Course Notes for Set Theory and Independence Proofs

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Acknowledgments

This is a work in progress. So far much of the material is taken from Kunen and Jech's textbooks.

Contents

Part I

Working with Objects in Set Theory

In this chapter, we will discuss the ways in which objects can be manipulated from the perspective of set theory. This includes a precise formulation of the axioms and a survey of what can be done with them on a relatively simple level. We are assuming familiarity with the basics of mathematical logic, including the notions of languages, formulas, sentences, and so on, as well as Gödel's Incompleteness Theorems.

Chapter 1

The Axioms of Zermelo-Fraenkel Set Theory

To start with, we will develop a familiarity with the Zermelo-Fraenkel axioms and the way they can be used to understand relatively simple objects.

1.1 Stating the Axioms and Beginning to Work with Them

Without further ado, we state the axioms.

1. **Extensionality:** $\forall x \forall y(\forall z(z \in x \iff z \in y) \implies x = y)$. (Sets are uniquely defined by their elements.)

2. **Foundation:** $\forall x(\exists y(y \in x) \implies \exists y(y \in x \wedge \neg\exists z(z \in x \wedge z \in y)))$. (Every nonempty set has an $\in$-minimal element.)

3. **Comprehension Scheme:** Let φ be any formula whose free variables are among $\{x, z, w_1, w_2, \ldots, w_n\}$. Then

$$\forall z \forall w_1 \ldots \forall w_n \exists y \forall x(x \in y \iff (x \in z \wedge \varphi)).$$

   (Definable subsets of sets are sets.)

4. **Pairing:** $\forall x, y \exists z(x \in z \wedge y \in z)$. (For any two sets, there is a set with both of those sets as elements.)

5. **Union:** $\forall F \exists U \forall Y, x(x \in Y \wedge Y \in F \iff x \in U)$. (If we have a set which is a family of sets, its union is a set.)

6. **Replacement Scheme:** Let φ be any formula whose free variables are among $\{x, y, A, w_1, \ldots, w_n\}$. Then

$$\forall A \forall w_1 \ldots \forall w_n (\forall x \in A \exists! y \varphi \implies \exists Y \forall x \in A \exists y \in Y \varphi).$$

   (Images of sets under functions are sets.)

We introduce some notation to make the remaining axioms easier to read.

- $x \subseteq y \iff \forall z \in x(z \in y)$.
- $x = \emptyset \iff \neg \exists y(y \in x)$.
- $y = \mathrm{succ}(x) \iff \forall z \in y(z = x \vee z \in x)$.
- $y = v \cap w \iff \forall x(x \in y \iff (x \in v \wedge x \in w))$.
- $\mathrm{singleton}(x) \iff (\exists y \in x \wedge \exists y \forall z \in x(z = y))$.

7. **Infinity:** $\exists x(\emptyset \in x \wedge \forall y \in x(\mathrm{succ}(y) \in x))$. (There exists an infinite set, and in particular, the set of natural numbers is a set.)

8. **Powerset:** $\forall x \exists y(z \subseteq x \iff z \in y)$. (Every set has a power set.)

9. **Choice:** $\forall F(\forall x \in F(x \neq \emptyset) \wedge \forall x, y \in F(x \neq y \implies x \cap y = \emptyset)) \implies \exists C(\forall x \in F(\mathrm{singleton}(C \cap x))))$ (Any nonempty family of sets has a choice function.)

We may want to consider various sub-collections of the axioms.

Definition 1.1.1. We can define subsystems of these axioms.

- ZFC refers to all nine axioms.
- ZF refers to the first eight axioms (excluding choice).
- If X is a sub-collection of these axioms, then $X - P$ is that set minus the powerset axiom.

- And there are many more variations.

Note that we do not explicitly have an axiom asserting the existence of an empty set, but the infinity axiom implies that there is an empty set.

Here we can give a simple example of the usage of these axioms to get something "obvious".

Proposition 1.1.2. ZF *proves that any pair of sets has a union.*

Proof. Let x and y be sets. We want to show that

$$\mathsf{ZF} \vdash \text{“}\exists z \forall u (u \in z \iff (u \in x \vee u \in y))\text{”}.$$

Then by pairing, there is a set w such that $\{x, y\} \subseteq w$. (Notice the actual formulation of the pairing axiom that we are using.) Using the formula $v = x \vee v = y$, we can use the comprehension axiom to ensure that we have a set equal to $\{x, y\}$. Then the union axiom gives us the set $x \cup y = \bigcup\{x, y\}$.

Observe that the formula above uniquely defines the union by the extension axiom. □

1.2 Orders and Their Formalizations

The next task as to develop the notion of orderings. This will help us build towards ordinal and cardinal numbers. Let's clarify the notions that we want.

Definition 1.2.1. A *partial order* is a relation $\leq$ on a set X with the following properties:

1. $\forall a \in X, a \leq a$ (reflexivity),
2. if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry),
3. if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).

We refer to X is the *underlying set* of $\leq$, and we may refer to X itself as a partial order and use the notation $(X, \leq_X)$.

Example 2. Examples of partial orders:

- Let X be the set of closed subsets of $\mathbb{R}$ under the usual Euclidean topology. For $A, B \in X$, we let $A \leq B$ if and only if $A \subseteq B$. We could just as easily let $A \leq B$ if and only if $A \supseteq B$.

- Let $X = \{0,1\}^{<\omega}$, the set of finite sequences of 0's and 1's. Let $s \leq t$ if and only if t end-extends s, i.e. if $\operatorname{dom} s = \{0, \dots, n\}$ then $s = t \restriction \{0, \dots, n\}$.

Definition 1.2.3. A *linear order* or *total order* is a partial order $(L, \leq_L)$ such that for all $a, b \in L$, one of the following three hold: $a = b$, $a \leq_L b$, or $a \geq_L b$.

Example 4. $\mathbb{R}$ under its standard ordering, or $\omega + 1 \cong \mathbb{N} \cup \{\infty\}$.

Definition 1.2.5. Let $(L, \leq)$ be a linear ordering. Then $(L, \leq)$ is a *well-ordering* if every nonempty subset $X \subseteq L$ has a $\leq$-least element.

Example 6. Any finite linear ordering is a well-ordering, and so are $\mathbb{N}$ and $\omega + 1$. Both $\mathbb{Q}$ and $\mathbb{R}$ are *not* well-orderings.

Now that we have some definitions in mind, let's show that they can be formalized in terms of ZFC.

Definition 1.2.7. Given sets x and y, the *ordered pair* $\langle x, y\rangle$ is the set $\{\{x\}, \{x, y\}\}$.

Proposition 1.2.8. $\forall x \forall y \forall x' \forall y' (\langle x, y\rangle = \langle x', y'\rangle \iff (x = x' \wedge y = y'))$. *(In other words, ordered pairs define a pair in a specific order.)*

Proof. We will show the forward direction by considering two cases. The other direction is fairly simple.

Suppose $x = y$. Then $\langle x, y\rangle = \{\{x\}\} = \{\{x'\}, \{x', y'\}\}$. By extension we have $\{x\} = \{x'\} = \{x', y'\}$. The first equality gives $x = x'$ and the second gives $x' = y'$. Transitivity of equality gives $x = y = x' = y'$ and so $x = x'$ and $y = y'$.

Suppose $x \neq y$. Then $\langle x, y\rangle = \{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$ (in which the apparent two-element sets are actual two-element sets). Therefore $\{x, y\} = \{x', y'\}$ and $x' \neq y'$ and $\{x'\} = \{x\}$. Therefore $x = x'$ and $y = y'$. $\square$

Proposition 1.2.9. *The Cartesian product of two sets is a set.*

Proof. Given sets A and B, we want to show that $\{\langle x, y\rangle : x \in A \wedge y \in B\}$ is a set.

If $x \in A$ and $y \in B$, then observe that $\langle x, y\rangle = \{\{x\}, \{x, y\}\} \in \mathcal{P}(\mathcal{P}(A \cup B))$ (where $\mathcal{P}$ is indicating the powerset operation). To see this, observe that $\{x\} \in \mathcal{P}(A) \subseteq \mathcal{P}(A \cup B)$ and $\{x, y\} \in \mathcal{P}(A \cup B)$.

Then apply comprehension to obtain

$$C = \{z \in \mathcal{P}(\mathcal{P}(A \cup B)) : \exists x \exists y (x \in A \wedge y \in B \wedge z = \langle x, y\rangle\}$$

and we are done. □

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1.3 Ordinals and Transfinite Induction

Proposition 1.3.1. *The following are equivalent:*

1. *L is a well-ordering, i.e. all subsets have a minimal element.*
2. *L does not contain any infinite descending sequences.*

Proof. *(1) $\implies$ (2)*: Suppose contrapositively that $\langle x_n : n < \omega\rangle$ is an infinite descending sequence through L, i.e. $x_0 > x_1 > x_2 > \ldots$. Then this is a subset of L without a minimal element. *(2) $\implies$ (1)*: Suppose contrapositively that $X \subseteq L$ has no minimal element. Let $x_0 \in X$ be arbitrary, then inductively choose $x_1 < x_0$ and so on. □

But how were we allowed to use induction there? We should to work towards some justification of induction, which is the next task. In order to do this, we first establish some facts about well-ordered sets using our starting definition.

Definition 1.3.2. If L is well-ordered and $x \in L$, then $\{y \in L : y <_L x\}$ is an *initial segment* of L.

Proposition 1.3.3. *No well-ordered set can be isomorphic to an initial segment of itself.*

Proof. First we argue that if $(W, <)$ is well-ordered and $f : W \to W$ is strictly increasing (i.e. $x < y$ implies $f(x) < f(y)$) then $f(x) \geq x$ for all $x \in W$: If $X = \{x \in W : f(x) < x\}$ is nonempty then it has a least element

z. If $w = f(z)$, then $f(w) = f(f(z)) < f(z) = w < z$, which contradicts minimality of z.

Now if $(W, <)$ were isomorphic to an initial segment $\{x : x < u\}$ via f, then $f(u) < u$, which is not possible. □

Proposition 1.3.4. *The only automorphism of a well-ordered set is the identity.*

Proof. Given f, apply Proposition 1.3.3 to both f and f^{-1}. □

Proposition 1.3.5. *If W_1 and W_2 are well-orderings and $f, g : W_1 \to W_2$ are isomorphisms, then $f = g$.*

Proof. If x is the W_1-least element such that $f(x) \neq g(x)$, then you can show that either f or g "misses a spot". □

Proposition 1.3.6. *If W_1 and W_2 are well-ordered, then exactly one of the following three cases will hold:*

1. *$W_1 \cong W_2$,*
2. *W_1 is isomorphic to an initial segment of W_2,*
3. *W_2 is isomorphic to an initial segment of W_1.*

Proof. The previous proposition shows that the cases are mutually exclusive.

If $x \in W_i$ for $i \in \{1, 2\}$ then let $W_i(x)$ denote the initial segment $\{y \in W_i : y <_{W_i} x\}$. Define

$$f = \{(x, y) : W_1(x) \cong W_2(y)\}.$$

We argue that f is a one-to-one function: The fact that it is a function and that fact that it is one-to-one both follow from Proposition 1.3.3.

If $\operatorname{dom} f = W_1$ and $\operatorname{range} f = W_2$ then $W_1 \cong W_2$: Using Proposition 1.3.5, we can argue that the isomorphisms from the $W_1(x)$'s to the $W_2(y)$'s can be unioned up to get an isomorphism from W_1 to W_2.

If $\operatorname{range} f \neq W_2$, then we can argue that W_1 is isomorphic to an initial segment of W_2. Observe that $\operatorname{range} f$ is downwards closed. Hence if $\operatorname{range}(f) \neq W_2$ and y is the least element of $W_2 \setminus \operatorname{range}(f)$ then $\operatorname{range}(f) = W_2(y)$. Then it must be the case that $\operatorname{dom}(f) = W_1$ since otherwise we would have $(x, y) \in f$ where x is the least element of $W_1 \setminus \mathrm{dom}(f)$, but then it would be absurd that this cannot be extended.

