DISTRIBUTIVITY AND MINIMALITY IN PERFECT TREE FORCINGS FOR SINGULAR CARDINALS

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ABSTRACT. Dobrinen, Hathaway and Prikry studied a forcing $\mathbb{P}_\kappa$ consisting of perfect trees of height λ and width κ where κ is a singular λ-strong limit of cofinality λ. They showed that if κ is singular of countable cofinality, then $\mathbb{P}_\kappa$ is minimal for ω-sequences assuming that κ is a supremum of a sequence of measurable cardinals. We obtain this result without the measurability assumption.

Prikry proved that $\mathbb{P}_\kappa$ is (ω, ν)-distributive for all $\nu < \kappa$ given a singular ω-strong limit cardinal κ of countable cofinality, and Dobrinen et al. asked whether this result generalizes if κ has uncountable cofinality. We answer their question in the negative by showing that $\mathbb{P}_\kappa$ is not $(\lambda, 2)$-distributive if κ is a λ-strong limit of uncountable cofinality λ and we obtain the same result for a range of similar forcings, including one that Dobrinen et al. consider that consists of pre-perfect trees. We also show that $\mathbb{P}_\kappa$ in particular is not $(\omega, \cdot, \lambda^+)$-distributive under these assumptions.

While developing these ideas, we address natural questions regarding minimality and collapses of cardinals.

1. INTRODUCTION AND BACKGROUND

The study of forcing in set theory is bound to the study of complete Boolean algebras through the fact that every forcing poset can be identified with its regular open algebra. Functions added in the forcing extension correspond to failures of distributivity in the regular open algebra, and subextensions correspond to complete subalgebras. Questions on distributivity and minimality thus motivated some of the early development of forcing [21]. Solovay asked the question: For which cardinals κ is there a Boolean algebra that is (ω, ν)-distributive for all $\nu < \kappa$ but not (ω, κ)-distributive? Namba [18] answered this question for regular cardinals $\kappa \geq \aleph_2$ using a perfect tree forcing and an assumption on cardinal exponentiation. In unpublished work, Prikry answered Solovay's question for singular cardinals of countable cofinality, also using a perfect tree forcing. Both forcings are minimal for ω-sequences. More recently, Dobrinen, Hathaway, and Prikry [9] developed this line of investigation further and posed an open-ended version of Solovay's question: Given a regular cardinal λ, for which cardinals κ is there a

complete Boolean algebra that is (λ, ν)-distributive for all $\nu < \kappa$ but not (λ, κ)-distributive? This paper focuses on the case where κ is singular and $\omega < \lambda = \mathrm{cf}(\kappa)$ by ruling out many natural candidates.

In pursuing this question, the most natural forcings to consider are higher tree forcings of the sort originally studied by Kanamori using additional requirements on splitting that do not pertain to the countable cases [15]. This includes the "vertical" requirement that the trees are both closed under ascending sequences of splitting nodes and the "horizontal" requirement that they split into sufficiently closed filters. Dobrinen et al. point out that the main obstacle to studying higher tree forcings in full generality has to do with the fusion sequences typically used to work with tree forcings—namely, that fusion sequences of length $\mathrm{cf}(\kappa)$ may break down at the first countable step. Our work here considers what happens both with and without Kanamori's splitting requirements. We show that even with the splitting requirements, a variety of perfect tree forcings for singular cardinals κ of uncountable cofinality fail to be $(\mathrm{cf}(\kappa), 2)$-distributive. This answers a question of Dobrinen et al. regarding the generalization of Prikry's perfect tree forcing. We also show countable distributivity fails for all versions but the ones consisting of trees that obey both the vertical and horizontal requirements. In other words, our findings suggest that failures of countable distributivity predominate in the higher cases when one does not deliberately ensure the viability of fusion sequences.

Most of the introduction to this paper will be a review of definitions of the concepts we are working with. We consider singulars of countable cofinality in Section 2, in which we present a short proof of Prikry's theorem on (ω, ν)-distributivity and obtain ω-minimality for $\mathbb{P}_\kappa$ given an ω-strong limit κ of countable cofinality. In Section 3 we move to the case of uncountable cofinality and present a technique whereby, given a perfect tree T of singular width κ and height $\lambda > \omega$, we consider Bernstein sets on the cofinal branches $[T \restriction \delta]$ for some $\delta < \lambda$ to define useful functions $\lambda \to \mathrm{On}$ in the extensions. We use this technique to prove that $\mathbb{P}_\kappa$ is not $(\mathrm{cf}(\kappa), 2)$-distributive. Then we settle some natural questions on the closed versions of perfect tree forcings and obtain a non-minimality result for one of the versions that is not closed. In Section 4 we define a concept of slow fusion that allows us to work with higher tree forcings that are not as tame as those usually considered in the literature. We use this together with an old trick of Balcar and Simon to obtain various failures of countable distributivity. In Section 5 we obtain failures of three-parameter distributivity. A curiosity resulting from this section is that the regular open algebra of the poset of perfect binary trees in a regular uncountable cardinal κ (a higher perfect tree forcing without Kanamori's restrictions) is isomorphic to that of the Lévy collapse $\mathrm{Coll}(\omega, \kappa^+)$ if $2^\kappa = \kappa^+$. Generally, Sections 4 and 5 are relevant to the cases where κ is regular despite this paper's title.

1.1. **Distributivity and Boolean Algebras.** Here we lay down our most important definitions pertaining to Boolean algebras. More details can be found in Jech's textbook [13] and in the Handbook of Boolean Algebras [12].

Definition 1.1. Given a poset $\mathbb{P}$, the *regular open algebra* ro($\mathbb{P}$) of $\mathbb{P}$ consists of subsets $u \subseteq \mathbb{P}$ such that:

(i) $\forall p \in u, q \leq p, q \in u$;
(ii) $\forall p \notin u, \exists q \leq p, \forall r \leq q, r \notin u$.

Expanded with suitable interpretations of $(\wedge, \vee, {}^c, 0, 1)$, the set ro($\mathbb{P}$) is a complete Boolean algebra, which is denoted by ro($\mathbb{P}$). For a Boolean algebra B, we denote by $B^+ = B \setminus \{0_B\}$.

Fact 1.2. *If $\mathbb{P}$ is a poset, then* ro($\mathbb{P}$) *is a complete Boolean algebra and $\mathbb{P}$ embeds densely into* ro($\mathbb{P}$).

Definition 1.3 ([2, Def.17])**.** Let B be a Boolean algebra. Let θ, μ, ν be cardinals such that $\theta, \nu \geq \omega$ and $\mu \geq 2$.

- A collection $\mathcal{P} \subseteq \mathcal{P}(B^+)$ is called a *matrix* if each member of $\mathcal{P}$ is a maximal disjoint subset of B^+.
- We say that B is (θ, ν, μ)-*distributive* if for every matrix $\mathcal{P} = \{P_\alpha : \alpha \in \theta\}$ such that $|P_\alpha| \leq \nu$ there is some maximal disjoint set $Q \subseteq B^+$ such that for each $q \in Q$ and $\alpha \in \theta$, $|\{p \in P_\alpha : p \wedge q \neq 0_B\}| < \mu$.
- If B is $(\theta, \nu, 2)$-distributive, then we say that B is (θ, ν)-distributive.
- Finally, B is called θ-distributive if it is (θ, ν)-distributive for any ν.
- If $\mathbb{P}$ is a notion of forcing, then for each of the mentioned distributivity properties, we say $\mathbb{P}$ has a given property if and only if ro($\mathbb{P}$) has that property.

Fact 1.4. *A forcing poset $\mathbb{P}$ is θ-distributive if and only if every function $f : \theta \to \mathrm{On}$ in $V[\mathbb{P}]$ is already contained in V. A poset $\mathbb{P}$ is (θ, ν)-distributive if and only if every function $f : \theta \to \nu$ in $V[\mathbb{P}]$ is already contained in V.*

Definition 1.5. Two posets $\mathbb{P}_1$ and $\mathbb{P}_2$ are called *forcing equivalent*, denoted $\mathbb{P}_1 \simeq \mathbb{P}_2$, if ro($\mathbb{P}_1$) $\cong$ ro($\mathbb{P}_2$). We write $\mathbb{P}_1 \lessdot \mathbb{P}_2$ if ro($\mathbb{P}_1$) is isomorphic to a complete subalgebra of ro($\mathbb{P}_2$).

1.2. **Basic Notions of Tree Forcings.** We recall some notions pertaining to trees.

Definition 1.6. Let κ be the supremum of an increasing sequence $\vec{\kappa} = \langle \kappa_\alpha : \alpha < \delta \rangle$ of regular infinite cardinals where δ is an ordinal less than or equal to κ.

(1) The set $N = N(\vec{\kappa}) = \bigcup_{\alpha < \delta} \prod_{\beta < \alpha} \kappa_\beta$ consists of all functions t such that $\mathrm{dom}(t) < \delta$ and $(\forall \alpha \in \mathrm{dom}(t))(t(\alpha) < \kappa_\alpha)$. We typically refer to such functions as *nodes*. We drop the notation for $\vec{\kappa}$ when the sequence is fixed in context.
(2) A non-empty subset $T \subseteq N$ that is closed under initial segments is called a *tree*.

(3) For an ordinal α, the set $T(\alpha)$ is the set of $t \in T$ with $\mathrm{dom}(t) = \alpha$.
(4) The *height of a tree* T is $\mathrm{ht}(T) = \min\{\alpha : T(\alpha) = \emptyset\}$.
(5) We let $[T] = \{f : \mathrm{ht}(T) \to \kappa : \forall \alpha < \mathrm{ht}(T), f \upharpoonright \alpha \in T\}$. Elements of $[T]$ are called *cofinal branches*.
(6) For t_1, $t_2 \in N \cup [N]$ we write $t_1 \sqsubseteq t_2$ if $t_2 \upharpoonright \mathrm{dom}(t_1) = t_1$. The tree order is the relation $\sqsubseteq$. If $t = s \cup \{(\mathrm{dom}(s), \beta)\}$, we write $t = s \frown \langle \beta \rangle$.
(7) A branch of T is an element of $T \cup \{f \in N \cup [T] : \forall \alpha \in \mathrm{dom}(f), f \upharpoonright \alpha \in T\}$ that does not have a $\sqsupseteq$-extension.
(8) $T \upharpoonright \alpha = \bigcup_{\beta < \alpha} T(\beta)$.
(9) $T \upharpoonright t = \{s \in T : s \sqsubseteq t \vee t \sqsubseteq s\}$.
(10) For $t \in T(\alpha)$ we let

$$\mathrm{Succ}_T(t) = \{c : c \in T(\alpha + 1) \wedge c \sqsupseteq t\}$$

denote the *set of immediate successors of* t, and

$$\mathrm{OSucc}_T(t) = \{\beta : t \frown \langle \beta \rangle \in T(\alpha + 1)\}$$

denote the *ordinal successor set of* t.
(11) We call $t \in T$ a *splitting node* if $|\mathrm{Succ}_T(t)| > 1$. We call $t \in T$ a κ-*splitting node* if $|\mathrm{Succ}_T(t)| \geq \kappa$.
(12) If it exists, $\mathrm{Stem}(T)$ is the unique splitting node of T that is comparable to all other nodes.

Definition 1.7. Throughout this paper we will need to consider different manners of branching. Once again we fix $\vec{\kappa} = \langle \kappa_\alpha : \alpha < \delta \rangle$.

(1) A non-empty tree $T \subseteq N(\vec{\kappa})$ is called *ever-branching* if for each $\alpha < \delta$ and for each $s \in T$ there is a node $t \sqsupseteq s$ such that

(1.1) $$|\mathrm{Succ}_T(t)| \geq \kappa_\alpha.$$

Note that $\mathrm{dom}(t) > \alpha$ is possible.
(2) T is called $<\delta$-*closed* if T has no branch of length $<\delta$.
(3) A non-empty tree $T \subseteq N(\vec{\kappa})$ is called *perfect* (in $N(\vec{\kappa})$) if it is $<\delta$-closed and ever-branching.
(4) A non-empty tree $T \subseteq N$ is called *non-stopping* (in $N(\vec{\kappa})$) if for each node $t \in T$ there is a cofinal branch b such that $t \in b$.
(5) A non-empty tree $T \subseteq N$ is called *pre-perfect* (in $N(\vec{\kappa})$) if it is ever-branching and non-stopping.

The property of being ever-branching is not influenced by the choice of the increasing cofinal sequence $\langle \kappa_\alpha : \alpha < \delta \rangle$. We use these notions for various ordinals δ.

Our predominating cardinal arithmetic assumption in the context of perfect tree forcings is as follows:

Definition 1.8. We say that κ is a δ-*strong limit* if for all $\tau < \kappa$, $\tau^\delta < \kappa$.

1.3. **Versions of Tree Forcing.** We fix $\vec{\kappa} = \langle \kappa_\alpha : \alpha < \lambda \rangle$ for a strictly increasing sequence of regular cardinals with $\omega \leq \lambda$ and let $\kappa = \sup(\vec{\kappa})$. In this paper we would like to cover a wide range of perfect tree forcings $\mathbb{P}_\kappa$ involving trees $p \subseteq N(\vec{\kappa})$ of height λ without being completely encyclopedic. For this purpose, we will cover three main schemes in which tree forcings might vary:

(1) *The type of tree: Miller versus Laver.* Miller-style forcings allow for fairly chaotic branching behavior, whereas Laver-style forcing allows a stem after which all nodes are splitting.

(2) *The degree of closure of the trees that serve as conditions.* This is regarding the closure of the trees themselves as well as the closure of splitting nodes.

(3) *A specification of the manner of splitting.* We consider variations of Condition 1.1, meaning that we will sometimes ask that the splitting sets themselves take a certain form, possibly a club or a stationary set.

Now we introduce precise definitions, beginning with the types of trees.

Definition 1.9. Let $\vec{\kappa} = \langle \kappa_\alpha : \alpha < \lambda \rangle$ be an increasing sequence of cardinals with limit κ and $\mathrm{cf}(\kappa) = \lambda \geq \omega$.

Conditions in the forcing $\mathbb{M}$ are perfect trees in $N(\vec{\kappa})$. If the forcing $\mathbb{M}$ is defined with respect to a cardinal κ, we may use the notation $\mathbb{M}_\kappa$.

Subtrees are stronger conditions: $q \leq p$ if $q \subseteq p$.

Remark 1.10. *If κ is any singular cardinal, then the poset $\mathbb{M}_\kappa$ is the poset $\mathbb{P}_\kappa$ of Dobrinen, Hathaway, and Prikry [9] referred to in the abstract.*

We recall a Laver-style relative $\mathbb{L}_\kappa$ of $\mathbb{M}_\kappa$ to which our results also apply.

Definition 1.11. Let $\vec{\kappa} = \langle \kappa_\alpha : \alpha < \lambda \rangle$ be an increasing sequence of regular cardinals that converges to κ, and $\lambda < \kappa$.

Singular Laver-Namba forcing $\mathbb{L}_\kappa$ is defined as follows: Conditions are $(<\lambda)$-closed trees $p \subseteq N$ with a unique stem s such that

$$(1.2) \qquad \forall t \in p (t \sqsupseteq s \rightarrow |\mathrm{Succ}_p(t)| = \kappa_{\mathrm{dom}(t)}).$$

In $\mathbb{L}_\kappa$ subtrees are stronger conditions.

Next, we give definitions regarding the closure of the trees that serve as conditions.

Definition 1.12. If conditions are $(<\lambda)$-closed trees p, we just write $\mathbb{M}_\kappa$ or $\mathbb{L}_\kappa$. If we weaken perfect to pre-perfect, we denote these posets ${}^{\mathrm{pre}}\mathbb{M}_\kappa$ or ${}^{\mathrm{pre}}\mathbb{L}_\kappa$.

Definition 1.13. If conditions are "$(<\lambda)$-splitting-closed" trees p, i.e., any increasing $<\lambda$-sequence $\langle t_\alpha : \alpha < \delta \rangle$, $\delta < \lambda$, of κ_α-splitting nodes t_α in p has a limit (the union) which is a κ_δ-splitting node in p. We write ${}^{\mathrm{scl}}\mathbb{M}$ for the $(<\lambda)$-splitting-closed version of $\mathbb{M}$. In the case of Laver forcings $(<\lambda)$-closure implies $(<\lambda)$-splitting closure, so there is no meaningful distinction.

