A NOTE ON THE CUT AND CHOOSE GAME

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Abstract. Jech popularized the study of the cut and choose game $\mathcal{G}^{\mathsf{c\&c}}_{\lambda}(\mathbb{B})$ for a complete Boolean algebra $\mathbb{B}$, where λ indicates the length of the game, establishing an interplay between the existence of winning strategies and distributivity properties of the given Boolean algebra, and Zapletal furthered this program. We extend a theorem of Zapletal, and thereby answer one of his questions [Zap95, Question 2], by proving that, up to the consistency of a supercompact cardinal, it is consistent that for all $(\omega_1, 2)$-distributive Boolean algebras $\mathbb{B}$, the choosing player has a winning strategy in $\mathcal{G}^{\mathsf{c\&c}}_{\omega}(\mathbb{B})$.

1. Introduction

This note is intended as an early presentation of material that will be included in a longer paper.

1.1. Definitions and Notation.

Before proving the results, we will establish some definitions and notation. The reader is assumed to be familiar with the basics of forcing and large cardinals (see e.g. [Jec03, Kun14]). Some additional definitions that are well-known but do not appear in introductory sources will be given throughout.

We should also comment on some particularities of notation. The notation λ-distributive indicates that functions with domain λ are not added (see below). If $\dot{x}$ is a $\mathbb{P}$-name and G is $\mathbb{P}$-generic over V, we will not emphasize that $x = \dot{x}_G$ when the notation makes this obvious. As for images of functions, we write $f[X] = \{f(z) \mid z \in X\}$.

For a Boolean algebra $\mathbb{B}$, we let $\wedge$ and $\vee$ be the meet and join operations, respectively, and we let $\sim$ indicate the complement. We let $0_{\mathbb{B}}$ and $1_{\mathbb{B}}$ be the respective minimal and maximal elements of $\mathbb{B}$. If $b \wedge c = 0_{\mathbb{B}}$, we write $b \perp c$. Boolean algebras are endowed with a natural order $\leq_{\mathbb{B}}$ where $b \leq_{\mathbb{B}} c$ if and only if $c \wedge b = b$. We say that $\mathbb{B}$ is $<\nu$-complete if for all $X \subseteq \mathbb{B}$ with $|X| < \nu$, $\bigvee X \in \mathbb{B}$ and $\bigwedge X \in \mathbb{B}$. We say that $\mathbb{B}$ is *complete* if it is $<\nu$-complete for all ν. See the Handbook of Boolean Algebras to refer to elementary concepts [Kop89].

Definition 1.1 ([BS89, Def.17])**.** Let $\mathbb{B}$ be a Boolean algebra. Let θ, μ, ν be cardinals such that $\theta, \nu \geq \omega$ and $\mu \geq 2$.

1. A collection $X \subseteq \mathcal{P}(\mathbb{B}^+)$ is called a *matrix* if each member of X is a maximal disjoint subset of $\mathbb{B}^+$.
2. We say that $\mathbb{B}$ is (θ, ν, μ)-*distributive* if for every matrix $X = \{X_\alpha \mid \alpha \in \theta\}$ such that $|X_\alpha| \leq \nu$ there is some maximal disjoint set $W \subseteq \mathbb{B}^+$ such that for each $q \in W$ and $\alpha \in \theta$, $|\{p \in X_\alpha \mid p \wedge q \neq 0_{\mathbb{B}}\}| < \mu$.
3. If $\mathbb{B}$ is $(\theta, \nu, 2)$-distributive, then we say that $\mathbb{B}$ is (θ, ν)-distributive.
4. Finally, $\mathbb{B}$ is called θ-distributive if it is (θ, ν)-distributive for any ν.

We stated the three-parameter version of distributivity to orient the reader, but this paper will only focus on two-parameter distributivity.

Fact 1.2. *The following are equivalent for non-atomic $\mathbb{B}$.*

(1) $\mathbb{B}$ is (θ, ν)-distributive,

(2) forcing with $\mathbb{B}$ does not add new functions $f : \theta \to \nu$.