If $\mathrm{dom}\, f \neq W_1$, then we can similarly argue that W_2 is isomorphic to an initial segment of W_1. $\square$

Definition 1.3.7. A set X is *transitive* if $\forall y \in X (z \in y \implies z \in X)$.

Example 8. Observe that $\emptyset$ is vacuously transitive. The set $\mathcal{P}(\mathcal{P}(\emptyset))$ is also transitive. However, the set $\{\emptyset, \{\{\emptyset\}\}\}$ is not transitive.

Definition 1.3.9. We say that α is an *ordinal* if it α is transitive and well-ordered by $\in$.

Definition 1.3.10. A *successor* ordinal is an ordinal of the type $\alpha = \beta \cup \{\beta\} := \beta + 1$. A *limit* ordinal α takes the form $\alpha = \bigcup \alpha$.

Example 11. Every natural number can be represented as an ordinal: $0 = \emptyset$, $1 = \{\emptyset\}$, $2 = \{\emptyset, \{\emptyset\}\}$, $3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, etc. We write the set of natural numbers as the limit ordinal $\omega = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}, \ldots\}$. $\omega + 1 := \omega \cup \{\omega\}$ is an infinite successor.

Proposition 1.3.12. *The following hold:*

1. $0 := \emptyset$ *is an ordinal.*
2. *If* α *is an ordinal and* $\beta \in \alpha$*, then* β *is an ordinal.*
3. *If* α, β *are ordinals,* $\alpha \neq \beta$*, and* $\alpha \subseteq \beta$*, then* $\alpha \in \beta$*.*
4. *If* α *and* β *are ordinals, then either* $\alpha \subseteq \beta$ *or* $\beta \subseteq \alpha$*.*
5. *If* α *and* β *are ordinals, then exactly one of the following hold:* $\alpha = \beta$*,* $\alpha \in \beta$*, or* $\beta \in \alpha$*.*
6. *If* X *is a set of ordinals, then* $\bigcup X$ *is an ordinal.*
7. *Every ordinal is either a successor or a limit.*

Proof. *(1)* is immediate. *(2)* will actually follow from *(3)* and *(4)*.

(3): Let γ be the $\in$-minimal element of $\beta \setminus \alpha$. If $\delta \in \gamma$ then $\delta \in \alpha$ by minimality. If $\delta \in \alpha$ then $\delta \in \beta$ by $\alpha \subseteq \beta$. Hence $\alpha = \gamma$.

(4): It is immediate that $\alpha \cap \beta$ is an ordinal. Let $\alpha \cap \beta = \gamma$. Then if $\alpha \neq \beta$ and if $\gamma \neq \alpha, \beta$, then it follows that $\gamma \in \alpha, \beta$. This implies that $\gamma \in \gamma$, which is not possible (see homework).

(5): Follows from *(4)* and Proposition 1.3.6.

(6): If $\alpha \in \beta \in \bigcup X$, and $\gamma \in X$ is such that $\beta \in \gamma$, then $\alpha \in \gamma$ so $\alpha \in \bigcup X$. If $Y \subseteq X$ choose some $\alpha \in Y$. Then α is an ordinal, so either $Y \cap \alpha = \emptyset$, in which case α is minimal, or we choose a minimal element of $Y \cap \alpha$. Since we now know that all ordinals are comparable we are done. □

Theorem 1.3.13. *Every well-ordered set is isomorphic to a unique ordinal number.*

Proof. We already have uniqueness.

If W is a well-order and $x \in W$, then let $F(x)$ be the (unique) α (if it exists) such that $\{y \in W : y <_W x\} \cong \alpha$. To show that F is defined for all $x \in W$, suppose for contradiction that x is the $<_W$-least element of W for which W is undefined. Then we can argue that $\{y \in W : y <_W x\} \cong \bigcup_{y <_W x} F(y)$, which is an ordinal, so this is a contradiction.

Now we can argue that $F[W] = \{F(x) : x \in W\}$ is a set by the replacement schema. By the same argument as in the previous paragraph, $F[W]$ is an ordinal α and $F : W \cong \alpha$. □

Now we want to start reasoning about the ordinals as a whole.

Proposition 1.3.14. *There is no set of all ordinals.*

Proof. If X were the set of all ordinals, then it would itself be an ordinal α. But then we would have $\alpha \in \alpha$, and this is not possible (see homework). □

A *class* refers to a collection of sets that is definable with some parameters that is not itself necessarily a set. If a class is not a set, then it is referred to as a *proper class*. By the above proposition, the collection of ordinals is a class which we will denote ON.

There are ways of formalizing classes, such as Bernays-Gödel set theory, but we will not emphasize that here. We want to make sure that whenever we refer to a class, our statements are shorthand for statements made in terms of sets.

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Theorem 1.3.15 (Transfinite Induction)**.** *Suppose C is a subclass of the ordinals such that:*

1. *$0 \in C$ holds,*

2. *for all ordinals α, if $\beta \in C$ for all $\beta < \alpha$, then $\alpha \in C$.*

Then C equals the class of all ordinals.

Proof. This is equivalent to saying if C is a nonempty class of ordinals and $C \neq \mathrm{ON}$, then there is a minimal $\alpha \in \mathrm{ON} \setminus C$.

To see that this is the case, let $\beta \notin C$. Then $\beta \cap C$ is a set by separation. (Remember that our "set operations" on classes are just abbreviations.) If $\beta \cap C = \emptyset$ then β is our witness. Otherwise, by the foundation axiom, there is some minimal $\alpha \in \beta \cap C$, so $\alpha \cap C = \emptyset$. Then α is the witness. □

A *class function* is a class taking the form of a function, i.e. a class of ordered pairs.

Theorem 1.3.16 (Transfinite Recursion)**.** *F be a class function that is defined on all sets. Then there exists a unique class function G that is defined on all ordinals such that $\forall \alpha \in \mathrm{ON}(G(\alpha) = F(G \restriction \alpha)$.*

Proof. If G_1, G_2 are two functions satisfying this description, then we can prove that $\forall \alpha \in \mathrm{ON}, G_1(\alpha) = G_2(\alpha)$ by transfinite induction.

For existence, call g a *δ-approximation* if g is a (set) function with domain δ and
$$\forall \alpha < \delta(g(\alpha) = F(g \restriction \alpha)).$$
If g is a δ-approximation and g' is a δ'-approximation, then $g \restriction (\delta \cap \delta') = g' \restriction (\delta \cap \delta')$: This uses the fact that δ can be compared with respect to the ordering on ordinals. By transfinite induction, we can argue that there is a δ-approximation for each δ. Then let $G(\alpha)$ be the value $g(\alpha)$ where g is a δ approximation for some (equivalently, any) $\delta > \alpha$. □

Example 17 (Schweber)**.** Let $\mathbb{N}^\omega = \mathbb{N} \times \mathbb{N} \times \ldots$ be the countable product of the natural numbers with itself where the copies themselves are ordered as in the natural numbers. In other words, there is $\mathbb{N}_0$, $\mathbb{N}_1$, and so on where $\mathbb{N}_m < \mathbb{N}_n$ if $m < n$. Let α be the ordinal that is isomorphic to this ordering.

We will define a function $\alpha \to \omega$ by transfinite induction. For $\beta < \alpha$, if $p : \beta \to X$ is a function, let $F(p)(\beta) = 0$ if β is a limit ordinal and let $F(p)(\gamma) = p(\gamma) + 1$ if $\beta = \gamma + 1$.

Then the function G produced by transfinite inductions "mods out" the copies of $\mathbb{N}$.

We have essentially started doing ordinal arithmetic.

1.4 Ordinal Arithmetic

Recall that *successor operation*: $\beta = \mathrm{succ}(\alpha) = S(\alpha) \iff \forall z(z \in \beta \iff (z = \alpha \vee z \in \alpha))$.

Proposition 1.4.1. *Every ordinal is either a successor ordinal or a limit ordinal.*

Proof. We want to show that for all ordinals α, either there is some β such that $S(\beta) = \alpha$ or else $\alpha = \bigcup\{\beta : \beta \in \alpha\}$. (Note that if $\alpha = S(\beta)$, then $\bigcup\{\beta : \beta \in \alpha\} = \bigcup(\{\beta\} \cup \{\gamma : \gamma \in \beta\}) = \beta \neq \alpha$.)

Suppose that α is not a successor. We can see that $\bigcup\{\beta : \beta \in \alpha\} \subseteq \alpha$, so the task is to show that there are no missing points. For all $\beta \in \alpha$, $S(\beta) \neq \alpha$. This implies that $S(\beta) \in \alpha$, and so $\beta \in S(\beta) \subseteq \bigcup \alpha$. □

According to the definition, 0 counts as a limit ordinal, but it is usually best not to think of it as one.

From now on, if α and β are ordinals, we will (usually) write $\alpha < \beta$ for $\alpha \in \beta$. We will write $\sup_{\xi<\eta} \alpha_\xi$ for $\bigcup_{\xi<\eta} \alpha_\xi$. We are ready to define the rules of ordinal arithmetic.

Remember that we proved that every well-ordered set is isomorphic to an ordinal.

Definition 1.4.2. If X is well-ordered, the *order-type* of X is the unique ordinal α such that $(X, <_X) \cong \alpha$. It is often denoted $\mathrm{ot}(X)$.

Definition 1.4.3. Suppose α and β are ordinals. Let R be

$$\{\langle\langle\xi, 0\rangle, \langle\eta, 0\rangle\rangle : \xi < \eta < \alpha\} \cup \\ \cup \{\langle\langle\xi, 1\rangle, \langle\eta, 1\rangle\rangle : \xi < \eta < \beta\} \cup [(\alpha \times \{0\}) \times (\beta \times \{1\})].$$

Then $\alpha + \beta := \mathrm{ot}(\alpha \times \{0\} \cup \beta \times \{1\}, R)$.

Example 4. $\omega + 1 \neq 1 + \omega$.

Proposition 1.4.5. *Let α, β, γ be ordinals.*

1. $\alpha + 0 = \alpha$.

2. $\alpha + 1 = S(\alpha)$.

3. $\alpha + S(\beta) = S(\alpha + \beta)$.

4. *If β is a limit ordinal, then $\alpha + \beta = \sup\{\alpha + \xi : \xi < \beta\}$.*

5. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

Proof. For *1.*, check that the definition of R above yields some empty sets as components. For *2.*, note that $S(\alpha)$ adds one extra point, which essentially gives *3.* as well.

For *4.*, we can inductively define an order-isomorphism as follows: Remember that $\sup\{\alpha+\xi : \xi < \beta\} = \bigcup_{\xi<\beta} \alpha+\xi$. So we will inductively define f_ξ to have domain $\alpha + \xi$. Define $f_0(\zeta) = \zeta$ for $\zeta < \alpha$. If we have defined f_ξ, let $f_{\xi+1}$ be such that $f_{\xi+1}(\zeta) = \zeta$ for $\zeta \in \operatorname{dom} f_\xi$. Then let $f_{\xi+1}(\xi)$ be the point in $\alpha + \beta$ that corresponds to ξ in the copy of β. If ξ is a limit, let $f_\xi = \bigcup_{\eta<\xi} f_\xi$.

Point *5.* can be proved by induction using the previous points where *3.* gives the base case. $\square$

Definition 1.4.6. Let α and β be ordinals and let R be the lexicographic order on $\beta \times \alpha$, i.e.

$$\langle \xi, \eta \rangle R \langle \xi', \eta' \rangle \iff (\xi < \xi' \vee (\xi = \xi' \wedge \eta < \eta')).$$

Then $\alpha \cdot \beta = \mathrm{ot}(R)$.

Example 7. $2 \cdot \omega = \omega < \omega \cdot 2$.

Proposition 1.4.8. *Let α, β, γ be ordinals.*

1. $\alpha \cdot 0 = 0$.

2. $\alpha \cdot 1 = \alpha$.

