CLASSICAL NAMBA FORCING CAN HAVE THE WEAK COUNTABLE APPROXIMATION PROPERTY

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Abstract. We show that it is consistent from an inaccessible cardinal that classical Namba forcing has the weak ω_1-approximation property. In fact, this is the case if $\aleph_1$-preserving forcings do not add cofinal branches to $\aleph_1$-sized trees. The exact statement we obtain is similar to Hamkins' Key Lemma. It follows as a corollary that MM implies that there are stationarily many indestructibly weakly ω_1-guessing models that are not internally unbounded. This answers a question of Cox and Krueger and partially answers another. Our result on MM gives a short proof of a weakening of Cox and Krueger's main result by removing their use of higher Namba forcings, but we find another application of their ideas by answering a question of Adolf, Apter, and Koepke on preservation of successive cardinals by singularizing forcings.

1. Background

Research in infinitary combinatorics has shown that the specific cardinals $\aleph_0$, $\aleph_1$, $\aleph_2$, etc. exhibit distinct properties. One way to look at this is to examine to what extent these cardinals can be turned into each other by forcing. Bukovský and Namba independently showed that $\aleph_2$ can be turned into an ordinal of cofinality $\aleph_0$ without collapsing $\aleph_1$, and this forcing and its variants for other cardinals are now known as Namba forcing [10]. This paper is about the functions added by variants of Namba forcing.[1]

The conditions in *classical Namba forcing*, which we denote $\mathbb{P}_{\mathsf{CNF}}$, are perfect trees of height ω and width $\aleph_2$. Specifically, $p \in \mathbb{P}_{\mathsf{CNF}}$ if and only if: $p \subseteq {}^{<\omega}\aleph_2$; $t \in p$ and $s \sqsubseteq t$ implies $s \in p$; for all $t \in p$, $|\{\alpha < \aleph_2 : t^\frown\langle\alpha\rangle \in p\}| \in \{1, \aleph_2\}$; and for all $t \in p$ there is some $s \sqsupseteq t$ such that $\{\alpha < \aleph_2 : s^\frown\langle\alpha\rangle \in p\}$ is unbounded in $\aleph_2$. The particularities of the width $\aleph_2$ will be significant for this paper.

Following work of Viale, Weiss, and others, Cox and Krueger introduced the weak guessing model property to explore questions around guessing models. Their approach required an analysis of higher Namba forcings—meaning variations of Namba forcing that add sequences to cardinals above $\aleph_2$—and a demonstration that they have the weak countably approximation property [3]. As they point out, CH implies that classical Namba forcing cannot have the weak ω_1-approximation property. (A forcing $\mathbb{P}$ has the *weak ω_1-approximation property* if it does not add new functions with domain ω_1 whose initial segments are in the ground model.) Since it collapses ω_2^V to ω_1, there is a new subset of ω_1, and since CH implies that Namba forcing does not add reals [8, Chapter 28], it follows that initial segments of this new subset are in the ground model. They asked whether classical Namba

[1]The reader is assumed to have familiarity with the basics of forcing theory [8].

forcing could have the weak ω_1-approximation property [3, Question 5], and we provide an affirmative answer here.

Theorem 1. *Suppose that every ω_1-sized tree cannot gain a cofinal branch from an ω_1-preserving forcing. Then it follows that if $\mathbb{P} = \mathbb{P}_{\mathsf{CNF}}$ is the classical Namba forcing and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a countably closed forcing, then $\mathbb{P} * \dot{\mathbb{Q}}$ has the weak ω_1-approximation property.*

Note that $\dot{\mathbb{Q}}$ can name the trivial forcing. The reader will also be able to observe from the proof Theorem 1 that it is sufficient for $\dot{\mathbb{Q}}$ to be countably strategically closed. Theorem 1 is optimal in a sense because Krueger showed that any forcing that has the countable approximation property also has the countable covering property [9]. Hence classical Namba forcing cannot have the ω_1-approximation property.

Theorem 1 also can be seen as a variation on Hamkins' Key Lemma [6], which was used to show how large cardinal properties are preserved in certain forcing extensions. Many variations have appeared since, notably by Usuba [13]. These ideas were used by Viale and Weiss, who introduced guessing models to derive information about the consistency strength of the proper forcing axiom (PFA) [15]. Guessing models have since become a major topic of research.

Let us introduce some of the basic terminology of guessing models that we will use here. Suppose $M \in P_{\omega_2}(X)$ with $\omega_1 \subseteq M$. We say that M is *weakly ω_1-guessing* if for all $f : \omega_1 \to \mathrm{ON}$ such that $f \upharpoonright i \in M$ for all $i < \omega_1$, it follows that $f \in M$. We say that M is *indestructibly weakly ω_1-guessing* if this property holds in all ω_1-preserving extensions. Such an M is internally unbounded if for all $x \in P_{\omega_1}(M)$, there is some $y \in M \cap P_{\omega_1}(M)$ such that $x \subseteq y$. Internal unboundedness and its variations were studied extensively by Krueger and are important for the properties of guessing models.