For $\lambda = \omega$, the "vertical" requirements coincide and describe the trees without maximal nodes.

Finally, we give definitions regarding the manner of splitting.

Definition 1.14. We enumerate the versions we consider below.

(i) We write $\mathbb{L} = \mathbb{L}^{\text{stat}}$ if for any splitting node $t \in p \in \mathbb{P}$, we have $\text{OSucc}_p(t)$ is a stationary subset of $\{\beta \in \kappa_{\text{dom}(t)} : \text{cf}(\beta) = \omega\}$. For $p \in \mathbb{M}^{\text{stat}}$ we require that for any node there is an extension t that splits into a stationary subset of $\{\beta \in \kappa_{\text{dom}(t)} : \text{cf}(\beta) = \omega\}$.[1]

This is the case of I-positive sets for an ideal I whose additivity matters.

(ii) We write $\mathbb{L} = \mathbb{L}^{\text{club}}$ if $\text{OSucc}_p(t)$ contains a club subset of $\kappa_{\text{dom}(t)}$. For $p \in \mathbb{M}^{\text{club}}$ we require that for any node there is an extension t that splits into a superset of a club in $\kappa_{\text{dom}(t)}$. Here splitting is into a highly closed filter.

(iii) If splitting is defined over a cardinal-wise criterion as in Clauses (1.1) or (1.2) we write just $\mathbb{M}$ and $\mathbb{L}$.

Remark 1.15. *In the case of countable λ, the forcing $\mathbb{L}_\kappa^{\text{stat}}$ is the one studied by Cummings and Magidor* [8].

Each version can be combined in various ways with each of the three named splitting criteria, e.g. $^{\text{pre}}\mathbb{L}_\kappa^{\text{stat}}$, $\mathbb{L}_\kappa^{\text{club}}$.

Observation 1.16. *The forcing notions $^{\text{scl}}\mathbb{M}_\kappa^{\text{club}}$ and $\mathbb{L}_\kappa^{\text{club}}$ are $(<\lambda)$-closed.*

Proof. We show that for a decreasing sequence $\langle p_\alpha : \alpha < \delta \rangle$ of conditions of length $\delta < \lambda$, the intersection $q = \bigcap_{\alpha < \delta} p_\alpha$ is a lower bound. In the Miller case, we have to show that $q \in {}^{\text{scl}}\mathbb{M}_\kappa^{\text{club}}$. The tree q is not empty, because $\emptyset \in p_\alpha$ for each $\alpha < \delta$. Since any p_α is a $(<\lambda)$-closed tree, q is also a $(<\lambda)$-closed tree. Now we show that q is ever-branching. Given $s \in q$, we choose an $\sqsubseteq$-increasing sequence $\langle t_\alpha : \alpha < \delta \rangle$ such that $t_\alpha \in \text{Split}_\alpha(p_\alpha)$ is above s. We let $t_\delta = \bigcup \{t_\alpha : \alpha < \delta\}$. Then for any $\alpha < \delta$, the limit t_δ is a κ_δ-splitting node in p_α, because p_α fulfills that any $(<\lambda)$-limit of splitting nodes is a splitting node of the fitting degree. Now we use $\delta < \lambda \leq \kappa_\delta$ and the $(<\kappa_\delta)$-completeness of the club filter in order to intersect $\text{OSucc}_{p_\alpha}(t_\delta)$, $\alpha < \delta$, and find that t_δ is a κ_δ-splitting node in q. The proof for club Laver forcing $\mathbb{L}_\kappa^{\text{club}}$ is simpler. $\square$

In the literature there are analogs of this closure for regular κ (in this case $\lambda = \kappa$): Baumgartner's higher Silver forcing [3, Section 6], Kanamori's seminal work [15] on forcings with higher Sacks trees, Landver's [16] versions of higher Sacks trees, and club κ-Miller forcing, e.g., in [10].

Observation 1.16 is sharp in the case of uncountable λ. We will prove in Section 4 that the other named variations are not ω-distributive. Waiving

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[1] $\mathbb{L}^{\text{stat}}$ has been used to add exact upper bounds in the context of PCF theory [8]. We are not aware of any use of $\mathbb{M}^{\text{stat}}$ yet.

the "vertical condition" that any $(<\lambda)$-limit of splitting nodes is a splitting node (see Subsection 4.2) or waiving the closure of set of ordinal splitting sets (see Subsections 4.3, 4.4) allows us to show the failure of ω-distributivity. All pre-perfect versions strongly violate ω-distributivity (see Subsection 5.1). For uncountable λ, $(<\lambda)$-closure implies ω-distributivity. Hence in the uncountable case, the versions in Observation 1.16 are the only $(<\lambda)$-closed posets among the named forcings, and in fact they are the only ω-distributive posets among them.

1.4. **Notable Features of Perfect Tree Forcing.**

A notable feature of perfect tree forcings in general is that of minimality, which motivated Sack's early work [19]. For our purposes, we will consider a relaxed version of minimality.

Definition 1.17 ([19])**.** If λ is a cardinal, a forcing $\mathbb{P}$ is *minimal for λ-sequences* if for any transitive model M such that $V \subseteq M \subseteq V[\mathbb{P}]$, if $A \in M$ is a function $A : \lambda \to \text{On}$ such that $A \notin V$, then $M = V[\mathbb{P}]$.

In the case of countable cofinality, minimality is often understood through the rendering of the generic branch. This is also justified in the higher cases where we are dealing with $(<\lambda)$-closed notions of forcing.

Proposition 1.18. *Let $\mathbb{P}$ be $^{\mathrm{scl}}\mathbb{M}_\kappa^{\mathrm{club}}$ or $\mathbb{P} = \mathbb{L}_\kappa^{\mathrm{club}}$ if $\mathrm{cf}(\kappa) > \omega$ or any version if $\mathrm{cf}(\kappa) = \omega$. Then the so-called generic branch*

$$g_G = \bigcup \{\text{Stem}(p) \,:\, p \in G\}$$

is an element of $\prod_{\alpha<\lambda} \kappa_\alpha$ and $V[g_G] = V[G]$.

Proof. The $(<\lambda)$-closure of the forcing notion implies that $g_G \in \prod_{\alpha<\lambda} \kappa_\alpha$. Now we show $V[G] \subseteq V[g_G]$. Suppose $q \Vdash g_G \in [p]$. Then $q \leq p$, since otherwise we could pick $t \in q \setminus p$ and have $q \upharpoonright t \Vdash g_G \notin [p]$. We claim $G = \{p \in \mathbb{P} \,:\, g_G \in [p]\}$. It is obvious that $p \in G$ implies $g_G \in [p]$. Conversely, suppose that $g_G \in [p]$. Then there is some $q \in G$ such that $q \Vdash g_G \in [p]$. We have $q \leq p$ and thus $p \in G$. $\qquad\square$

In all of our minimality results, we will work with the generic branch. For uncountable λ, we will show that the closed versions are non-minimal, as is at least one of the non-closed versions (see Theorem 3.10 below).

Another notable property of tree forcings is that of fusion.

Definition 1.19. We fix a sequence $\langle \kappa_\alpha \,:\, \alpha < \lambda \rangle$ of regular cardinals converging to κ. The notions $\text{Split}_\alpha(p)$ and $\leq_\alpha$ depend on this choice. Let $\mathbb{P}$ be any of our forcings. For convenience, we restrict all our forcing notions to the dense sets of conditions in which $\text{Stem}(p)$ is κ_0-splitting. We write $\lambda = \mathrm{cf}(\kappa)$.

(1) Take $p \in \mathbb{M}_\kappa$. By induction on $\alpha \in \lambda$, we define

$$\mathrm{Split}_\alpha(p) = \{t \in p : t \text{ is } \sqsubseteq\text{-minimal in } p \text{ such that}$$

$$|\mathrm{Succ}_p(t)| \geq \kappa_\alpha \wedge \exists \langle t_\beta : \beta < \alpha \rangle,$$

$$(\forall \beta < \gamma < \alpha, t_\beta \sqsubset t_\gamma \sqsubset t,$$

$$\wedge \forall \beta < \alpha, t_\beta \in \mathrm{Split}_\beta(p))\}.$$

Observe that $\mathrm{Split}_\alpha(p)$ is an antichain in p.

For $p \in \mathbb{L}_\kappa$, we have $\mathrm{Split}_\alpha(p) = p(\mathrm{dom}(\mathrm{Stem}(p)) + \alpha)$ and that for $t \in \mathrm{Split}_\alpha(p)$, $\mathrm{OSucc}_p(t) \subseteq \kappa_{\mathrm{dom}(\mathrm{Stem}(p))+\alpha}$ and $|\mathrm{OSucc}_p(t)| = \kappa_{\mathrm{dom}(\mathrm{Stem}(p))+\alpha}$.

We define $\mathrm{Split}_\alpha(p)$ analogously for the various other cases.

(2) Let $p, q \in \mathbb{P}$, $\alpha < \lambda$. We let $q \leq_\alpha p$ if $q \leq p$ and $\mathrm{Split}_\alpha(p) = \mathrm{Split}_\alpha(q)$.

(3) A sequence $\langle p_\alpha : \alpha < \delta \rangle$ such that $\delta \leq \lambda$ and for $\alpha < \gamma < \delta$, $p_\gamma \leq_\alpha p_\alpha$ is called a *fusion sequence*.

We recall the so-called fusion lemma:

Lemma 1.20. *Suppose that either* $\mathrm{cof}(\kappa) = \lambda = \omega$ *and that* $\mathbb{P}$ *is any of the perfect versions or that* $\mathrm{cof}(\kappa) = \lambda > \omega$ *and that* $\mathbb{P}$ *is one of the closed versions,* $\mathbb{L}_\kappa^{\mathrm{club}}$ *or* $^{\mathrm{scl}}\mathbb{M}_\kappa^{\mathrm{club}}$. *Then for a fusion sequence* $\langle p_\alpha : \alpha < \delta \rangle$, $1 \leq \delta \leq \lambda$, *we have* $q_\delta = \bigcap\{p_\alpha : \alpha < \delta\} \in \mathbb{P}$ *and* $q_\delta \leq_\alpha p_\alpha$ *for all* $\alpha < \delta$.

Considerations of fusion become more complicated when we consider the cases of higher cofinality. In particular, the fusion lemma implies some $(<\lambda)$-closure: For this look at a limit $\delta < \lambda$ and consider a fusion sequence $\langle p_\alpha : \alpha < \delta \rangle$. If $q = \bigcap\{p_\alpha : \alpha < \delta\}$ is a condition, then $\mathrm{Split}_\delta(q) \neq \emptyset$, and for any $t \in \mathrm{Split}_\delta(q)$ we have that $q \upharpoonright t$ is a lower bound of $\langle p_\alpha \upharpoonright t : \alpha < \delta \rangle$; the latter is just an ordinary $\leq$-descending sequence. In Section 4 we will explore situations in which closure of the forcing cannot be assumed.

2. Improvements to the Countable Case

2.1. **A Short Proof of Prikry's Theorem.** We begin by giving a proof of Prikry's theorem (see [9, Theorem 3.5]) that harmonizes with the proof from Jech's textbook [13] that under CH the classical Namba forcing is (ω, ω_1)-distributive. However, we do not use a rank function. This approach would also work for the classical Namba forcing.

Theorem 2.1 (Prikry). *Suppose that* $\mathrm{cf}(\kappa) = \omega$, *and assume* $\forall \nu < \kappa$, $\nu^\omega < \kappa$. *Then for any* $\nu < \kappa$, $\mathbb{M}_\kappa$ *is* (ω, ν)-*distributive.*

The following lemma introduces a relative of the Cantor-Bendixson rank.

Lemma 2.2. *Suppose that* κ *is the supremum of* $\vec{\kappa} = \langle \kappa_\alpha : \alpha < \delta \rangle$ *where* δ *is an ordinal of countable cofinality less than* κ, *and that* $\tau^\delta \leq \kappa$ *for all* $\tau < \kappa$. *Let* $T \subset N(\vec{\kappa})$ *be a tree with no branches of length* $< \delta$. *Then the following are equivalent:*

(1) T has a perfect subtree.

(2) The cardinality of $[T]$ is equal to the cardinality of $\prod_{\alpha<\delta}\kappa_\alpha$.

(3) The cardinality of $[T]$ is greater than κ.

Proof. First we show that *(1)* implies *(2)*. It is enough to define an injection $\prod_{\alpha<\delta}\kappa_\alpha \hookrightarrow [T'] \subseteq [T]$ where T' is a perfect subtree of T. First we will define an injection $\bar{\varphi} \colon N(\vec{\kappa}) \to T'$ by induction on β. We let $\bar{\varphi}(\emptyset)$ be a κ_0-splitting node of p. For the successor case, suppose that $\bar{\varphi}$ is defined on $\prod_{\alpha<\gamma}\kappa_\alpha$ and $\beta = \gamma + 1$. Then for all $t \in \prod_{\alpha<\gamma}\kappa_\alpha$, find a κ_γ-splitting node $s \sqsupseteq \bar{\varphi}(t)$, enumerate $\mathrm{Succ}_{T'}(s) = \langle t_\xi \ : \ \xi < \kappa_\gamma \rangle$, and then define $\bar{\varphi}(t^\frown\langle\gamma,\xi\rangle)$ to be t_ξ for all $\xi < \kappa_\gamma$. If β is a limit, for $t \in \prod_{\alpha<\beta}\kappa_\alpha$ we let $\bar{\varphi}(t)$ be the shortest κ_β-splitting node extending $\bigcup\{\bar{\varphi}(t\restriction\gamma) \ : \ \gamma < \beta\}$. Finally, we define $\varphi \colon \prod_{\alpha<\delta}\kappa_\alpha \to [T']$ by letting $\varphi(f)$ be the branch given by $\langle\bar{\varphi}(f\restriction\beta) \ : \ \beta < \delta\rangle$.

The fact that *(2)* implies *(3)* is because of König's Theorem.

Most of the proof consists in arguing that *(3)* implies *(1)*. Let $\langle\delta_n \ : \ n < \omega\rangle$ by a strictly increasing sequence such that $\sup_{n<\omega}\delta_n = \delta$. By induction on α we define:

$$A_\alpha = \{t \in T \ : \ \exists i < \omega, \forall t' \sqsupseteq t, |\,\mathrm{Succ}_T(t') \setminus \bigcup\{A_\beta \ : \ \beta < \alpha\}| < \kappa_{\delta_i}\}.$$

Observe that each A_β is upwards closed by definition. Moreover, using upwards closure, it follows by induction that $\beta \leq \gamma$ implies $A_\beta \subseteq A_\gamma$. Since $|T| \leq \kappa$, it follows that there must be some $\theta < \kappa^+$ such that $A_\beta = A_\theta$ for all $\beta \geq \theta$. Hence we define $A := \bigcup\{A_\beta : \beta \leq \theta\}$.

For this proof let $\mathrm{rk}(t) = \beta$ if $t \in A_\beta \setminus \bigcup\{A_\gamma \ : \ \gamma < \beta\}$.

We argue that $A = T$ implies that $[T]$ has cardinality $\leq \kappa$: If $b \in [T]$, then let $\mathrm{rk}(b)$ be the minimal α such that there is some $t \in b$ with $\mathrm{rk}(t) = \alpha$. (This must exist because otherwise $\{\mathrm{rk}(t) : t \in b\}$ would contain an infinite descending sequence of ordinals.) This means that if $B_\alpha = \{b \in [T] : \mathrm{rk}(b) = \alpha\}$, then $[T] = \bigcup_{\alpha<\theta} B_\alpha$. Moreover, we can argue that $|B_\alpha| \leq \kappa$ for each $\alpha < \theta < \kappa^+$: For each $t \in A_\alpha$, the set of $b \in B_\alpha$ such that $t \sqsubseteq b$ has cardinality $\leq \kappa$ because this set is a $< \kappa_{\delta_i}$-branching tree for some $i < \omega$. Hence by (3) we have $A \subsetneq T$.