3. $\alpha \cdot S(\beta) = \alpha \cdot \beta + \alpha$,

4. *If β is a limit then $\alpha \cdot \beta = \sup\{\alpha \cdot \xi : \xi < \beta\}$.*

5. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.

6. $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.

Proof. Homework. ☐

Definition 1.4.9. Let $\alpha \neq 0$ and β be ordinals.

1. $\alpha^0 = 1$.

2. $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$.

3. If β is a limit then $\alpha^\beta = \sup\{\alpha^\xi : \xi < \beta\}$.

Note: I was indeed wrong about needing transfinite induction to define the ordinals. As suggested in the lecture, we have a solution here: `https://math.stackexchange.com/questions/149158/ordinal-exponentiation-and-transfinite-induction`.

Example 10. $2^\omega = \omega$ in terms of ordinal exponentiation, but not in terms of cardinal exponentiation!

Definition 1.4.11. The natural numbers (in this context) are the set of finite ordinals.

Proposition 1.4.12. *Let α be an ordinal.*

1. *If $\beta < \gamma$ then $\alpha + \beta < \alpha + \gamma$.*

2. *If $\beta < \gamma$ and $\alpha > 0$ then $\alpha \cdot \beta < \alpha \cdot \gamma$.*

3. *If $\alpha < \beta$ then there is a unique ordinal δ such that $\alpha + \delta = \beta$.*

4. *If $\alpha > 0$ and γ is arbitrary, then there exist a unique β and a unique $\rho < \alpha$ such that $\gamma = \alpha \cdot \beta + \rho$.*

5. *If $\beta < \gamma$ and $\alpha > 1$ then $\alpha^\beta < \alpha^\gamma$.*

Proof. The first two points can be proved by induction. Point *3* lets δ be the order type of $\{\gamma \in \beta : \gamma \geq \alpha\}$, and its uniqueness is from the first point.

For *4*, we argue first that there is a greatest ordinal β such that $\alpha \cdot \beta \leq \gamma$. This is because the least ordinal β' such that $\alpha \cdot \beta' > \gamma$ is a successor: if it were a limit, then we would have $\alpha \cdot \epsilon \leq \gamma$ for all $\epsilon < \beta'$ and then our definition would give us $\alpha \cdot \epsilon \leq \gamma$. Once we have defined β, we let ρ be as given from *3*.

For *5* we do an induction similar to *1* and *2*. ☐

Theorem 1.4.13 (Cantor Normal Form)**.** *Every ordinal $\alpha > 0$ can be represented uniquely in the form*

$$\alpha = \omega^{\beta_1} \cdot k_1 + \ldots + \omega^{\beta_n} \cdot k_n,$$

where $n \geq 1$, $\alpha \geq \beta_1 > \ldots > \beta_n$, and $k_1, \ldots, k_n$ are nonzero natural numbers.

Proof. We prove this by induction. If $\alpha = 0$ then there is technically nothing to prove. If $\alpha = 1$ then $\alpha = \omega^0 \cdot 1$.

Suppose that $\alpha > 1$ is an arbitrary ordinal. Therefore there is a maximal β such that $\omega^\beta \leq \alpha$ (as in the argument for *4* in Proposition 1.4.12). By *4* of Proposition 1.4.12 there are unique δ and $\rho < \omega^\beta$ such that $\omega^\beta \cdot \delta + \rho$. By minimality of β, δ must be finite. Then plug in the inductive statement for ρ. □

1.5 The Notion of Cardinality

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Next we will develop the notion of cardinality, and the aspects of it that do not depend on the axiom of choice.

Definition 1.5.1. A *cardinal* is an ordinal κ such that for all $\alpha < \kappa$, there is no surjection $f : \alpha \to \kappa$.

Definition 1.5.2. If (emphasis on if!) A is well-ordered, then the *cardinality* $|A|$ is the least ordinal α such that there is a bijection $f : \alpha \to A$.

We have some structure without necessarily assuming the axiom of choice.

Theorem 1.5.3 (Bernstein-Cantor-Schröder)**.** *If there is an injection $f_1 : A \to B$ and an injection $f_2 : B \to A$ then there is a bijection from A to B.*

Proof. Homework. □

Proposition 1.5.4. *If $|\alpha| \leq \beta \leq \alpha$ then $|\beta| = |\alpha|$.*

Proof. We have an injection $\beta \to \alpha$ since $\beta \subseteq \alpha$. We have an injection $\alpha \to \beta$ via $\alpha \to |\alpha| \to \beta$. □

Definition 1.5.5. Let κ and λ be cardinals.

1. $\kappa + \lambda = |(\kappa \times \{0\}) \cup (\lambda \times \{1\})|$.
2. $\kappa \cdot \lambda = |\kappa \times \lambda|$.

Proposition 1.5.6. *Cardinal addition and multiplication are associative and commutative.*

Definition 1.5.7. Gödel's canonical well-ordering on ON $\times$ ON is defined as follows: $(\alpha, \beta) < (\gamma, \delta)$ if and only if

- either $\max\{\alpha, \beta\} < \max\{\gamma, \delta\}$ or
- $\max\{\alpha, \beta\} = \max\{\gamma, \delta\}$ and $\alpha < \gamma$ or
- $\max\{\alpha, \beta\} = \max\{\gamma, \delta\}$, $\alpha = \gamma$ and $\beta < \delta$.

Clearly, this a linear ordering.

Proposition 1.5.8. *Gödel's ordering is a well-ordering.*

Proof. Homework. □

Theorem 1.5.9. *If κ is a cardinal then $\kappa \cdot \kappa = \kappa$.*

Proof. It is clear that $|\kappa \cdot \kappa| \geq \kappa$, so we will prove the other direction by transfinite induction.

Each $\langle \alpha, \beta \rangle \in \kappa \times \kappa$ has $\leq |(\max\{\alpha, \beta\} + 1) \times (\max\{\alpha, \beta\} + 1)| < \kappa$ (which can be checked by cases). Therefore $\mathrm{ot}(\kappa \times \kappa, \triangleleft) \leq \kappa$ where $\triangleleft$ is the Gödel ordering (which can be seen by considering the contrapositive). Therefore $|\kappa \times \kappa| \leq \kappa$. □

Theorem 1.5.10. *If κ and λ are cardinals, then $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$.*

Proof. We have $\kappa, \lambda \leq \kappa + \lambda \leq \kappa \cdot \lambda$. If WLOG $\kappa \leq \lambda$ then $\kappa \cdot \lambda \leq \lambda \cdot \lambda = \lambda$. □

Theorem 1.5.11. *If κ is an infinite cardinal, then $[\kappa]^{<\omega}$ (the set of finite subsets of κ) has cardinality κ.*

Proof. By induction we can use the previous proof to obtain an injection $f : [\kappa]^{<\omega} \to \omega \times \kappa$. And $|\omega \times \kappa| = \kappa$. □

Since the ordinals are ordered, the cardinals are ordered.

Definition 1.5.12. α^+ is the least cardinal larger than α.

Definition 1.5.13. We define $\aleph_\alpha$ be transfinite recursion:

- $\aleph_0 = \omega$,
- $\aleph_{\alpha+1} = \aleph_\alpha^+$,
- if α is a limit then $\aleph_\alpha = \sup_{\beta<\alpha} \aleph_\beta$.

Definition 1.5.14. If α is an ordinal, the *cofinality* of α is the least ordinal β such that there is an increasing unbounded function $f : \beta \to \alpha$. Let $\mathrm{cf}(\alpha)$ denote the cofinality of α.

Definition 1.5.15. A cardinal κ is *singular* if $\mathrm{cf}(\kappa) < \kappa$. Otherwise κ is *regular*.

Example 16. $\aleph_\omega$ is singular.

Example 17. $\mathrm{cf}(\omega + \omega) = \mathrm{cf}(\omega \cdot \omega) = \omega$.

Proposition 1.5.18. *For all ordinals* α, $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$.

Proof. Composition of functions. □

Proposition 1.5.19. *If* α *is an ordinal, then* $\mathrm{cf}(\alpha)$ *is a regular cardinal.*

Proof. Let $f : \mathrm{cf}(\alpha) \to \alpha$ be unbounded.

Suppose for contradiction that $\kappa = |\mathrm{cf}(\alpha)| < \mathrm{cf}(\alpha)$ (i.e. that $\mathrm{cf}(\alpha)$ is not a cardinal). Let $g : \kappa \to \mathrm{cf}(\alpha)$ be a surjection. Observe that $f \circ g : \kappa \to \alpha$ is unbounded. This is a contradiction because $\mathrm{cf}(\alpha)$ is supposed to be minimal for the existence of such an unbounded function.

Suppose for contradiction that there is an unbounded function $g : \kappa \to \mathrm{cf}(\alpha)$ and then run a similar argument. □

1.6 The Axiom of Choice and Cardinal Arithmetic

We can quickly review some equivalent statements of the axiom of choice.

Definition 1.6.1. Let $(P, <_P)$ be a partially ordered.

1. A subset $X \subseteq P$ is a *chain* if it is totally ordered, i.e. for all $y, z \in X$, either $y <_P z$, $z <_P y$, or $z = y$.

2. An *upper bound* of a chain X is an element $y \in P$ such that for all $z \in X$, $y <_P z$.

3. An element $x \in P$ is *maximal* if for all $y \in P$, if $y \geq_P x$ then $y = x$.

Theorem 1.6.2. *The following are equivalent:*

1. *For every family F of nonempty sets, there is a function $C : F \to \bigcup F$ such that for all $x \in F$, $F(x) \in x$. (The Axiom of Choice,* AC*)*

2. *If P is a partially ordered set such that every chain has an upper bound, then P has a maximal element. (Zorn's Lemma)*

3. *Every set can be well-ordered. (The Well-Ordering Theorem)*

Proof. *3. $\Longrightarrow$ .1* Let $<_F$ be a well-ordering of $\bigcup F$. For all $x \in F$, let $<_F$ be a well-ordering of x. Let $F(x)$ be the $<_x$-least element of x.

1. $\Longrightarrow$ 2. Suppose for contradiction that AC holds and that P is a partially ordered set such that every chain has an upper bound, but that P has no maximal element. Using the choice function we can define a $<_{\mathbb{P}}$-increasing sequence that has no upper bound.

2. $\Longrightarrow$ .3 Let X be a set. Let P consist of all functions f into the ordinals such that $\operatorname{dom}(f) \subseteq X$. Every such chain clearly has an upper bound. Let $g \in P$ be a maximal element. Then it must necessarily have $\operatorname{dom} g = X$. □

For the remainder of the section assume AC.

Proposition 1.6.3. *If there is surjection from Y onto X then there is an injection from X to Y.*

Proof. Let $f : Y \to X$ be a surjection. For each $b \in X$ let $a_b \in f^{-1}(a)$ (using the axiom of choice). Then let $g(b) = a_b$. □

Proposition 1.6.4. *Every infinite cardinal is a limit ordinal.*

Proof. $|\alpha + 1| = |\alpha|$. □

Theorem 1.6.5 (Cantor)**.** *If X is a nonempty set then there is no surjection from X onto $\mathcal{P}(X)$.*

Proof. Suppose for contradiction that there is a surjection from X onto $\mathcal{P}(X)$. Let $Y = \{z \in X : z \notin f(z)\}$. Let z be such that $f(z) = Y$. If $z \in Y \iff z \notin f(z) \iff z \notin Y$. $\square$

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Proposition 1.6.6. *A countable union of countably infinite sets is countable.*

Proof. Let $\langle A_n : n < \omega \rangle$ be a sequence of countably infinite sets. Its union will be at least countable.

For each n we *choose* an enumeration $\langle a_k^n : k < \omega \rangle$. Then there is a surjection $f : \omega \times \omega \to \bigcup_{n<\omega} A_n$ given by $(n, k) \mapsto a_k^n$. This shows that the union is at most countable. $\square$

Proposition 1.6.7. *If S is a set then $|\bigcup S| \leq |S| \cdot \sup\{|X| : X \in S\}$.*

Proof. Let $\kappa = |S|$ and let $\lambda = \sup\{|X| : X \subseteq S\}$.

