\beta < \alpha : \nVdash_{\mathbb{P}_\beta} p(\beta) = 1\}$ is the *support* of p.

Lemma 7.2.3. *Let κ be regular and let $\mathbb{P}_\alpha$ be an iteration with $\leq \kappa$-support. Then $\mathbb{P}_\alpha$ is $\leq \kappa$-closed.*

Proof. Let $\langle p_\xi : \xi < \kappa\rangle$ be a $\leq_{\mathbb{P}_\alpha}$-decreasing sequence. Let s_ξ be the support of p_ξ and let $s = \bigcup_{\xi<\kappa} s_\xi$. Of course, $|s| \leq \kappa$.

We will define a lower bound p_* such that the support of p_* is a subset of s. This will be done by defining $p_*(\beta)$ by induction on $\beta < \alpha$.

Let $p(0)$ be a lower bound of $\langle p_\xi(0) : \xi < \kappa\rangle$. Suppose we have defined $p_*(\gamma)$ for $\gamma < \beta$ and that $p_*(\gamma) \leq p_\xi(\gamma)$ for all $\xi < \kappa$. Then if $p_* \restriction \beta = \langle p_*(\beta) : \beta < \gamma\rangle$, then $p_* \restriction \beta$ forces that $\langle p_\xi(\beta) : \xi < \kappa\rangle$ has a lower bound, so we let $p_*(\gamma)$ be a name for this lower bound. □

Proposition 7.2.4. *If $\alpha = \beta + 1$ and $\mathbb{P}$ is an iteration of length α defined from $\langle \dot{\mathbb{Q}}_\beta : \beta < \alpha\rangle$. Then $\mathbb{P}_\alpha \cong \mathbb{P}_\beta \cdot \dot{\mathbb{Q}}_\beta$.*

Theorem 7.2.5. *Let κ be a regular uncountable cardinal. Let $\mathbb{P}_\alpha$ be an iteration with finite support such each iterand has the κ-chain condition. Then $\mathbb{P}_\alpha$ satisfies the κ-chain condition.*

Proof. We prove the theorem by induction on α. If $\alpha = \beta + 1$, then $\mathbb{P}_\alpha = \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$ has the κ-cc by induction because of Lemma 7.1.6.

Now we assume that α is a limit. For each $p \in \mathbb{P}_\alpha$, let $s(p)$ denote the support of p.

Let $W = \{p_\xi : \xi < \kappa\} \subseteq \mathbb{P}_\alpha$ be a subset of size κ. We consider two cases.

The first case is where $\operatorname{cf} \alpha \neq \kappa$. Then we argue that there is some $\beta < \alpha$ and some $Z \subseteq W$ of size κ such that $s(p) = \beta$ for all $p \in Z$: Let $\beta_\xi < \alpha$ be such that $s(p_\xi) \subseteq \beta_\xi$. First, if $\operatorname{cf} \alpha > \kappa$, then $\beta = \sup_{\xi<\kappa} \beta_\xi < \alpha$. (In this case, $Z = W$.) Otherwise, if $\operatorname{cf} \alpha < \kappa$, then then apply the Pigeonhole Principle to obtain $Z \subseteq W$ of cardinality κ and some β such that $\beta_\xi = \beta$ for all $\xi \in Z$.

Now we showed that $\{p \restriction \beta : \beta \in Z\} \subseteq \mathbb{P}_\beta$. Since $\mathbb{P}_\beta$ satisfies the κ-chain condition, there exist $p, q \in Z$ such that $p \restriction \beta, q \restriction \beta$ are compatible in $\mathbb{P}_\beta$. Choose some $r \in \mathbb{P}_\beta$ such that $r \leq p \restriction \beta, q \restriction \beta$. Then let $r^* \in \mathbb{P}_\alpha$ be a condition such that $r^*(\gamma) = r(\gamma)$ for $\gamma < \beta$ and $r^*(\gamma)$ is a name for the trivial condition for all $\gamma \geq \beta$.

The second (more difficult) case is where $\operatorname{cf} \alpha = \kappa$. Let $D \subseteq \alpha$ be a club of order-type κ enumerated as $D = \langle \alpha_\xi : \xi < \kappa\rangle$. Let $C \subseteq \kappa$ be the closed unbounded set of η such that $s(p_\xi) \subseteq \alpha_\eta$ for all $\xi < \eta$. By Fodor's

Theorem, there is a stationary set $S \subseteq C$ and some $\gamma < \kappa$ such that for all $\xi \in S$, $s(p_\xi) \cap \alpha_\xi \subseteq \alpha_\gamma$.