We will momentarily give a precise definition of the cut and choose game.

Definition 1.3. Given a Boolean algebra $\mathbb{B}$ and $b \in \mathbb{B}$, we say that $\{b_0, b_1\}$ is a *partition* of b if $b_0 \vee b_1 = b$ and $b_0 \wedge b_1 = 0_{\mathbb{B}}$. We let $\text{part}(b)$ denote the set of partitions of b, and we let $\text{part}(\mathbb{B}) = \{\text{part}(b) \mid b \in \mathbb{B}\}$.

First, let us define the game informally. The cut and choose game of length λ begins with cut making an opening move $\bar{b} \in \mathbb{B}$ and selecting a partition $\{b_0^0, b_1^1\} \in \text{part}(\bar{b})$. Then choose selects an element of the partition $b_0^{\epsilon_0}$ where $\epsilon_0 \in \{0,1\}$. (Typically the game is written with choose's moves rendered as digits, but the constructions below seem to call for different notation.) Then cut plays another partition, and so on. At limits, if there is a nonzero infimum of choose's choices, then the game proceeds with cut partitioning the infimum. If the infimum is zero, choose loses. If λ-many rounds occur without choose losing, then choose wins at the end if $\bigwedge_{i<\lambda} b_\xi^{\epsilon_\xi} \neq 0_{\mathbb{B}}$.

cut	$\bar{b}, \{b_0^0, b_0^1\}$		$\{b_1^0, b_1^1\}$	$\dots$	$\{b_\xi^0, b_\xi^1\}$		
choose		$b_0^{\epsilon_0}$		$\dots$		$b_\xi^{\epsilon_\xi}$	$\dots$

Now let us present a more formal definition.

Definition 1.4. Fix a $<\lambda^+$-complete Boolean algebra $\mathbb{B}$. We describe *the cut and choose game of length* λ, denoted $\mathcal{G}_\lambda^{\mathsf{c\&c}}(\mathbb{B})$. This is a game of perfect information between two players, denoted cut and choose

(1) A partial play of $\mathcal{G}_\lambda^{\mathsf{c\&c}}(\mathbb{B})$ of length $\eta \leq \lambda + 1$ takes the form

$$\vec{s} = \bar{b}^\frown \langle \{b_\xi^0, b_\xi^1\}, b_\xi^{\epsilon_\xi} \mid \xi < \eta \rangle$$

if it is cut's turn, or

$$\vec{s} = \bar{b}^\frown \langle \{b_\xi^0, b_\xi^1\}, b_\xi^{\epsilon_\xi} \mid \xi < \eta \rangle^\frown \{b_\eta^0, b_\eta^1\}$$

if it is choose's turn (where $\epsilon_\xi \in \{0,1\}$ in both cases). For all $\xi \leq \eta$,

$$\{b_\xi^0, b_\xi^1\} \in \text{part}\left(\bigwedge_{\zeta<\xi} b_\zeta^{\epsilon_\zeta}\right).$$

We refer to $\bar{b}$ as the *opening move*, we refer to $\{b_\xi^0, b_\xi^1\}, b_\xi^{\epsilon_\xi}$ as the *ξ'th round.*

(2) The player choose wins the game if the play has length $\lambda + 1$, i.e. where after λ moves it holds that

$$\bar{b} \wedge \bigwedge_{\xi \leq \lambda} b_\xi^{\epsilon_\xi} \neq 0_{\mathbb{B}}$$

where $\bar{b}$ is the opening move.

If we want to indicate a game of length λ that does not have a λ'th round, we write $\mathcal{G}_{<\lambda}^{\mathsf{c\&c}}(\mathbb{B})$. In this formulation, choose wins if and only if they can play at all rounds up to but not including λ. In our notation, $\mathcal{G}_\omega^2(\mathbb{B})$ is the standard cut and choose game of length ω.