The findings here may be useful for Viale and Weiss' proposal to work on guessing models for fragments of Martin's Maximum (MM). Some indication for this possibility comes from an application to another question of Cox and Krueger, who ask whether MM suffices to prove their main theorem [3, Question 1]. We are able to obtain ω_1-guessing:

Corollary 2. MM *implies that for all $\theta \geq \omega_2$, there is a stationary set of $M \in P_{\omega_2}(H(\theta))$ such that $\omega_1 \subset M$, and M is indestructibly weakly ω_1-guessing and not internally unbounded.*

This in particular uses a weaker large cardinal hypothesis than Cox and Krueger, who obtain their statement from a supercompact and countably many measurables above it. The higher Namba forcings that they use are not needed for obtaining Corollary 2. However, we will demonstrate an alternative application for their work:

Theorem 3. *Assume the consistency of class-many supercompact cardinals. Then there is a forcing extension in which, for all double successor cardinals λ^{++}, there is a further set forcing extension adding a cofinal ω-sequence to λ^{++} without collapsing μ for $\mu \leq \lambda^+$.*

Hence we can have a sense of to what extent these higher versions of Namba forcing generalize the behavior of classical Namba forcing. This answers the first question of a paper by Adolf, Apter, and Koepke [1], who also indicate that a substantial cardinal hypothesis is necessary in order to obtain the statement of Theorem 3.

2. Classical Namba Forcing and Weak Approximation

Because we will refer to the hypothesis of Theorem 1 repeatedly, we will say that the *Baumgartner Freezing Property* holds if for all ω_1-preserving forcings $\mathbb{P}$ and all ω_1-sized trees T, $\mathbb{P}$ does not add a cofinal branch to T. We denote this BFP. (The attribution will be clarified later.) It consistently holds in the presence of certain specializing functions, but we will refer to its abstract form [2].

An ω_1-sized tree T is *B-special* if there is a function $f : T \to \omega$ such that for all $x, y, z \in T$, if $x \leq y, z$ and $f(x) = f(y) = f(z)$, then y and z are compatible. In this case f is called a *B-specializing function.*

Proposition 4. *If all ω_1-sized trees have a B-specializing function, then* BFP *holds.*

Baumgartner obtained the consistency of BFP using a model in which there are no Kurepa trees.

Fact 5. [2, Section 8] BFP *is consistent from an inaccessible cardinal.*

We underscore an important fact that does not depend on CH. The proof of this fact will to some extent be imitated in the proof of Theorem 1:

Fact 6. $\mathbb{P}_{\mathsf{CNF}}$ *preserves ω_1.*

Now we are ready to actually prove the main theorem.

Proof of Theorem 1. Suppose for contradiction that $(p', \dot{c}') \in \mathbb{P} * \dot{\mathbb{Q}}$ forces that $\dot{F} : \omega_1 \to \mathrm{ON}$ is a $\mathbb{P} * \dot{\mathbb{Q}}$-name for a function whose initial segments are in V.

Claim 7. *One of the following holds:*

(1) BFP *fails.*
(2) For all $(p, \dot{c}) \leq (p', \dot{c}')$ and all $X \in [V]^{\leq \omega_1} \cap V$ it is not the case that $(p, \dot{c}) \Vdash$ "$\{\dot{F} \restriction i : i < \omega_1\} \subseteq X$".

Proof. Suppose that *(2)* does not hold. It follows that there is some $(p, \dot{c}) \leq (p', \dot{c}')$ and some $X \in [V]^{\leq \omega_1} \cap V$ such that $(p, \dot{c}) \Vdash$ "$\{\dot{F} \restriction i : i < \omega_1\} \subseteq X$". Without loss of generality, there is a large enough τ such that all elements of X are functions $\sigma : \gamma \to \tau$ for some $\gamma < \omega_1$. We can assume that $|X| > \aleph_0$ since otherwise it would be implied that $(p, \dot{c})$ forces $\dot{F}$ to have domain bounded in ω_1. Therefore, X is a ω_1-sized tree where the ordering is end-extension, i.e. if $y, z \in X$ then $y \leq_X z$ if and only if $z \restriction \operatorname{dom} y = y$. Since $\dot{F}$ is forced to be new, $(p, \dot{c}) \Vdash$ "X has a cofinal branch not in V". Therefore we have shown that $\mathbb{P} * \dot{\mathbb{Q}}$, which preserves ω_1 (Fact 6 and then countable closure) and adds a cofinal branch to an $\aleph_1$-sized tree, and hence BFP fails. □

We are assuming that BFP holds, so for the rest of the proof we will argue that Case *(2)* in Claim 7 leads to a contradiction.

Now we introduce some standard notation: If $t \in p \in \mathbb{P}$ then $\mathrm{succ}_p(t) = \{t' \in p : t' \sqsupseteq t, |t'| = |t| + 1\}$ and $\mathrm{osucc}_p(t) = \{\alpha < \aleph_2 : t^\frown\langle\alpha\rangle \in p\}$. If $p \in \mathbb{P}$, then $\operatorname{stem} p$ is the $\sqsubseteq$-minimal node such that for all $t' \sqsubseteq t$, $|\mathrm{succ}_p(t')| = 1$.