Now we show that $\kappa \cdot \lambda \geq |\bigcup S|$. We have an enumeration $S = \langle X_\alpha : \alpha < \kappa \rangle$. For each $X_\alpha \in S$, we have an enumeration $X_\alpha = \langle a_\xi^\alpha : \xi \leq \lambda_\alpha \rangle$ where $\lambda_\alpha \leq \lambda$. Now we have the mapping $\kappa \times \lambda \to \bigcup S$ given by $(\alpha, \xi) \mapsto a_\xi^\alpha$. $\square$

Proposition 1.6.8. *If α is a successor ordinal, then $\aleph_\alpha$ is regular.*

Proof. Suppose $f : \delta \to \aleph_{\alpha+1}$ where $\delta < \aleph_{\alpha+1}$. Then $f(\xi)$ is an ordinal of cardinality $\leq \aleph_\alpha$. Therefore $|\bigcup \operatorname{im} f| = |\bigcup_{\xi \in \delta} f(\xi)| = |\delta| \cdot \aleph_\alpha = \aleph_\alpha$ by the previous proposition. $\square$

Remark: This is consistently false! The *Axiom of Determinacy* implies that $\aleph_3$ is singular.

1.7 Cardinal Arithmetic with Exponentiation

Definition 1.7.1. If $\kappa \geq \lambda$ are cardinals, then $[\kappa]^\lambda = \{X \subseteq \kappa : |X| = \lambda\}$.

Definition 1.7.2. If A and B are sets, then $A^B = {}^B A = \{f | f : B \to A\}$.

Definition 1.7.3 (Using AC)**.** If κ and λ are cardinals then $\kappa^\lambda = |{}^\lambda \kappa|$ (where the former is indicating the exponential operation and the latter is indicating the cardinality of that particular set of functions).

Proposition 1.7.4. $2^\kappa = |\mathcal{P}(\kappa)|$.

Proof. Use the characteristic functions χ_X where $\chi_X(\alpha) = 1$ if $\alpha \in X$ and $\chi_X(\alpha) = 0$ otherwise. $\square$

Proposition 1.7.5. *If $\kappa \leq \lambda$ then $2^\kappa \leq 2^\lambda$.*

Proof. From $\mathcal{P}(\kappa) \subseteq \mathcal{P}(\lambda)$. $\square$

Proposition 1.7.6. *If κ, λ, μ are cardinals, then $\mu^{\kappa+\lambda} = \mu^\kappa \cdot \mu^\lambda$ and $\mu^{\kappa \cdot \lambda} = (\mu^\kappa)^\lambda$.*

Proof. Check the definitions in terms of functions. For the first assertion, observe that the set of functions from $\kappa \times \{0\} \cup \lambda \times \{1\}$ to μ is in bijection with the disjoint union of the functions from κ to μ and the functions from λ to μ.

You can also argue by cases where, without loss of generality $\lambda \geq \kappa$ and both are assumed to be infinite (since the finite cases are fairly immediate). $\square$

Proposition 1.7.7. *If $2 \leq \kappa \leq \lambda$ and λ is infinite then $\kappa^\lambda = 2^\lambda$.*

Proof. $2^\lambda \leq \kappa^\lambda \leq (2^\kappa)^\lambda = 2^{\kappa \cdot \lambda} = 2^\lambda$. $\square$

Sometimes infinite sums and products are used.

Definition 1.7.8. Given an indexed set of cardinals $\{\kappa_i : i \in I\}$ choose $\{X_i : i \in I\}$ such that $|X_i| = \kappa_i$ for all $i \in I$. We define

$$\sum_{i \in I} \kappa_i = \left| \bigcup_{i \in I} X_i \right|$$

and

$$\prod_{i \in I} \kappa_i = \left| \prod_{i \in I} X_i \right|.$$

Observe that assuming AC, these definitions do not depend on the X_i's.

Proposition 1.7.9. *If λ is infinite and $\kappa_i > 0$ for all $i < \lambda$, then*

$$\sum_{i<\lambda} \kappa_i = \lambda \cdot \sup_{i<\lambda} \kappa_i.$$

Theorem 1.7.10 (König's Theorem)**.** *If $\kappa_i < \lambda_i$ for every $i \in I$ then*

$$\sum_{i\in I} \kappa_i < \prod_{i \in I} \lambda_i.$$

Proof. Choose Y_i's such that $|Y_i| = \lambda_i$. Suppose that X_i's are subsets of $\prod_{i\in I} Y_i$ such that $|X_i| \leq \kappa_i$. We want to show that $\bigcup_{i\in I} X_i \neq \prod_{i\in I} Y_i$.

Let $Z_i = \{f(i) : f \in X_i$. Since $|X_i| < |Y_i|$, we have that $Z_i \neq Y_i$. Let f be a function so that $f(i) \notin Z_i$ for all $i \in I$. Then f does not belong to any of the X_i's.

Hence have shown that $\bigcup_{i\in I} X_i \neq \prod_{i\in I} Y_i$. □

Corollary 1.7.11. *If κ is an infinite cardinal then* $\mathrm{cf}(2^\kappa) > \kappa$*.*

Proof. It suffices to show that if $\kappa_i < 2^\kappa$ for all $i < \kappa$, then $\sum_{i<\kappa} \kappa_i < 2^\kappa$. Let $\lambda_i = 2^\kappa$. Then

$$\sum_{i<\kappa} \kappa_i < \prod_{i<\kappa} \lambda_i = (2^\kappa)^\kappa = 2^\kappa.$$

□

Corollary 1.7.12. *If κ is an infinite cardinal, $\kappa^{\mathrm{cf}\,\kappa} > \kappa$.*

Proof. Write $\kappa = \sum_{i<\mathrm{cf}\,\kappa} \kappa_i$ with $\kappa_i < \kappa$. Then

$$\kappa = \sum_{i<\mathrm{cf}\,\kappa} \kappa_i < \prod_{i<\mathrm{cf}\,\kappa} \kappa = \kappa^{\mathrm{cf}\,\kappa}.$$

□

Definition 1.7.13. The *continuum hypothesis*, denoted CH, is the assertion that $2^{\aleph_0} = \aleph_1$. The *generalized continuum hypothesis*, denoted GCH, as the assertion that for all infinite cardinals κ, $2^\kappa = \kappa^+$.

Definition 1.7.14. By transfinite induction, define the *beth function* as follows:

- $\beth_0 = \aleph_0$,
- $\beth_{\alpha+1} = 2^{\beth_\alpha}$,
- $\beth_\alpha = \sup\{\beth_\beta : \beta < \alpha\}$ if α is a limit.

So GCH holds if $\beth_\alpha = \aleph_\alpha$ for all ordinals α.

Proposition 1.7.15. *If* GCH *holds and* κ *and* λ *are infinite cardinals, then the following are true:*

1. *If* $\kappa \leq \lambda$ *then* $\kappa^\lambda = \lambda^+$,
2. *If* $\operatorname{cf}\kappa \leq \lambda < \kappa$ *then* $\kappa^\lambda = \kappa^+$,
3. *If* $\lambda < \operatorname{cf}\kappa$ *then* $\kappa^\lambda = \kappa$.

Proof. *1.* $\kappa^\lambda = 2^\lambda$.

2. We know that $\kappa^\lambda \geq \kappa^{\operatorname{cf}\kappa} \geq \kappa^+$. We also have $\kappa^\lambda \leq \kappa^\kappa \leq (2^\kappa)^\kappa = 2^\kappa = \kappa^+$.

3. Write $\kappa^\lambda = \bigcup_{\alpha<\kappa} \alpha^\lambda$ using the fact that functions from λ into κ are bounded. For $\alpha < \kappa$ we have $|\alpha^\lambda| \leq (2^{|\alpha|})^\lambda = 2^{|\alpha|\cdot\lambda} = (|\alpha| \cdot \lambda)^+ \leq \kappa$ (where the second-to-last relation is using GCH). This then implies that $\kappa^\lambda = \kappa$. ∎

Chapter 2

Filters, Ideals, and Algebras

2.1 The Basics of Filters and Ideals

Definition 2.1.1. Let X be a set. A *filter* on X is a set $F \subseteq \mathcal{P}(X)$ such that:

1. $\forall A, B \in F, A \cap B \in F$,
2. $\forall A \in F, B \supseteq A, B \in F$,
3. $\emptyset \notin F$.

Some sources omit the requirement that $\emptyset \notin F$ and instead call a filter nontrivial if this holds.

Example 2.
- The trivial filter $\{X\}$.
- Principal filters: Take $z \in X$, then $F = \{Y \subseteq X : z \in Y\}$.
- The Frechet filter, e.g. on ω let $F = \{Y \subseteq \omega : |\omega \setminus Y| < \omega\}$.

Definition 2.1.3. A family G of sets has the *finite intersection property* if for every finite $\{X_1, \ldots, X_n\} \subset G$ has a nonempty intersection, i.e. $X_1 \cap \ldots \cap X_n \neq \emptyset$.

Proposition 2.1.4.
1. *Every filter has the finite intersection property.*
2. *If X is a set and $G \subseteq \mathcal{P}(X)$ has the finite intersection property, then there is a filter F such that $G \subseteq F$.*

Proof. The first assertion follows easily by induction. For the second assertion, let F be the set of all $Y \subseteq X$ such that for some finite $\{X_1, \ldots, X_n\} \subseteq G$, $X_1 \cap \ldots \cap X_n \subseteq Y$. $\square$

Definition 2.1.5. A filter F on a set X is an *ultrafilter* if for all $Y \subseteq X$, either $Y \in F$ or $X \setminus Y \in F$.

Proposition 2.1.6. *The following are equivalent for a set X:*

1. *F is a maximal filter on X, i.e. if $F' \supseteq F$ is a filter then $F' = F$.*

2. *F is an ultrafilter on X.*

Proof. If U is an ultrafilter is is maximal: If we have $F \supsetneq U$ with $Y \in F \setminus U$, then $X \setminus Y \in U$, contradicting that F is a filter.

Suppose F is not an ultrafilter and that $Y, X \setminus Y \notin F$. We will show that there is filter extending F. We can use the finite intesection property to show that either there is a filter containing $F \cup \{Y\}$ or a filter extending $F \cup \{X \setminus Y\}$.

So we want to show that either $F \cup \{Y\}$ has the finite intersection property or else $F \cup \{X \setminus Y\}$. Suppose this is true for neither. Then there is some $A_1, \ldots, A_m \in F$ such that $A_1 \cap \ldots \cap A_m \cap Y = \emptyset$ and $B_1, \ldots, B_n \in F$ such that $B_1 \cap \ldots \cap B_n \cap (X \setminus Y) = \emptyset$. But we know that $A_1 \cap \ldots \cap A_m \cap B_1 \cap \ldots B_n \neq \emptyset$, so this is impossible. $\square$

Theorem 2.1.7. *Every filter can be extended to an ultrafilter.*

Proof. Use Zorn's lemma and the fact that maximal filters are ultrafilters. The main point is to show that if $\langle F_i : i < \theta \rangle$ is a sequence of filters so that $F_i \subseteq F_j$ for $i < j$, then $\bigcup_{i<\theta} F_i$ is a filter (in other words, unions of chains of filters are filters). $\square$

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Definition 2.1.8. A filter F on X is *uniform* if $|Y| = |X|$ for all $Y \in F$.

Example 9. The Frechet filter is uniform, but principal filters on sets of cardinality > 1 are not.

Theorem 2.1.10. *If κ is an infinite cardinal then there are 2^{2^κ}-many uniform filters on κ.*

Proof. We call a family $\mathcal{A}$ of subsets of κ *independent* if for distinct sets $X_1, \ldots, X_n, Y_1, \ldots, Y_m \in \mathcal{A}$, the intersection

$$X_1 \cap \ldots X_n \cap (\kappa \setminus Y_1) \cap \ldots \cap (\kappa \setminus Y_m)$$

has cardinality κ.