Fact 1.5. *Let $\mathbb{B}$ be a complete Boolean algebra and λ a cardinal. The existence winning strategy for* cut *or, respectively,* choose*, is equivalent for the following games:*

(1) $\mathcal{G}^{\mathsf{c\&c}}_{\lambda}(\mathbb{B})$,

(2) an alternation of $\mathcal{G}^{\mathsf{c\&c}}_{\lambda}(\mathbb{B})$ *which is identical except that partial plays take the form*

$$\vec{s} = \bar{b}^\frown\langle \{b^0_\xi, b^1_\xi\}, b^{\epsilon_\xi}_\xi \mid \xi < \eta\rangle$$

where for each $\xi < \eta$,

$$\{b^0_\xi, b^1_\xi\} \in \operatorname{part}\left(\bigwedge_{\zeta<\xi} b^{\epsilon_\zeta}_\zeta\right).$$

Much of the literature considers other parameters for the cut and choose game, for example allowing the partitions to have cardinality greater than 2, and possibly allowing choose to select more than one element of the partitions. One could use the notation $\mathcal{G}^2_\lambda(\mathbb{B})$ for the cut and choose game discussed here, and $\mathcal{G}^\tau_\lambda(\mathbb{B})$ for a cardinal τ for larger partitions. Sometimes, notation similar to $\mathcal{G}^\infty_\lambda(\mathbb{B})$ is used to indicate that cut may play partitions of arbitrary cardinality. The Banach-Mazur game of length κ on Boolean algebras can be written as $\mathcal{G}^\infty_\kappa(\mathbb{B})$ for the given Boolean algebra $\mathbb{B}$ because work of Jech and Veličković ([Jec84, Theorem 2] and [Vel86, Theorem 2.1]) shows that there is an equivalence.

2. Extending a Theorem of Zapletal

Our point of departure is a theorem of Zapletal, who proved that, assuming the consistency of a supercompact cardinal, there is a model of set theory in which choose has a winning strategy in $\mathcal{G}^{\mathsf{c\&c}}_{\omega}(\mathbb{B})$ for every $\aleph_1$-distributive Boolean algebra $\mathbb{B}$.

We want to obtain a slight improvement:

Theorem 2.1. *Suppose κ is a supercompact cardinal. Then if G is* $\mathrm{Col}(\omega_1, {<}\kappa)$*-generic over V, then in $V[G]$, for every Boolean algebra $\mathbb{B}$ that does not add new subsets of ω_1, the player* choose *has a winning strategy in* $\mathcal{G}^{\mathsf{c\&c}}_{\omega}(\mathbb{B})$.

This answers Question 2 in Zapletal's paper "More on the Cut and Choose Game" [Zap95].

The proof of Theorem 2.1 will serve as a template for the following results in this paper, both in terms of how the lifting arguments work, and in terms of how the quotient is treated—although the later quotient lemmas will be more complex.

2.1. Proving the Theorem.

Note: For this section, we will use the version of the cut and choose game in which cut partitions the meet of choose's moves in every round. In other words, we are using characterization *(2)* from Fact 1.5.

We obtain our first preservation lemma now.

Lemma 2.2. *Suppose $\mathbb{P}$ is ${<}\mu$-closed, $\lambda < \mu$, and $\mathbb{B}$ is a complete Boolean algebra. If $\mathbb{P}$ forces that* choose *has a winning strategy for* $\mathcal{G}^{\mathsf{c\&c}}_{\lambda}(\mathbb{B})$, *then* choose *has a winning strategy for* $\mathcal{G}^{\mathsf{c\&c}}_{\lambda}(\mathbb{B})$ *in the ground model.*

Proof. Let $\dot{\sigma}$ be a $\mathbb{P}$-name that is forced to be a winning strategy for choose in $\mathcal{G}^{\mathsf{c\&c}}_{\lambda}(\mathbb{B})$. We fix some $\bar{b}$ and describe a winning strategy for a run of the game where $\bar{b}$ is cut's opening move. Let $\Theta = |\mathbb{B}|$, and for $b \in \mathbb{B}$, let $\langle \{b^{\xi,0}, b^{\xi,1}\} \mid \xi < \Theta\rangle$ be a fixed enumeration of partitions of b (allowing repetitions).