We will repeatedly use the following in the remainder of the argument:

Claim 8. *Suppose* $p \Vdash$ "$\dot{c} \in \dot{\mathbb{Q}}$" *where* $\operatorname{osucc}_p(\operatorname{stem}(p)) = Z$ *and there is a sequence* $\langle (q_\alpha, \dot{d}_\alpha) : \alpha \in Z \rangle$ *such that* $(q_\alpha, \dot{d}_\alpha) \leq (p \restriction \operatorname{stem}(p)^\frown \langle \alpha \rangle, \dot{c})$ *for all* $\alpha \in Z$. *If* $q = \bigcup_{\alpha \in Z} q_\alpha$, *then there is some* $\dot{d}$ *such that* $q \Vdash$ "$\dot{d} \leq \dot{c}_\alpha$" *and for all* $\alpha \in Z$, $q_\alpha \Vdash$ "$\dot{d} = \dot{c}_\alpha$".

Proof. This is an application of the proof of the Mixing Principle, since $\langle q_\alpha : \alpha \in Z \rangle$ is a maximal antichain below q. $\square$

Now we define the main idea of the rest of the proof. Let $\varphi(i, (q, \dot{d}))$ denote the formula

$$\begin{aligned} &i < \omega_1 \wedge (q, \dot{d}) \in \mathbb{P} * \dot{\mathbb{Q}} \wedge (q, \dot{d}) \leq (p', \dot{c}') \wedge \exists \langle a_\alpha : \alpha \in \operatorname{osucc}_q(\operatorname{stem}(q)) \rangle \text{ s.t.} \\ &\qquad \forall \alpha \in \operatorname{osucc}_q(\operatorname{stem}(q)), (q \restriction (\operatorname{stem}(q)^\frown \langle \alpha \rangle), \dot{d}) \Vdash \text{`}\dot{F} \restriction i = a_\alpha\text{'} \wedge \\ &\qquad \forall \alpha, \beta \in \operatorname{osucc}_q(\operatorname{stem}(q)), \alpha \neq \beta \implies a_\alpha \neq a_\beta. \end{aligned}$$

Claim 9. $\forall j < \omega_1, (p, \dot{c}) \leq (p', \dot{c}') \in \mathbb{P} * \dot{\mathbb{Q}}, \exists i \in (j, \omega_1), (q, \dot{d}) \leq (p, \dot{c})$ *s.t.* $\operatorname{stem}(p) = \operatorname{stem}(q) \wedge \varphi(i, (q, \dot{d}))$.

Proof of Claim. First we establish a slightly weaker claim: $\forall j < \omega_1, (p, \dot{c}) \in \mathbb{P} * \dot{\mathbb{Q}}, t \in p$ s.t. $|\operatorname{succ}_p(t)| > 1$, there is an $\aleph_2$-sized set $W \subset \operatorname{osucc}_p(t)$ and a sequence $\langle (q_\alpha, \dot{c}_\alpha), i_\alpha : \alpha \in W \rangle$ s.t. :

- $\forall \alpha \in \operatorname{osucc}_p(t)$,
  - $i_\alpha \in (j, \omega_1)$,
  - $(q_\alpha, \dot{c}_\alpha) \leq (p \restriction (t^\frown \langle \alpha \rangle), \dot{c})$,
  - $(q \restriction t^\frown \langle \alpha \rangle, \dot{c}_\alpha) \Vdash$ "$\dot{F} \restriction i_\alpha = a_\alpha$),
- $\alpha \neq \beta \implies a_\alpha \neq a_\beta$.

Consider $(p, \dot{c}) \in \mathbb{P} * \dot{\mathbb{Q}}$ and $t := \operatorname{stem}(p)$. Inductively choose a sequence $\langle (q_{\alpha_\xi}, \dot{d}_{\alpha_\xi}), i_{\alpha_\xi} : \xi < \aleph_2 \rangle$ where $\langle \alpha_\xi : \xi < \aleph_2 \rangle \subseteq \operatorname{osucc}_p(t)$ as follows: Suppose $\langle (q_{\alpha_\xi}, \dot{d}_{\alpha_\xi}), i_{\alpha_\xi} : \xi < \eta \rangle$ is defined. Then if possible, choose $\beta \in \operatorname{osucc}_p(t) \setminus (\sup_{\xi < \eta} \alpha_\xi)$ and $(q_\beta, \dot{d}_\beta)$ such that $q_\beta \leq p \restriction t^\frown \langle \beta \rangle$ and $(q_\beta, \dot{d}_\beta) \Vdash$ "$\dot{F} \restriction i_\beta = a_\beta$" for some a_β with $\alpha_\beta \notin \{\alpha_{\alpha_\xi} : \xi < \eta\}$. Moreover let i_β be minimal such that $((q_\beta, \dot{d}_\beta), i_\beta)$ fitting this description can be found. Then set $\alpha_\eta := \beta$. If it is not possible to find such a $\langle q_\beta, \dot{d}_\beta \rangle$ and i_β, then halt the construction.

