GOOD SCALES AND NON-COMPACTNESS OF SQUARES

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Abstract. Cummings, Foreman, and Magidor investigated the extent to which square principles are compact at singular cardinals. The first author proved that if κ is a singular strong limit of uncountable cofinality, all scales on κ are good, and $\square^*_\delta$ holds for all $\delta < \kappa$, then $\square^*_\kappa$ holds. In this paper we will present a strongly contrasting result for $\aleph_\omega$. We construct a model in which $\square_{\aleph_n}$ holds for all $n < \omega$, all scales on $\aleph_\omega$ are good, but in which $\square^*_{\aleph_\omega}$ fails and some weak forms of internal approachability for $[H(\aleph_{\omega+1})]^{\aleph_1}$ fail. This requires an extensive analysis of the dominating and approximation properties of a version of Namba forcing. We also prove some supporting results.

1. Introduction

There is less independence exhibited in the behavior of singular cardinals than there is with regular cardinals. Moreover, the circumstances depend on the cofinality of the singular cardinal. One early example of this phenomenon had to do with the behavior of the continuum function. Magidor proved that the general continuum hypothesis (GCH) can fail for the first time at $\aleph_\omega$ [Mag77], while Silver proved that GCH cannot fail for the first time at a singular of uncountable cofinality [Sil75]. Questions in this area often take a form pertaining to compactness: How much do the configurations below a singular cardinal affect the configuration *at* the singular cardinal?

The theory behind these phenomena has developed considerably. Shelah introduced PCF theory in the late 1980's to study the behavior of singular cardinals like $\aleph_\omega$. Using these revolutionary methods, Shelah was able to obtain surprising ZFC theorems for the cardinal arithmetic of singular cardinals that do not have analogs for regular cardinals. In the early 2000's, Cummings, Foreman, and Magidor wrote a series of papers connecting PCF theory to the combinatorial properties of canonical inner models. They focused notably on varieties of good scales, which are the most typical tame objects in PCF theory, and variants of Jensen's square principle, which embody the combinatorial properties of Gödel's model L. In this manner, they established much of the basic language and theoretical tools that continue to be used in the investigations of these objects.

This paper will consider the question of the extent to which good scales can be used to construct variants of the square principle for successors of singular cardinals. There is some evidence in the positive direction: that these principles are to some extent compact. On one hand, Cummings, Foreman, and Magidor proved that if $\square_{\aleph_n}$ holds for $n < \omega$, then the good points of a scale on $\aleph_\omega$ can be used to construct a square-like sequence of length $\aleph_{\omega+1}$ [CFM04, Theorem 3.5]. On the other hand, Cummings et al. proved that it is consistent that $\square_{\aleph_n}$ holds for all $n < \omega$ while the

Date: December 20, 2024.

canonical principle $\square_{\aleph_\omega}$ fails [CFM03].[1] This result was strengthened by Krueger, who showed that a similar model can be obtained with no good scales on $\aleph_\omega$ [Kru13]. Nonetheless, the first author proved that if κ is a singular strong limit cardinal of uncountable cofinality such that $\square^*_\delta$ (a weak version of $\square_\delta$) holds for all $\delta < \kappa$, and all scales on κ are good, then $\square^*_\kappa$ holds [Lev22]. Hence it is natural to further investigate the interplay between squares below a singular cardinal and what they imply for the successor of that singular cardinal in the presence of good scales.

We will prove a consistency result as our main theorem, and this will lead us to a careful analysis of a version of Namba forcing. This method was originally used to demonstrate that $\aleph_2$ can be singularized without collapsing $\aleph_1$, and to address questions about Boolean algebras, but later it became apparent that Namba forcing is bound up with the study of singular cardinals as such (see e.g. [BCH90, FM95, FT05, CK18]).