We will construct an assignment on the tree ${}^{<\lambda}\Theta$ of the form $t \mapsto (b_t^{\bar{b}}, p_t^{\bar{b}}) \in \mathbb{B}^+ \times \mathbb{P}$ for $t \in {}^{<\lambda}\Theta$. Technically the superscripts give information about the opening move, but let us suppress this notation and instead write (b_t, p_t) for $t \in {}^{<\lambda}\Theta$.

This assignment will have the property that $b_{t^\frown\langle\xi\rangle} \in \{b_t^{\xi,0}, b_t^{\xi,1}\}$ for all t. Therefore, an element t with domain $\mathrm{dom}(t) = j$ can be associated with a play of $\mathcal{G}_\lambda^{\mathsf{c\&c}}(\mathbb{B})$ of the form

$$\vec{s}_t = \langle \{b_{t\restriction i}^{t(i),0}, b_{t\restriction i}^{t(i),1}\}, b_{t\restriction(i+1)} \mid i < j\rangle$$

where it is cut's turn to play.

We will construct the assignment such that the following properties hold:

(1) $b_\emptyset = \bar{b}$,
(2) if $u \sqsupseteq t$ then $p_u \leq_{\mathbb{P}} p_t$,
(3) if $\mathrm{dom}(t) = i+1$ and $t = u^\frown\langle\xi\rangle$, then $p_t \Vdash$ "$\dot{\sigma}(\vec{s}_u{}^\frown\{b_u^{\xi,0}, b_u^{\xi,1}\}) = b_t$".

The assignment is constructed by induction on $\mathrm{dom}(t)$.

Suppose that $\mathrm{dom}(t) = i+1$, that $t = u^\frown\langle\xi\rangle$, and we have defined b_u and p_u. Then choose $p' \leq p_u$ such that p' forces "$\dot{\sigma}(\vec{s}_u{}^\frown\{b_u^{\xi,0}, b_u^{\xi,1}\}) = b'$". Then let $p_t = p'$ and let $b_t = b'$.

Suppose that $\mathrm{dom}(t)$ is a limit i. Then let p_t be a lower bound of $\langle p_{t\restriction i} \mid i < \mathrm{dom}(t)\rangle$ and let $b_t = \bigwedge_{i<\mathrm{dom}(t)} b_{t\restriction i}$. Since p_t forces that $\vec{s}_t$ is a run of the game in which choose plays according to $\dot{\sigma}$, and that b_t is the meet of choose's selections, it follows that $b_t \neq 0_{\mathbb{B}}$.

Now we will define a winning strategy $\tilde{\sigma}$ for choose in V. Fix an opening move $\bar{b}$; we will denote $\tilde{\sigma}$ with respect to this opening move given the assignment $t \mapsto (b_t, p_t)$ with $\bar{b}$ implicit. In particular, we will define $\tilde{\sigma}$ such that any play of the game (with $\bar{b}$ opening) in which it is choose's turn will take the form $\vec{s}_t{}^\frown\{b_t^{\xi,0}, b_t^{\xi,1}\}$ for some $t \in {}^{<\lambda}\Theta$ and $\xi < \Theta$. Given such a position, choose will select $b_{t^\frown\langle\xi\rangle}$. To see that this is a winning strategy, note that any full run of the game in which choose plays according to $\tilde{\sigma}$, minus choose's last move, can be represented as

$$\vec{s} = \langle \{b_{f\restriction i}^{f(i),0}, b_{f\restriction i}^{f(i),1}\}, b_{f\restriction(i+1)} \mid i < \lambda\rangle$$

for some $f : \lambda \to \Theta$. Let p_f be a lower bound of $\langle p_{f\restriction i} \mid i < \lambda\rangle$. Then p_f forces that $\vec{s}$ is a run of the game in which choose plays according to the strategy $\dot{\sigma}$, and therefore an extension of p_f forces that $\langle b_{f\restriction i} \mid i < \lambda\rangle$ has a nonzero lower bound b_f. Hence $\langle b_{f\restriction i} \mid i < \lambda\rangle$ has a lower bound in V. □

Let us clarify a definition that could be taken for granted.