Now we show that $A \subsetneq T$ implies that T has a perfect subtree. Suppose $t \in T \setminus A$. We let $F_{n-1} = \{t\}$. By induction on n, we show that there in antichain $F_n \subseteq T \setminus A$ such that

(a) any $s \in F_n$ is $\geq \kappa_n$-splitting in $T \setminus A$ and

(b) such that for any $s \in F_{n-1}$ there is some $s' \in F_n$ with $s \sqsubset s'$.

Let $F_0 = \{t\}$. Suppose that $n > 0$ and F_{n-1} is defined. For any $s \in F_{n-1}$, we do the following. For each $s' = s^\frown\langle\alpha\rangle \in T \setminus A_\theta$, we have $s' \notin A = A_\theta = A_{\theta+1}$. Hence by the definition of the set A_θ, for $\alpha = n$ we find some $s'' = s''(s') \sqsupseteq s', |\,\mathrm{Succ}_T(s'') \setminus A_\theta| \geq \kappa_{\delta_n}$. We let

$$F_n = \{s''(s') \ : \ \exists s \in F_{n-1}, s' \in \mathrm{Succ}_T(s) \setminus A_\theta\}.$$

Now the F_n's are defined for $n \in \omega$. We let T' be the subtree of T equal to $\{t \in T : \exists t' \sqsupseteq t, t' \in \bigcup\{F_n \ : \ n < \omega\}\}$. $\qquad\square$

Now we can prove Theorem 2.1.

Proof. Let $p \Vdash$ "$\dot{f} : \omega \to \nu$". We will find some $r \leq p$ and $g : \omega \to \nu$ such that $r \Vdash$ "$\dot{f} = \check{g}$".

We define antichains A_n in the tree $(p, \sqsubseteq)$ by induction on $n < \omega$. At the same time, we define sets $\{q_s : s \in \bigcup_{n<\omega} A_n\}$ and $\{\alpha_s : s \in \bigcup_{n<\omega} A_n\}$ by induction on the length of s such that the following hold:

(i) All elements of A_n lie $\sqsubseteq$-below elements of A_{n+1}.
(ii) If $s_1 \sqsubseteq s_2$, then $q_{s_2} \leq q_{s_1}$.
(iii) If $s \in A_n$, then $q_s \Vdash$ "$\dot{f}(n) = \alpha_s$".
(iv) $s = \mathrm{Stem}(q_s)$.
(v) If $s \in A_n$, then $\mathrm{Stem}(q_s)$ is a κ_n-splitting node.

The construction goes as follows: Let $q_\emptyset \leq p$ be a condition deciding $\dot{f}(0)$ such that if $s = \mathrm{Stem}(q_\emptyset)$ then s is κ_0-splitting. Then let $A_0 = \{s\}$. Now suppose A_n has been defined. Given $s \in A_n$, for each successor t of $\mathrm{Stem}(q_s)$, choose $q_t \leq q_s$ and α_t such that $t \sqsubseteq \mathrm{Stem}(q_t)$, $q_t \Vdash$ "$\dot{f}(n+1) = \alpha_t$", and $\mathrm{Stem}(q_t)$ is a κ_{n+1}-splitting node. Then collect all such t's and include them in A_{n+1}.

Let q be the downward closure of $\bigcup_{n<\omega} A_n$, which is a perfect tree and hence a condition in $\mathbb{M}_\kappa$. For any $g : \omega \to \nu$, we define $q(g) \leq q$ (not necessarily perfect) as follows: If $\bar{\beta} = \langle \beta_0, \ldots, \beta_n \rangle \in \nu^{<\omega}$, let $q(\bar{\beta}) = \bigcup\{q_s : s \in A_n, \bar{\beta} = \langle \alpha_\emptyset, \ldots, \alpha_{s\upharpoonright k}, \ldots, \alpha_s \rangle\}$ and $q(g) = \bigcap_{n<\omega} q(g\upharpoonright n)$. It is immediate that $q(g) \Vdash$ "$\dot{f} = \check{g}$" if $q(g)$ is a condition in $\mathbb{M}_\kappa$.

Claim: There is some $g : \omega \to \nu$ such that $q(g)$ is a perfect tree.

To prove the claim, suppose otherwise for contradiction. Then for each $g : \omega \to \nu$, Lemma 2.2 implies that $[q(g)]$ has cardinality at most κ. Since $[q]$ has cardinality $> \kappa$, we choose $b \in [q] \setminus \bigcup_{g:\omega\to\nu} [q(g)]$. If $b = \langle s_n : n \in \omega \rangle$, let h be given by $h(n) = \alpha_{s_n}$ for $n < \omega$. But then $s_n \in q(h)$ for all $n < \omega$, which is a contradiction of the choice $b \notin [q(h)]$.

If $g : \omega \to \nu$ witnesses the claim, then $q(g) \in \mathbb{M}_\kappa$ and $q(g) \Vdash$ "$\dot{f} = \check{g}$". $\square$

This style of argumentation works for other tree forcings for singulars of countable cofinality. In particular, we obtain a new proof of the same result for the forcing $\mathbb{L}_\kappa^{\mathrm{stat}}$ studied by Cummings and Magidor. Previous arguments, like the ones in [22, Ch. XI, Section 3, especially 3.3., 3.4 and 3.6] and [6, Theorem 2.3], used an intermediate step in which regular trees are sought (see Theorem 2.2 in [6]). A tree p is called regular if there is a set $\{\ell_n : n < \omega\} \subseteq \omega$, such that for any n, any $t \in p(\ell_n)$ is κ_n-splitting.

Theorem 2.3. *If* $\mathrm{cf}(\kappa) = \omega < \kappa$ *and* κ *is an* ω-*strong limit, then* $\mathbb{L}_\kappa^{\mathrm{stat}}$ *is* (ω, ν)-*distributive for all* $\nu < \kappa$.

We will use an important aspect of the Cummings-Magidor arguments, which to some extent derives from Laver's original forcing [17].

Fact 2.4 (See, e.g., [8, Fact one, page 3341])**.** *If* $\mathrm{cf}(\kappa) = \omega$, $\mathbb{L}_\kappa^{\mathrm{stat}}$ *is defined on the product* $N(\vec{\kappa})$, $p \in \mathbb{L}_\kappa^{\mathrm{stat}}$ *has a stem of length at least* n, *and* p *forces*

that $\dot\gamma$ is an ordinal below μ for some $\mu < \kappa_n$, then there is some $q \leq_0 p$ such that q decides a value for $\dot\gamma$.

Proof of Theorem 2.3. Again we use a modification of the Cantor-Bendixson rank to analyze the conditions in $\mathbb{L}^{\mathrm{stat}}$. Let $T \in \mathbb{L}^{\mathrm{stat}}$. Let $s_0 = \mathrm{Stem}(T)$. We induct on $\alpha \in \mathrm{On}$ to define

$$A_\alpha = \{t \in T \ : \exists t', s_0 \sqsubseteq t' \sqsubseteq t, \mathrm{Succ}_T(t') \setminus \bigcup\{A_\beta \ : \ \beta < \alpha\}$$

$$\text{is not stationary in } \kappa_{\mathrm{dom}(t')} \cap \mathrm{cof}(\omega)\}.$$

Again, we have the properties that the A_β's are upwards closed, that $\beta \leq \gamma$ implies $A_\beta \subseteq A_\gamma$, and that there is some $\theta < \kappa^+$ such that $A_{\theta+1} = A_\theta$, so we let $A := A_\theta$. We define $\mathrm{rk}(t) = \beta$ if and only if $t \in A_\beta \setminus \bigcup\{A_\gamma : \gamma < \beta\}$.

We show that if $s_0 \notin A$, then T contains a condition in $\mathbb{L}^{\mathrm{stat}}$ as a subtree, namely letting $T' = T \setminus A$ we have that $T' \in \mathbb{L}^{\mathrm{stat}}$. Specifically, we argue by induction on $|t|$ that if $s_0 \sqsubseteq t$ and $t \notin A$, then there is a stationary subset of $\zeta \in \kappa_{\mathrm{dom}(t)}$ such that $t^\frown \zeta \notin A$. If this is not the case for some such t, then it follows that $t \in A_{\theta+1} = A_\theta = A$, contradicting the assumption that $t \notin A$.

Now we can commence with the distributivity proof.

Suppose $p \Vdash$ "$f : \omega \to \nu$" and use Fact 2.4 to assume without loss of generality that p has a stem of length $\mathrm{dom}(\mathrm{Stem}(p)) = m_0 \geq m$ where $\nu^\omega < \kappa_m$ and p decides $\dot f(n)$ for all $n \leq m$.

Using Fact 2.4, we choose $q_{m_0} \leq_0 p$ such that q_{m_0} decides $\dot f \restriction m_0$, say to t_f.

Now by induction on $n \geq m_0$ we choose q_n and for $s \in q_n(n+1)$ we choose $\alpha_s \in \nu$. The beginning is q_{m_0} as above. For $n \geq m_0$, given q_n we define for $s \in q_n(n+1)$ a condition q_s and α_s with the following properties:

(i) $q_n = \bigcup\{q_s \ : \ \mathrm{dom}(s) = n\}$.
(ii) $\forall s, \forall s^\frown \langle \alpha \rangle \in q_s, q_{s^\frown \langle \alpha \rangle} \leq_0 q_s \restriction (s^\frown \langle \alpha \rangle)$.
(iii) If $|s| = n$ then $q_s \Vdash \dot f(n) = \alpha_s$.

This construction works using Fact 2.4. We let $q = \bigcap\{q_n \ : \ m_0 \leq n < \omega\}$ be the fusion limit. We also define $q(g)$ for $g : \omega \to \nu$ with $g \restriction m_0 = t_f$ by

$$q(g) = \text{ the tree in } q \text{ spanned by } \{s \in q \ : \ \forall k \in [m_0, \mathrm{dom}(s)], g(k) = \alpha_{s \restriction k}\}.$$

We also denote $s_0 = \mathrm{Stem}(q)$ as above.

To finish the proof, it suffices to prove the following:

Claim: There is some $g : \omega \to \nu$, $g \supseteq t_f$, such that $q(g)$ is a condition in $\mathbb{L}_\kappa^{\mathrm{stat}}$. Indeed, $q(g) \leq_0 q$.

To prove the claim, suppose otherwise for contradiction. Then for each $g : \omega \to \nu$ with $g \sqsupseteq t_f$, there is by the above analysis a sequence $\langle A_\alpha^g \ : \ \alpha < \theta(g) \rangle$ of subsets of $q(g)$ and we have $A^g = A_{\theta(g)}^g = q(g)$ as well as a rank function rk^g. It follows from the assumption that $q(g) \notin \mathbb{L}_\kappa^{\mathrm{stat}}$ that $s_0 \in A^g$ and therefore $t \in A^g$ for all $g : \omega \to \nu$ and $t \sqsupseteq s_0$. Now construct a $\sqsubseteq$-increasing sequence $\langle s_n \ : \ n < \omega \rangle$ where $s_n \in q(n)$, $n \geq m_0$ as follows. The set

$$S(s_n) := \{s \in \mathrm{Succ}_q(s_n) : \exists g \colon \omega \to \nu, s_n \in A^g = q(g),$$
$$s \in q(g) \wedge \mathrm{rk}^g(s) = \mathrm{rk}^g(s_n)\}$$

is not stationary in $\kappa_n \cap \mathrm{cof}(\omega)$ while $\mathrm{Succ}_q(s_n)$ is. More precisely, $S(s_n)$ is the union of strictly fewer than κ_n-many nonstationary sets. So choose $s_{n+1} \in q(n+1)$ such that $s_{n+1} \sqsupset s_n$ and s_{n+1} has no predecessor in $S(s_n)$.

Now let h be given by $h(n) = \alpha_{s_n}$ for $n < \omega$, hence $s_n \in q(h)$ for all $n < \omega$. By construction, the sequence $\langle \mathrm{rk}_{q(h)}(s_n) : n \in [m, \omega) \rangle$, is an infinite descending sequence of ordinals. Thus we have a contradiction and the proof of the claim. $\qquad\square$

2.2. **Minimality Without Measurables.** As before, $\mathbb{P}$ is $\mathbb{M}_\kappa$ where κ is a singular of countable cofinality. We prove the following theorem, which waives the assumption in Theorem 6.6 from [9] that the κ_n's are measurable. The key is that our proof uses the (ω, κ_n)-distributivity of $\mathbb{P}$ for $n < \omega$.

Theorem 2.5. *Suppose $p \in \mathbb{P}$ and $p \Vdash \dot{A} \colon \omega \to \check{V} \wedge \dot{A} \notin \check{V}$. Then $p \Vdash$ "$\Gamma(\mathbb{P}) \in \check{V}[\dot{A}]$". In other words, $\mathbb{P}$ is minimal for ω-sequences.*

Here we use $\Gamma(\mathbb{P})$ to refer to the canonical $\mathbb{P}$-name for a $\mathbb{P}$-generic filter over V.

Definition 2.6 (See [9, Definition 6.1] and [5, Lemma 11]). Let $\mathbb{P}$ be either $\mathbb{M}_\kappa$ or $\mathbb{L}_\kappa$ for a singular κ. Let $\dot{A}$ be a $\mathbb{P}$-name such that $1_{\mathbb{P}} \Vdash \dot{A} \colon \omega \to \check{V} \wedge \dot{A} \notin \check{V}$. For each condition $p \in \mathbb{P}$ we let $\psi_p \colon p \to {}^{<\omega}V$ be the function that assigns to each node $t \in p$ the longest sequence $s = \psi_p(t)$ such that $p \restriction t \Vdash \check{s} \sqsubseteq \dot{A}$.

Given $q \leq p$, a splitting node $t \in \mathrm{Split}(q)$ is called $\dot{A}$-*sharp* (with respect to q) if $\{\psi_p(c) : c \in \mathrm{Succ}_q(t)\}$ consists of pairwise incompatible finite sequences. A condition $q \leq p$ is $\dot{A}$-sharp if all of its splitting nodes are $\dot{A}$-sharp.

Lemma 2.7 ([5, Lemma 9] or [9, Lemma 6.2]). *For $\mathbb{P} = \mathbb{M}_\kappa$ or $\mathbb{P} = \mathbb{L}_\kappa$ for $\omega = \mathrm{cf}(\kappa) < \kappa$, suppose that $p \Vdash \dot{A} \colon \omega \to \check{V} \wedge \dot{A} \notin \check{V}$ and that each splitting node of p is $\dot{A}$-sharp. Then $p \Vdash \Gamma(\mathbb{P}) \in \check{V}[\dot{A}]$.*

The basic idea of Lemma 2.7 is that the generic sequence, and therefore the generic filter, can be reconstructed from $\dot{A}_G$ using Proposition 1.18 by determining a condition with the sharp coding.

Thanks to Lemma 2.7, Theorem 2.5 is implied by the following lemma.

Lemma 2.8. *Suppose that $p \Vdash \dot{A} \colon \omega \to \check{V} \wedge \dot{A} \notin \check{V}$. Then there are densely many $q \leq p$ in $\mathbb{M}_\kappa$ that are $\dot{A}$-sharp.*

Lemma 2.8 will be proved via a fusion sequence $(p_n)_{n < \omega}$ that is defined inductively. The following lemma focuses on a technical point that allows us define p_{n+1} given p_n.

Lemma 2.9. *Given $q \leq p$, $n \in \omega$, and $s \in q$, there is some $t \sqsupseteq s$ and $q' \leq q$ such that $\text{Stem}(q') = t$, t is κ_n-splitting, and t is $\dot{A}$-sharp with respect to q'.*

Proof. The procedure consists of the following five steps:

Step 1: We choose $t \sqsupseteq s$, $t \in q$ such that $|\text{Succ}_q(t)| \geq \kappa_n$.

Step 2: We fix an injective enumeration $\langle t_\alpha : \alpha < \kappa_n \rangle$ of (possibly a subset) of $\text{Succ}_q(t)$. By induction on $\alpha < \kappa_n$ choose a condition p_α and a finite set A_α with the following properties:

(1) $p_\alpha \leq q \restriction t_\alpha$.
(2) $\exists i_\alpha < \omega$ such that $p_\alpha \Vdash$ " $\text{range}(\dot{A} \restriction \check{i}_\alpha) = \check{A}_\alpha$".
(3) $A_\alpha \not\subseteq \bigcup_{\beta < \alpha} A_\beta$.