For technical reasons that will become clear in the course of the paper, we found it useful to define a variation of the notion of internal approachability:

Definition 1.1. Let θ and λ be regular cardinals and let $M \prec H(\theta)$. We say that M is *sup-internally approachable at* λ if there is a sequence $\langle M_i : i < \omega_1 \rangle$ of countable sets such that

(1) for all $j < \omega_1$, $\langle M_i : i < j \rangle \in M_{j+1} \cap M$,
(2) $\sup_{i<\omega_1} \sup(M_i \cap \lambda) = \sup(M \cap \lambda)$.

Our theorem presents a contrast to the situation with compactness of weak squares for singulars of uncountable cofinality.

Theorem 1.2. *Assuming the consistency of a cardinal κ that is $\kappa^{\omega+1}$-supercompact, it is consistent that there is a model of set theory in which the following are true:*

(1) $\aleph_\omega$ is a strong limit,
(2) all scales on $\aleph_\omega$ are good,
(3) $\square_{\aleph_n}$ holds for all $n < \omega$,
(4) $\square^*_{\aleph_\omega}$ fails,[2]
(5) there are stationarily-many $N \prec H(\aleph_{\omega+1})$ of cardinality $\aleph_1$ that are not sup-internally approachable at $\aleph_{\omega+1}$.

The use of a large cardinal assumption is necessary for our result. The failure of $\square_\kappa$ for singular κ implies the consistency of substantial large cardinals [Sar14]. An exact lower bound for failure of $\square_\kappa$ for singular κ is unknown.

Theorem 1.2 depends on the approximation and dominating properties (roughly-stated) of a version of Namba forcing, which we analyze in Section 2. We will also use a simple poset for forcing the existence of good scales in the construction. The interaction of the Namba forcing and the good scale forcing will be the crux of the proof, which we provide in Section 3. Our construction will be shaped in a way that gives an analogy with guessing models, so that our work can be connected to more recent research.

In Section 4, we will prove some elaborating results that more or less consider possible variations on Theorem 1.2. First, we show that it as possible to obtain

[1] See from the definition of $\square_\kappa$ below that it is really an assertion about κ^+.

[2] The consistency of the conjunction of the first four points in Theorem 1.2 was claimed the first arXiv version of the compactness of weak square paper [Lev22], but the proof in the original source was flawed.

a model in which $\square_{\aleph_n}$ holds for all $n < \omega$ while $\square(\aleph_{\omega+1}, \aleph_1)$ fails. This square principle would fail under PFA, so this result indicates possible tension between good scales and square principles. We will also verify that it is possible for models to be sup-internally approachable without being internally unbounded. This lends extra force behind point *(5)* of Theorem 1.2. Finally, we show that PFA is consistent with all scales on $\aleph_\omega$ being good. This makes use of our good scale-forcing poset, and it stands in contrast with a result of Cummings and Magidor, which states that Martin's Maximum (MM, see [FMS89]) implies that if λ is a singular cardinal of countable cofinality, then all scales on λ are bad [CM11].

We are assuming that the reader is familiar with the basics of cardinal arithmetic, forcing, and large cardinals (see [Jec03]).

1.1. **Basic Combinatorial Notions.** Here we will define some PCF-theoretic notions and recall some fundamental facts. All definitions and facts due to Shelah [She94]. For the sake of readability, we will give more recent citations and short proofs where possible.

Definition 1.3.

(1) If τ is a cardinal and $f, g : \tau \to \mathrm{ON}$, then we write $f <^* g$ if there is some $j < \tau$ such that $f(i) < g(i)$ for all $i \geq j$.
(2) Given a singular cardinal κ, we say that a strictly increasing sequence $\langle \kappa_i : i < \mathrm{cf}\,\kappa \rangle$ of regular cardinals converging to κ is a *product* when we regard $\prod_{i<\mathrm{cf}\,\kappa} \kappa_i$ as a space, and $f \in \prod_{i<\mathrm{cf}\,\kappa} \kappa_i$ means that $\mathrm{dom}\, f = \mathrm{cf}\,\kappa$ and $f(i) < \kappa_i$ for all $i < \mathrm{cf}\,\kappa$.
(3) Given a product $\vec{\kappa} = \prod_{i<\mathrm{cf}\,\kappa} \kappa_i$, a sequence $\langle f_\alpha : \alpha < \nu \rangle$ is a *scale of length* κ^+ on $\vec{\kappa}$ if:
    (a) for all $\alpha < \kappa^+$, $f_\alpha \in \vec{\kappa}$;
    (b) for all $\alpha < \beta < \kappa^+$, $f_\alpha <^* f_\beta$;
    (c) for all $g \in \vec{\kappa}$, there is some $\alpha < \kappa^+$ such that $g <^* f_\alpha$.