Suppose for contradiction that the slightly weaker claim fails. This means that the construction in the above paragraph must halt. Let $\eta < \aleph_2$ be the least ordinal where it is not possible to continue the construction in the paragraph above and let $B := \{a_{\alpha_\xi} : \xi < \eta\}$. Then for all $\alpha \in \operatorname{osucc}_p(t) \setminus (\sup_{\xi < \eta} \alpha_\xi)$ we have $(p \restriction t^\frown \langle \alpha \rangle, \dot{c}) \Vdash$ "$\bigcup_{i < \omega_1} \dot{F} \restriction i \subseteq B$". Let $q := \bigcup_{\alpha \in \operatorname{osucc}_p(t) \setminus (\sup_{\xi < \eta} \alpha_\xi)} p \restriction (t^\frown \langle \alpha \rangle)$. Then $q \in \mathbb{P}$ and $q \leq p$. Furthermore, it is the case that $(q, \dot{d}) \Vdash$ "$\bigcup_{i < \omega_1} \dot{F} \restriction i \subseteq B$" where $\dot{d}$ is obtained by applying Claim 8. But B has size $\leq \aleph_1$, contradicting the assumption that we are working in Case *(2)* of Claim 7.

Now we have established the slightly weaker claim. Choose a $\aleph_2$-sized subset $W \subseteq \operatorname{osucc}_p(t)$ and some $i \in (j, \omega_1)$ such that $i_\alpha = i$ for all $\alpha \in W$. Then let $q := \bigcup_{\alpha \in W} q_\alpha$ and apply Claim 8 to obtain $\dot{d}$. $\square$

We plan to build a fusion sequence using Claim 9. To this end, we define a game $\mathcal{G}_k$ for $k < \omega_1$.[2]

Suppose round n of the game is being played where $n = 0$ is the first round. If $n = 0$ then let $((q_*, \dot{c}_*), i_*)$ be $((p', \dot{c}'), 0)$ (recall that $(p', \dot{c}')$ is the condition from the beginning of the proof of Theorem 1). Otherwise if $n > 0$ let $((q_*, \dot{c}_*), i_*)$ be $((q_n, \dot{c}_n), i_n)$. First Player I chooses an $\aleph_1$-sized subset $Z_n \subseteq \mathrm{osucc}_{q_*}(\mathrm{stem}(q_*))$ and some $\delta_n < k$. Then Player II chooses some $\alpha \in \mathrm{osucc}_{q_*}(\mathrm{stem}(q_*)) \setminus Z_n$ and some condition $(q_n, \dot{d}_n) \leq (q_* \restriction \mathrm{stem}\, q_* {}^\frown \langle \alpha \rangle, \dot{c}_*)$ and some $i_n \in (\delta_n, k)$ such that $\varphi((q_n, \dot{d}_n), i_n)$ holds. Hence we have the following diagram:

Player I	Z_0, δ_0		Z_1, δ_1		Z_2, δ_2	
Player II		$(q_0, \dot{d}_0), i_0$		$(q_1, \dot{d}_1), i_1$		$(q_2, \dot{d}_2), i_2$

Player II loses at some stage n if they cannot find appropriate $(q_n, \dot{c}_n)$ and i_n witnessing $\varphi(i_n, (q_n, \dot{c}_n))$ for some $i < k$, i.e. if they cannot in particular find such $i_n \in (\delta_n, k)$. Otherwise Player II wins.

Claim 10. *There is some $k < \omega_1$ such that Player II has a winning strategy in $\mathcal{G}_k$.*[3]

Proof. Suppose this is not the case. For all $i < \omega_1$, $\mathcal{G}_i$ is an open game. Therefore by the Gale-Stewart Theorem, there is a winning strategy σ_i for Player I in $\mathcal{G}_i$.

Let θ be large enough for $H(\theta)$ to contain the sets mentioned in the upcoming argument, so we consider the structure $\mathcal{H} := (H(\theta), \in, <_\theta, \mathbb{P}, \dot{F}, \langle \sigma_i : i < \omega_1 \rangle)$ where $<_\theta$ is a fixed well-ordering of $H(\theta)$ that allows for Skolem functions. Let $M \prec \mathcal{H}$ be a countable elementary submodel. Then $M \cap \omega_1 \in \omega_1$, so let us denote $k := M \cap \omega_1$.

We will construct a run of the game $\mathcal{G}_k$ such that Player I uses the strategy σ_k but nonetheless loses the game. We will define the sequence

$$(Z_0, \delta_0), ((q_0, \dot{d}_0), i_0), (Z_1, \delta_1), ((q_1, \dot{d}_1), i_1), \ldots$$

so that Player II's moves are all in M even though $\sigma_k \notin M$.

Assume for notational convenience that the game is defined with the opening move $((p', \dot{c}'), 0)$ by Player II. Suppose we are considering stage n of the game and that $((q_n, \dot{d}_n), i_n)$ is defined (we let it be the opening move if $n = 0$). Let (Z_{n+1}, δ_{n+1}) be obtained by σ_k as applied to the previous moves. Then let W be the set of indices i for which $((p', \dot{c}'), 0), \ldots, ((q, \dot{d}_n), i_n)$ is a sequence of Player II's moves in the game $\mathcal{G}_i$ where Player I is using σ_i. This set is nonempty and $W \in M$. Therefore if

$$Y = \bigcup \{ Z_i : i \in W, \sigma_i(((q_{-1}, \dot{d}_{-1}), i_{-1}), \ldots, ((q, \dot{d}_n), i_n)) = (Z_i, \delta^*) \}$$

then $Y \in M$, and moreover $|Y| < \aleph_2$ because it is the union of at most $\aleph_1$-many $\aleph_1$-sized sets. Choose $\alpha \in \mathrm{osucc}_{q_n}(\mathrm{stem}(q_n)) \setminus Y$ and apply Claim 9 to $(q_n \restriction \mathrm{stem}(q_n) {}^\frown \langle \alpha \rangle, \delta_{n+1})$ in order to obtain $((q_{n+1}, \dot{d}_{n+1}), i_{n+1})$. By elementarity, we can obtain $((q_{n+1}, \dot{d}_{n+1}), i_{n+1}) \in M$, so $i_{n+1} < k$, hence we can continue the construction.