Now we consider the set $\{p_\xi \restriction \alpha_\gamma : \xi \in S\}$. This is a subset of $\mathbb{P}_{\alpha_\gamma}$ of size κ. Since $\mathbb{P}_{\alpha_\gamma}$ is κ-cc there are $\xi, \eta \in S$ with $\gamma < \xi < \eta$ such that $p_\xi \restriction \alpha_\gamma$ and $p_\eta \restriction \alpha_\gamma$ are compatible. Let $q \in \mathbb{P}_{\alpha_\gamma}$ be such that $q \leq p_\xi \restriction \alpha, p_\eta \restriction \alpha_\gamma$.

Then define

$$r(\beta) = \begin{cases} q(\beta) & \text{if } \beta < \alpha_\gamma \\ p_\xi(\beta) & \text{if } \alpha_\gamma \leq \beta < \alpha_\eta \\ p_\eta(\beta) & \text{if } \alpha_\eta \leq \beta < \alpha. \end{cases}$$

We can check that $r \in \mathbb{P}_\alpha$ and that $r \leq p_\xi, p_\eta$. This proves that W is not an antichain. $\square$

7.3 A Short Digression on Suslin Trees

Fact 7.3.1. *The linear order* $(\mathbb{R}, <)$ *is the unique linear order* L *such that:*

1. *L is a dense linear order without endpoints,*
2. *L has a countable dense subset,*
3. *L has the least upper bound property.*

Definition 7.3.2. A *tree* is a partially ordered set T with an order $<_T$ such that for every $x \in T$, $\{y \in T : y \leq x\}$ is well-ordered.

- The α^{th} *level* of T is the set $T_\alpha = \{x \in T : \text{ot}\{y \in T : y < x\} = \alpha\}$.
- The *height* of T is $\sup\{\alpha : \exists x \in T, \text{ot}\{y \in T : y < x\} = \alpha\}$.

Example 3. The set T of functions $f : \alpha \to \{0, 1\}$ for $\alpha < \omega_1$ is a tree where $g \leq_T f$ if $g = f \restriction \operatorname{dom} g$. If $\operatorname{dom} f = \alpha$, then f is in the α^{th} level of T. T has height ω_1.

Definition 7.3.4. An *antichain* of a tree T is a subset $A \subseteq T$ such that for all distinct $s, t \in A$, $s \not\leq_T t$ and $t \not\leq_T s$.

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Definition 7.3.5. A *Suslin tree* T is a tree such that:

1. T has height ω_1,

2. every branch of T is at most countable,

3. every antichain of T is at most countable.

Definition 7.3.6. The *Suslin hypothesis* is the statement that there are no Suslin trees.[1]

Theorem 7.3.7. *There is a forcing* $\mathbb{P}$ *such that* $\Vdash_{\mathbb{P}}$ "*There is a Suslin tree*". *Hence* $\mathrm{Con}(\mathsf{ZFC}) \implies \mathrm{Con}(\mathsf{ZFC} \wedge \neg\mathsf{SH})$.

Proof. Let $\mathbb{P}$ be a poset of T such that for some $\alpha < \omega_1$:

1. T consists of functions $t : \beta \to \omega$ for $\beta < \alpha$,

2. T is a tree under the ordering $t_1 \leq_T t_2$ if and only if $t_2 \restriction \operatorname{dom} t_1 = t_1$,

3. T has height α,

4. T is at most countable,

5. if $\beta + 1 < \alpha$ and $t \in T$ with $t : \beta \to \omega$, then $t^\frown\langle \beta, n\rangle \in T$ for all $n < \omega$,

6. if $\beta < \alpha$ and $t : \beta \to \omega$ is in T, then for all γ with $\beta \leq \gamma < \alpha$, there is some $s : \gamma \to \omega$ in T such that $t \leq_T s$.

Let $T_1 \leq_{\mathbb{P}} T_2$ if and only if $\exists \alpha < \operatorname{hgt}(T_1), T_2 = \{t \restriction \alpha : t \in T_1\}$.

Observe that if $T_1, T_2 \in \mathbb{P}$ are compatible, then either $\operatorname{hgt}(T_1) \leq \operatorname{hgt}(T_2)$ or else $\operatorname{hgt}(T_2) \leq \operatorname{hgt}(T_1)$.