Condition (3) is possible since otherwise we would have $q \restriction t_\alpha \Vdash$ " $\text{range}(\dot{A}) \subseteq \bigcup_{\beta < \alpha} \check{A}_\beta$". Since the cardinality of $\bigcup_{\beta < \alpha} A_\beta$ is less than κ_n, this would imply that $q \restriction t_\alpha \Vdash$ "$\dot{A} \in \check{V}$" by (ω, κ_n)-distributivity, which contradicts our assumption that $p \Vdash$ "$\dot{A}$ is a new sequence".

Step 3: Choose $i < \omega$ and $B \subseteq \kappa_n$ with $|B| = \kappa_n$ such that $\forall \alpha \in B, i_\alpha = i$. This is possible since $\text{cf}(\kappa_n) > \omega$. Now $\{A_\alpha : \alpha \in B\}$ are pairwise distinct sets of size i.

Step 4: We let $q' = \bigcup_{\alpha \in B} p_\alpha$. Then $\text{Stem}(q') = t$ and t is $\dot{A}$-sharp with respect to q'. □

Now we return to Lemma 2.8.

Proof. We use a fusion sequence and the fact that $\mathbb{M}_\kappa$ is (ω, ν)-distributive for any $\nu < \kappa$.

Observe that if $s \in q$ is $\dot{A}$-sharp, $q' \leq q$, $s \in q'$, and $\text{Succ}_q(s) = \text{Succ}_{q'}(s)$, then s remains $\dot{A}$-sharp.

We define a fusion sequence $(p_n)_{n<\omega}$ as follows: Let $p_0 = p$. Suppose that $n \geq 1$ and p_{n-1} is constructed. We then construct p_n such that $p_n \leq_{n-1} p_{n-1}$ and such that each $s^+ \in \text{Split}_n(p_n)$ is $\dot{A}$-sharp. Explicitly, for all $r \in \text{Split}_{n-1}(p_{n-1})$ and $s \in \text{Succ}_{p_{n-1}}(r)$, we find $t \sqsupseteq s$ and $q(r,s)$ witnessing Claim 2.9. Then we let

$$p_n = \bigcup \{q(r,s) : r \in \text{Split}_{n-1}(p_{n-1}), s \in \text{Succ}_{p_{n-1}}(r)\}.$$

Finally, we let $\bar{p}$ be the fusion of $(p_n)_{n<\omega}$ so that $\bar{p}$ witnesses the statement of the lemma. This completes the proof of Lemma 2.8 and hence the proof of Theorem 2.5. □

3. THE BERNSTEIN TECHNIQUE

3.1. **Failures of $(\text{cf}(\kappa), 2)$-Distributivity and Collapsing κ to $\text{cf}(\kappa)$.** In this subsection we prove the following:

Theorem 3.1. *Suppose $\text{cf}(\kappa) = \lambda > \omega$ and κ is a λ-strong limit. Let $\mathbb{P}$ be any of $^\psi \mathbb{M}_\kappa^\phi$ or $\mathbb{L}_\kappa^\phi$ where $\psi \in \{\text{scl}, \emptyset\}$, $\phi \in \{\text{stat}, \text{club}, \emptyset\}$. Then $\mathbb{P}$ is not $(\lambda, 2)$-distributive.*

This answers Question 3.6 of [9]. While the second author was presenting a pre-print of this paper, Gitik observed that the failure of $(\mathrm{cf}(\kappa), 2)$-distributivity follows readily if one assumes the non-existence of $0^{\#}$ and uses Jensen's Covering Lemma, but we note that our result is a theorem of ZFC. Also, the reader may observe that the proof does not apply to the pre-perfect versions.

Throughout this section, κ is a singular cardinal of uncountable cofinality λ. Let $\mathbb{P}$ be any perfect version of $\mathbb{M}_{\kappa}$ or $\mathbb{L}_{\kappa}$ as indicated above. We show that our assumptions regarding cardinal arithmetic are sharp using a result that probably goes back to Solovay.

Remark 3.2. *Let $\mathbb{P}$ be any forcing poset. If $\tau^{\lambda} \geq \kappa$ and $\mathbb{P}$ is not (λ, κ)-distributive, then $\mathbb{P}$ is not (λ, τ)-distributive. In particular, $\mathbb{P}$ is not $(\lambda, 2)$-distributive if $2^{\lambda} \geq \kappa$.*

Proof. Let $f : \kappa \to {}^{\lambda}\tau$ be an injective function, let $b : \lambda \to \lambda \times \lambda$ be a bijective function such that $b(\alpha) = (b_1(\alpha), b_2(\alpha))$ for $\alpha < \lambda$, and let $\dot{x}$ be a $\mathbb{P}$-name for a new function $\lambda \to \kappa$. Let $\dot{h}$ be a $\mathbb{P}$-name for the assignment $\alpha \mapsto f(\dot{x}(b_1(\alpha)))(b_2(\alpha)$. We will show that $\dot{h}$ is a name for a new function $\lambda \to \tau$ by showing that for all $p \in \mathbb{P}$, there are $q_1, q_2 \leq p$, $\alpha < \lambda$, and $\beta_1 \neq \beta_2$ such that $q_1 \Vdash \dot{h}(\check{\alpha}) = \check{\beta}_1$ and $q_2 \Vdash \dot{h}(\check{\alpha}) = \check{\beta}_2$. Choose $q_1, q_2 \leq p$ and $\gamma < \lambda$ such that $q_1 \Vdash \dot{x}(\check{\gamma}) = \check{\delta}_1$ and $q_2 \Vdash \dot{x}(\check{\gamma}) = \check{\delta}_2$ for $\delta_1 \neq \delta_2$. Then $f(\delta_1) \neq f(\delta_2)$, and so $f(\delta_1)(\delta) \neq f(\delta_2)(\delta)$ for some $\delta < \lambda$. If $\alpha < \lambda$ is such that $(b_1(\alpha), b_2(\alpha)) = (\gamma, \delta)$, then $q_1 \Vdash \dot{h}(\check{\alpha}) = \check{f}(\check{\delta}_1)(\check{\delta})$ and $q_2 \Vdash \dot{h}(\check{\alpha}) = \check{f}(\check{\delta}_2)(\check{\delta})$. $\qquad\square$

We recall the classical result of Bernstein for completeness.

Lemma 3.3 (The Existence of $\mathcal{A}$-Bernstein Sets.). *Let μ be an infinite cardinal and let $\mathcal{A} \subseteq \mathcal{P}(\mu)$, be such that $|\mathcal{A}| \leq \mu$ and such that for any $A \in \mathcal{A}$ we have $|A| = \mu$. Then there is a set $\{B_{\alpha} : \alpha < \mu\}$ with the following properties:*

(1) for each $\alpha < \mu$, $B_{\alpha} \subseteq \mu$ and $|B_{\alpha}| = \mu$,
(2) for $\alpha \neq \beta$, $B_{\alpha} \cap B_{\beta} = \emptyset$,
(3) for all $A \in \mathcal{A}$, for any $\alpha < \mu$, $A \cap B_{\alpha} \neq \emptyset$.

The B_{α}'s are sometimes called Bernstein sets for $\mathcal{A}$.

Proof. We enumerate $\mathcal{A}$ as $\langle A_{\beta} : \beta < \mu \rangle$. By induction on $\gamma < \mu$ we choose an element $a_{\beta,\gamma} \in \mu$ by induction on $\beta < \gamma$ such that

$$a_{\beta,\gamma} \in A_{\beta} \setminus \{a_{\beta',\gamma'} : (\beta' < \gamma' < \gamma) \vee (\beta' < \beta < \gamma' = \gamma)\}.$$

Then we define $D_{\alpha} = \{a_{\alpha,\beta} : \alpha < \beta < \mu\}$. Now D_{α}, $\alpha < \mu$, is a disjoint refinement of $\mathcal{A}$. That means $D_{\alpha} \in [A_{\alpha}]^{\mu}$ and for $\beta \neq \alpha$, $D_{\alpha} \cap D_{\beta} = \emptyset$. We enumerate each D_{α} as $\langle d_{\alpha,\beta} : \beta < \mu \rangle$. We let $B_{\beta} = \{d_{\alpha,\beta} : \alpha < \mu\}$. $\qquad\square$

The following can be seen as a variation on Silver's famous theorem for singular cardinals of uncountable cofinality.

Lemma 3.4. *Let κ be a singular λ-strong limit cardinal of uncountable cofinality λ. Let $\langle \kappa_\alpha : \alpha < \lambda \rangle$ be a sequence of regular cardinals converging to κ, and for $\alpha < \lambda$ let μ_α be the cardinality of $\prod_{\beta < \alpha} \kappa_\beta$.*

If T is a perfect tree in the space $\prod_{\alpha < \lambda} \kappa_\alpha$, then the set

$$\{\alpha < \lambda : \operatorname{cf}(\alpha) = \omega, \ |[T \restriction \alpha]| = \mu_\alpha\}$$

contains an unbounded set.

Proof. Let $\langle \delta_\alpha : \alpha < \lambda \rangle$ be the club such that for all $\alpha < \lambda$, $\delta_{\alpha+1} = \kappa_{\alpha+1}$, and such that for all limits $\alpha < \lambda$, $\delta_\alpha := \sup_{\beta < \alpha} \kappa_\alpha$. There is a club $D \subseteq \lambda$ such that for all $\alpha \in D$, α is a limit and δ_α is a λ-strong limit. Observe that for all $\alpha < \lambda$, $T \restriction \alpha$ has no branches of length less than α. Then observe that by Lemma 2.2, if $\alpha \in D$ and $\operatorname{cf}(\alpha) = \omega$, then $|[T \restriction \alpha]| \neq \mu_\alpha$ implies that $|[T \restriction \alpha]| \leq \delta_\alpha$. Therefore, if the conclusion of the lemma we are trying to prove is false, then $S := \{\alpha \in D : \operatorname{cf}(\alpha) = \omega, \ |[T \restriction \alpha]| \leq \delta_\alpha\}$ is stationary in λ.

Now we can demonstrate the basic idea behind our definition of S. For all $\alpha \in S$, enumerate $[T \restriction \alpha]$ as $\langle t_\xi^\alpha : \xi < \delta_\alpha \rangle$. If $b \in [T]$ is a cofinal branch, then for $\alpha \in S$ let $F(\alpha)$ be the least $\beta < \alpha$ such that there is some $\xi < \delta_\beta$ such that $b \restriction \alpha = t_\xi^\alpha$. This is a regressive function on S, so Fodor's Lemma implies that there is a stationary subset $S' \subseteq S$ and some $\beta < \lambda$ such that for all $\alpha \in S'$, $b \restriction \alpha = t_\xi^\alpha$ for some $\xi < \delta_\beta$.

We can use this idea to show that there are at most κ-many cofinal branches of T. Suppose for contradiction that there are at least κ^+-many cofinal branches of T. Then because $2^\lambda < \kappa$, there is a stationary subset $S' \subset S$ and some $\beta < \lambda$ such that are κ^+-many branches b with the property that for all $\alpha \in S'$, $b \restriction \alpha = t_\xi^\alpha$ for some $\xi < \delta_\beta$. Then since $\delta_\beta \leq \kappa_\beta$ and $\kappa_\beta^\lambda < \kappa$, it follows that there is a function $h : S' \to \delta_\beta$ such that for κ^+-many distinct branches b, $b \restriction \alpha = t_{h(\alpha)}^\alpha$ for all $\alpha \in S'$. But these branches are determined cofinally, so this is impossible.

Since T has at most κ-many branches, it cannot be a perfect tree. $\square$

Now we turn to the proof of Theorem 3.1:

Proof. Again let μ_α denote $|\prod_{\beta < \alpha} \kappa_\beta|$. For a limit $\alpha < \lambda$, let $\mathcal{A}(\alpha)$ be the collection of subsets X of $\prod_{\beta < \alpha} \kappa_\beta$ of size μ_α (i.e. the subsets with "full" size) that are the branches of a perfect tree in $\bigcup_{\gamma < \alpha} \prod_{\beta < \gamma} \kappa_\beta$. Note that there are at most $2^{\delta_\alpha} = 2^{\lim_{\beta < \alpha} \kappa_\beta} = \mu_\alpha$-many perfect trees in $\bigcup_{\gamma < \alpha} \prod_{\beta < \gamma} \kappa_\beta$ and hence $|\mathcal{A}(\alpha)| \leq \mu_\alpha$. Let $\{B_{\alpha, \gamma} : \gamma < \mu_\alpha\}$ be a set of pairwise disjoint $\mathcal{A}(\alpha)$-Bernstein sets. This means that if $X \subset \prod_{\beta < \alpha} \kappa_\beta$ is in $\mathcal{A}(\alpha)$, then for $\gamma < \mu_\alpha$, $B_{\alpha, \gamma} \cap X \neq \emptyset$ and $X \cap (\prod_{\beta < \alpha} \kappa_\beta \setminus B_{\alpha, \gamma}) \neq \emptyset$.

Now we define a $\mathbb{P}$-name for a function from $\operatorname{Lim}(\lambda)$, the set of limit ordinals in λ, to 2. We define $\dot{f}$ as the following name:

$$(3.1) \quad \begin{aligned} &\{\langle(\check{\alpha},0),p\rangle \,:\, \alpha \in \mathrm{Lim}(\lambda), \mathrm{dom}(\mathrm{Stem}(p)) \geq \alpha, \mathrm{Stem}(p){\upharpoonright}\alpha \in B_{\alpha,0}\}\cup \\ &\{\langle(\check{\alpha},1),p\rangle \,:\, \alpha \in \mathrm{Lim}(\lambda), \mathrm{dom}(\mathrm{Stem}(p)) \geq \alpha, \mathrm{Stem}(p){\upharpoonright}\alpha \notin B_{\alpha,0}\}. \end{aligned}$$

Assume for contradiction that there is some $g \in {}^{\mathrm{Lim}(\lambda)}2 \cap V$ and there is some $p \in \mathbb{P}$ such that $p \Vdash \dot{f} = g$.

By Lemma 3.4, there is some limit ordinal $\alpha > \mathrm{dom}(\mathrm{Stem}(p))$ such that $|[p{\upharpoonright}\alpha]| = \mu_\alpha$, and therefore $[p{\upharpoonright}\alpha] \in \mathcal{A}(\alpha)$. If $g(\alpha) = 1$, then we strengthen p to some condition q such that that $\mathrm{Stem}(q){\upharpoonright}\alpha \in B_{\alpha,0}$. This is possible, since $B_{\alpha,0}$ is an $\mathcal{A}(\alpha)$-Bernstein set and $[p{\upharpoonright}\alpha] \in \mathcal{A}(\alpha)$, so $[p{\upharpoonright}\alpha] \cap B_{\alpha,0} \neq \emptyset$. If $g(\alpha) = 0$, then we strengthen p to some condition q such that $\mathrm{Stem}(q){\upharpoonright}\alpha \notin B_{\alpha,0}$. This is again possible, since $\{B_{\alpha,\gamma} \,:\, \gamma < \mu_\alpha\}$ is a set of disjoint $\mathcal{A}(\alpha)$-Bernstein sets and $[p{\upharpoonright}\alpha] \in \mathcal{A}(\alpha)$, so $[p{\upharpoonright}\alpha] \not\subseteq B_{\alpha,0}$.

So in any case we have $q \leq p$ and $q \Vdash \dot{f}(\alpha) \neq g(\alpha)$, and hence a contradiction. Thus, forcing with $\mathbb{P}$ adds a new function $\dot{f}_G \in {}^{\mathrm{Lim}(\lambda)}2$ and hence $\mathbb{P}$ is not $(\lambda,2)$-distributive. $\qquad\square$

This Bernstein argument applies equally to the Laver versions of $\mathbb{P}$ (see Definition 1.11(2)) and to the versions splitting into particular filter sets and also to the versions in which each limit of splitting nodes is a splitting node. Of course, Lemma 3.4 is not necessary for the Laver versions.