Fact 1.4. *If κ is singular of cofinality λ, then there is a product $\prod_{i<\lambda} \kappa_i$ on κ that carries a scale of length κ^+* [AM10, Section 2].

Fact 1.4 is only nontrivial if $2^\kappa > \kappa^+$.

Definition 1.5. Fix a product $\prod_{i<\mathrm{cf}\,\kappa} \kappa_i$ on a singular κ.

(1) If $\vec{f} = \langle f_\beta : \beta < \gamma \rangle$ is a $<^*$-increasing subsequence of $\vec{\kappa}$, then a function h is an *exact upper bound* (or *eub* for short) of $\vec{f}$ if
    (a) for all $\beta < \gamma$, $f_\beta <^* h$,
    (b) for all $g <^* h$, there is some $\beta < \gamma$ such that $g <^* f_\beta$.
(2) Given a $<^*$-increasing sequence $\vec{f} = \langle f_\beta : \beta < \gamma \rangle$ on a product $\prod_{i<\mathrm{cf}\,\kappa} \kappa_i$, we say that $\alpha \leq \gamma$ is *good* if there is some unbounded $A \subset \alpha$ with $\mathrm{ot}\, A = \mathrm{cf}\,\alpha$ and some $j < \mathrm{cf}\,\kappa$ such that for all $i \geq j$, $\langle f_\beta(i) : \beta \in A \rangle$ is strictly increasing.
(3) If there is a club $D \subset \kappa^+$ such that every $\alpha \in D$ with $\mathrm{cf}\,\alpha > \mathrm{cf}\,\kappa$ is a good point of $\vec{f}$, then $\vec{f}$ is a *good scale.*

It is an exercise to obtain:

Fact 1.6. *If $\vec{f} = \langle f_\beta : \beta < \gamma \rangle$ is a $<^*$-increasing subsequence of $\vec{\kappa}$, then an exact upper bound of $\vec{f}$ is in particular a least upper bound.*

Fact 1.7. *Let $\langle f_\beta : \beta < \alpha\rangle$ be a sequence of functions in a product $\prod_{i<\mathrm{cf}\,\kappa} \kappa_i$ where $\mathrm{cf}\,\kappa < \kappa_0$. Then the following are equivalent if $\mathrm{cf}(\alpha) > \mathrm{cf}(\kappa)$:*

(1) α is a good point.
(2) There is a $<$-increasing sequence $\langle h_\gamma : \gamma < \mathrm{cf}(\alpha)\rangle$ such that:
   (a) for all $i < \mathrm{cf}\,\kappa$ and $\gamma < \gamma'$, $h_\gamma(i) < h_{\gamma'}(i)$,
   (b) for all $\gamma < \mathrm{cf}(\alpha)$, there is $\beta < \alpha$ such that $h_\gamma <^* f_\beta$, and for all $\beta < \alpha$, there is $\gamma < \mathrm{cf}(\alpha)$ such that $f_\beta <^* h_\gamma$.
(3) There is an exact upper bound h of $\langle f_\beta : \beta < \alpha\rangle$ such that for some $j < \mathrm{cf}\,\kappa$, $\mathrm{cf}(h(i)) = \mathrm{cf}(\alpha)$ for $i \geq j$.