Lemma 7.3.8. $\mathbb{P}$ *is countably closed.*

Proof. Suppose that $\langle p_n : n < \omega\rangle$ is $\leq_{\mathbb{P}}$-decreasing. (We are switching the notation a bit.) We claim that $\bar{p} = \bigcup_{n<\omega} p_n$ is a condition in $\mathbb{P}$. Let $\alpha_n = \operatorname{hgt}(p_n)$. Assume without loss of generality that the α_n's are strictly increasing.

We check each requirement:

1. immediate,

2. immediate,

[1] There is no indication that Suslin expected SH to be a theorem of ZFC.

3. we can consider α where $\alpha = \sup_{n<\omega}(p_n)$.

4. Countable unions of countable sets are countable.

5. Suppose $\beta + 1 < \alpha$. Then there is some $n < \omega$ such that $\beta + 1 < \alpha_n$, so the requirement is fulfilled because p_n is a condition.

6. Supposing $\beta < \alpha$ and γ such that $\beta \leq \gamma < \alpha$, we have α_n such that $\alpha_n \in (\gamma, \alpha_n)$. Then the requirement is fulfilled because p_n is a condition.

□

Now we know that $\aleph_1$ is preserved by $\mathbb{P}$. Then next step is to show that if G is $\mathbb{P}$-generic over V, then $\bigcup G$ is a Suslin tree.

Lemma 7.3.9. *If G is $\mathbb{P}$-generic over V, then $\bigcup G$ has height ω_1.*

Proof. It is enough to that if $\delta < \omega_1$, then

$$D_\delta := \{p \in \mathbb{P} : \operatorname{hgt} p \geq \delta\}$$

is dense. We argue that D_δ is dense by induction on $\delta < \omega_1$.

Of course, $\delta = 0$ is not a problem.

Suppose that $\delta = \delta' + 1$ where δ' is a successor. Fix $r \in \mathbb{P}$ of height δ'. Then let p be defined to by r together with $t^\frown\langle\delta, n\rangle$ for all $t \in r$ and $n < \omega$. It can be checked that all requirements are met.

Suppose that $\delta = \delta' + 1$ where δ' is a limit. Let r be a condition of height δ'. Then for each $t \in r$ we can inductively find a branch containing t of height δ': Take a sequence $\langle\delta_k : k < \omega\rangle$ converging to δ'. Take an $\leq_r$ increasing sequence $\langle t_k : k < \omega\rangle$ such that t_k has height δ_k. Then let u be the limit of the t_k's. Include all of these u's to build p.

Now suppose that δ is a limit. Let $\langle\delta_k : k < \omega\rangle$ be a sequence of ordinals converging to δ. Let $\langle p_k : k < \omega\rangle$ be a $\leq_{\mathbb{P}}$-decreasing sequence of conditions such that $p_k \in D_{\delta_k}$. Then $\bigcup_{k<\omega} p_k$ is a condition in D_δ. □

Now we need to show that the generic tree has no maximal antichains. To do this, we will prove a couple preliminary points.

If $A \subseteq T$ is an antichain, we say that it is *maximal* if it is maximal in the usual sense.

Lemma 7.3.10. *If $A \subseteq p$ is maximal as a subset of p and there is some $\beta < \mathrm{hgt}(p)$ such that $\mathrm{hgt}(t) \leq \beta$ for all $t \in A$, then for all $q \leq p$, A is maximal in q.*

Proof. Let $\mathrm{hgt}(p) = \alpha$. Use the fact that any element of q has a predecessor in p . □

Lemma 7.3.11. *If α is a countable limit ordinal and $A \subseteq p$ is maximal, then there is some $q \leq p$ of height $\alpha + 1$ such that A is maximal in q.*

Proof. Use the same trick from the density argument to build q such that every $u \in q$ at level α is above some element of A. □

Lemma 7.3.12. *$\bigcup G$ does not have maximal antichains.*

Proof. Now suppose $p \Vdash$ "$\dot{A}$ is an antichain". Our plan is to find $q \leq p$ and some A' such that A' is maximal in q and $q \Vdash A' \subseteq \dot{A}$. Then we can apply the previous two lemmas.

Then build a sequence of conditions $\langle p_k : k < \omega \rangle$ below p in such a way that:

$$\forall s \in p_k, \exists t_s \in p_{k+1}, s \| t_s, p_{k+1} \Vdash t_s \in \dot{A}.$$

Then let q be a lower bound of the p_k's. □

Finally, we can conclude the theorem by showing that the generic tree does not have any chains of length ω_1.