The Bernstein technique shows us that Lemma 3.4 leads to a stronger negation of $\mathrm{cf}(\kappa)$-distributivity: there is a name for a collapsing function.

Theorem 3.5. *If* $\mathrm{cf}(\kappa) = \lambda > \omega$ *and* κ *is a* λ-*strong limit, then* $\mathbb{P}$ *adds a surjection from* λ *to* κ.

Proof. For each $\alpha < \lambda$, let $\mathcal{A}(\alpha)$ be as in the proof of Theorem 3.1 and again choose a set $\{B_{\alpha,\beta} \,:\, \beta < \mu_\alpha\}$ of disjoint Bernstein sets.

We define a $\mathbb{P}$-name for a function from $\mathrm{Lim}(\lambda)$ onto κ:

$$\begin{aligned} \dot{f} = \{\langle(\check{\alpha},\gamma),p\rangle \,:\, &\alpha \in \mathrm{Lim}(\lambda), \gamma < \mu_\alpha, \\ &\mathrm{dom}(\mathrm{Stem}(p)) \geq \alpha, \mathrm{Stem}(p){\upharpoonright}\alpha \in B_{\alpha,\gamma}\}. \end{aligned}$$

The argument that $\Vdash_\mathbb{P}$ "$\dot{f} : \mathrm{Lim}(\lambda) \twoheadrightarrow \kappa$" is analogous to the one presented in the proof of Theorem 3.1. For $\gamma < \kappa$ the set

$$D_\gamma = \{p \in \mathbb{P} \,:\, \exists \alpha \in \mathrm{Lim}(\lambda), p \Vdash \dot{f}(\check{\alpha}) = \check{\gamma}\}$$

is a dense subset of $\mathbb{P}$. $\qquad\square$

Observe that the Bernstein technique does not work for pre-perfect trees. In Section 5.1 we will obtain failure of $(\lambda,2)$-distributivity for the pre-perfect versions using a completely different approach.

3.2. **Non-Minimality and Cardinal Preservation for the** $(< \mathrm{cf}(\kappa))$**-Closed Versions.** In this section we settle natural questions of minimality and cardinal collapses for $^{\mathrm{scl}}\mathbb{M}_\kappa^{\mathrm{club}}$ and $\mathbb{L}_\kappa^{\mathrm{club}}$ given singular λ-strong limits of cofinality $\lambda > \omega$.

The first observation is a corollary to Theorem 3.5. For $\lambda = \kappa = \omega$, Groszek [11, Theorem 5] proved an analogous result. For $\lambda = \omega < \kappa$, the technique of Brendle et al. [4, Proposition 77] shows that $^{\mathrm{scl}}\mathbb{M}_\kappa^{\mathrm{club}}$ and $\mathbb{L}_\kappa^{\mathrm{club}}$ add an ω-Cohen real and hence are not minimal for ω-sequences. Note that Theorem 2.5 does not generalize to uncountable λ because we do not have the (λ, ν)-distributivity needed for Step 2 of Claim 2.9.

Corollary 3.6. *Suppose that $\mathbb{P}$ is either $^{\mathrm{scl}}\mathbb{M}_\kappa^{\mathrm{club}}$ or $\mathbb{L}_\kappa^{\mathrm{club}}$, that $2^{<\lambda} = \lambda$, and that κ is a singular λ-strong limit of cofinality $\lambda > \omega$. Then $\mathrm{Add}(\lambda, 1) \lessdot \mathbb{P}$, and so $\mathbb{P}$ is not minimal for λ-sequences.*

Proof. This follows from two folklore lemmas. The first is that for homogenous forcings $\mathbb{Q}_1$ and $\mathbb{Q}_2$, there is a complete embedding $\iota : \mathbb{Q}_1 \to \mathrm{ro}(\mathbb{Q}_2)$ if and only if every filter G that is $\mathbb{Q}_2$-generic over V induces a filter H that is $\mathbb{Q}_1$-generic over V. The second is that if $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are such that $V[\mathbb{Q}_2] \models$ "$\mathbb{Q}_1$ is $(< \lambda)$-closed and $|\mathcal{P}^V(\mathbb{Q}_1)| = \lambda$", then any $\mathbb{Q}_2$-generic induces a $\mathbb{Q}_1$-generic.

By Theorem 3.5, we have $V[\mathbb{P}] \models$ "$|\mathcal{P}^V(\mathrm{Add}(\lambda, 1))| = |(2^\lambda)^V| = \lambda$", so we conclude that forcing with either $^{\mathrm{scl}}\mathbb{M}_\kappa^{\mathrm{club}}$ or $\mathbb{L}_\kappa^{\mathrm{club}}$ induces a generic on $\mathrm{Add}(\lambda, 1)$. $\square$

The rest of the section is dedicated to proving that the closed versions preserve κ^+ as well as the cardinals up to and included λ.

We prove a lemma on $\leq_\alpha$-stronger decision into boundedly-many possibilities. There are analogs used in many places, e.g. [15].

Theorem 3.7. *Let $\mathbb{P}$ be either $^{\mathrm{scl}}\mathbb{M}_\kappa^{\mathrm{club}}$ or $\mathbb{L}_\kappa^{\mathrm{club}}$. Let $p \in \mathbb{P}$ force that $\dot{f}$ be a $\mathbb{P}$-name for a function from λ into κ^+. Then there are a set x of size at most κ and a condition $q \leq p$ such that*

$$q \Vdash \mathrm{range}(\dot{f}) \subseteq \check{x}.$$

Proof. First we establish a claim that works for any of the ever-splitting versions:

Claim: Let $p \in {}^{\mathrm{scl}}\mathbb{M}_\kappa^{\mathrm{club}}$ force that $\dot{\beta}$ is an ordinal and let $\alpha < \lambda$. Then there is some $q \leq_\alpha p$ and some set x of size at most $|\mathrm{Split}_\alpha(p)| \cdot \kappa_\alpha$ such that $q \Vdash \dot{\beta} \in x$.

To prove the claim, consider $\alpha < \kappa$. By definition, $\mathrm{Split}_\alpha(p)$ is a maximal antichain. For each $s \in \mathrm{Split}_\alpha(p)$ we proceed as in the step $\alpha = 0$, i.e., for each $t \in \mathrm{Succ}_p(s)$ we choose some $q(t, s) \leq p \restriction t$ that decides $\dot{\beta}$ and we let x_s be a set containing all these decisions. We let $x = \bigcup\{x_s : s \in \mathrm{Split}_\alpha(p)\}$. The size of $\mathrm{Split}_\alpha(p)$ is bounded by $\sup_{\beta < \alpha} \mu_\beta \leq \kappa$ by our assumptions on cardinal exponentiation in Definitions 1.9 and 1.11. Each x_s has size at most

$\kappa_{\mathrm{dom}(s)} < \kappa$. Thus $|x| \leq |\operatorname{Split}_\alpha(p)| \cdot \kappa \leq \kappa$ We let

$$q = \bigcup \{q(t,s) \,:\, s \in \operatorname{Split}_\alpha(p) \wedge t \in \operatorname{Succ}_p(s)\}.$$

Then $q \leq_\alpha p$ and $q \Vdash \check{\beta} \in \check{x}$. This proves the claim.

Given p, assume without loss of generality that $p \Vdash \dot{f} \colon \lambda \to \kappa^+$. Then construct a fusion sequence $\langle q_\alpha \,:\, \alpha < \lambda \rangle$ such that there are x_α of size at most κ such that $q_\alpha \Vdash \dot{f}(\alpha) \in x_\alpha$.

We choose the q_α be recursion on α. In the successor steps we proceed according to the claim. In the limit steps $\alpha < \kappa$, we first let $q'_\alpha = \bigcap \{q_\beta \,: \beta < \alpha\}$. By closure, q'_α is a condition. Now we perform a successor step to go from q'_α to q_α.

The fusion limit $q = \bigcap \{q_\alpha \,:\, \alpha < \lambda\}$ has the desired properties. $\square$

Although the previous theorem applies to $\mathbb{L}_\kappa^{\mathrm{club}}$, there is an even simpler way of obtaining the preservation of κ^+ for this poset:

Proposition 3.8. *The singular club-Laver-Namba forcing $\mathbb{L}_\kappa^{\mathrm{club}}$ has the κ^+-chain condition.*

Proof. Observe that two conditions in the club-Laver-Namba forcing are compatible if they have the same stem, and there are only κ-many stems. $\square$

Theorem 3.5 on collapsing κ to λ pertains also to the $(<\lambda)$-closed versions of the forcings from this section. Combining Theorem 3.7 with Theorem 3.5 about collapsing κ to λ (for which we used λ-strong limits) we get:

Corollary 3.9. *For any λ-strong limit κ and $\mathbb{P}_\kappa$ being $^{\mathrm{scl}}\mathbb{M}_\kappa^{\mathrm{club}}$ or $\mathbb{L}_\kappa^{\mathrm{club}}$, we have*

$$\mathbb{P}_\kappa \Vdash |\kappa| = \lambda \wedge \lambda^+ = (\kappa^+)^V$$

and all cardinals $\leq \lambda$ are preserved.

3.3. **The Non-Miniminality of a Non-Closed Version.** The aim of this section is to show that we do not necessarily need closure to obtain non-minimality results.

Theorem 3.10. *Let κ be a singular cardinal of cofinality ω_1. Let $\mathbb{P}$ be any of $\mathbb{L}_\kappa$, $\mathbb{L}_\kappa^{\mathrm{stat}}$, and $\mathbb{L}_\kappa^{\mathrm{club}}$. Then $\operatorname{Add}(\omega_1, 1) \lessdot \mathbb{P}$.*

The hypothesis that $\mathrm{cf}(\kappa) = \omega_1$ is used in Lemma 3.13, in which the limit steps of the construction work for ordinals of countable cofinality. For the rest of this section we fix κ, a singular $\aleph_1$-strong limit cardinal of cofinality $\aleph_1$. The proof is redundant for $\mathbb{L}_\kappa^{\mathrm{club}}$ and would work for $\mathbb{L}_\kappa^{\mathrm{stat}}$ with some extra care. We will give the proof for $\mathbb{L}_\kappa$ for simplicity.

Definition 3.11. Let $\dot{f}$ be the binary function 3.1 introduced in the proof of Theorem 3.1. Given $p \in \mathbb{L}_\kappa$, let $b(p)$ be defined so that $\alpha \in \mathrm{dom}(b(p))$ and $b(p)(\alpha) = i$ if and only if $p \Vdash \dot{f}(\omega \cdot \alpha) = i$.

Proposition 3.12. *If $p \in \mathbb{L}_\kappa$, then $b(p) \in \operatorname{Add}(\omega_1, 1)$.*

Proof. This is the observation that if α is the length of the stem of p, then for all limit ordinals γ in (α, λ), $p \upharpoonright \gamma$ is perfect and therefore p does not decide $\dot{f}(\gamma)$ by the reasoning in the proof of Theorem 3.1. $\qquad\square$

Lemma 3.13. *Suppose we are given $\bar{p} \in \mathbb{L}_\kappa$. If $\bar{b} = b(\bar{p})$, then given any $b \in \mathrm{Add}(\omega_1, 1)$ such that $b \leq \bar{b}$, there is some $p \leq \bar{p}$ such that $b(p) = b$.*

Proof. Fix $\bar{p}, \bar{b}$, and b. Let α be the length of the stem $\bar{s}$ of $\bar{p}$. Let $\delta = \mathrm{dom}(b)$. For $t \in \bar{p}$, we say that t is *good* if for all $\beta \in \mathrm{Lim}((\mathrm{dom}(t)+1)) \cap \delta$, $t \upharpoonright \beta \in B_{\beta,0}$ if and only if $b(\tilde{\beta}) = 0$ where $\omega \cdot \tilde{\beta} = \beta$.

We will show by induction on limits $\gamma \in [\alpha + \omega, \delta]$ that:

> for all good nodes $s \sqsupseteq \bar{s}$ in $\bar{p}$, there is some $t \sqsupseteq s$ such that t is good and $\gamma \in \mathrm{dom}(t) + 1$.

The successor case of the induction pertains to ordinals of the form $\gamma + \omega$. Let $s \sqsupseteq \bar{s}$ be a good node of p such that $\mathrm{dom}(s) = \bar{\gamma} < \gamma + \omega$. If $\gamma > \alpha$, apply the inductive hypothesis to find $s' \sqsupseteq s$ with domain γ. Because $p \in \mathbb{L}_\kappa$, $(p \upharpoonright s') \upharpoonright (\gamma + \omega)$ is perfect. Therefore we can find some $t \sqsupseteq s$ such that the set of predecessors of t is a cofinal branch of $(p \upharpoonright s) \upharpoonright (\gamma + \omega)$ and $t \upharpoonright (\gamma + \omega) \in B_{\gamma+\omega,0}$ if $b(\tilde{\gamma} + 1) = 0$ where $\omega \cdot (\tilde{\gamma} + 1) = \gamma + \omega$ and $t \upharpoonright (\gamma + \omega) \notin B_{\gamma+\omega,0}$ otherwise. Then t is as sought.

Now suppose that γ is a limit of limit ordinals, and in particular that $\langle \gamma_n : n < \omega \rangle$ is a sequence of limit ordinals above α converging to γ. We pick a good node s and define a perfect subtree T of $(p \upharpoonright s) \upharpoonright \gamma$ by induction on n. Suppose we have defined $T \upharpoonright \gamma_n$. Then for each cofinal branch s' representing a cofinal branch of $T \upharpoonright \gamma_n$, choose κ_{γ_n+1}-many *immediate* successors s'_ξ, $\xi < \kappa_{\gamma_n+1}$ of s' in p (again using that $p \in \mathbb{L}_\kappa$) which are still good nodes, and then choose good nodes $s''_\xi \sqsupseteq s_\xi$ such that $\gamma_{n+1} \in \mathrm{dom}\, s''_\xi$ for all $\xi < \kappa_{\gamma_n+1}$ using the inductive hypothesis. Then collect all such s''_ξ's for each s' representing a cofinal branch of $T \upharpoonright \gamma_n$ and let this form $T \upharpoonright \gamma_{n+1}$. Once we are done defining T, we choose a node t of length $\gamma + 1$ defining a cofinal branch of T such that $t \upharpoonright \gamma \in B_{\gamma,0}$ if and only if $b(\tilde{\gamma}) = 0$ where $\omega \cdot \tilde{\gamma} = \gamma$.

Finally, we choose some $t \in \bar{p}$ witnessing the statement for δ and we let $p = \bar{p} \upharpoonright t$. $\qquad\square$

We proceed with the proof of Theorem 3.10.

Proof. We show that there is a complete embedding $\iota : D \to \mathrm{ro}(\mathbb{L}_\kappa)$ where D is a dense subset of $\mathrm{Add}(\omega_1, 1)$. Specifically, let D be the dense subset of $\mathrm{Add}(\omega_1, 1)$ consisting of conditions b such that $\mathrm{dom}(b) = \gamma$ for some $\gamma < \omega_1$. Given $b \in D$, define

$$\iota(b) := \{p \in \mathbb{L}_\kappa : \forall i \in 2, \alpha \in \mathrm{dom}(b) \wedge b(\alpha) = i \Rightarrow p \Vdash \text{``}\dot{f}(\omega \cdot \alpha) = i\text{''}\}.$$

Observe that $\iota(b) \neq \emptyset$ by Lemma 3.13.

The basic properties of complete embeddings hold for ι. We see that $\iota(b)$ is a regular open set, meaning that if $p \notin \iota(b)$, then there is some $q \leq p$ such

that $\{r \in \mathbb{L}_\kappa : r \le q\} \cap \iota(b) = \emptyset$. Moreover, $b' \le b$ implies that $\iota(b') \subseteq \iota(b)$ and $b' \perp b$ implies that $\iota(b') \cap \iota(b) = \emptyset$ by definition.

To prove completeness, suppose that $u \in \text{ro}(\mathbb{L}_\kappa)$ and let $p \in u$. Let $b = b(p)$. We claim that b witnesses completeness. If $b' \le b$, then let $p' \le p$ witness Lemma 3.13. Then $\iota(b') \cap u \ne \emptyset$ because this intersection contains p'. $\qquad\qquad\square$

The fact that this construction abandons notions of closure leads us to the next section.