Proof. (See [CFM04, Lemma 2.1], [Cum05, Section 13].)

For *(1)* $\Rightarrow$ *(2)*:, let $A \subseteq \alpha$ and $j < \mathrm{cf}\,\kappa$ witness goodness. For each $\gamma < \mathrm{cf}(\alpha)$, let $h_\gamma(i) := \sup\{f_\beta(i) : \beta \in A, \mathrm{ot}(A \cap \beta) < \gamma\}$. For *(2)* $\Rightarrow$ *(3)*: Given $\langle h_\gamma : \gamma < \mathrm{cf}(\alpha)\rangle$ as in *(2)*, let $h(i) := \sup_{\gamma<\mathrm{cf}(\alpha)} h_\gamma(i)$. For *(3)* $\Rightarrow$ *(2)*: take such an exact upper bound and observe that by the assumption $\mathrm{cf}\,\kappa < \kappa_0$ we can assume that $\mathrm{cf}(h(i)) = \mathrm{cf}(\alpha)$ for *all* $i < \mathrm{cf}\,\kappa$. Let $\langle \beta^i_\xi : \xi < \mathrm{cf}(\alpha)\rangle$ be cofinal in $h(i)$ for $i < \mathrm{cf}\,\kappa$. Then let $h_\xi : i \mapsto \beta^i_\xi$.

(2) $\Rightarrow$ *(1)*:[3] Fix such a $\langle h_\gamma : \gamma < \mathrm{cf}(\alpha)\rangle$. Choose $\langle \beta_\xi, \gamma_\xi : \xi < \mathrm{cf}(\alpha)\rangle$ cofinal such that for all h, $h_{\gamma_\xi} <^* f_{\beta_\xi} <^* h_{\gamma_{\xi+1}}$. For all ξ, let j_ξ be such that for all $i \geq j_\xi$, $h_{\gamma_\xi}(i) < f_{\beta_\xi}(i) < h_{\gamma_{\xi+1}}(i)$. There is some j and some unboudned $A' \subseteq \mathrm{cf}(\alpha)$ such that for all $\xi \in A'$, $j_\xi = j$. Then $A := \{\beta_\xi : \xi \in A\}$ and j witness goodness for α. ☐

Fact 1.7 is particularly useful because of the uniqueness of exact upper bounds:

Fact 1.8. *If g and h are exact upper bounds of $\langle f_\beta : \beta < \alpha\rangle$, then $g =^* h$.*

Proof. Suppose g and h are eub's of $\langle f_\beta : \beta < \alpha\rangle$ in a product $\prod_{i<\mathrm{cf}\,\kappa} \kappa_i$ but that $g \not\leq^* h$, so there is an unbounded $X \subseteq \mathrm{cf}\,\kappa$ such that for all $i \in X$, $h(i) < g(i)$. Let $h'(i)(= h(i)$ for $i \in X$ and $h'(i) = 0$ for $i \notin X$. Then $h' <^* g$, so there is some $\beta < \alpha$ such that $h' <^* f_\beta$ since g is an exact upper bound. If j is such that $i > j$ implies $h'(i) <^* f_\beta(i)$, then this means that for all $i \in X \setminus j$, $h(i) < f_\beta(i)$, so $f_\beta <^* h$, contradicting that h is an eub. ☐

We will also give a definition of the square principle.

Definition 1.9. If κ is a cardinal, we say the $\square_\kappa$ holds if there is a sequence $\langle C_\alpha : \alpha < \kappa^+\rangle$ such that the following hold for all $\alpha < \kappa^+$:

(1) C_α is a club in α,
(2) for all $\beta \in \lim C_\alpha$, $C_\alpha \cap \beta = C_\beta$,
(3) $\mathrm{ot}(C_\alpha) \leq \kappa$.

We also make some use of variations in internal approachability. These variations were introduced by Foreman and Todorčević [FT05] and were proved distinct by Krueger [Kru07, Kru08].