Definition 2.3. Let $\mathbb{B}$ be a λ^+-complete Boolean algebra.

(1) We say that $\mathcal{F} \subseteq \mathcal{P}(\mathbb{B})$ is an *ultrafilter* if the following hold: for all $b \in \mathbb{B}$, either $b \in \mathcal{F}$ or $\sim b \in \mathcal{F}$; for all $b, c \in \mathcal{F}$, $b \wedge c \in \mathcal{F}$; $b \in \mathcal{F}$ and $c \geq_{\mathbb{B}} b$ implies $c \in \mathcal{F}$; and $\emptyset \notin \mathcal{F}$.
(2) Let $\mathcal{F}$ be an ultrafilter on a complete Boolean algebra $\mathbb{B}$ and let λ be a regular cardinal. We say that $\mathcal{F}$ is λ-complete if for all $\tau < \lambda$ and $\langle b_\xi \mid \xi < \tau\rangle \subseteq \mathcal{F}$, we have $\bigwedge_{\xi<\tau} b_\xi \in \mathcal{F}$.

Proposition 2.4. *Suppose that $\mathbb{B}$ is a complete Boolean algebra and λ is a regular cardinal. If for all $b \in \mathbb{B}^+$ there is a λ^+-complete ultrafilter $\mathcal{F}$ on $\mathbb{B}$ with $b \in \mathcal{F}$, then it follows that* choose *has a winning strategy in the game $\mathcal{G}_\lambda^{\mathsf{c\&c}}(\mathbb{B})$.*

Proof. We define a winning strategy σ for choose with respect to an opening move $\bar{b}$ played by cut. Suppose $\vec{s}$ is a partial play where it is choose's turn and the last move of cut is $\{b_0, b_1\}$. Then choose selects b_ϵ such that $b_\epsilon \in \mathcal{F}$. We argue that σ is well-defined and is a winning strategy by observing that for all partial plays $\vec{s}$, if $\{b_\xi \mid \xi < \eta\}$ is the set of choose's choices, then $b_* := \bigwedge_{\xi<\eta} b_\xi$ is in $\mathcal{F}$. Therefore, for any partition $\{c_0, c_1\} \in \mathrm{part}(b_*)$, there is some ϵ such that $c_\epsilon \wedge b_* \neq 0_{\mathbb{B}}$. □

Proof of Theorem 2.1. We can actually prove the theorem for λ^+ in place of ω_1 for any cardinal $\lambda < \kappa$.

Let $\dot{\mathbb{B}}$ be a $\mathrm{Col}(\lambda^+, {<}\kappa)$-name for a complete $(\lambda^+, 2)$-distributive Boolean algebra of cardinality ν, let $j : V \to M$ be a ν-supercompact embedding with critical point κ, and let G be $\mathrm{Col}(\lambda^+, {<}\kappa)$-generic over V. Write $\mathbb{B} = \dot{\mathbb{B}}_G$.

As in standard arguments, we write

$$j(\mathrm{Col}(\lambda^+, {<}\kappa)) = \mathrm{Col}(\lambda^+, {<}\kappa) \times \prod_{\kappa \leq \alpha < j(\kappa)} \mathrm{Col}(\lambda^+, \alpha) = \mathrm{Col}(\lambda^+, {<}\kappa) \times \mathbb{P}$$

where $\mathbb{P}$ is our notation for the countably closed remainder term.