[2]This is where the proof of Fact 6 is being imitated (see [4, Section 2.1, Fact 5]). Preservation of ω_1 for classical Namba forcing (and for various other Namba forcings) is achieved by showing that the forcing preserves stationary subsets of ω_1, in which case the game involves deciding ordinals in a club subset of ω_1.

[3]Technically we get club-many such $k < \omega_1$.

Since Player II does not lose at any initial stage, they win the run of the game $\mathcal{G}_k$. Hence we have obtained our contradiction. □

Now we will build a condition $(q, \dot{d}) \in \mathbb{P} * \dot{\mathbb{Q}}$ by a fusion process in such a way that any stronger condition deciding $\dot{F} \upharpoonright k$ will also code a cofinal sequence in ω_2 that exists in the ground model V, thus obtaining a contradiction.

For this we define a bit more standard notation. For all $n < \omega$, define the n^{th}-*order splitting front* of some $p \in \mathbb{P}$ as the set of $t \in p$ such that $|\operatorname{osucc}_p(t)| > 1$ and such that there are at most n-many $t' \sqsubseteq t$ with $t' \neq t$ that have this property.

Fix a sequence $\langle \delta_n : n < \omega \rangle$ converging to k. We will define $\langle (p_n, \dot{c}_n) : n < \omega \rangle$ by induction on $n < \omega$ in such a way that:

(1) $\langle p_n : n < \omega \rangle$ is a fusion sequence,
(2) for all $n < \omega$, $p_{n+1} \Vdash$ "$\dot{c}_{n+1} \leq \dot{c_n}$",
(3) For all $n < \omega$, if A_n is the n^{th}-order splitting front of p_n, then for all $t \in A_n$, the following is the case: Let $s_0 \sqsubseteq s_1 \sqsubseteq \ldots \sqsubseteq s_n = t$ be the sequence of all splitting nodes up to and including t. Then there is a sequence $Z_0^t, \ldots, Z_n^t$ such that

$$(Z_0^t, \delta_0), ((p_0 \upharpoonright s_0, \dot{d}_0), i_0), \ldots, (Z_n^t, \delta_n), ((p_n \upharpoonright s_n, \dot{d}_n), i_n)$$

is a run of the game $\mathcal{G}_k$ in which Player II's moves are determined by the winning strategy obtained in Claim 10.

Note that the third point implies the following: For all positive $n < \omega$, if A_n is the n^{th}-order splitting front of p_n, then for all $t \in A_n$, there is $i_t \in (\delta_n, k)$ and a sequence $\langle a_s : s \in \operatorname{succ}_{p_n}(t) \rangle$ witnessing that $\varphi(i_t, (p_n \upharpoonright t, \dot{c}_n))$ holds.

We do this as follows: Start with stage -1 for convenience and let $(p_{-1}, \dot{c}_{-1}) = (p, \dot{c})$. Then $A_{-1} = \operatorname{stem}(p_{-1})$. Now assume we have defined p_{n-1}, that A_n is its n^{th}-order splitting front, and we are considering $t \in A_n$. Let $s_0 \sqsubseteq s_1 \sqsubseteq \ldots \sqsubseteq s_{n-1} = t$ be the sequence of splitting nodes up to and including t. Let S_t be the set of $\alpha \in \operatorname{osucc}_{p_{n-1}}(t)$ such that for some Z_n^α, the winning strategy for Player II applied to the sequence

$$(Z_0^t, \delta_0), ((p_0 \upharpoonright s_0, \dot{d}_0), i_0), \ldots, (Z_{n-1}^t, \delta_{n-1}), ((p_{n-1} \upharpoonright s_{n-1}, \dot{d}_{n-1}), i_{n-1}), (Z_n^\alpha, \delta_n)$$

produces some $((q_n, \dot{d}_n), i_n)$ where $q_n \leq p_{n-1} \upharpoonright t^\frown \langle \alpha \rangle$. We claim that $|S_t| = \aleph_2$. Otherwise Player I would have a winning move for the sequence

$$(Z_0^t, \delta_0), ((p_0 \upharpoonright s_0, \dot{d}_0), i_0), \ldots, (Z_{n-1}^t, \delta_{n-1}), ((p_{n-1} \upharpoonright s_{n-1}, \dot{d}_{n-1}), i_{n-1})$$

by playing S_t as the subset-of-$\aleph_2$-component of their move. For each such $t \in A_n$ and $\alpha \in S_t$, choose $(q_{t,\alpha}, \dot{d}_{t,\alpha})$ to be produced by the winning strategy for Player II as the $\mathbb{P} * \dot{\mathbb{Q}}$ component of their move. Now let $p_n = \bigcup \{q_{t,\alpha} : t \in A_n, \alpha \in S_t\}$. Then use Claim 8 to obtain $\dot{c}_n$ from the $\dot{c}_{t,\alpha}$'s.