Definition 1.10. Given an uncountable regular κ and a set $N \in [H(\theta)]^\kappa$, we say:

- N is *internally unbounded* if $\forall x \in P_\kappa(N), \exists M \in N, x \subseteq M$,
- N is *internally stationary* if $P_\kappa(N) \cap N$ is stationary in $P_\kappa(N)$,
- N is *internally club* if $P_\kappa(N) \cap N$ is club in $P_\kappa(N)$,

[3]This part is often known as the "sandwich argument."

- N is *internally approachable* if there is an increasing and continuous chain $\langle M_\xi : \xi < \kappa\rangle$ such that $|M_\xi| < \kappa$ and $\langle M_\eta : \eta < \xi\rangle \in M_{\xi+1}$ for all $\xi < \kappa$ such that $N = \bigcup_{\xi<\kappa} M_\xi$.

Definition 1.11. A model $M \prec H(\theta)$ is *tight* with respect to K if $\langle \kappa_i : i \in I\rangle$ means $M \cap \prod_i \kappa_i$ is cofinal in $\prod_i(\kappa_i \cap M)$ in the $<^*$-ordering.

1.2. **The Namba Forcing.** Here we will define the most important part of the proof of Theorem 1.2.

Definition 1.12. Let T be a tree.

(1) For an ordinal α, the set T_α is the set of $t \in T$ with $\mathrm{dom}(t) = \alpha$.
(2) The *height* $\mathrm{ht}(T)$ of a tree T is $\min\{\alpha : T(\alpha) = \emptyset\}$.
(3) We let $[T] = \{f\colon \mathrm{ht}(T) \to \kappa : \forall \alpha < \mathrm{ht}(T), f \restriction \alpha \in T\}$. Elements of $[T]$ are called *cofinal branches*.
(4) For $t_1, t_2 \in T \cup [T]$ we write $t_1 \sqsubseteq t_2$ if $t_2 \restriction \mathrm{dom}(t_1) = t_1$. The tree order is the relation $\sqsubseteq$. If $t = s \cup \{(\mathrm{dom}(s), \beta)\}$, we write $t = s^\frown\langle\beta\rangle$.
(5) $T \restriction \alpha = \bigcup_{\beta<\alpha} T_\beta$.
(6) $T \restriction t = \{s \in T : s \sqsubseteq t \vee t \sqsubseteq s\}$.
(7) For $t \in T_\alpha$ we let $\mathrm{succ}_T(t) = \{c : c \in T_{\alpha+1} \wedge c \sqsupseteq t\}$ denote the *set of immediate successors of* t, and $\mathrm{osucc}_T(t) = \{\beta : t^\frown\langle\beta\rangle \in T_{\alpha+1}\}$ denote the *ordinal successor set of* t.
(8) We call $t \in T$ a *splitting node* if $|\mathrm{succ}_T(t)| > 1$.
(9) $\mathrm{stem}(T)$ is the $\sqsubseteq$-minimal splitting node.

Definition 1.13. Fix a bijection $d : \omega \to \omega \setminus \{0, 1\}$ such that:

(1) For all $m \geq 2$, there are infinitely many n such that $d(n) = m$;
(2) If n is the least number such that $d(n) = m$, then for all $k < n$, $d(k) < m$.

The poset $\mathbb{L}$ will consist of conditions p such that the following hold:

(1) p is a tree consisting of finite sequences t.
(2) For all $t \in p$ and $n \in \mathrm{dom}(t)$, $t(n) \in \aleph_{d(n)}$.
(3) Let $t \in p$ be the unique node maximal in the ordering of p such that for all $s \in p$, either $t \sqsubseteq s$ or $s \sqsubseteq t$. Then for all $t \in p$ with $t \sqsupseteq \mathrm{stem}(p)$, if $n = \mathrm{dom}(t)$, then $\{\eta : t^\frown\eta \in p\}$ is a stationary subset of $\aleph_{d(n)} \cap \mathrm{cof}(\omega_1)$.