We will argue that in $M[G][H]$, there is a λ^+-complete ultrafilter (i.e. a countably complete ultrafilter) $\mathcal{F}$ on $\mathbb{B}$. By Proposition 2.4, this implies that in $V[G][H]$, choose has a winning strategy in $\mathcal{G}^{\mathsf{c\&c}}_\lambda(\mathbb{B})$. By Lemma 2.2 and the fact that $\mathbb{P}$ is countably closed, this implies that choose has a winning strategy for $\mathcal{G}^{\mathsf{c\&c}}_\lambda(\mathbb{B})$ in $V[G]$.

We establish some conditions for defining a generic element of $j(\mathbb{B})$. Let $\langle \dot{b}_\xi \mid \xi < \nu\rangle$ be a sequence of $\mathrm{Col}(\lambda^+, {<}\kappa)$-names forced to be an enumeration of $\dot{\mathbb{B}}$. Then $\langle j(\dot{b}_\xi) \mid \xi < \nu\rangle \in M$ by $M^\nu \subseteq M$, and since $G * H \in M[G * H]$, it follows that we have $\langle j(b_\xi) \mid \xi < \nu\rangle \in M[G][H]$ where $b_\xi = (\dot{b}_\xi)_G$. Then $\langle \{j(b_\xi), \sim j(b_\xi)\} \mid \xi < \nu\rangle \in M[G][H]$ is a sequence of partitions of $j(\mathbb{B})$. Also, in $M[G][H]$, ν has cardinality λ^+: if $D_\alpha \subseteq j(\mathrm{Col}(\lambda^+, {<}\kappa))$ is the open dense subset of conditions p with $\langle i, \nu\rangle \in \mathrm{dom}(p)$ and $p(i, \nu) = \alpha$ for some $i < \lambda^+$ (using that $j(\kappa) > \nu$), then $D^M_\alpha = D^V_\alpha$ and $D_\alpha \in M$ for all $\alpha < \nu$, so $\langle D_\alpha \mid \alpha < \nu\rangle \in M$, so we can define a surjection $\lambda^+ \to \nu$ in $M[G][H]$ by choosing $p_\alpha \in D_\alpha \cap (G * H)$ and taking $\bigcup_{\alpha\in\nu} p_\alpha$. Also, $M[G][H] \models$ "$j(\mathbb{B})$ is $(j(\lambda^+), j(2)) = (\lambda^+, 2)$-distributive" by elementarity.

Now we will define the object from which we define the filter: applying $(\lambda^+, 2)$-distributivity in $M[G][H]$, we find some $b_{\mathrm{gen}} \in j(\mathbb{B}) \cap M[G][H]$ with $b_{\mathrm{gen}} \neq 0_{j(\mathbb{B})}$ such that for all $\xi < \nu$, $b_{\mathrm{gen}} \wedge c \neq 0_{\mathbb{B}}$ for exactly one of $c \in \{j(b_\xi), {\sim} j(b_\xi)\}$; in other words, either $b_{\mathrm{gen}} \leq_{j(\mathbb{B})} j(b_\xi)$ or $b_{\mathrm{gen}} \perp_{j(\mathbb{B})} j(b_\xi)$.

We can therefore define

$$\mathcal{F} = \{b \in \mathbb{B} \mid b_{\mathrm{gen}} \leq_{j(\mathbb{B})} j(b)\}$$