4. The Balcar-Simon Technique and Failures of Countable Distributivity

Here are some examples of failures of countable distributivity in the non-closed versions of forcings of perfect tree forcings. We demonstrate the ways in which an idea of Balcar and Simon can be used to obtain failures of countable distributivity by finding a way to degrade a structure through countably many steps (see [1], [2]), mainly by picking out a non-stationary set and discarding limit points at each stage. All of these involve an ordering, and we cover $\prec^{\text{vert}}$, $\prec^{\text{horiz}}$, and $\prec^{\text{stat}}$. For forcing with the pre-perfect versions we refer the reader to Subsection 5.1.

4.1. **Slow Fusion.** Each of our results in the section will require a lemma stating that for each condition p in the forcing we are interested in, there is some $q \prec p$ where $\prec$ is one of the orders mentioned above. These conditions q will be provided by fusion sequences in which conditions are strengthened locally and only slightly. We call these "slow fusion" sequences. These will in the end be used to prove that the usual sequences in the non-closed versions are not viable because they break down at intermediate steps.

We describe the basic idea of slow fusion here. It is simpler in the Laver case, which by definition satisfies Kanamori's vertical requirement that limits of splitting nodes are splitting nodes. In this case, a $\le$-decreasing sequence $\langle p_\alpha : \alpha < \lambda \rangle$ is slow if p_α differs from $\bigcap \{p_\beta : \beta < \alpha\}$ only in the level $p_\alpha(\text{dom}(\text{Stem}(p_0)) + \alpha + 1)$, and for any $\beta < \alpha$ and $t \in p_\alpha(\text{dom}(\text{Stem}(p_0)) + \alpha + 1)$, we have $p_\alpha \upharpoonright t = p_\beta \upharpoonright t$. In other words, only one level is altered at a time.

The Miller case is analogous, but we must keep track of the antichains $\text{Split}_\alpha(p)$ for $\alpha < \lambda$. In this case, in order to slowly descend from $\bigcap \{p_\beta : \beta < \alpha\}$ to p_α, a "horizontal stripe" of $\bigcap \{p_\beta : \beta < \alpha\}$ that starts after $\text{Split}_\alpha(\bigcap \{p_\beta : \beta < \alpha\})$ and is bounded in the tree ordering is thinned out in the transition to p_α, whereas the other areas of $\bigcap \{p_\beta : \beta < \alpha\}$ are just copied into p_α. By the disjointness of the stripes shrinkage to a non-condition in an intersection over infinitely many previous steps shall be precluded.

The existence of the intermediate limit $\bigcap\{p_\beta : \beta < \alpha\}$ is proved rigorously in Lemma 4.2 below. The final intersection, meaning the eventual limit of the fusion sequence at stage λ, is justified in the same manner.

Definition 4.1. We fix a sequence $\langle \kappa_\alpha : \alpha < \lambda \rangle$ of regular cardinals converging to κ. Let $\mathbb{P}$ be any of our forcings where $\lambda = \mathrm{cf}(\kappa)$.

For $\alpha \leq \gamma$ we write $q \leq_{\gamma,\alpha} p$ if the following hold:

(1) $q \leq p$,
(2) $\mathrm{Split}_\alpha(p) = \mathrm{Split}_\alpha(q)$,
(3) $\forall t \in \mathrm{Split}_\alpha(p), \mathrm{Succ}_p(t) = \mathrm{Succ}_q(t)$,
(4) $\forall s \in \mathrm{Split}_\gamma(q), \forall t \in \mathrm{Succ}_q(s), q{\restriction}t = p{\restriction}t$.

We say that a sequence $\langle p_\alpha : \alpha < \delta \rangle$ is a *slow fusion sequence* (of length δ) if $\delta \leq \lambda$ and if for $\alpha < \gamma < \delta$, we have $p_\gamma \leq_{\gamma,\alpha} p_\alpha$.

Note that this definition depends on the choice of $\langle \kappa_\alpha : \alpha < \lambda \rangle$.

Lemma 4.2. *For all the forcings in Definitions 1.9 and 1.11 and for any slow fusion sequence $\langle p_\alpha : \alpha < \delta \rangle$ with $1 \leq \delta \leq \lambda$, we have that $q_\delta = \bigcap\{p_\alpha : \alpha < \delta\} \in \mathbb{P}$ and that the following equation from the standard fusion lemma holds:*

$$q_\delta = \{t \in p_0 : \exists \alpha < \delta, \exists s \in \mathrm{Split}_\alpha(p_\alpha), t \sqsubseteq s\} \cup$$

(4.1)
$$\bigcup\{\bigcap_{\alpha < \delta} p_\alpha {\restriction} t : t \in \bigcap\{\mathrm{Split}_{\geq \delta}(p_\alpha) : \alpha < \delta\})\}.$$

where $\mathrm{Split}_{\geq \delta}(p_\alpha)$ has the natural interpretation. If $\delta = \lambda$, then the set in the second line is empty. If $\delta < \lambda$, the right-hand side of the first line is redundant since it is a subset of the set in the second line.

Moreover, $q_\delta \leq_{\delta,\alpha} p_\alpha$ for $\alpha < \delta$, and for $\delta < \lambda$, $\langle p_\alpha : \alpha < \delta \rangle ^\frown \langle q_\delta \rangle$ is a slow fusion sequence.

Proof. We show that the fusion does not break down at intermediate steps or at step λ by induction. Specifically, we prove that $q_\delta := \bigcap_{\alpha<\delta} p_\alpha \in \mathbb{P}$ and that Equation (4.1) holds by induction on $\delta \leq \lambda$. We give the argument for the Miller versions. The argument for the Laver versions is simpler because the antichains $\mathrm{Split}_\alpha(p_\gamma)$ are levels.

We must show that q_δ as defined is a condition. In the case of $\delta = \varepsilon + 1$ we have $q_\delta = p_\varepsilon$ is a condition and the verification of Equation (4.1) is easy. Now suppose that $\delta < \lambda$ is a limit. Since the empty node is in any p_α, we have $\emptyset \in q_\delta$. The main task is then to show that for each node $s \in q_\delta$, for any $\varepsilon \in [\delta, \lambda)$, there is a κ_ε-splitting node in q_δ above t.

To find this splitting, first let δ be the limit of a strictly increasing sequence δ_ι, $\iota < \mathrm{cf}(\delta)$. By the induction hypothesis and an inspection of Equation (4.1), there is a strictly increasing sequence $\langle s_\iota : \iota < \mathrm{cf}(\delta) \rangle$ such that the following hold:

(1) $s_\iota \in \mathrm{Split}_{\delta_\iota}(p_{\delta_\iota})$;
(2) for any $s \in \mathrm{Split}_{\delta_\iota}(p_{\delta_\iota})$ and any $t \in \mathrm{Succ}_{p_{\delta_\iota}}(s)$, $p_{\delta_\iota} {\restriction} t = p_0 {\restriction} t$;
(3) for any $\varepsilon \in [\delta_\iota, \delta)$ we have

(a) $\mathrm{Split}_{\delta_\iota}(p_{\delta_\iota}) = \mathrm{Split}_{\delta_\iota}(p_\varepsilon)$,

(b) for any $s \in \mathrm{Split}_{\delta_\iota}(p_{\delta_\iota})$, $\mathrm{Succ}_{p_{\delta_\iota}}(s) = \mathrm{Succ}_{p_\varepsilon}(s)$.

By $(< \lambda)$-closure of the conditions (not of $\mathbb{P}$ itself) we have $r := \bigcup\{s_\iota : \iota < \mathrm{cf}(\delta)\} \in q_\delta$. We take g_ι to be the $\sqsubseteq$-minimal node in $\mathrm{Split}_{\geq\delta}(q_{\delta_\iota})$ that is $\sqsupseteq r$. For any $\iota < \mathrm{cf}(\delta)$, the fourth condition in Definition 4.1 of $p_{\delta_\iota} \leq_{\delta_\iota,\delta_0} p_{\delta_0}$ tells us that we have $p_{\delta_\iota} \upharpoonright r = p_{\delta_0} \upharpoonright r$. Hence g_ι does not depend on ι and its set of immediate successors also does not depend on ι, so we write $g = g_\iota$ for any $\iota < \mathrm{cf}(\delta)$. Then $g \in \mathrm{Split}_{\geq\delta}(q_\delta)$, $\mathrm{Succ}_{q_\delta}(g) = \mathrm{Succ}_{p_{\delta_\iota}}(g)$, and for any $t \in \mathrm{Succ}_{q_\delta}(g)$, $q_\delta \upharpoonright t = p_{\delta_\iota} \upharpoonright t$. By $\sqsubseteq$-minimality, we have $g \in \mathrm{Split}_\delta(q_\delta)$.

By construction we have $\mathrm{Split}_\alpha(p_\alpha) = \mathrm{Split}_\alpha(q_\delta)$ for $\alpha < \delta$, and for $t \in \mathrm{Split}_\alpha(q_\delta)$, $\mathrm{Succ}_{p_\alpha}(t) = \mathrm{Succ}_{q_\delta}(t)$. Since for $g \in \mathrm{Split}_\delta(q_\delta)$ $p_0 \upharpoonright g = q_\delta \upharpoonright g$, the higher part is a perfect tree, as required. So q_δ is indeed a condition and Equation (4.1) holds. Then $q_\delta \leq_{\delta,\alpha} p_\alpha$ for all $\alpha < \delta$ and the so condition q_δ can serve as a prolongation of the slow fusion sequence $\langle p_\delta : \alpha < \delta\rangle$. $\square$

The next lemma shows that there are particular prolongations $\langle p_\alpha : \alpha < \delta\rangle ^\frown \langle p_\delta\rangle$ in which p_δ is stronger than the condition q_δ from Equation (4.1).

Lemma 4.3. *Let $\langle p_\alpha : \alpha < \delta\rangle$, $\delta \in \mathrm{Lim}(\kappa)$, be a slow fusion sequence and let $q_\delta = \bigcap\{p_\alpha : \alpha < \delta\}$. In the Miller case, we can strengthen q_δ to a condition p_δ, such that for any $t \in \mathrm{Split}_\delta(q_\delta)$, $|\mathrm{Succ}_{p_\delta}(t)| = 1$ and such that $\langle p_\alpha : \alpha \leq \delta\rangle$ is a slow fusion sequence. In the Laver case, we can allow for $t \in \mathrm{Split}_\delta(q_\delta) = \mathrm{Split}_\delta(p_\delta)$ that $\mathrm{Succ}_{p_\delta}(t)$ is a subset of $\mathrm{Succ}_{q_\delta}(t)$.*

Proof. For each $t \in \mathrm{Split}_\delta(q_\delta)$ we choose just one $f(t) \in \mathrm{Succ}_{q_\delta}(t)$ and some $\beta > \delta$ and some extension $f(t) \sqsubseteq g(t) \in \mathrm{Split}_\beta(q_\delta)$. Then we choose $S(g(t)) \subseteq \mathrm{Succ}_{q_\delta}(g(t))$ that meets the κ_δ-splitting criterion and let

$$p_\delta = \bigcup\{q_\delta \upharpoonright s : \exists t \in \mathrm{Split}_\delta(q_\delta), s \in S(g(t))\}.$$

Then we have for $\alpha < \delta$, $\mathrm{Split}_\alpha(p_\alpha) = \mathrm{Split}_\alpha(p_\delta)$, for any $t \in \mathrm{Split}_\alpha(p_\alpha)$, $\mathrm{Succ}_{p_\alpha}(t) = \mathrm{Succ}_{p_\delta}(t)$, and for any $s = g(t) \in \mathrm{Split}_\delta(p_\delta)$ for any $r \in \mathrm{Succ}_{p_\delta}(s) = S(s)$, $p_\delta \upharpoonright r = q_\delta \upharpoonright r = p_0 \upharpoonright r$ and $\langle p_\alpha : \alpha \leq \delta\rangle$ is a slow fusion sequence. In the Laver case we let $\beta = \delta$ and $t = f(t) = g(t) \in \mathrm{Split}_\delta(q_\delta)$ and we choose a subset $\mathrm{Succ}_{p_\delta}(t)$ of $\mathrm{Succ}_{q_\delta}(t)$ that meets the splitting criterion. $\square$

4.2. **Miller Forcing and the Vertical Balcar-Simon Trick.** Let $\mathbb{P}$ be $\mathbb{M}_\kappa$, $\mathbb{M}_\kappa^{\mathrm{stat}}$, $\mathbb{M}_\kappa^{\mathrm{club}}$. The following does not work either for the Laver case or for the splitting-closed Miller versions.

Definition 4.4 (The Vertical Balcar-Simon Trick). Let $p \in \mathbb{P}$. We let $q \prec^{\mathrm{vert}} p$ if:

(i) $q \leq p$,

(ii) $\forall t \in \mathrm{Split}_{\alpha+1}(p) \cap q$, $\mathrm{Succ}_q(t) = \mathrm{Succ}_p(t)$,

(iii) for $\alpha = 0$ or limit $\alpha < \lambda$, $\forall t \in \mathrm{Split}_\alpha(p)$, $|\mathrm{Succ}_q(t)| = 1$.

Lemma 4.5. *For any $p \in \mathbb{P}_\kappa$ there is some $q \prec^{\mathrm{vert}} p$.*

Proof. By induction on α we choose a slow fusion sequence $\langle p_\alpha : \alpha < \lambda \rangle$ using Lemma 4.3. We let $p_0 = p$. If $\alpha = \beta + 1$ is a successor, we let $p_\alpha = p_\beta$. If α is a limit we first let $q_\alpha = \bigcap \{p_\beta : \beta < \alpha\}$. By Lemma 4.2, $q_\alpha \in \mathbb{P}_\kappa$. Now for any $t \in \mathrm{Split}_\alpha(q_\alpha)$ we pick some $f(q_\alpha, t) \in \mathrm{Succ}_{q_\alpha}(t)$ and some $g(q_\alpha, t) \sqsupseteq f(q_\alpha, t)$ such that $g(q_\alpha, t) \in \mathrm{Split}_{\alpha+1}(q_\alpha)$. We let

$$p_\alpha = \bigcup \{q_\alpha \restriction g(q_\alpha, t) : t \in \mathrm{Split}_\alpha(q_\alpha)\}.$$

Then $\mathrm{Split}_\alpha(p_\alpha) = \mathrm{Split}_{\alpha+1}(q_\alpha)$ and $\langle p_\beta : \beta \leq \alpha \rangle$ is an initial segment of a slow fusion sequence. By Lemma 4.2 $q = \bigcap\{p_\alpha : \alpha < \lambda\}$ is a condition and for any limit α and any $t \in \mathrm{Split}_\alpha(p) \cap q$, $|\mathrm{Succ}_q(t)| = 1$ by construction. Thus we have $q \prec^{\mathrm{vert}} p$. $\qquad\square$

Theorem 4.6. $\mathbb{M}_\kappa$, $\mathbb{M}_\kappa^{\mathrm{club}}$, $\mathbb{M}_\kappa^{\mathrm{stat}}$ *are not ω-distributive.*

Proof. Let $\mathcal{A}_0$ be any maximal antichain in $\mathbb{P}_\kappa$. Given $\mathcal{A}_n$, choose for any $p \in \mathcal{A}_n$, a maximal antichain of conditions $q \prec^{\mathrm{vert}} p$ and let $\mathcal{A}_{n+1}$ be the union of these q's. Then we claim that there is no $q \in \mathbb{P}$ such that for any n there is $p_n \in \mathcal{A}_n$ with $q \leq p_n$. Suppose otherwise, and we have $\langle p_n : n < \omega \rangle$ such that $q \leq p_n \in \mathcal{A}_n$ for all $n < \omega$. Since the $\mathcal{A}_n$'s are antichains, we have $p_{n+1} \leq p_n$ for all $n < \omega$, which entails $p_{n+1} \prec^{\mathrm{vert}} p_n$. By the definition of $\prec^{\mathrm{vert}}$, there are no splitting nodes in $\bigcap\{p_n : n < \omega\}$, hence q cannot be a condition. $\qquad\square$

4.3. **Laver Forcing and the Horizontal Balcar–Simon Trick.** Let $\mathbb{P}$ be $\mathbb{M}_\kappa$, $^{\mathrm{scl}}\mathbb{M}_\kappa$ or $\mathbb{L}_\kappa$. Here the $(<\mathrm{cf}(\kappa))$-closed versions are of course excluded. The version with stationary ordinal successor sets is also excluded and will be handled in the next subsection.