Lemma 7.3.13. *If G is $\mathbb{P}$-generic over V, then $\bigcup G$ has no chains of height ω_1.*

Proof. If $\bigcup G$ had an uncountable chain b, then we could build an uncountable antichain: For each $t \in b$, if $t^\frown \epsilon \in b$, include $t^\frown |1 - \epsilon|$ in the antichain. □

This completes the proof. □

Fact 7.3.14. *$L \models \neg \mathsf{SH}$.*

7.4 Proof of the Consistency of MA with the Failure of CH

Definition 7.4.1. $\mathsf{MA}(\kappa)$ is the statement that if $\mathbb{P}$ has the countable chain condition and $\mathcal{D} = \{D_\alpha : \alpha < \kappa\}$ is a collection of dense subsets of $\mathbb{P}$, then there is a filter $F \subseteq \mathbb{P}$ such that for all $\alpha < \kappa$, $D_\alpha \cap F \neq \emptyset$. *Martin's Axiom*, denoted MA (without reference to a cardinal) is the statement that $\mathsf{MA}(\kappa)$ holds for all $\kappa < 2^{\aleph_0}$.

Proposition 7.4.2. $\mathsf{MA}(\aleph_0)$ *is true (and does not even need* $\mathbb{P}$ *to have the countable chain condition).*

Proof. Let $\langle D_n : n \in \mathbb{N}\rangle$ enumerate the family of dense sets in question. Define a $\leq$-decreasing sequence $\langle p_n : n \in \mathbb{N}\rangle$ so that $p_n \in D_n$. Choose $p_0 \in D_0$. Given p_n and D_{n+1}, since D_{n+1} is dense there is some $p_{n+1} \leq p_n$, $p_{n+1} \in D_n$. Then $G := \{q \in \mathbb{P} : \exists n \in \mathbb{N}, p_n \leq q\}$ works. □

Corollary 7.4.3. CH *implies* MA.

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Proposition 7.4.4. $\mathsf{MA}(2^{\aleph_0})$ *is false.*

Proof. Let $\mathcal{P}(\mathbb{N})$ be enumerated as $\langle A_\alpha : \alpha < 2^{\aleph_0}\rangle$. Let χ_{A_α} be the characteristic function of A_α, i.e., $\chi_A(n) = 1$ if $n \in A$ and $\chi_A(n) = 0$ if $n \in \mathbb{N} \setminus A$. We let $\mathbb{P}$ be the *Cohen forcing*, that is

$$\mathbb{P} = \{p \colon F \to \{0,1\} : F \subseteq \mathbb{N} \text{ is finite}\}, \qquad q \leq p :\Leftrightarrow q \supseteq p.$$

We let for $n \in \mathbb{N}$, $E_n = \{p \in \mathbb{P} : n \in \mathrm{dom}(p)\}$. For $\alpha < 2^{\aleph_0}$ we let

$$D_\alpha = \{p \in \mathbb{P} : \exists n \in \mathrm{dom}(p), p(n) \neq \chi_{A_\alpha}(n)\}$$

We let $\mathcal{D} = \{D_\alpha : \alpha < 2^{\aleph_0}\} \cup \{E_n : n \in \mathbb{N}\}$. It is easy to see that each E_n and each D_α is dense in $\mathbb{P}$. For a contradiction, we assume that G is $\mathcal{D}$-generic. Then $\bigcup G \colon \mathbb{N} \to 2$ Then $\bigcup G \neq \chi_{A_\alpha}$, $\alpha < 2^{\aleph_0}$. But such a G cannot exist. □

Definition 7.4.5. A tree T of height ω_1 is *normal enough* if:

1. T has a unique root,
2. each level of T is at most countable,

3. for all $x \in T$ and $\alpha < \omega_1$ above the height of x, there is a descendent of x of height α,

4. nodes at limit points are determined by their predecessors.

Proposition 7.4.6. *If there is a Suslin, then there is a normal enough Suslin tree.*

Proof. (This will be more of a sketch.) Let T be a Suslin tree. This means it has height ω_1 and countable levels (because an uncountable level would be an uncountable antichain).

For each $x \in T$, let T_x be the set of descendants of x. Let $T_1 = \{x \in T : |T_x| > \omega\}$. We can observe that T_1 is a Suslin tree. For example, if it has countable height, then it can be argued that T has uncountable height. This gives us the fourth condition since the original tree has countable levels.