The ordering on $\mathbb{L}$ is given by inclusion: $p \leq q$ (i.e. p contains more information than q) if and only if $p \subseteq q$.

Definition 1.14. If $t \in p$ is the unique node such that for all $s \in p$, either $t \sqsubseteq s$ or $s \sqsubseteq t$, then t is called the *stem* of p and is denoted $\mathrm{stem}(p)$.

We also have the requisite notion of fusion, which will be familiar to readers who have seen tree forcings.

Definition 1.15. If $p \in \mathbb{L}$, we write $n(p) := |\,\mathrm{stem}(p)|$. If $S, p \in \mathbb{L}$ and $n < \omega$, we write $q \leq_n p$ if $q \leq p$, $\mathrm{stem}(q) = \mathrm{stem}(p)$, and for all t with $|t| \leq n(q) + n$, $t \in q$ if and only if $t \in p$.

We say that $\langle p_n : n < \omega\rangle$ is a *fusion sequence* if $p_n \geq_n p_{n+1}$ for all $n < \omega$.

Fact 1.16. *If $\langle p_n : n < \omega\rangle$ is a fusion sequence of conditions in $\mathbb{L}$, then $\bigcap_{n<\omega} p_n \in \mathbb{L}$.*

Our poset is similar to a number of singular Namba forcings that appear in the literature [BCH90, CFM03, Kru13], but the particular properties of $\mathbb{L}$ will be important for the proof of Theorem 1.2.

First, there is the fact that $\mathbb{L}$ is a "Laver-style" poset in which there is one stem as in Definition 1.14. In fact, a similar model of Krueger has a non-Laver style Namba forcing in order *not* to have any good scales [Kru13]. This is necessary for Proposition 2.2 below and will be used in Subsection 2.2 below to derive exact upper bounds that are added by the forcing. In Subsection 2.2, we will use the fact that the splitting sets are stationary rather than merely cofinal because we need a normal ideal for an application of Fodor's Lemma. The function d is used to ensure that the $\aleph_n$'s are singularized to have cofinality ω, which will be used to apply idea of Cummings et al. to get $\square_{\aleph_n}$-sequences in the final model. Finally, we need our forcing to split into sets concentrating on cofinality ω_1, because the exact upper bounds that are added must stabilize to output points of cofinality ω_1, so that the characterization of goodness from Fact 1.7 can be applied.

Now we can collect some properties of our Namba forcing for which existing arguments suffice without alteration.

Fact 1.17. *For all n such that $1 < n < \omega$, $\mathbb{L}$ forces that $\mathrm{cf}(\aleph_n^V) = \omega$.*

Proof. Observe that for all $m < \omega$ and $t \in T$, there are infinitely many n with $d(m) = n$ such that for some $t' \sqsupseteq t$, $\{\eta : t'^\frown\eta \in T\}$ has cardinality $\aleph_m$. This then comes from the fact that there are infinitely many k such that $d(k) = n$: A genericity argument defines a cofinal function whose domain consists of these k's. $\square$

Fact 1.18. *$\mathbb{L}$ forces that $\aleph_{\omega+1}^V$ is an ordinal of cardinality and cofinality $\geq \aleph_1$.*

Sketch of Proof. This comes from a fusion argument using e.g. Proposition 2.3 below, where we build $p' \leq p$ with at most $|p'| = \aleph_\omega$-many possible decisions for $\dot{\alpha}$. $\square$

Fact 1.19. *Then $\mathbb{L}$ preserves stationary subsets of $\aleph_1$.*

This is a variation of the arguments presented by Cummings-Magidor [CM11] and Krueger [Kru13] using an open game and the fact that the splitting nodes all split into sets of size $> \aleph_1$. The component of Theorem 2.1 below starting with Claim 2.5 is a variation of this argument.

2. Some Technical Ideas Needed for the Proof

This section will present the main technical ideas that are more or less new to this paper.