Claim. *There is an independent family of subsets of κ of cardinality 2^κ.*

Assuming the claim, we can obtain the rest of the theorem: Let $\mathcal{A}$ be the family witnessing the claim. For each $\mathcal{A} \to \{0,1\}$, consider

$$F_f = \{X : f(X) = 1\} \cup \{\kappa \setminus X : f(X) = 0\} \cup \{X : |\kappa \setminus X| < \kappa\}.$$

We can see that F_f has the finite intersection property: Suppose $X_1, \ldots, X_\ell$ are such that $f(X_i) = 1$, $Y_1, \ldots, Y_m$ are such that $f(\kappa \setminus Y_i) = 0$, and $Z_1, \ldots, Z_n$ are sets that are co-bounded (i.e. they are from the last component of F_f). Then $W_0 := \bigcap_{1 \le i \le \ell} X_i \cap \bigcap_{1 \le i \le m} Y_i$ has cardinality κ by the definition of independence. It is reasonably easy to see (just from De Morgan's Laws) that $W_1 := \bigcap_{1 \le i \le n} Z_i$ also has the property that $|\kappa \setminus W_1| < \kappa$, so there is some $\alpha < \kappa$ such that $(\alpha, \kappa) = \{\beta < \kappa : \alpha \le \beta < \kappa\} \subseteq W_1$. Therefore $W_0 \cap W_1$ has cardinality equal to κ.

We can also see that if $f \neq g$ then $F_f \neq F_g$. Therefore if U_f's are the respective ultrafilters extending the F_f's, then we have 2^{2^κ}-many U_f's because $\{f | f \to \{0,1\}\}$ has cardinality κ.

Proof of Claim. Let P be the set of all pairs (s, F) where s is a finite subset of κ and F is a finite set of finite subsets of κ. Since $|P| = |[\kappa]^{<\omega}| = \kappa$, it is sufficient to find an independent family of subsets of P of cardinality 2^κ.

For each $u \subseteq \kappa$, let

$$X_u = \{(s, F) \in P : s \cap u \in F\}$$

and let $\mathcal{A} = \{X_u : u \subseteq \kappa\}$. If u and v are distinct subsets of κ, then $X_u \neq X_v$: if (WLOG) $\alpha \in u$ and $\alpha \notin v$, then let $s = \{\alpha\}$, $F = \{s\}$, and $(s, F) \in X_u$. Then $(s, F) \in X_u$ and $(s, F) \notin X_v$.

To say that $\mathcal{A}$ is independent, let $u_1, \ldots, u_n, v_1, \ldots, v_m$ be distinct subsets of κ. For each $i \le n$ and $j \le m$, let $\alpha_{i,j} \in u_i \Delta v_j$. Now let s be any finite subset of κ such that $s \supseteq \{\alpha_{i,j} : i \le n, j \le m\}$. We have $s \cap u_i \neq s \cap v_j$ for all $i \le n$ and $j \le m$. Thus if we let $F = \{s \cap u_i : i \le n\}$ then $(s, F) \in X_{u_i}$ for $i \le n$ and $(s, F) \notin X_{v_j}$ for $j \le m$.

Hence we have chosen that every such finite set s is an element of

$$X_{u_1} \cap \ldots \cap X_{u_n} \cap (P \setminus X_{v_1}) \cap \ldots \cap (P \setminus X_{v_m}),$$

and since there are κ-many such s, the intersection has cardinality κ. □

Having proven the claim, we are done with the proof. □

2.2 Clubs and Stationary Sets

Definition 2.2.1. A function f whose domain is a subset of the ordinals is *regressive* if $f(\alpha) < \alpha$ for all $\alpha \in \mathrm{dom}(f) \setminus \{0\}$.

Remark 2.2.2. Obviously we have an injective regressive function f with domain ω: Just let $f(n) = n - 1$. But can we get an injective regressive function with domain $\aleph_1$?

Definition 2.2.3. Let κ be an uncountable regular cardinal. A subset $C \subseteq \kappa$ is *club* in κ (or *a* club in κ) if:

1. C is unbounded in κ, i.e. $\forall \beta < \kappa, \exists \alpha \in C$, $\alpha > \beta$;
2. C is *closed*, i.e. if $\langle \alpha_\xi : \xi < \lambda \rangle \subset C$ with $\lambda < \kappa$, then $\sup_{\xi<\lambda} \alpha_\xi \in C$.

The set $\{X \subset \kappa : X \text{ contains a club}\}$ is called *the club filter on* κ.

Example 4. Consider (1) the set of limit ordinals in κ or perhaps (2) $\kappa \setminus \alpha$ for any $\alpha < \kappa$.

Remark 2.2.5. We can define clubs in limit ordinals that are not cardinals.

Proposition 2.2.6. *The club filter is κ-complete. In other words, if $\langle C_\xi : \xi < \lambda \rangle$ are clubs in κ and $\lambda < \kappa$, then $\bigcap_{\xi<\lambda} C_\xi$ is a club in κ. (In particular, the club filter is a filter.)*

Proof. Closure of $\bigcap_{\xi<\lambda} C_\xi$ is straightforward from the definitions.

For unboundedness, we will first argue that the intersection of any two clubs C and D in κ is unbounded. Fix $\delta < \kappa$. Using the unboundedness of C and D, define by induction sequences $\langle \alpha_n : n < \omega \rangle \subset C$ and $\langle \beta_n : n < \omega \rangle \subset D$ such that $\alpha_0 \geq \delta$ and $\alpha_n < \beta_n < \alpha_{n+1}$ for all $n < \omega$. Then we can see that $\sup_{n<\omega} \alpha_n = \sup_{n<\omega} \beta_n = \gamma$. (This is known as "interleaving.") By closure of C, we know that $\gamma = \sup_{n<\omega} \alpha_n \in C$, and by closure of D, we know that $\sup_{n<\omega} \beta_n = \gamma \in D$, and thus $\gamma \in C \cap D$.

Now let us do the general argument. We will argue that $\bigcap_{\xi<\eta} C_\xi$ is unbounded in κ by induction on $\eta < \kappa$.

- The statement is of course trivial if we are taking only one club, so that gives us the base case.
- Suppose that we are considering

$$\bigcap_{\xi<\eta+1} C_\xi = \left(\bigcap_{\xi<\eta} C_\xi\right) \cap C_{\xi+1}.$$

  The first part is a club by our inductive hypothesis, and the intersection of everything is a club by the same argument we used for two clubs.
- Now suppose we are considering $\bigcap_{\xi<\eta} C_\xi$ where η is a limit ordinal. By induction, $\bigcap_{\xi<\zeta} C_\xi$ is a club for all $\zeta < \eta$. Therefore we can assume without loss of generality that $C_\zeta \subseteq C_\xi$ for all $\xi < \zeta$, i.e. the clubs are "nested." Now define a sequence $\langle \alpha_\xi : \xi < \eta \rangle$ to be an increasing sequence above some fixed $\delta < \kappa$ such that $\alpha_\xi \in C_\xi$ for all $\xi < \eta$. If $\beta = \sup_{\xi<\eta} \alpha_\xi$, then $\beta < \kappa$ by regularity. Because of nestedness, $\alpha_\xi \in C_\zeta$ for all $\zeta \leq \xi$, and so $\beta = \sup_{\zeta\leq\xi<\eta} \alpha_\xi \in C_\zeta$ for all $\zeta < \eta$.

This finishes the proof. □

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Definition 2.2.7. Let κ be an uncountable regular cardinal and let $\langle X_\alpha : \alpha < \kappa \rangle$ be a collection of subsets of κ. Then $\triangle_{\alpha<\kappa} X_\alpha := \{\alpha < \kappa : \alpha \in \bigcap_{\beta<\alpha} X_\beta\}$ is the *diagonal intersection* of this collection. A filter F on κ is *normal* if for all $\langle X_\alpha : \alpha < \kappa \rangle \subset F$, $\triangle_{\alpha<\kappa} X_\alpha \in F$.

Remark 2.2.8. We do not necessarily have $\triangle_{\alpha<\kappa} X_\alpha \subseteq X_\alpha$ for all $\alpha < \kappa$: Consider the example where $X_\alpha = \kappa \setminus \alpha$ for all $\alpha < \kappa$.

Proposition 2.2.9. *If κ is an uncountable regular cardinal and $\langle C_\alpha : \alpha < \kappa \rangle$ is a collection of clubs in κ, then $\triangle_{\alpha<\kappa} C_\alpha$ is a club in κ. (In other words, the club filter is normal.)*

Proof. Notice that the diagonal intersection is the same if we replace each C_α with $\bigcap_{\beta\leq\alpha} C_\beta$. Hence, as in the last proof, we can assume without loss of generality that $C_\beta \subseteq C_\gamma$ for $\gamma \leq \beta$.

Closure: Consider $\langle \gamma_\xi : \xi < \eta \rangle \subset \triangle_{\alpha<\kappa} C_\alpha$ be a strictly increasing sequence where η is a limit ordinal, and let $\sup_{\xi<\eta} \gamma_\xi = \gamma^*$. By the definition of the diagonal intersection, we need to show that $\gamma^* \in \bigcap_{\beta<\gamma^*} C_\beta$.

The definition of diagonal intersections already tells us that $\gamma_\xi \in \bigcap_{\beta<\gamma_\xi} C_\beta$ for all $\xi < \eta$. Using nestedness, this means that $\gamma_\zeta \in C_{\gamma_\xi}$ for all $\zeta \in (\xi, \eta)$, which implies that $\gamma^* = \sup_{\zeta<\eta} \gamma_\zeta = \sup_{\xi\leq\zeta<\eta} \gamma_\zeta \in C_{\gamma_\xi}$ for all $\xi < \eta$. Again using nestedness, we conclude that $\gamma^* \in C_\beta$ for all $\beta < \gamma^*$.

Unboundness: Given $\beta < \kappa$, we will inductively define a sequence $\langle \gamma_n : n < \omega \rangle$ as follows: Let γ_0 be any ordinal in the interval (β, κ). Given γ_n, choose $\gamma_{n+1} \in (\gamma_n, \kappa)$ to be an element of $\bigcap_{\alpha<\gamma_n} C_\alpha$, which we know is a club. Then let $\gamma^* = \sup_{n<\omega} \gamma_n$.

Of course, γ^* is larger than β, so we just need to show that $\gamma^* \in \triangle_{\alpha<\kappa} C_\alpha$, i.e. that $\gamma^* \in C_\alpha$ for all $\alpha < \gamma^*$. Given some particular $\alpha < \gamma^*$, there is some n such that $\alpha < \gamma_n$. Then we see that $\gamma_m \in C_\alpha$ for all $m > n$. As in our previous reasoning, $\gamma^* \in C_\alpha$. □

Definition 2.2.10. Let κ be regular uncountable. We say that $S \subseteq \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for all clubs $C \subset \kappa$.

Example 11. Given a regular uncountable κ, all clubs in κ are stationary. Also, $\{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$ is stationary.

Proposition 2.2.12. *If $S \subset \kappa$ is stationary, then S is unbounded in κ.*

Theorem 2.2.13 (Fodor's Lemma)**.** *Let κ be regular uncountable and let $S \subset \kappa$ be stationary. If f is a regressive function with domain S, then there is a stationary subset $S' \subseteq S$ and some $\gamma < \kappa$ such that for all $\alpha \in S'$, $f(\alpha) = \gamma$.*

Proof. Suppose otherwise. Then for all $\gamma < \kappa$, there is some club C_γ such that for all $\alpha \in C_\gamma \cap S$, $f(\alpha) \neq \gamma$. (We are sort of jumping past a step here.) Now take $C := \triangle_{\gamma<\kappa} C_\gamma$, which we now know is a club. Let $\delta \in C \cap S \neq \emptyset$, and let $f(\delta) = \gamma < \delta$. By the definition of diagonal intersections, $\delta \in \bigcap_{\alpha<\delta} C_\alpha$, meaning that $\delta \in C_\gamma$, but this contradicts the way we defined C_γ. □

Theorem 2.2.14. *If κ is an uncountable regular cardinal, then every stationary subset of $\{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$ is the union of κ-many disjoint stationary sets.*

Proof. Let $S \subseteq E^\kappa_\omega := \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$. For each $\alpha \in S$, let $\langle \delta^\alpha_n : n < \omega\rangle$ be a sequence of ordinals less than α that is cofinal in α.