Now for every limit level $\alpha < \omega_1$ of T, take only on element of the level per sequence defining it. This gives T_2, which has the fifth condition while preserving the fourth condition. □

Proposition 7.4.7. $\mathsf{MA} \wedge \neg\mathsf{CH} \implies \mathsf{SH}$.

Proof. Suppose that there is a normal enough Suslin tree T of height ω_1. (We assume it is normal because of the previous proposition.)

Let $(\mathbb{P}, \leq_{\mathbb{P}}) = (T, \geq_T)$ (where we are flipping the ordering). Then $\mathbb{P}$ has the countable chain condition as a poset. Let $D_\alpha = \{t \in \mathbb{P} : \operatorname{dom} t \geq \alpha\}$. Then each D_α is dense by normality. A filter intersecting these is an uncountable branch. □

Theorem 7.4.8. $\mathrm{Con}(\mathsf{ZFC}) \implies \mathrm{Con}(\mathsf{MA} \wedge \neg\mathsf{CH})$.

Lemma 7.4.9. *The following are equivalent:*

1. MA,

2. $\mathsf{MA}(< 2^{\aleph_0})$, *i.e. for all ccc posets $\mathbb{P}$ such that $|\mathbb{P}| < 2^{\aleph_0}$, and for all sequences of open dense sets $\langle D_\alpha : \alpha < \kappa \rangle$ with $\kappa < 2^{\aleph_0}$, there is some filter $F \subseteq \mathbb{P}$ such that for all $\alpha < \kappa$, $D_\alpha \cap F \neq \emptyset$.*

Proof. The definition of MA is strictly more general than the second statement, so we only need to take the second statement as a premise and then prove MA.

Take some ccc poset $\mathbb{P}$ and a sequence $\langle D_\alpha : \alpha < \kappa\rangle$ of dense subsets of $\mathbb{P}$ with $\kappa < 2^{\aleph_0}$. Fix each $\alpha < \kappa$, let $A_\alpha \subseteq D_\alpha$ be a maximal antichain. (We used this early in our study of forcing.)

We construct a subset $\mathbb{Q} \subseteq \mathbb{P}$ of cardinality $< 2^{\aleph_0}$ such that $A_\alpha \subseteq \mathbb{Q}$ for all $\alpha < \kappa$ and such that for all $p, q \in \mathbb{Q}$, if there is some $r' \in \mathbb{P}$ with $r' \leq p, q$, then there is some $r \in \mathbb{Q}$ such that $r \leq p, q$. We can do this by defining a $\subseteq$-increasing sequence $\langle \mathbb{Q}_n : n < \omega\rangle$ by letting $\mathbb{Q}_0 = \bigcup_{\alpha<\kappa} A_\alpha$, and by letting $\mathbb{Q}_{n+1}$ to be including some $r \leq p, q$ for each $p, q \in \mathbb{Q}_n$ where we can find such an r. Then each $\mathbb{Q}_n$ has cardinality κ and we can let $\mathbb{Q} = \bigcup_{n<\omega} \mathbb{Q}_n$.

For each $\alpha < \kappa$, let $E_\alpha = \{q \in \mathbb{Q} : \exists p \in A_\alpha, q \leq p\}$. Then E_α is dense in $\mathbb{Q}$: Fix some $r \in \mathbb{Q}$. Since A_α is a maximal antichain, there is some $q \in A_\alpha$ such that there is some $p' \leq r, q$, so we let $p \in \mathbb{Q}$ be such that $p \leq r, q$. So $p \in E_\alpha$.

Now we can apply the assumption and let $F \subseteq \mathbb{Q}$ be generic for the sequence $\langle E_\alpha : \alpha < \kappa\rangle$. Let $F' = \{p \in \mathbb{P} : \exists q \in F, q \leq p\}$. Then F' is a filter on $\mathbb{P}$ such that $F' \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$. $\square$

Proof of Theorem 7.4.8. Specifically, we will prove that if GCH holds up to κ and $\kappa > \aleph_1$ is a regular uncountable cardinal, then there is a ccc forcing $\mathbb{P}$ such that in the extension, MA holds and $2^{\aleph_0} = \kappa$.

Fix a function $\pi : \kappa \to \kappa \times \kappa$ be a surjection such that if $\pi(\alpha) = (\beta, \gamma)$ then $\beta \leq \alpha$. For example, let $g : \kappa \to \kappa \times \kappa \times \kappa$ be a surjection. If $g(\xi) = \langle \eta, \gamma, \delta\rangle$, let $f(\xi)$ be $\langle \eta, \gamma\rangle$ if $\eta \leq \xi$ and $\langle 0, 0\rangle$ if $\eta > \xi$.