Now let q be the fusion limit of $\langle p_n : n < \omega \rangle$ and let $\dot{d}$ be the name canonically forced by q to be a lower bound for $\langle \dot{c}_n : n < \omega \rangle$. Then $(q, \dot{d})$ forces that the generic sequence for $\mathbb{P}$ can be recovered from $\dot{F} \upharpoonright k$ as follows: Let $(r, \dot{e}) \leq (q, \dot{d})$ force $\dot{F} \upharpoonright k = g \in V$. We can inductively choose a cofinal branch $b \subset r$ such for all $t \in b$, for some $i < k$, $g \upharpoonright i_t = a_t$. Specifically, we construct b by defining a sequence $\langle s_n : n < \omega \rangle$ of splitting nodes as follows: Let $s_0 = \operatorname{stem} q$. Given s_n, let $s_{n+1}^* \sqsupseteq s_n$ be the next splitting node. Then since $\varphi(i_t, (r \upharpoonright s_{n+1}^*, \dot{e}))$ holds for some $i_t \in (\delta_n, k)$, there is some $\alpha \in \operatorname{osucc}_r(s_{n+1}^*)$ such that $(r \upharpoonright s_{n+1}^* {}^\frown \langle \alpha \rangle, \dot{e}) \Vdash$ "$g \upharpoonright i_t = a_t$". Then lelt $s_{n+1} = s_{n+1}^* {}^\frown \langle \alpha \rangle$. Then let $b = \{t \in r : \exists n < \omega, t \sqsubseteq s_n\}$. This implies that

$(r, \dot{e})$ forces that the generic object is equal to b, i.e. that $\bigcap \Gamma(\mathbb{P}) = b \in V$, but this is not possible.

Hence $(q, \dot{d}) \Vdash$ "$\dot{F} \restriction k \notin V$" lest we obtain the contradiction from the previous paragraph. This contradicts the premise from the beginning of the proof that initial segments of $\dot{F}$ are in V. □

Now we can work on proving Corollary 2. This is mostly a matter of noting some statements in the established theory. First we state the fact that is most frequently used to produce various sorts of guessing models:

Fact 11 (Woodin)**.** *(Implicit in* [16, Theorem 2.53]*) Suppose that $\mathbb{P}$ is a forcing poset and that for all sequences $\mathcal{D} = \langle D_\alpha : \alpha < \omega_1 \rangle$ of dense subsets of $\mathbb{P}$, there is a $\mathcal{D}$-generic filter. Then for any regular cardinal θ such that $\mathbb{P} \in H(\theta)$, there are stationarily many $M \in P_{\omega_2}(H(\theta))$ with $\omega_1 \subseteq M$ for which there exists an $(M, \mathbb{P})$-generic filter.*

Then we use a property to violate internal unboundedness. The following is a partial weakening of Cox-Krueger [3, Lemma 5.2].

Fact 12. *Let $\mathbb{P}$ be a forcing poset such that $\mathbb{P} \in H(\theta)$ for a regular cardinal $\theta \geq \omega_2$. Let $M \in P_{\omega_2}(H(\theta))$ be such that $\omega_1 \subseteq M$ and $M \prec (H(\theta), \in, \mathbb{P}, \tau)$. Suppose that $\mathbb{P}$ forces that there is a countable set of ordinals in τ not covered by any countable set in V. If G is an M-generic filter on $\mathbb{P}$, then there is a countable subset of M that is not covered by any countable set* in *M.*

Then we have another weaking of Cox-Krueger [3, Proposition 5.4] which an analog of the lemma used by Viale and Weiss to produce guessing models from Woodin's result [15, Lemma 4.6].

Fact 13. *Fix a regular cardinal $\theta \geq \omega_2$. Assume that the poset $\mathbb{P}$ has the weak ω_1-approximation property and forces that 2^θ has size ω_1. Then there exists a set w and a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for an ω_1-cc forcing poset such that the following holds: For any regular χ with $\mathbb{P}, \dot{\mathbb{Q}}, w \in H(\chi)$ and any $M \in P_{\omega_2}(H(\chi))$ such that $\omega_1 \subseteq M$ and $M \prec (H(\chi), \in, \mathbb{P} * \dot{\mathbb{Q}}, \theta, w)$, if there exists an M-generic filter on $\mathbb{P} * \dot{\mathbb{Q}}$, then $M \cap H(\theta)$ is indestructibly ω_1-weakly guessing.*