We argue that there is some $N < \omega$ such that for all $\gamma < \kappa$, $\{\alpha \in S : \gamma \geq \delta^\alpha_N\}$ is stationary. Suppose otherwise. Then for all $n < \omega$, there is some C_n and γ_n such that for all $\alpha \in C_n \cap S$, $\delta^\alpha_n < \gamma_n$. Consider the stationary set $S \cap \bigcap_{n<\omega} C_n$, let $\gamma^* = \sup_{n<\omega} \gamma_n$, and take $\alpha \in S \cap \bigcap_{n<\omega} C_n$ such that $\alpha > \gamma^*$. Then we should be able to choose n such that $\delta^\alpha_n > \gamma^*$ because $\langle \delta^\alpha_n : n < \omega\rangle$ is supposed to be cofinal in α, but this contradicts the choice of C_n and γ_n.

Fix N as in the previous paragraph. Define S_ξ and γ_ξ for $\xi < \kappa$ as follows: Apply Fodor's Lemma with the function $\alpha \mapsto \delta^\alpha_N$ and let $S_0 \subseteq S$ be a stationary set such that this function is constant with value γ_0. Then we continue in this way. Given S_ζ and γ_ζ defined similarly for $\zeta < \xi$, apply $\alpha \mapsto \delta^\alpha_N$ to the set $\{\alpha \in S : \delta^\alpha_N > \sup_{\zeta<\xi} \gamma_\zeta\}$ to get a stationary set S_ξ such that this function is constant with value γ_ξ.

Because κ is regular, we can continue until we have defined $\langle S_\xi : \xi < \kappa\rangle$. These sets are distinct because if $\delta^\alpha_n \neq \delta^\beta_n$ then $\alpha \neq \beta$. (We can "fill in the complement" with S_0.) □

2.3 Boolean Algebras

Definition 2.3.1. A *Boolean algebra* is a set B with at least two elements, 0 and 1, and endowed with two binary operations, $+$ and $\cdot$, is well is a unary operation $-$.

The operations satisfy the following axioms:

1. $u + v = v + u$ and $u \cdot v = v \cdot u$ (commutativity),
2. $u + (v + w) = (u + v) + w$ and $u \cdot (v \cdot w) = (u \cdot v) \cdot w$ (associativity),
3. $u \cdot (v + w) = u \cdot v + u \cdot w$ and $u + (v \cdot w) = (u + v) \cdot (u + w)$ (distributivity),
4. $u \cdot (u + v) = u$ and $u + (u \cdot v) = u$ (absorption),
5. $u + (-u) = 1$ and $u \cdot (-u) = 0$.

Example 2. Let X be any nonempty set. Then there is a Boolean algebra on $\mathcal{P}(X)$ where $0 = \emptyset$, $1 = X$, $+$ is $\cup$, $\cdot$ is $\cap$, and $-$ is $\setminus$.

Proposition 2.3.3. *If B is a Boolean algebra, then for all $x, y \in B$:*

1. $x + x = x$ *and* $x \cdot x = x$,
2. $x + y = y$ *if and only if* $x \cdot y = x$.

Proof. $x + x = x + x \cdot (x + x) = x$ for the first part of the first item.
If $x + y = y$ then $x \cdot y = x \cdot (x + y) = x$. □

Definition 2.3.4. If B is a Boolean algebra, then let $a \leq_B b$ if and only if $a + b = b$.

Proposition 2.3.5. *If B is a Boolean algebra, $\leq_B$ is a partial order. The greatest lower bound of $\{x, y\}$ is $x \cdot y$ and the least upper bound $\{x, y\}$ is $x + y$.*

Proof. Reflexivity is by the previous proposition. For transitivity: Let $x \leq y$ and $y \leq z$. Then

$$x + z = x + (y + z) = (x + y) + z = y + z = z.$$

For antisymmetry: Let $x \leq y$ and $y \leq x$; then $y = x + y = y + x = x$.
To see that, e.g. $x+y$ is the *least* upper bound of x and y, then $z \geq x, y$. Then $(x + y) + z = x + (y + z) = x + z = z$. □

Proposition 2.3.6. *For all x:*

1. $0 \leq x \leq 1$,
2. $x + 0 = x$ *and* $x \cdot 1 = x$,
3. $x \cdot 0 = 0$ *and* $x + 1 = 1$.

Proof. $\text{glb}\{-x, x\} = x \cdot (-x) = 0$, so $0 \leq x$. Hence $0 + x = x$ by definition of $\leq$. Also, $x \cdot 0 = 0$ since $0 + x = x$. You can do the rest with duality. □

Proposition 2.3.7. *The following can be derived from the Boolean algebra axioms:*

1. $u + u = u$,

2. $u \cdot u = u$,
3. $u + 1 = 1$,
4. $-(u + v) = -u \cdot -v$,
5. $-(u \cdot v) = -u + -v$.

Example 8. Let $\mathcal{L}$ be a first-order language and let S be the set of sentences of $\mathcal{L}$. Consider the equivalence relation $\vdash \varphi \iff \psi$. Let B be the set of equivalence classes $[\varphi]$ is a Boolean algebra with: $[\varphi] + [\psi] = [\varphi \vee \psi]$, $[\varphi] \cdot [\psi] = [\varphi \wedge \psi]$, $-[\varphi] = [\neg\varphi]$, $0 = [\varphi \wedge \neg\varphi]$, $1 = [\varphi \vee \neg\varphi]$.

Definition 2.3.9. A *homomorphism* of Boolean algebras will preserve $0, 1$, and the operations. An *isomorphism* (of course) is a bijective homomorphism.

Theorem 2.3.10 (Stone's Representation Theorem)**.** *Every Boolean agebra is isomorphic to an algebra of sets.*

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Proof. Let B be a Boolean algebra. Let $S = \{p : p \text{ is an ultrafilter on } B\}$. For every $u \in B$, let X_u be the set of $p \in S$ such that $u \in p$. Let $\mathcal{S} = \{X_u : u \in B\}$.

Consider the map $\pi : B \to \mathcal{S}$ be given by $\pi(u) = X_u$. We have $\pi(1) = S$ and $\pi(0) = \emptyset$. From the definition of an ultrafilter: $\pi(u \cdot v) = \pi(u) \cap \pi(v)$, $\pi(u + v) = \pi(u) \cup \pi(v)$, $\pi(-u) = S - \pi(u)$. So we have that π is a homomorphism.

We see that π is a surjection by definition. To see that π is injective: If $u \neq v$, then find an ultrafilter p on B containing one element but not the other. $\square$

We will sketch some material briefly with the goal of reaching Theorem 2.3.16.

Definition 2.3.11. If B is a BA then $B^+ = B \setminus \{0\}$.

Definition 2.3.12. Suppose B is a Boolean algebra.

1. Suppose $X \subseteq B$. Then $\sum X = \sup X$ and $\prod X = \inf X$ if these values exist.

2. A Boolean algebra B is *complete* if $\sum X \in B$ and $\prod X \in B$ for all $X \subseteq B$.

3. If B is a Boolean algebra and $A \subseteq B$ is a subalgebra, then A is *dense* in B if for every $u \in B^+ = B \setminus \{0\}$, there is some $0 \neq v \in A$ such that $v \leq u$.

4. If B is a Boolean algebra, then a *completion* C of B is a complete Boolean algebra such that $B \subseteq C$.

Proposition 2.3.13. *If B is a Boolean algebra and $B \subseteq C, D$ where C and D are completions of B, then $C \cong D$.*

Proof. Define a map $\pi : C \to D$ by

$$\pi(c) = \sum_D \{u \in B : u \leq c\}$$

where the supremum is taken within D. Then argue that π is an isomorphism. □

Theorem 2.3.14. *Every Boolean algebra has a completion*

Proof. Let B be a Boolean algebra.

A set $U \subseteq B^+$ is a *cut* if $p \leq q$ and $q \in U$ implies $p \in U$.

For some $p \in B^+$, let $U_p = \{x : x \leq p\}$.

A cut U is *regular* if whenever $p \notin U$, there is some $q \leq p$ such that $U \cap U_q = \emptyset$.

Consider the operations $U \cdot V = U \cap V$ and $U + V = \overline{U \cup V}$, and also $-U = \{p : U_p \cap U = \emptyset\}$.

One can argue that the algebra of regular cuts is complete, and that the assignment $p \mapsto U_p$ shows that B is dense in the algebra of regular cuts. □

Definition 2.3.15. Let B be a Boolean algebra.

- A set $W \subseteq B^+$ is an *antichain* if $u \cdot v = 0$ for all $u, v \in W$ with $u \neq v$.

- If W is an antichain and $\sum W = u$, then W is a *partition* of u. A partition of 1 is called a *partition of B* or a *maximal antichain.*

- B is κ-saturated if there is no partition W of B such that $|W| = \kappa$.

- $\mathrm{sat}(B)$ equals the least κ such that B is κ-saturated.

Theorem 2.3.16. *If B is an infinite complete Boolean algebra, then $\mathrm{sat}(B)$ is a regular uncountable cardinal.*

Proof. Given a Boolean algebra B, we will let cB denote

$$\sup\{|X| : X \text{ is an antichain of } B\}$$

(this is called the *cellularity*). Also, if $x \in B^+$, we let $B \restriction x = \{y \in B : y \leq x\}$. If B is a Boolean algebra, we let $c_B(x) = c(B \restriction x)$. Observe that if $x \in B$ and $c_B(x) > \mu$, then there is some antichain Y of $B \restriction x$ such that $|Y| = \mu$.

Suppose that $\lambda = cB$ is a singular cardinal. Then our goal is to show that there is some partition/antichain of B of cardinality λ.

Let $\lambda = \sup_{\alpha<\kappa} \lambda_\alpha$ where $\kappa = \mathrm{cf}(\lambda)$.

There are three cases to consider.

Case 1: There is some $z \in B$ such that $c(x) = \lambda$ for all $0 < x \leq z$. Since $\kappa < \lambda$, we have an antichain $\{b_\alpha : \alpha < \kappa\}$. For each $\alpha < \kappa$, we have some antichain Z_α in $B \restriction b_\alpha$ of cardinality λ_α. Then if $Z = \bigcup_{\alpha<\kappa} Z_\alpha$, we have an antichain of cardinality λ.

Now assume that the first case does not hold. Let

$$S = \{a \in B^+ : c(a) < \lambda\}$$

and let X be maximal among the antichains of B included in S. This uses Zorn's Lemma: observe that unions of antichains are antichains.

Case 2: Suppose $\sup_{z\in X} c(z) = \lambda$. Inductively choose a sequence of elements $\langle y_\alpha : \alpha < \kappa\rangle$ such that for all $\alpha < \kappa$, $\lambda_\alpha < c(y_\alpha)$. Let Z_α be an antichain of cardinality λ_α in $B \restriction y_\alpha$. Then $Z = \bigcup_{\alpha<\kappa} Z_\alpha$ is an antichain of cardinality λ.

Case 3: Suppose $\sup_{z\in X} c(z) < \lambda$. We will argue that $|X| = \lambda$. Suppose otherwise.

Let $\mu = \sup_{z\in X} c(z)$ and let $\mu' = (\max\{|X|, \mu\})^+$. Then $\mu' < \lambda$ since λ is a limit cardinal. Since $\lambda = cB$ we find an antichain Y of cardinality μ'.

For $z \in X$, let

$$Y_z = \{w \in Y : w \cdot z > 0\}.$$

Since X is maximal, we have that $Y = \bigcup_{z\in X} Y_z$. The set $\{z \cdot y : y \in Y_z\}$ is pairwise disjoint in $B \restriction z$, so $|Y_z| \leq c(z) \leq \mu$. It follows that $|Y| \leq \mu \cdot |X| < \mu'$, which is a contradiction of the choice of Y. $\square$

Definition 2.3.17. Let κ be a regular uncountable cardinal.