in $V[G][H]$. That $\mathcal{F}$ is a filter is immediate from elementarity of j. To see that $\mathcal{F}$ is an ultrafilter, suppose that $\{c_0, c_1\}$ partitions $\mathbb{B}$. Then let $\xi, \eta < \nu$ be such that $c_0 = b_\xi$ and $c_1 = b_\zeta$. Then $\{j(b_\xi), j(b_\zeta)\}$ partitions $j(\mathbb{B})$ in $M[G][H]$, so either $b_{\mathrm{gen}} \wedge j(b_\xi) \neq 0_{j(\mathbb{B})}$ or $b_{\mathrm{gen}} \wedge j(b_\zeta) \neq 0_{j(\mathbb{B})}$. Hence b_{gen} is $\leq_{j(\mathbb{B})}$-below either $j(b_\xi)$ or $j(b_\zeta)$. To see that $\mathcal{F}$ is λ^+-complete, consider a sequence $\langle b_{\xi_i} \mid i < \lambda\rangle \subseteq \mathcal{F}$. Since $\mathbb{P}$ is λ^+-closed, the sequence $x := \langle b_{\xi_i} \mid i < \lambda\rangle$ is an element of $V[G]$. Then $M[G][H] \models$ "$j(x)$ has a greatest lower bound in $j(\mathbb{B})$ above b_{gen}", since $j(b_{\xi_i}) \geq_{j(\mathbb{B})} b_{\mathrm{gen}}$ for all $i < \lambda$. Therefore, by elementarity, $V[G] \models$ "x has a nonzero greatest lower bound in $\mathbb{B}$". Let $\bar{b}$ be the greatest lower bound of x in $\mathbb{B}$ as computed in

$V[G]$. Then by elementarity, since $b_{\text{gen}} \leq_{j(\mathbb{B})} j(b_{\xi_i})$ for all $i < \lambda$, it follows that $b_{\text{gen}} \leq \bigwedge_{i<\lambda} j(b_{\xi_i}) = j(\bar{b})$, and so $\bar{b} \in \mathcal{F}$.

As indicated, the fact that $\mathcal{F}$ is a λ^+-complete complete ultrafilter on $\mathbb{B}$, living in $V[G][H]$, is enough to finish the proof. □

2.2. **A Couple Elaborations.** This is a good moment to sketch some possible extensions of Theorem 2.1. If we remove the forcing from Theorem 2.1, we essentially obtain the following theorem.

Theorem 2.5. *If κ is weakly compact, then $\mathcal{G}^{\mathsf{c\&c}}_{<\kappa}(\mathbb{B})$ is determined for all complete Boolean algebras of cardinality κ.*

Proof. The forward direction of **??** reduces the problem to arguing that if $\mathbb{B}$ is a $<(\kappa, 2)$-distributive Boolean algebra of cardinality κ, then choose has a winning strategy in $\mathcal{G}^{\mathsf{c\&c}}_{<\kappa}(\mathbb{B})$. Recall the characterization of weak compactness which states that for any transitive M of cardinality κ and $\kappa \in M$, there is a transitive N and an elementary embedding $j : M \to N$ with critical point κ. Given a $<\kappa$-distributive Boolean algebra $\mathbb{B}$, let M be a κ-model with $\mathbb{B} \in M$ and let $j : M \to N$ be a weakly compact embedding with critical point κ. Let $\vec{b} = \langle \{b_\xi, \sim b_\xi\} \mid \xi < \kappa \rangle$ enumerate the partitions of $\mathbb{B}$ compatible with some $c \in \mathbb{B}$. Then $N \models$ "$j(\mathbb{B})$ is $<(j(\kappa), 2)$-distributive", so there is some $b^c_{\text{gen}} \in N \cap j(\mathbb{B})$ meeting each of the partitions in $\vec{b}$. Then $\{b^c_{\text{gen}} \mid c \in \mathbb{B}\}$ provides a winning strategy for choose in $\mathcal{G}^{\mathsf{c\&c}}_{<\kappa}(\mathbb{B})$ as in Theorem 2.1. □

It is also fairly straightforward to obtain a global version of Theorem 2.1.

Theorem 2.6. *Assume the consistency of a proper class of supercompact cardinals. Then it is consistent that, for all cardinals λ and every complete $(\lambda^+, 2)$-distributive Boolean $\mathbb{B}$, choose has a winning strategy in $\mathcal{G}^{\mathsf{c\&c}}_{\lambda}(\mathbb{B})$.*

Proof. We will show how to set up the argument so that it reduces to the proof of Theorem 2.1.