Theorem 4.7. *If $\mathbb{P}$ is one of $\mathbb{M}_\kappa$, $^{\mathrm{scl}}\mathbb{M}_\kappa$, or $\mathbb{L}_\kappa$, then $\mathbb{P}$ is not countably distributive.*

Rather than changing splitting nodes to non-splitting nodes, we thin out successor sets. In this way, we prove Theorem 4.7, adapting Balcar and Simon's descending technique of their Theorem [1, Theorem 5.16].

Definition 4.8. A pair (p, f_p) is called a *labeled tree* if it has the following properties:

(1) $p \in \mathbb{P}$.
(2) $f_p = \langle f_{p,t} : t \in \mathrm{Split}(p)\rangle$.
(3) For for each $\alpha < \lambda$ for each $s \in \mathrm{Split}_\alpha(p)$ the function

$$f_{p,s} \colon \mathrm{OSucc}(s) \to \kappa_\alpha$$

is one-to-one and strictly increasing.

We let L be the set of all labeled trees.

Definition 4.9. Let (p, f_p) and (q, f_q) be two labeled trees such that $q \leq p$. We write $f_q < f_p$ if

$$\forall s \in \mathrm{Split}(q), \forall \xi \in \mathrm{dom}(f_{q,s}), f_{q,s}(\xi) < f_{p,s}(\xi).$$

We write $(q, f_q) \prec^{\text{horiz}} (p, f_p)$ if $q \leq p$ and $f_q < f_p$.

Lemma 4.10. *Let (p, f_p) be a labeled tree. Then there is some (q, f_q) such that $q \leq_0 p$ and $(q, f_q) \prec^{\text{horiz}} (p, f_p)$.*

Proof. We let $q_0 = p$, $f_{q_0} = f_p$ and we construct q along with $f_{q,s}$, $s \in \text{Split}_\alpha(q_\alpha)$, by a slow fusion sequence $\langle p_\alpha : \alpha < \lambda \rangle$. In step α the function $f_{p_\alpha,s}$ will be defined for $s \in \text{Split}_\alpha(p_\alpha)$.

In successor steps, given p_α we define for $s \in \text{Split}_\alpha(p_{\alpha+1})$ a successor set $\text{Succ}_{p_{\alpha+1}}(s) \subseteq \text{Succ}_{p_\alpha}(s)$ and a function $f_{p_{\alpha+1},s}$ as follows: Let the range of $f_{p_\alpha,s}$ be enumerated increasingly as $\langle \eta_i : 0 \leq i < \kappa_\alpha \rangle$. We define the lower part of $p_{\alpha+1}$ such that for each $s \in \text{Split}_{\alpha+1}(p_\alpha) = \text{Split}_{\alpha+1}(p_{\alpha+1})$ we have

$$(4.2) \qquad \text{Succ}_{p_{\alpha+1}}(s) = \{s \frown \xi : \exists i < \kappa_\alpha, f_{p,s}(\xi) = \eta_{i+1}\}.$$

The $f_{p,s}$ is not a misprint here: This is the only stage of the construction when we refer to direct successors of s. Moreover, we define $p_{\alpha+1}$ such that $\text{Succ}_{p_{\alpha+1}}(s) = \text{dom}(f_{p_{\alpha+1},s})$ is the set of these ξ's and for $i < \kappa_\alpha$, we let $f_{p_{\alpha+1},s}(\xi) = \eta_i < \eta_{i+1} = f_{p,s}(\xi)$. The function $f_{p_{\alpha+1},s}$ is also one-to-one and increasing. We define the higher part of the condition $p_{\alpha+1}$ by letting $p_{\alpha+1} \upharpoonright t = p_\alpha \upharpoonright t$ for $t \in \text{Succ}_{p_{\alpha+1}}(s)$, $s \in \text{Split}_{\alpha+1}(p_{\alpha+1})$.

In the limit steps $\alpha \leq \lambda$, we first take the intersection $q_\alpha = \bigcap \{p_\beta : \beta < \alpha\}$ and invoke Lemma 4.2. For $\alpha < \lambda$, we strengthen q_α to p_α as in the successor step using $\text{Split}_\alpha(q_\alpha)$ as a starting point. According to Lemma 4.3, $\langle p_\alpha : \alpha < \lambda \rangle$ is a slow fusion sequence.

In the end we let q be the fusion of p_α, $\alpha < \lambda$, and for $s \in \text{Split}_\alpha(p_\alpha)$ we let $f_{q,s} = f_{p_\alpha,s}$. $\qquad\square$

Definition 4.11. A subset $\mathcal{A} \subseteq L$ is called an L-antichain if for any two $(p, f_p), (q, f_q) \in \mathcal{A}$, p and q are incompatible.

Lemma 4.12. *Suppose $(p, f_p) \in L$. Suppose that $\mathcal{A}$ is a maximal L-antichain in $L_{(p,f_p)} = \{(q, h) \in L : (q, h) \prec^{\text{horiz}} (p, f_p)\}$. Then the set $\pi_1(\mathcal{A})$ of first components of $\mathcal{A}$ is a maximal antichain in $\mathbb{P}$.*

Proof. Suppose not. Then we may take $q \leq p$ such that q is incompatible with any member of $\pi_1(\mathcal{A})$. By Lemma 4.10 there is $(r, f_r) \prec^{\text{horiz}} (q, f_p \upharpoonright q)$. Now (r, f_r) violates the L-maximality of $\mathcal{A}$. $\qquad\square$

Now we prove Theorem 4.7.

Proof. In the Miller case we work with the dense set of conditions p that have a layered sequence $\langle \text{Split}_\alpha(p) : \alpha < \lambda \rangle$ of splitting maximal antichains and no other splitting. These are called weakly splitting normal conditions in [9, Def. 2.8]. These conditions correspond to the T''s and $\text{Split}_\alpha(T') = \{\bar{\varphi}(t \upharpoonright (\kappa_\alpha + 1)) : t \in \prod_{\beta \leq \alpha+1} \kappa_\beta\}$ of the proof of $(1) \to (2)$ in Lemma 2.2. Nodes in the maximal antichain $\text{Split}_\alpha(T')$ have κ_α immediate successors in the condition.

By induction on $n < \omega$ we choose a sequence of maximal L-antichains $\langle \mathcal{A}_n : n < \omega \rangle$ such that

$$\forall m < n < \omega, \forall (r, f_r) \in \mathcal{A}_n, \exists (q, f_q) \in \mathcal{A}_m, (r, f_r) \prec^{\text{horiz}} (q, f_q).$$

We let $\mathcal{A}_0 = \{(p, \text{id})\}$ where id is the naturally defined identity function.

We carry out the successor step: For each $(q, f_q) \in \mathcal{A}_n$ we take a maximal L-antichain $\mathcal{A}_{n+1}^{(q,f_q)} \prec^{\text{horiz}}$-below (q, f_q) and we let $\mathcal{A}_{n+1} = \bigcup \{\mathcal{A}_{n+1}^{(q,f_q)} : (q, f_q) \in \mathcal{A}_n\}$.

Claim: There is no condition $r \in \mathbb{P}$ such that

$$\forall n < \omega, \exists q_n \in \pi_1(\mathcal{A}_n), r \leq q_n.$$

Suppose the contrary and fix such an r and a witnessing sequence $\langle q_n : n < \omega \rangle$ such that $(q_n, f_{q_n}) \in \mathcal{A}_n$ and $(q_{n+1}, f_{q_{n+1}}) \prec^{\text{hor}} (q_n, f_{q_n})$ for all $n < \omega$. For $s \in \text{Split}(r)$ and $s ^\frown \xi \in r$ let

$$f'_{r,s}(\xi) = \min\{f_{q_n,s}(\xi) : n < \omega\}.$$

This is countable-to-one. Now focus on $\text{Stem}(r) = s$. It is κ_0-splitting, and for any $n < \omega$, the function $f_{q_n,s}$ is defined. For $n < \omega$ let

$$X_n = \{\xi : s ^\frown \xi \in \text{Succ}_r(s) \wedge f'_{r,s}(\xi) = f_{q_n,s}(\xi)\}.$$

There is some $n \in \omega$ such that $|X_n| = \kappa_0$ (recall that we assume $\text{cf}(\kappa_0) > \omega$). Then for such a $\xi \in X_n$,

$$f_{q_n,s}(\xi) = f'_{r,s}(\xi) = \min\{f_{q_m,s}(\xi) : m < \omega\} \leq f_{q_{n+1},s}(\xi).$$

However, this contradicts $f_{q_{n+1},s}(\xi) < f_{q_n,s}(\xi)$. This proves the claim.

Thus the maximal antichains $\{\pi_1(\mathcal{A}_n) : n < \omega\}$ witness the failure of ω-distributivity. $\square$

4.4. **The Stationary Balcar–Simon Trick.** Now we present a version of the horizontal Balcar-Simon trick for $\mathbb{L}_\kappa^{\text{stat}}$ where we can degrade stationary sets, whereas a construction along the lines of Equation (4.2) would contradict Fodor's lemma. We refute ω-distributivity instead via an analysis of Solovay's proof of splitting a stationary subset $S \subseteq \mu \cap \text{cof}(\omega)$ into μ mutually disjoint stationary sets. The main point is that we use a fixed ladder system.

Let us first recall a component of Solovay's argument [14, Prop 1.9].

Proposition 4.13. *Let μ be an uncountable regular cardinal with a ladder system $\langle a_n^\xi : n < \omega, \xi \in \mu \cap \text{cof}(\omega) \rangle$, and let $S \subseteq \mu \cap \text{cof}(\omega)$ be a stationary set. Then there is some $n < \omega$ such that for all $\eta < \mu$, the set $\{\xi \in S : a_n^\xi \geq \eta\}$ is stationary. Moreover, if $n_{\min}(S)$ is defined as the minimal such n for a stationary set S, then there is a stationary subset $T \subset S$ such that $n_{\min}(T) > n_{\min}(S)$.*

Proof. To prove that $n_{\min}(S)$ exists for any stationary $S \subset \mu \cap \text{cof}(\omega)$, first assume otherwise. Then there is some η_n such that $a_n^\xi \geq \eta_n$ for only a nonstationary set of ξ's in S. From the $(< \mu)$-completeness of the ideal of

non-stationary subsets of μ, it follows that for all but a nonstationary set of ξ's, the sequences $\langle a_n^\xi : n < \omega \rangle$ are bounded by $\sup\{\eta_n : n < \omega\}$. This is a contradiction.

Now fix $n = n_{\min}(S)$. For each η, the function $f(\xi) = a_n^\xi$, defined on $S(\eta, n) := \{\xi \in S : a_n^\xi \geq \eta\}$, is regressive and so, by Fodor's Theorem, there is some $\gamma(\eta) \geq \eta$ such that $\{\xi \in S(\eta, n) : a_n^\xi = \gamma(\eta)\}$ is stationary. We collect all these $\gamma(\eta)$'s into $\{\eta_i : i < \mu\}$ and let $S_i = \{\xi \in S : a_n^\xi = \eta_i\}$. Each S_i is such that $n_{\min}(S) < n_{\min}(S_i)$. $\hfill\square$

Definition 4.14 (The Stationary Balcar-Simon Trick).
  (1) Let $\langle \kappa_\alpha : \alpha < \lambda \rangle$ be an increasing sequence of regular cardinals converging to κ and $\lambda = \operatorname{cf}(\kappa) < \kappa_0$. For each $\alpha < \lambda$ we fix a ladder system $\overline{a}_\alpha = \langle \langle a_{\alpha,n}^\xi : n < \omega \rangle : \xi \in \kappa_\alpha \cap \operatorname{cof}(\omega) \rangle$. Let $S \subseteq \kappa_\alpha$ be stationary. We let $n_\alpha(S) = n_{\min}(S)$ as in Proposition 4.13 above.
  (2) For stationary subsets $S, T \subseteq \kappa_\alpha \cap \operatorname{cof}(\omega)$, we write $T \subseteq_{\alpha,\mathrm{str}} S$ if $n_\alpha(T) > n_\alpha(S)$.
  (3) Finally, for $p, q \in \mathbb{L}_\kappa^{\mathrm{stat}}$ we let $q \prec^{\mathrm{stat}} p$ if for $\alpha < \lambda$ and any $t \in \operatorname{Split}(q)$ with $\operatorname{dom}(t) = \alpha$, $\operatorname{Succ}_q(t) \subseteq_{\alpha,\mathrm{str}} \operatorname{Succ}_p(t)$.

Lemma 4.15. *For any $p \in \mathbb{L}_\kappa^{\mathrm{stat}}$ there is $q \prec^{\mathrm{stat}} p$.*

The lemma is proved via a suitable slow fusion sequence, similar to Lemma 4.10.

Theorem 4.16. *The forcing $\mathbb{L}_\kappa^{\mathrm{stat}}$ is not ω-distributive.*

Now the rest of the proof is as in Theorem 4.7. We use a matrix $\langle \mathcal{A}_n : n < \omega \rangle$ such that for any $p \in \mathcal{A}_n$ and any $q \in \mathcal{A}_{n+1}$, $q \leq p$ implies that $q \prec^{\mathrm{stat}} p$. If $\langle p_n : n < \omega \rangle$ is any descending sequence such that $p_n \in \mathcal{A}_n$ for all $n < \omega$, then $\langle p_n : n < \omega \rangle$ has no lower bound q, because for such a q the number $n_{\operatorname{dom}(\operatorname{Stem}(q))}(\operatorname{OSucc}_q(\operatorname{Stem}(q)))$ would not exist.

5. Failures of Three-Parameter Distributivity

5.1. **A Cofinal Function from ω to $\operatorname{cf}(\kappa)$ in the Pre-Perfect Case.** Let $\mathbb{P}$ be any of the pre-perfect versions and let λ be uncountable. In the Miller case, we restrict the forcing to the dense set of conditions p in which only the nodes in $\bigcup\{\operatorname{Split}_\alpha(p) : \alpha < \lambda\}$ have more than one successor.

We use the technical notion of a minimal branch.

Definition 5.1. Let $\chi > (2^\kappa)^+$ be a regular cardinal and let $<_\chi$ be a well-ordering of $H(\chi)$, the set of sets of hereditary cardinality $< \chi$.

For any $t \in p \in {}^{\mathrm{pre}}\mathbb{P}_\kappa$ we let $b_{\min}(p, t)$ be the $<_\chi$-least branch b of p such that $t \in b$.

Definition 5.2. For each $p \in {}^{\mathrm{pre}}\mathbb{P}_\kappa$ we define its standardized pre-perfect subtree p^{pre} as follows. By induction on $\alpha < \lambda = \operatorname{cf}(\kappa)$ we define subsets $S(p, \alpha) \subseteq p$ and $M(p, \alpha) \subseteq \operatorname{Split}_\alpha(p)$:
  (1) $M(p, 0) = \{\operatorname{Stem}(p)\}$.

(2) For $\beta < \lambda$, we let $S(p, \beta) = \bigcup\{\mathrm{Succ}_p(t) : t \in M(p, \beta)\}$.

(3) We let
$$M(p, \alpha + 1) = \{t \in \mathrm{Split}_{\alpha+1}(p) : \exists s \sqsubseteq t, s \in S(p, \alpha)\}.$$

(4) For $\delta < \lambda$ limit we let
$$M(p, \delta) = \{t \in \mathrm{Split}_\delta(p) : \exists \alpha < \delta, \exists s \in S(p, \alpha), \exists \varepsilon < \lambda,$$
$$t = (b_{\min}(p, s) \restriction \varepsilon) \in \mathrm{Split}_\delta(p)\}.$$

We let
$$p^{\mathrm{pre}} = \{s \in p : \exists \alpha < \lambda, \exists t \in S(p, \alpha), s \sqsubseteq t\}.$$

Then $p^{\mathrm{pre}} \in {}^{\mathrm{pre}}\mathbb{P}_\kappa$ and $p^{\mathrm{pre}} \subseteq p$. Note that $(p^{\mathrm{pre}})^{\mathrm{pre}} = p^{\mathrm{pre}}$ and no splitting antichains disappear, i.e.,
$$\emptyset \neq M(p^{\mathrm{pre}}, \alpha) \subseteq \mathrm{Split}_\alpha(p).$$

In the Laver case, $p^{\mathrm{pre}}(\mathrm{dom}(\mathrm{Stem}(p^{\mathrm{pre}})) + \alpha) = M(p^{\mathrm{pre}}, \alpha)$. In addition, for any $q \leq p^{\mathrm{pre}}$ we have $q = q^{\mathrm{pre}}$.