- The *nonstationary ideal* on κ is the subset of nonstationary subsets of κ and is often denoted NS_κ.
- The *saturation* of the nonstationary ideal is the saturation of the Boolean algebra $P(\kappa)/\mathrm{NS}_\kappa$, where element are equivalence classes under the relation $X \sim Y$ if and only if $X \Delta Y$ is nonstationary.

Corollary 2.3.18 (of Solovay's Splitting Theorem)**.** *The saturation of the nonstationary ideal on ω_1 is $\geq \omega_2$.*

Of course, this is generalized to κ for the "non-baby" version of Solovay's theorem.

Corollary 2.3.19. *Let κ be a regular uncountable cardinal. Then the saturation of the nonstationary ideal is regular.*

Part II

Working with Models of Set Theory

Chapter 3

Some Examples of Models and Easy Independence Proofs

3.1 The Von Neumann Hierarchy

Definition 3.1.1. The Von Neumann hierarchy is defined by induction on $\alpha \in \mathrm{ON}$ as follows:

- $V_0 = \emptyset$,
- $V_{\alpha+1} = \mathcal{P}(V_\alpha)$,
- if α is a limit then $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$.

Proposition 3.1.2. *For all ordinals α:*

1. *V_α is transitive,*
2. *$\forall \beta < \alpha$, $V_\beta \subset V_\alpha$.*

Proof. Prove *1.* by induction on α. This holds vacuously for $\emptyset$. If we know that V_α is transitive and $y \in x \in V_{\alpha+1} = \mathcal{P}(V_\alpha)$, then $y \in V_\alpha \in \mathcal{P}(V_\alpha) = V_{\alpha+1}$. If α is a limit and $y \in x \in V_\alpha$, then there is some $\beta < \alpha$ so that $x \in V_\beta$, so $y \in V_\beta \subseteq V_\alpha$.

Again, this is vacuous for V_0. If the statement holds for V_α and $\beta < \alpha+1$, then either $\beta < \alpha$ and $V_\beta \subseteq V_\alpha \subseteq \mathcal{P}(V_\alpha)$ or $\beta = \alpha$ we have the same think. The same kind of argument applies to the limit case. $\square$

Definition 3.1.3. If $x \in V$, then $\mathrm{rank}(x)$ if the least α such that $x \in V_{\alpha+1}$.

Proposition 3.1.4. *1. If $x \in y$ then* $\mathrm{rank}(x) < \mathrm{rank}(y)$.

2. $\mathrm{rank}(y) = \sup\{\mathrm{rank}(x) + 1 : x \in y\}$.

Proof. Suppose $\alpha = \mathrm{rank}(y) \leq \mathrm{rank}(x)$. This means that $x \notin V_\alpha$. If it were the case that $x \in y \in \mathcal{P}(V_\alpha)$, then we would have $x \in V_\alpha$.

The second point is clarified if we consider the difference between successor and limit cases. □

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Proposition 3.1.5. *For all $\alpha \in \mathrm{ON}$,*

1. $\alpha \in V$,
2. $\mathrm{rank}(\alpha) = \alpha$,
3. $V_\alpha \cap \mathrm{ON} = \alpha$.

Proof. These are all straightforward with transfinite induction. Observe that $\mathrm{rank}(\emptyset) = 0$. □

Proposition 3.1.6. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \in V_{\omega+\omega}$.

Proof. We have that $\mathbb{N} = \omega \in V_{\alpha+1}$. For the rest, check that we have enough room when ordered-pairs are used. □

3.2 Transitive Closures, Well-Founded Relations and the Mostowski Collapse

Proposition 3.2.1. *If x is a set, there is a transitive set T such that $x \subseteq T$. Moreover, we can show that if T' is transitive and $x \subseteq T'$, then $T \subseteq T'$.*

Proof. Define S_n by induction on $n \in \omega$ as follows: $S_0 = x$ and for $n \in \omega$, $S_{n+1} = \bigcup S_n$. Then let $T = \bigcup_{n<\omega} S_n$. Obviously we have that $T \supseteq S$. To see that T is transitive, suppose that $z \in y \in T$. Then there is some $n \in \omega$ such that $y \in S_n$. Then $z \in S_{n+1} \subseteq T$. To see that T is the smallest transitive set in the sense claimed by the proposition, observe that if A is transitive then $\bigcup A \subseteq A$, so we can argue inductively that if $S_n \subseteq T'$ where T' is as in the statement of the proposition, then $S_{n+1} \subseteq T'$. □

Definition 3.2.2. Let x be a set. The *transitive closure* of x, denoted $\mathrm{tc}(x)$, is the smallest transitive set containing x as a subset.

Proposition 3.2.3.

1. *If x is transitive, then $\mathrm{tc}(x) = x$.*

2. *If $x \subseteq y$ then $\mathrm{tc}(x) \subseteq \mathrm{tc}(y)$.*

3. $\mathrm{tc}(x) = x \cup \{\mathrm{tc}(y) : y \in x\}$.

Proof. *1.* By definition.

2. Use the fact that if $A \subseteq T$ and T is transitive then $\mathrm{tc}(A) \subseteq T$.

3. For the forward direction, it is enough to show that the set on the right hand side is transitive, which is clear.

For the other direction, suppose that $z \in x \cup \{\mathrm{tc}(y) : y \in x\}$. If $z \in x$ then we are done by definition. If there is some $y \in x$ such that $z \in \mathrm{tc}(y)$, then we have $z \in \mathrm{tc}(x)$ because $\mathrm{tc}(y) \subseteq \mathrm{tc}(x)$. □

Proposition 3.2.4. *If C is a class, then there is an $\in$-minimal member of C.*

Proof. We can denote C to be the class of x such that $\varphi(x, \bar{p})$ holds (for some φ). Let $S \in C$, so S is in particular a set. Then $S \cap C$ is a set by the separation schema. If $S \cap C = \emptyset$ then we are done. If $S \cap C \neq \emptyset$ then we let $X = T \cap C$ where $T = \mathrm{tc}(S)$. If $z \in X$ is $\in$-minimal element (of which there must be an instance) then z is an $\in$-minimal element of C. □

Proposition 3.2.5. *For all sets x there is some α such that $x \in V_\alpha$.*

Proof. Let C be the hypothetical class of x such that there is no $\alpha \in \mathrm{ON}$ such that $x \in V_\alpha$. Then let x be an $\in$-minimal element of C. Then for all $z \in x$ there is α_z such that $z \in V_{\alpha_z}$, so there is some $\beta = \sup_{z \in x} \alpha_z$, and we see that $x \in V_{\beta+1}$, a contradiction. □

Theorem 3.2.6 ($\in$-induction)**.** *Let T be a transitive class and let Φ be a property. Assume that:*

1. $\Phi(\emptyset)$ *holds,*

2. *if $x \in T$ and $\Phi(z)$ holds for every $z \in x$, then $\Phi(x)$ holds.*

Then it follows that every $x \in T$ has property Φ.

Proof. Let C be the class of all $x \in T$ that do not have the property Φ. If C is nonempty, then it has an $\in$-minimal element. If it is $\emptyset$ we contradict the first point, otherwise we contradict the second point. $\square$

Theorem 3.2.7 ($\in$-recursion)**.** *Let T be a transitive class and let G be a function defined for all x. Then there is a function F on T such that $F(x) = G(F \restriction x)$ for every $x \in T$. Moreover, F is the unique such function.*

Proof. For every $x \in T$, we let $F(x) = y$ if and only if there exists a function f such that $\mathrm{dom}(f)$ is a transitive subset of T and *(i)* $(\forall z \in \mathrm{dom}\, f) f(z) = G(f \restriction z)$ and *(ii)* $f(x) = y$. As in ordinal induction, we can argue that this definition does not depend on f. $\square$

Definition 3.2.8. A class C is *extensional* if for all $x, y \in C$, if $x \neq y$ then there is some $z \in x \Delta y$ such that $z \in C$.

Theorem 3.2.9 (Mostowski's Collapsing Theorem)**.**

1. *For every extensional class C, there is a transitive class M and an isomorphism π between $(C, \in)$ and $(M, \in)$. Both the class M and the map π are unique.*

2. *If C is an extensional class and $T \subseteq C$ is transitive, then $\pi(x) = x$ for all $x \in T$.*

Proof. We define π by $\in$-induction. Let $\pi_C(x) = \{\pi_C(y) : y \in x \cap C\}$. Let $M := \mathrm{im}\, \pi_C$. To see that M is transitive, suppose that $z \in y \in M$. Then $y = \pi_C(a)$ for some a, and so $z = \pi_C(b)$ for some $b \in a$. Hence $z \in M$.

To see that π_C is one-one, suppose for contradiction that z is of minimal rank such that $z = \pi(x) = \pi(y)$ for some $x \neq y$. It must be the case that x and y are nonempty and that (without loss of generality) there is some u such that $u \in x \setminus y$. Then $\pi(u) \in \pi(x) = \pi(y)$. Therefore there is $v \in y$ such that $\pi(u) = \pi(v)$. But $v \neq u$ so this is a contradiction of minimality of z.

Next we argue that $x \in y$ if and only if $\pi_C(x) \in \pi_C(y)$. First, if $x \in y$ then $\pi_C(x) \in \pi_C(y)$ by definition. If $\pi_C(x) \in \pi_C(y)$ then $\pi_C(x) = \pi_C(z)$ for some $z \in y$. By injectivity, $z = x$.

It can be argued that an isomorphism between transitive classes must be the identity. If π_1, π_2 are isomorphisms from C to M_1 and M_2, then $\pi_2 \circ \pi_1^{-1}$ is an isomorphism and therefore the identity.

Lastly, suppose that C is transitive. Then we can argue by $\in$-induction that $\pi_C(x) = x$ for all $x \in C$: First, $\pi_C(\emptyset) = \emptyset$. Furtherore, $x \subseteq C$ for all $x \in C$, so by induction we have $\pi_C(x) = \{\pi_C(y) : y \in x \cap C = x\} = \{y : y \in x\} = x$. □

Example 10.
- Suppose that X is a set of ordinals. Then the Mostowski collapse of X equals α where $\alpha = \mathrm{ot}(X)$.
- Consider V_α and consider $\gamma > \alpha$. Then the Mostowski collapse of $V_\alpha \cup \{\gamma\} = V_\alpha \cup \{\alpha\}$.

3.3 Hereditary Sets

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Definition 3.3.1. Let κ be an infinite cardinal. Then $H(\kappa)$ denotes the set of sets x such that $|\operatorname{tc}(x)| < \kappa$.

Example 2.

- $\{\aleph_2, \aleph_3\} \notin H(\aleph_1)$.
- We have $\omega \subset H(\omega)$ but $\omega \notin H(\omega)$.
- Interpret the set of rationals $\mathbb{Q}$ as pairs of natural numbers modded out be an equivalence relation. Then $\mathbb{Q} \in H(\aleph_1)$.

Proposition 3.3.3. *Let κ be an infinite cardinal. Then the following are true:*

1. *$H(\kappa)$ is transitive,*
2. *$H(\kappa) \cap \mathrm{ON} = \kappa$.*
3. *If $x \in H(\kappa)$ then $\bigcup x \in H(\kappa)$.*
4. *If $x, y \in H(\kappa)$ then $\{x, y\} \in H(\kappa)$.*
5. *If $x \in H(\kappa)$ and $y \subseteq x$ then $y \in H(\kappa)$.*

Proof. 1. First we argue that $x \in y$ implies that $\mathrm{tc}(x) \subseteq \mathrm{tc}(y)$: It is sufficient to argue that $\mathrm{tc}(y) \supseteq x$ by minimality of $\mathrm{tc}(x)$. Suppose that $z \in x$. Since $z \in x \in y$, it follows that z is an element of any transitive set containing y, so $z \in \mathrm{tc}(y)$.