Proof of Corollary 2. Fix the θ such that we want stationarily many indestructibly weakly ω_1-guessing models in $P_{\omega_2}(H(\theta))$ that are not internally unbounded. Let $\mathbb{P}$ denote the classical Namba forcing and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for the Lévy collapse $\mathrm{Col}(\omega_1, (2^\theta)^+)$. Let $\chi > \theta$ be large enough for $H(\chi)$ to contain the needed objects referred to in the statement of Fact 13 and then apply MM to Fact 11 to find a stationary set $S' \subseteq H_{\omega_2}(H(\chi))$ of $M \supseteq \omega_1$ for which there is an $(M, \mathbb{P} * \dot{\mathbb{Q}})$-generic filter. Then the set $S := \{M \cap H(\theta) : M \in S'\}$ is the stationary set we are looking for. Since $\mathbb{P}$ adds a countable sequence in ω_2 not covered by any set in V and since we can restrict to find an $(M, \mathbb{P})$-generic filter, it follows by Fact 12 that for all $M \in S'$, M is not internally unbounded as witnessed by a subsequence of ω_2, so this is also the case for $M \cap H(\theta)$. Since MM implies BFP, Theorem 1 implies that $\mathbb{P} * \dot{\mathbb{Q}}$ has the weak ω_1-approximation property, so it follows by Fact 13 that all $M \in S$ are indestructibly weakly guessing. □

3. Higher Namba Forcing and Cardinal Preservation

Now we work towards proving Theorem 3. The work uses higher Laver-Namba forcings employed by Cox and Krueger to obtain a stationary set of guessing models.

We retain the crux of their argument: A deft use of the pigeonhole principle, where $\dot{f}$ is a name for a function, $\langle D_\xi : \xi \in \mathrm{dom}\, \dot{f} \rangle$ is a sequence of open dense sets deciding values $\dot{f}$ and a stem length $n < \omega$ is isolated so that unboundedly many values of the function can be decided with a single condition. In Cox and Krueger's case they argue that a certain Laver-Namba forcing has a weak approximation property, but in our case we will be showing that certain cardinals are preserved.

We let $\mathbb{P}_{\mathsf{LNF}}(I)$ be the Laver version of Namba forcing with respect to an ideal I on some cardinal λ. This means that $p \in \mathbb{P}_{\mathsf{LNF}}$ if and only if $p \subseteq {}^{\omega}\lambda$ and there is some $t \in p$ such that for all $s \sqsupseteq t$, $\mathrm{osucc}_p(s) \in I^+$. The main feature of the Laver version to consider is that we can use direct extensions. Given $p, q \in \mathbb{P}_{\mathsf{LNF}}(I)$, we write $p \leq^* q$ if $p \leq q$ and $\mathrm{stem}(p) = \mathrm{stem}(q)$. We will also use a principle found by Laver, for which we will suggest some useful notation:

Definition 14. [12, Chapter X,Definition 4.10] Given a successor cardinal κ^+, we write $\mathsf{LIP}(\kappa^+)$ if there is a κ^+-complete ideal $I \subset P(\kappa^+)$ such that there is a set $\mathcal{D} \subseteq I^+$ that is κ-closed subset and dense in itself, i.e. for all $A \in I^+$, there is some $B \subseteq A$ with $B \in I^+$ such that $B \in \mathcal{D}$.

Fact 15 (Laver)**.** *If $\kappa < \mu$ where κ is regular and μ is measurable, then* $\mathrm{Col}(\kappa, <\mu)$ *forces* $\mathsf{LIP}(\mu)$*.*

Laver's proof of Fact 15 is unpublished, but the argument is similar to the one found by Galvin, Jech, and Magidor for obtaining a certain precipitous ideal on $\aleph_2$ [5]. Some additional details appear in Shelah [12, Chapter X]. We will use a version of this result for successive cardinals that also comes from unpublished work of Laver:

Theorem 16 (Laver)**.** *Assume the consistency of class-many supercompact cardinals. Then there is a forcing extension in which* $\mathsf{LIP}(\lambda^{++})$ *holds for all infinite cardinals λ. (See* [7, Page 89] *and* [11] *for related arguments.)*

The following facts are covered in detail by Cox and Krueger, though we state them in some specificity. They are fairly standard arguments for Namba forcings, and the first two facts to some extent go back to Laver's proof of the consistency of the Borel Conjecture (see also Cummings-Magidor [4] and Shelah [12]).

Fact 17. [3, Lemma 6.5],[4, Section 2.1, Fact 1] *Suppose $I \subseteq \kappa^+$ is a κ^+-complete ideal and suppose $p \in \mathbb{P}_{\mathsf{LNF}}(I)$ forces that $\dot{\gamma}$ is a name for an ordinal below κ. Then there is some $q \leq^* p$ and $\delta \in \mathrm{ON}$ such that $q \Vdash$ "$\dot{\gamma} = \delta$".*

Fact 18. [3, Lemma 6.4],[4, Section 2.1, Fact 2] *Suppose $I \subseteq \kappa^+$ is a κ^+-complete ideal. Let $D \subseteq \mathbb{P}_{\mathsf{LNF}}(I)$ be dense open. Then for each $p \in \mathbb{P}_{\mathsf{LNF}}(I)$, there is some $q \leq^* p$ and some $n < \omega$ such that for any $t \in q$ with $|t| = \mathrm{stem}(p) + n$, $q \restriction t \in D$.*

Finally, we have closure of the direct extension, the argument for which is much easier than the one for the last two facts. Lower bounds can be obtained by inductively defining splitting sets via the lower bounds for $\mathsf{LIP}(\kappa^+)$.