Definition 5.3. We call p *standardized* if $p = p^{\mathrm{pre}}$.

The construction in Definition 5.2 implies the following:

Fact 5.4. *The set of standardized pre-perfect trees is an open dense subset of* $\mathbb{P}^{\mathrm{pre}}$.

For $s \sqsubseteq t \in p$ we write $[s, t) = \{r \in p : s \sqsubseteq r \sqsubset t\}$. We let $t_{-1} = \emptyset$.

We use the following, which in the special case of allowing only eventually zero nodes is proved in [9, Lemma 7.1]:

Proposition 5.5. *For $p = p^{\mathrm{pre}} \in \mathbb{P}$, any node $t \in p$ has only finitely many different $b_{\min}(p, s)$ for $s \sqsubseteq t$.*

Proof. This is shown by reading the recursive definition of $p = p^{\mathrm{pre}}$. By induction on α, we see that $\forall t \in p(\alpha)$, $\{b_{\min}(s, p) : s \sqsubseteq t\}$ is finite: For $\alpha = 0$ and for successor steps, there is nothing to show. Now we assume that $\alpha < \lambda$ is a limit ordinal and $t \in p(\alpha)$. We fix the construction stage, i.e., the limit ordinal $\delta \leq \alpha$ such that $t \in M(p, \delta)$. By Definition 5.2(4) there are $\delta' < \delta$, $t' \in S(p, \delta')$, and $\varepsilon < \lambda$ such that $t = b_{\min}(p, t') \restriction \varepsilon$. Then we have that $\{b_{\min}(s, p) : s \sqsubseteq t\} = \{b_{\min}(s, p) : s \sqsubseteq t'\}$ is finite. The equality holds, because for $s \in p$, $t' \sqsubseteq s \sqsubseteq t$, we have $b_{\min}(p, s) = b_{\min}(p, t') = b_{\min}(p, t)$, since otherwise $b_{\min}(p, t')$ could not have been the minimal branch through t'. $\square$

Definition 5.6. Fix $p = p^{\mathrm{pre}}$. We let $t_{i-1} = \emptyset$.
$$\dot{f}_p = \{\langle (i, \check{\alpha}_i), q \rangle : i < n, q \leq p, \ \exists t_0 \ldots \exists t_{n-1}$$
$$\bigwedge_{0 \leq i < n} (t_i \sqsubseteq \mathrm{Stem}(q) \wedge \mathrm{dom}(t_i) = \alpha_i \wedge$$
$$\forall t \in [t_{i-1}, t_i), b_{\min}(p, t_{i-1}) = b_{\min}(p, t) \neq b_{\min}(p, t_i))\}.$$

Lemma 5.7. *The condition p forces that $\dot{f}_p$ is a cofinal function from ω to λ.*

Proof. Let $\alpha \in \lambda$ and let $q \leq p \in \mathbb{P}$. There is some $n \in \omega$ such that $q \Vdash \dot{f}_p \upharpoonright n = r$ for some $r \in V \cap {}^n\lambda$ and q determines just $\dot{f} \upharpoonright n$, that means that there are t_i, $0 \leq i < n$, $t_i \sqsubseteq \text{Stem}(q)$, as in the definition of $\dot{f}$ and $\text{dom}(t_i) = r(i)$ and $b_{\min}(p, t_{i-1}) = b_{\min}(p, s)$ for $t_{i-1} \sqsubseteq s \sqsubset t_i$ and $b_{\min}(p, t_{n-1}) = b_{\min}(p, s)$ for $t_{n-1} \sqsubseteq s \sqsubseteq \text{Stem}(q)$. In other words, on $\{s : s \sqsubseteq \text{Stem}(q)\}$ we find exactly n different values of $b_{\min}(p, s)$, represented by $\{b_{\min}(q, t_i) : 0 \leq i < n\}$. We assume that $r(n-1) < \alpha$.

We choose $t \in b_{\min}(p, t_{n-1}) \cap \text{Split}(q)$ such that $t_{n-1} \sqsubseteq t$ and $\text{dom}(t) \geq \alpha$. Then we take $t_n \in \text{Succ}_q(t)$ with $b_{\min}(p, t_n) \neq b_{\min}(p, t)$. We let $q' = q \upharpoonright t_n$. Then $q' \Vdash \dot{f}(n) = \text{dom}(t_n)$ and $\text{dom}(t_n) \geq \alpha + 1$. $\square$

Remark 5.8. *By definition of $\dot{f}_p$, $\text{range}(\dot{f}_p)$ contains only successor ordinals. In all trees that appear as conditions in any of our forcings any increasing sequence of nodes has at most one limit node, namely their union. Sometimes, these trees are called set-theoretic trees.*

Now we take a maximal antichain A of $p \in \mathbb{P}$ such that $p = p^{\text{pre}}$ and let $\dot{f}$ be the mixing of $\{\dot{f}_p : p \in A\}$. Then we have:

Theorem 5.9. $\mathbb{P}$ *forces that $\dot{f}$ maps ω cofinally into λ.*

This means for that $\mathbb{P}$ collapses λ to ω if $\lambda = \omega_1$.

Corollary 5.10. *The forcing $\mathbb{P}$ is not $(\lambda, 2)$-distributive.*

Proof. Let f be a cofinal function from Theorem 5.9 and $B = \text{range}(f)$. The characteristic function of B is a new function from λ to 2. $\square$

5.2. **The Singular Perfect Case.** Now we further develop our non-distributivity results from Section 4.

Theorem 5.11. *Let κ be a singular cardinal of cofinality $\lambda > \omega$ and let $\mathbb{P}$ be $\mathbb{M}_\kappa$, $\mathbb{M}_\kappa^{\text{stat}}$, or $\mathbb{M}_\kappa^{\text{club}}$. Then $\mathbb{P}$ is not $(\omega, \cdot, \lambda^+)$-distributive.*

Lemma 5.12. *Let $p \in \mathbb{P}$ be a condition and suppose that $\mathcal{A} = \{q_\alpha : \alpha < \lambda\}$ is a maximal antichain below p. Then for all $t \in p$, there is some $s \in p \upharpoonright t$ and some $\alpha < \lambda$ such that $p \upharpoonright s \leq q_\alpha$.*

Proof. Fix such a $p \in \mathbb{P}$ and such an $\mathcal{A}$. Suppose the conclusion of the lemma is false. This means that

$$\exists \bar{t} \in p, \forall t \sqsupseteq \bar{t}, \forall \alpha < \lambda, p \upharpoonright t \not\leq q_\alpha.$$

This implies that there is some $\bar{t} \in p$ such that

$$\forall t \in p \upharpoonright \bar{t}, \forall \alpha < \lambda, \exists s \sqsupseteq t, p \upharpoonright s \cap q_\alpha \subseteq s,$$

where $p \upharpoonright s \cap q_\alpha \subseteq s$ clearly implies that $p \upharpoonright s \perp q_\alpha$. We will find some q such that $q \perp q_\alpha$ for all $\alpha < \lambda$, thus violating maximality of $\mathcal{A}$.

We define a slow fusion sequence $\langle p_\alpha \,:\, \alpha < \lambda \rangle$ below $p \upharpoonright \bar{t}$ with the property that $\forall \alpha < \lambda$, $p_\alpha \perp q_\beta$ for all $\beta < \alpha$. Let $p_0 = p \upharpoonright \bar{t}$. Suppose that $\alpha = \beta + 1$. For all $t \in \mathrm{Split}_\beta(p_\beta)$ and all $t' \in \mathrm{Succ}_{p_\beta}(t)$, we pick $s_{t'} \sqsupseteq t'$ such that $p \upharpoonright s_{t'} \cap q_\beta \subseteq s_{t'}$ (by slowness, $p \upharpoonright s_{t'} = p_\beta \upharpoonright s_{t'}$). Then define

$$p_\alpha = \bigcup \{p \upharpoonright s_{t'} \,:\, t \in \mathrm{Split}_\beta(p_\beta),\, t' \in \mathrm{Succ}_{p_\beta}(t)\}.$$

If α is a limit, let $p_\alpha = \bigcap_{\beta < \alpha} p_\beta$, which exists because $\langle p_\beta \,:\, \beta < \alpha \rangle$ is a slow fusion sequence. Note that $p_\alpha \perp q_\beta$ for $\beta < \alpha$. Finally, let $q = \bigcap_{\alpha < \lambda} p_\alpha$, which again exists due to slow fusion. This completes the proof. $\qquad\square$

Here is the proof of Theorem 5.11.

Proof. Let $\{\mathcal{A}_n \,:\, n \in \omega\}$ be a matrix that witnesses the failure of ω-distributivity. (The horizontal or vertical Balcar-Simon trick will work for $\mathbb{M}_\kappa$ or $\mathbb{M}_\kappa^{\mathrm{stat}}$, but only the vertical Balcar trick will work for $\mathbb{M}_\kappa^{\mathrm{club}}$.) Fix $p \in \mathbb{P}$ and suppose for contradiction that for all $n < \omega$, $|\{r \in \mathcal{A}_n : r \| p\}| \leq \lambda$.

Let $\mathcal{A}_n^* = \{r \cap p : r \in \mathcal{A}_n, r \| p\}$. Then $\mathcal{A}_n^*$ is a maximal antichain of size $\leq \lambda$ below p, and the downward closures of the $\mathcal{A}_n^*$'s have empty intersection. Now define an increasing sequence $\langle t_n \,:\, n < \omega \rangle$ of nodes through p as follows: Let t_0 be the root, and if t_n is defined apply Lemma 5.12 to t_n and $\mathcal{A}_n^*$ to obtain t_{n+1}. Let $t^* = \bigcup_{n < \omega} t_n$, where we see that $t^* \in p$ because $\lambda > \omega$. Then for all $n < \omega$, there is a unique $q_n \in \mathcal{A}_n^*$ such that $p \upharpoonright t^* \leq q_n$. But this is impossible, contradiction. $\qquad\square$

5.3. **Collapses for Naive Higher Sacks Forcings.** In this section we use failures of three parameter distributivity to obtain collapses for perfect tree forcings for regular cardinals.

We define a basic tree forcing that consists merely of perfect binary trees in a regular cardinal κ such that $\kappa^{<\kappa} = \kappa$. This will suffice to demonstrate an isomorphism to the Lévy collapse, but the reader can observe that the result can be generalized to other forms of splitting behavior.

Definition 5.13. Let κ be a regular cardinal. Then $\mathbb{T}$ will consist of conditions $p \subseteq 2^{<\kappa}$ such that:

(1) $\forall t \in p, \exists \alpha < \kappa, t \in 2^{<\alpha}$,
(2) $\forall t \in p$ and $s \sqsubseteq t$, then $s \in p$,
(3) $\forall t \in p, \exists s \sqsupseteq t, s^\frown 0, s^\frown 1 \in p$,
(4) if $\alpha < \kappa$ is a limit, $s \in 2^\alpha$, and for all $\beta < \alpha$, $s \upharpoonright \beta \in p$, then $s \in p$.

So there is no restriction on splitting as there is in Kanamori's version. The main theorem of this subsection is the following:

Theorem 5.14. *If* $2^\kappa = \kappa^+$, *then* $\mathbb{T} \simeq \mathrm{Coll}(\omega, \kappa^+)$.

We adapt the methods from the previous subsection and combine them with the theory amalgamated in the following:

Fact 5.15 (From the Prague School). *Suppose that $\mathbb{Q}$ is a poset of cardinality ν where ν is regular and that $\mathbb{Q}$ is not $(\omega, \cdot, \nu)$-distributive. Then $\Vdash_{\mathbb{Q}}$ "$|\nu| = \omega$". Moreover, $\mathbb{Q}$ is forcing-equivalent to $\mathrm{Coll}(\omega, \nu)$.*

Sketch. We include an outline of the concepts at play. It is well-known that any forcing of cardinality ν that collapses ν to have countable cofinality is forcing-equivalent to $\mathrm{Coll}(\omega, \nu)$ [13]. This gives us the last part of the statement.

Now we want to justify the assertion that ν will be collapsed under these circumstances. First of all, it is enough to show that $\Vdash_{\mathbb{Q}}$ "$\mathrm{cf}(\nu) = \omega$" because an argument of Sakai tells us that any ν-sized poset that singularizes ν to have cofinality ω will actually collapse ν to be countable [20]. Now suppose $\mathcal{A}_n, n < \omega$, witness the failure of $(\omega, \cdot, \nu)$-distributivity. Then one can observe that if we have enumerations $\mathcal{A}_n = \langle q_\xi^n : \xi < \nu \rangle$ for all $n < \omega$, then

$$\dot{c} := \{\langle (\check{n}, \check{\xi}), q_\xi^n \rangle \, : \, n < \omega, \xi < \nu\}$$

is a name for a function singularizing ν to have cofinality ω. $\square$

Proposition 5.16. $|\mathbb{T}| = 2^\kappa$.

Proof. Because $\kappa^{<\kappa} = \kappa$. $\square$

Now we prove Theorem 5.14.

Proof. By Fact 5.15, it is enough to show that $\mathbb{T}$ fails to be $(\omega, \cdot, \kappa^+)$-distributive. We employ the vertical Balcar–Simon trick. For every $p \in \mathbb{T}$, there is some $q \in \mathbb{T}$ with $q \prec^{\mathrm{vert}} p$. In this way define a sequence of maximal antichains $\langle \mathcal{A}_n : n < \omega \rangle$ such that for any n for any $p \in \mathcal{A}_n$ for any $q \leq p$, $q \in \mathcal{A}_{n+1}$ we have $q \prec^{\mathrm{vert}} p$. If $\langle p_n : n < \omega \rangle$ is $\prec^{\mathrm{vert}}$-decreasing and $q \leq p_n$ for all $n < \omega$, then q has no splitting. The fact that the sequence $\langle \mathcal{A}_n : n < \omega \rangle$ witnesses the failure of $(\omega, \cdot, \kappa^+)$-distributivity is exactly analogous to the argument for Lemma 5.12. $\square$

6. FURTHER DIRECTIONS

Many issues remain in the uncountable case in terms of the non-closed versions, and some of these issues may be independent of ZFC. We summarize some of the key results with a table. Assume κ is a singular strong limit cardinal of cofinality $\lambda = \omega_1$:

	${}^{\mathrm{scl}}\mathbb{M}_\kappa^{\mathrm{club}}$	$\mathbb{L}_\kappa^{\mathrm{club}}$	$\mathbb{M}_\kappa^{\mathrm{stat}}$	$\mathbb{M}_\kappa^{\mathrm{club}}$	$\mathbb{L}_\kappa^{\mathrm{stat}}$	$\mathbb{L}_\kappa$	${}^{\mathrm{pre}}\mathbb{M}_\kappa$
ω-closed	yes	yes	no	no	no	no	no
ω-distributive	yes	yes	no	no	no	no	no
$(\omega, \cdot, \lambda^+)$-distributive	yes	yes	no	no			no
forces $\mathrm{cf}(\lambda) = \omega$	no	no					yes
forces $|\kappa| = \lambda$	yes	yes	yes	yes	yes	yes	
embeds $\mathrm{Add}(\lambda, 1)$	yes	yes			yes	yes	

This leaves some general problems open.

Problem 6.1. What are the complete sub-forcings of those forcings we have studied here?

It is possible that given the different behavior in terms of regularity between the Miller and Laver cases (see 3.13), additional assumptions about diamond principles or ladder systems may be necessary to settle these questions.

Problem 6.2. To what extent do these forcings collapse cardinals?

Note that certain collapses of successors of singulars involve PCF theory and large cardinals [7].

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