Note: This was the part where the lecture was sketchy! Hopefully it's clear here.

Use that $x \in y$ implies that $\mathrm{tc}(x) \subseteq \mathrm{tc}(y)$ and hence $|\mathrm{tc}(x)| \leq |\mathrm{tc}(y)|$.

2. Use that $\mathrm{tc}(\alpha) = \alpha$ for ordinals α since they are transitive.

3. We can prove by $\in$-induction that $\mathrm{tc}(\bigcup x) = \bigcup_{y \in x} \mathrm{tc}(y)$. Therefore $|\mathrm{tc}(\bigcup x)| \leq \cdot |x| \cdot \sup_{y \in x} |\mathrm{tc}(y)| \leq |x| \cdot |x| = |x| < \kappa$.

4. $\mathrm{tc}(\{a, b\}) = \{a, b\} \cup \mathrm{tc}(a) \cup \mathrm{tc}(b)$ as in the previous point.

5. Uses that $y \subseteq x$ implies $\mathrm{tc}(y) \subseteq \mathrm{tc}(x)$. □

Proposition 3.3.4. *Suppose κ is a regular uncountable cardinal. Then $H(\kappa)$ satisfies the* ZFC *axioms besides powerset.*

Proof. Extensionality. We need to say that if $x \neq y$ and $x, y \in H(\kappa)$, then there is some $z \in x \Delta y$ such that $z \in H(\kappa)$. Since there is definitely some $z \in x \Delta y$, WLOG in x, we know $|\mathrm{tc}(z)| \leq |\mathrm{tc}(x)| < \kappa$.

Pairing. Done above.

Schema of Comprehension/Separation. Uses that if $x \subseteq y$ then $\mathrm{tc}(x) \subseteq \mathrm{tc}(y)$.

Union. Done above.

Infinity. ω is transitive.

Schema of Replacement. Let $\varphi(v, \bar{w})$ define a function with parameters $\bar{b}$ and domain A.

Regularity/Foundation. If $y \in x$ then $\mathrm{tc}(y) \subseteq \mathrm{tc}(x)$.

Axiom of Choice. Similar to previous items. □

We still get a fragment of the powerset axiom for $H(\kappa)$'s though.

Definition 3.3.5. We say that an uncountable cardinal κ is:

1. *weakly inaccessibe* if $\lambda^+ < \kappa$ for all $\lambda < \kappa$;
2. *strongly inaccessible* or just *inaccessible* if $2^\lambda < \kappa$ for all $\lambda < \kappa$.

Theorem 3.3.6. *Let κ be regular and uncountable. The following are equivalent:*

1. *κ is strongly inaccessible,*
2. *$H(\kappa) = V_\kappa$,*

3. $H(\kappa) \models \mathsf{ZFC}$.

Proof. 1. $\Longrightarrow$ 2. We know that $H(\kappa) \subseteq V_\kappa$. The other direction uses induction, the definition of inaccessibility, and the fact that if $2^\lambda < \kappa$ then $|\operatorname{tc}(\mathcal{P}(\lambda))| < \kappa$.

2. $\Longrightarrow$ 1. Suppose that $\lambda < \kappa$ and $2^\lambda \geq \kappa$. Then $\mathcal{P}(\lambda) \in V_\kappa \setminus H(\kappa)$.

1. $\Longrightarrow$ .3 We only need to check the powerset axiom because we already proved that we have the others: Suppose $X \in H(\kappa)$. Then $|X| \leq |\operatorname{tc}(X)| < \kappa$. Since $2^{|X|} < \kappa$, we have that $\mathcal{P}(X) \in H(\kappa)$.

3. $\Longrightarrow$ .1 Again, if the powerset axiom fails, this implies failure of inaccessibility. More precisely, if there is some $X \in H(\kappa)$ such that $\mathcal{P}(X) \notin H(\kappa)$, then this can only be the case if $2^{|X|} \geq \kappa$. $\square$

3.4 Reviewing Some Basic Model Theory

Recall some definitions from model theory, which we will summarize in very loose terms for the sake of haste:

1. A *language* is a set of symbols including constant, function, and relation symbols. In set theory we will only use the language $\mathcal{L} = \{=, \in\}$ (typically the notation for equality is suppressed).

2. Symbols from the language are built up into *terms*, which are build up with variables to create *formulas* using $\neg, \wedge, \vee$ and adding quantifiers $\exists, \forall$. Variables are *free* if they are not included in the scope of quantifiers. A formula with no free variables is called a *sentence* and sentences have truth values.

3. Given a language $\mathcal{L}$, an $\mathcal{L}$-*structure* is a set in which we can interpret truth values of sentences.

4. A *theory* is a set of sentences. It is satisfiable if it has a model M.

5. Two structures are *elementary equivalent*, denoted $M \equiv N$, if they satisfy the same sentences. If $M \subseteq N$, then we say M is an *elementary submodel* of N, denoted $M \prec N$, if for all $\bar{a} \in M$ and formulas φ, $M \models \varphi(\bar{a})$ if and only if $N \models \varphi(\bar{a})$.

6. We say that $a \in M$ is *definable* with a parameter $\bar{b}$ if there is a formula $\varphi(v, \bar{b})$ such that $M \models \varphi(a, \bar{b})$ and a is the only element with this property.

Theorem 3.4.1 (Tarski-Vaught Test)**.** *Let $M \subseteq N$ be $\mathcal{L}$-structures. Suppose it is the case that if $\bar{a} \in M$ and there is $b \in N$ such that $N \models \varphi(b, \bar{a})$, then there is $c \in M$ such that $M \models \varphi(c, \bar{a})$. Then it follows that $M \prec N$.*

Theorem 3.4.2 (Downward Löwenheim-Skolem)**.** *Let K be a structure and let $A \subset K$. Then there is $M \prec K$ such that $A \subset M$ and $|M| = |A| + \aleph_0$.*

3.5 Absoluteness and Reflection

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Definition 3.5.1. Let $\varphi(x_1, \ldots, x_n)$ be a formula and M a class. Then the *relativization* of φ to M, denoted φ^M, is defined via the following cases:

1. $(x \in y)^M \leftrightarrow x \in y$,
2. $(x = y)^M \leftrightarrow x = y$,
3. $(\varphi \wedge \psi)^M \leftrightarrow \varphi^M \wedge \psi^M$,
4. $(\neg\varphi)^M \leftrightarrow \neg(\varphi^M)$,
5. $(\exists v \varphi(v, \bar{x}))^M \leftrightarrow (\exists v \in M)\varphi(v, \bar{x})$.

Example 2. This can also affect the truth of the statement. Let φ state that there is an uncountable cardinal. Then φ is false in $H(\aleph_1)$.

Definition 3.5.3. Suppose $\varphi(\bar{x})$ is a formula with free variables among $\bar{x}$.

1. Suppose that $M \subseteq N$ are classes. Then φ is *absolute for M and N* if
$$\forall \bar{a} \in M(\varphi^M(\bar{a}) \iff \varphi^N(\bar{a})).$$
2. If M is a class, we say that φ is *absolute* (without explicit reference to another model) if it is absolute for M and V.

Proposition 3.5.4. *A formula Δ_0 if it is generated by the following rules:*

1. $x \in y$ and $x = y$ are Δ_0.

2. *If* φ, ψ *are* Δ_0 *then* $\varphi \wedge \psi$ *and* $\neg\varphi$ *are* Δ_0.

3. *If* φ *is* Δ_0, *then* $\exists v(v \in w \wedge \varphi(v, \bar{u}))$ *is* Δ_0.

Proposition 3.5.5. *Suppose* $M \subseteq N$ *are transitive classes. Then* Δ_0 *formulas are absolute with respect to* M *and* N.

Proof. We induct on formula composition. The only nontrivial case is bounded quantification. First, observe that if $\bar{a}, b \in M$ and $M \models \exists v(v \in b \wedge \varphi(v, \bar{a})$ as witnessed by c, then $c \in N$ and we have $N \models (c \in \wedge \varphi(c, \bar{a})$ by formula induction and therefore $N \models \exists v(v \in b \wedge \varphi(v, \bar{a})$.

Now suppose that $\bar{a}, b \in M$ and $N \models \exists v(v \in b \wedge \varphi(v, \bar{a})$ as witnessed by c. Since $b \in M$, transitivity implies that $c \in M$, and by formula induction we have that $M \models \exists v(v \in b \wedge \varphi(v, \bar{a})$. □

Proposition 3.5.6. *The following formulas are expressible as* Δ_0 *formulas for any model of* $\mathsf{ZF} - \{\text{powerset, foundation, infinity}\}$: *1.* $x \in y$, *2.* $x = y$, *3.* $x \subseteq y$, *4.* $\{x, y\}$ *(or* $z = \{x, y\}$*), 5.* $\{x\}$, *6.* $\langle x, y\rangle$, *7.* $\emptyset$, *8.* $x \cup y$, *9.* $x \cap y$, *10.* $x \setminus y$, *11.* $S(x) = x \cup \{x\}$, *12.* x *is transitive, 13.* $\bigcup x$, *14.* $\bigcap x$ *(where* $\cap\emptyset = 0$*).*

Proof. Since some of these are similar, we will present sufficiently many of the items to give a clear picture.

1,2 are by definition.

3. $\forall z \in x(z \in y)$.

4. $x \in z \wedge y \in z \wedge \forall w \in z(w = x \vee w = y)$.

6. $z = \langle x, y\rangle$ if and only if

$$\exists w \in z(w = \{x\}) \wedge \exists w \in z(w = \{x, y\}) \wedge \forall w \in z(w = \{x\} \vee w = \{x, y\}).$$

7. $z = \emptyset$ iff $\forall w \in z(w \neq w)$.

8. $z = x \cup y$ if and only if $\forall w \in z(w \in x \vee w \in y) \wedge x \subseteq z \wedge y \subseteq z$.

12. x is transitive if and only if $\forall v \in x \forall z \in v(z \in x)$.

13. $y = \bigcup x$ if and only if $\forall v \in x(v \subseteq y) \wedge \forall z \in y \exists v \in x(z \in v)$. □

Lemma 3.5.7. *Suppose* $M \subseteq N$ *and that* $\varphi(\bar{x})$ *and* $G_i(\bar{y})$ $(i = 1, \ldots, n)$ *are absolute for* M, N *(where absoluteness for functions has the natural definition). Then the formula*

$$\varphi(G_1(\bar{y}), \ldots, G_n(\bar{y}))$$

is absolute for M, N.

Proof. If $a \in M$, then with some notational simplifications,

$$(\varphi(G(a)))^M \iff \varphi^M(G^M(a)) \iff \varphi^N(G^N(a)) \iff (\varphi(G(a)))^N.$$

$\square$

Proposition 3.5.8. *The following are absolute for any model of* $\mathsf{ZF} - \{$*powerset, foundation, infinity*$\}$*: 1. z is an ordered pair. 2. $A \times B$. 3. R is a relation. 4.* $\operatorname{dom} R$. *5.* $\operatorname{range} R$. *6. R is a function. 7. $R(x)$. 8. R is a one-one function.*

Proof. Again, we will go through just some of the main points.

Saying that z is an ordered pair is equivalent to saying that

$$\varphi(z \mapsto \bigcup z, z \mapsto \bigcup z, z)$$

where $\varphi(a, b, c)$ is the formula

$$\exists x \in a \exists y \in b (c = \langle x, y \rangle).$$

The rest of the points clearly build on one another.

For example, R is a relation if and only if $\forall z \in R$, z is an ordered pair. $\square$

3.6 Relative Consistency

Chapter 4

The Consistency of the Continuum Hypothesis and the Axiom of Choice

4.1 Defining Definability

4.2 Defining Gödel's Constructible Universe

4.3 Relative Consistency of the Axiom of Choice

4.4 Condensation and the Continuum Hypothesis

Part III

Forcing

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The Fundamentals of Forcing

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The Independence of the Continuum Hypothesis

Chapter 7

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Chapter 8

Iterated Forcing and Martin's Axiom

Bibliography