Fact 19. [3, Lemma 6.13] *Suppose that* $\mathsf{LIP}(\kappa^+)$ *holds and is witnessed by I and that $\eta < \kappa$. If $\langle p_\xi : \xi < \eta \rangle$ is a $\leq^*$-decreasing sequence of conditions in $\mathbb{P}_{\mathsf{LNF}}(I)$, then there is $\bar{p}$ such that $\bar{p} \leq^* p_\xi$ for all $\xi < \eta$.*

Proof of Theorem 3. Using Theorem 16, assume that $\mathsf{LIP}(\lambda^{++})$ holds for all infinite λ. For each infinite cardinal μ such that $\mathsf{LIP}(\mu)$ holds, let I_μ be the witnessing ideal.

We will argue that for all $\mu = \lambda^{++}$, $\mathbb{P}_\mu := \mathbb{P}_{\mathsf{LNF}}(I_\mu)$ preserves the cofinalities of all regular $\nu \leq \lambda^+$. To do this, we fix some λ^{++} and choose some regular $\nu \leq \lambda^+$ for which we will prove the statement.

First we consider the possibility that $\nu > \aleph_0$ is forced by some $p \in \mathbb{P}_\mu$ to have countable cofinality. Then if $\dot{f} : \omega \to \nu$ is a name for an unbounded function we can use Fact 17 to choose a $\leq^*$-descending sequence $\langle p_n : n < \omega \rangle$ below p such that p_n forces "$\dot{f}(n) = \beta_n$" for some β_n. Then use Fact 19 to obtain a $\leq^*$-lower bound q for this sequence. Then $q \Vdash$ "$\sup f[\omega] = \sup_{n<\omega} \beta_n < \nu$", which is a contradiction.

Now suppose that ν is forced to have cofinality τ where $\aleph_0 < \tau < \nu$. Suppose that $p \in \mathbb{P}_\mu$ forces that $\dot{f}$ is a $\mathbb{P}_\mu$-name for a function $\tau^V \to \nu$. If $\dot{f}$ were a surjection, then there would be a strictly increasing and unbounded function, so assume without loss of generality that $\dot{f}$ is forced by p to be strictly increasing and unbounded.

For all $\xi < \tau$, let $D_\xi \subseteq \mathbb{P}_\mu$ be the open dense set of conditions deciding $\dot{f}(\xi)$. Now we define a $\leq^*$-decreasing sequence $\langle p_\xi : \xi < \tau \rangle$ as follows: Let $p_0 = p$. If p_ξ is defined, use Fact 18 to obtain some $p_{\xi+1} \leq^* p_\xi$ such that for some $n_\xi < \omega$, for any $t \in q$ with $|t| = \mathrm{stem}(p_\xi) + n_\xi$, we have $q \restriction t \in D_\xi$. If ξ is a limit and we have defined $\langle p_{\xi'} : \xi' < \xi \rangle$, then let p_ξ be a $\leq^*$-lower bound obtained using Fact 19. Once we have defined the sequence, apply Fact 19 again to obtain $\bar{p}$, a $\leq^*$-lower bound of $\langle p_\xi : \xi < \tau \rangle$ (if $\nu^+ = \mu$ then this is the best possible $\leq^*$-closure).

Now apply the Pigeonhole Principle to find an unbounded set $X \subseteq \tau$ and some $\ell < \omega$ such that for all $\xi \in X$, $n_\xi = \ell$. Take any $t \in \bar{p}$ such that $|t| = |\mathrm{stem}(\bar{p})| + \ell$ and let $q = \bar{p} \restriction t$. Then for all $\xi \in X$, $q \in D_\xi$ and therefore there is some β_ξ such that $q \Vdash$ "$\dot{f}(\xi) = \beta_\xi$". Let $\beta := \sup_{\xi \in X} \beta_\xi + 1 < \nu$. Then since q forces that $\dot{f}$ is strictly increasing, it follows that q forces that the range of $\dot{f}$ is bounded by β. This shows that ν is preserved by $\mathbb{P}_\mu$ and thus completes the proof. □

Remark 20. Note that the argument for the case of $\aleph_2$ in the proof of Theorem 3 essentially tells us that having a precipitous ideal on $\aleph_2$ implies CH: Modulo some details, $\mathbb{P}_{\mathsf{LNF}}(I)$ does not add reals if I is precipitous, yet failure of CH implies that $\mathbb{P}_{\mathsf{LNF}}(I)$ (and $\aleph_2$-splitting Namba forcing in general) adds reals. Hence the nice behavior obtained for $\mathbb{P}_{\mathsf{LNF}}(I)$ for higher cardinals does not seem adaptable to the situation in Theorem 1.

We close with some questions:

(1) What is the exact consistency strength of the statement that $\mathbb{P}_{\mathsf{CNF}}$ has the weak ω_1-approximation property?
(2) Suppose that I is just the bounded ideal on $\aleph_2$. Is it consistent that the Laver-Namba forcing $\mathbb{P}_{\mathsf{LNF}}(I)$ has the weak ω_1-approximation property? What about for other ideals I on $\aleph_2$? (See Remark 20.)
(3) More broadly, are there more specific applications of Theorem 1 to the program suggested by Viale and Weiss for studying guessing models under fragments of Martin's Maximum?

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