Let $\langle \kappa_\alpha \mid \alpha \in \mathrm{ON} \rangle$ be the sequence inductively defined so that κ_0 is the least supercompact, $\kappa_\alpha = \sup_{\beta<\alpha} \kappa_\beta$ for limits α, and $\kappa_{\alpha+1}$ is the least supercompact above κ_α. Let $\mathbb{P} = \langle \mathbb{P}_\alpha \mid \alpha \in \mathrm{ON} \rangle$ be the Easton-support class iteration such that $\mathbb{P}_0 = \mathrm{Col}(\omega_1, <\kappa_0)$, and if $\alpha = \beta + 1$, then $\mathbb{P}_\alpha = \mathbb{P}_\beta * \dot{\mathrm{Col}}(\aleph_{\alpha+1}, <\mu)$ where μ is the least supercompact above $\aleph_{\alpha+1}$ in the extension by $\mathbb{P}_\alpha$.

Let G be $\mathbb{P}$-generic over V. We will argue that $V[G]$ witnesses the statement of the theorem.

In $V[G]$, we argue by induction that $\kappa_\alpha = \aleph_{\alpha+2}$: The case $\alpha = 0$ being immediate. If $\alpha = \beta+1$, then it is the case that in $V[G_\beta]$, the induced $\mathbb{P}_\beta$-generic submodel, it holds that $\kappa_\beta = \aleph_{\beta+2} = \aleph_{\alpha+1}$. This is because the next step of the iteration is sufficiently distributive. So this case follows immediately from the definition of the forcing. Then the case for limit α follows easily by induction. Moreover, observe that GCH holds in $V[G]$.

Now let λ be a cardinal in $V[G]$ and let α be such that $\aleph_\alpha = \lambda$ in $V[G]$. Let $\mathbb{B}$ be a complete λ^+-distributive Boolean algebra of cardinality $\aleph_\Theta$. So $\lambda^+ \leq \aleph_\Theta$, and by GCH the set of possible plays in $\mathcal{G}^{\mathsf{c\&c}}_{\lambda}(\mathbb{B})$ has cardinality $\aleph_\Theta$. Hence we only need to argue in $V[G_\Theta]$, which contains $\mathbb{B}$ by closure. Write $\mathbb{P}_\Theta = \mathbb{P}_{\text{low}} * \mathrm{Col}(\check{\aleph}_{\alpha+1}, \mu) * \dot{\mathbb{P}}_{\text{high}}$ where $\mu := \kappa_\alpha$.

We work in $V' = V[G_{\text{low}}]$, the induced sub-extension by $\mathbb{P}_{\text{low}}$, where μ retains its supercompactness by smallness of $\mathbb{P}_{\text{low}}$. It is sufficient to show that $\text{Col}(\aleph_{\alpha+1}, < \mu) * \dot{\mathbb{P}}_{\text{high}}$ forces that choose has a winning strategy in $\mathcal{G}^{\mathsf{c\&c}}_{\lambda}(\mathbb{B})$. Take a κ_Θ-supercompact embedding $j : V' \to M$ with critical point μ. Then the quotient of $\mathbb{P}_\Theta$ with G_{low} takes the form $\text{Col}(\aleph_{\alpha+1}, < \mu) * \dot{\mathbb{P}}_{\text{high}}$ where $\dot{\mathbb{P}}_{\text{high}}$ is $\aleph_{\alpha+1}$-closed. Write $j(\text{Col}(\aleph_{\alpha+1}), < \mu) = \text{Col}(\aleph_{\alpha+1}, < \mu) \times \mathbb{T}$. Let H be $\text{Col}(\aleph_{\alpha+1}, < \mu)$-generic. By the absorption theorem (see [Cum10, Section 14]), there is a filter K which is $\mathbb{T}$-generic and a filter I which is $\mathbb{P}_{\text{high}}$-generic such that the embedding can be lifted to $j : V'[H][I] \to M[H][K][I']$. From here it is possible to reuse the exact lifting argument from the proof of Theorem 2.1. □

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