Our poset $\mathbb{P}$ will be a finite support iteration defined from $\langle \dot{\mathbb{Q}}_\alpha : \alpha < \kappa\rangle$ which we will define presently by induction: Suppose we have defined $\langle \dot{\mathbb{Q}}_\beta : \beta < \alpha\rangle$ so far. As an inductive hypothesis, assume that for all $\beta < \alpha$, $\Vdash_{\mathbb{P}_\beta} |\dot{\mathbb{Q}}_\beta| < \kappa$. Let $\pi(\alpha) = (\beta, \gamma)$. Because of GCH, we have that $|\mathbb{P}_\beta| < \kappa$. Assume we have a fixed enumeration of $\mathbb{P}_\beta$. Let $\dot{\mathbb{Q}}$ be the γ^{th} name from $\mathbb{P}_\alpha$ for a poset of cardinality $< \kappa$. Then let $\dot{\mathbb{Q}}_\alpha$ be a name for $\dot{\mathbb{Q}}$ if it is forced to be ccc or a name for the trivial poset otherwise.

Lemma 7.4.10. *If $\lambda < \kappa$ and $X \subseteq \lambda$ is in $V[G]$ then $X \in V[G_\alpha]$ for some $\alpha < \kappa$.*

Proof. Let $\dot{X}$ be a $\mathbb{P}$-name for X. For each $\xi < \lambda$, consider the open dense set $D_\xi = \{p \in \mathbb{P} : p \Vdash \xi \in \dot{X} \vee p \Vdash \xi \notin \dot{X}\}$ and take a maximal antichain $A_\xi \subseteq D_\xi$. Because $\mathbb{P}$ is ccc, there is some $\alpha < \kappa$ that for all $\xi < \lambda$ and all

$p \in A_\xi$, the support of p is a subset of α. Therefore we can rewrite $\dot{X}$ as a $\mathbb{P}_\alpha$-name. □

Lemma 7.4.11. *Let $(\mathbb{Q}, <_{\mathbb{Q}}) \in V[G]$ and let $\mathcal{D} \in V[G]$ be such that $(\mathbb{Q}, <_{\mathbb{Q}})$ is a ccc poset, $|\mathbb{Q}| < \kappa$, and $|\mathcal{D}| < \kappa$ (and $\mathcal{D}$ is a sequence of dense subsets of $\mathbb{Q}$). Then in $V[G]$ there is a $\mathcal{D}$-generic filter on $\mathbb{Q}$.*

Proof. By the previous lemma, we can choose some $\beta < \kappa$ such that both $(\mathbb{Q}, <_{\mathbb{Q}})$ and $\mathcal{D}$ are in $V[G_\beta]$. Take a V_β-name $\dot{\mathbb{Q}}$ in the extension by $\mathbb{P}_\beta$. Assume that $\dot{\mathbb{Q}}$ is the γ^{th} such name. Let α be such that $\pi(\alpha) = (\beta, \gamma)$. Since $\mathbb{Q}$ is ccc in $V[G]$, it is ccc in $V[G_\alpha]$. Therfore $Q = (\dot{\mathbb{Q}}_\alpha)_{G_\alpha}$.

In $V[G_{\alpha+1}]$, there is a generic filter H on $\mathbb{Q}$ over $V[G_\alpha]$. We have $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$. Hence H meets every dense subset of subset of Q in $V[G_\alpha]$, which includes every $D \in \mathcal{D}$. □

Lastly, let use argue the $\mathbb{P}$ forces $2^{\aleph_0} = \kappa$. On one hand, $2^{\aleph_0}$ by the fact that $|\mathbb{P}| \leq \kappa$. On the other hand, suppose $\mathbb{Q}$ is the forcing $\mathrm{Add}(\omega)$. For any $X \subseteq \{0,1\}^\omega$ of size $< \kappa$, let $\mathcal{D}_X = \{D_g : g \in X\}$ where $D_g = \{q \in \mathbb{Q} : q \not\subseteq g\}$. Any filter meeting these dense sets adds a subset of ω that is not included in X. Therefore $2^{\aleph_0} \geq \kappa$. □

Corollary 7.4.12. SH *is independent of* ZFC.

Bibliography