2.1. **Some Approximation for the Namba Forcing.**

The first technical idea that we will discuss is an approximation-like result that holds for our forcing $\mathbb{L}$. The approximation property originates in work of Hamkins [Ham01] and often comes up when obtaining various so-called compactness properties, like the tree property, square principles, and so on.

Theorem 2.1. *Assume $2^{\aleph_\omega} = \aleph_{\omega+1}$. Let $\dot{\mathbb{U}}$ be a $\mathbb{P}$-name for a countably closed forcing. Then if $\dot{F}$ is a $\mathbb{L} * \dot{\mathbb{U}}$-name for a function with domain ω_1 that is cofinal in ν where $\mathrm{cf}^V(\nu) \geq \aleph_{\omega+1}^V$, then $\mathbb{L} * \dot{\mathbb{U}}$ forces that there is some $i < \omega_1$ such that $\dot{F} \restriction i \notin V$.*

We are formulating the lemma as such—in terms of a two-step iteration—so it fits the analog that is already in the literature [Lev23]. This would be useful for applications where some guessing before is sought for substructures of some $H(\theta)$ where $\theta > \aleph_{\omega+1}$. The point is that $\mathbb{L}$ would not collapse $H(\theta)^V$ on its own, and would be paired with a Lévy collapse.

Since we are dealing with two-step iterations we will write $(p, \dot{c}) \leq_0 (q, \dot{d})$ if $(p, \dot{c}) \leq (q, \dot{d})$ and $p \leq_0 q$.

We need versions of the Cummings-Magidor facts that account for an iteration following the initial Namba forcing.

Proposition 2.2. *Let $\dot{\mathbb{U}}$ be a $\mathbb{L}$-name for a forcing poset. Suppose that $(p, \dot{c}) \in \mathbb{L} * \dot{\mathbb{U}}$ and that $\dot{\delta}$ is a name for an ordinal below ω_1. Then there is some $(q, \dot{d}) \leq_0 (p, \dot{c})$ such that $(q, \dot{d})$ decides a value for $\dot{\delta}$.*

The proof is essentially the same as in the version of the lemma omitting $\dot{\mathbb{U}}$, except that at a particular step we use a gluing argument.

Proof of Proposition 2.2. Suppose that the proposition is false. We say that a node t is *bad* if there is no $(q, \dot{d}) \leq (p, \dot{c})$ with $q \leq_0 p \restriction t$ such that $(q, \dot{d})$ decides a value for $\dot{\delta}$. Hence, we are working under the assumption that $\operatorname{stem} p$ is bad.

We will construct a fusion sequence through the following: If t is bad and I is the ideal that defines the splitting for t, then the set of $\alpha \in \operatorname{osucc}_p(t)$ such that $t^\frown\langle\alpha\rangle$ is bad is I-positive. Otherwise we have a set $W \in I^+$ such that $W \subseteq \operatorname{osucc}_p(t)$ and all $\alpha \in W$ are not bad. Then for each $\alpha \in W$ we choose some $(q_\alpha, \dot{d}_\alpha) \leq (p \restriction t, \dot{c})$ with $q_\alpha \leq_0 p \restriction t$ deciding a value β_α for $\dot{\delta}$. By ω_2-completeness, there is some $W' \in I^+$ with $W' \subseteq W$ and some $\gamma < \tau$ such that for all $\alpha \in W'$, $\beta_\alpha = \gamma$. Now let $q = \bigcup_{\alpha \in W'} q_\alpha$ and let $\dot{d}$ be the name that glues together the $\dot{d}_\alpha$'s below q. Then $(q, \dot{d})$ forces that $\dot{\delta} = \gamma$ and $q \leq_0 p \restriction t$, hence t is not in fact bad.

We are then able to construct some $r \leq_0 p$ such that all $t \in r$ are bad. Find some $(r', \dot{d}) \leq (r, \dot{c})$ deciding a value for $\dot{\delta}$. If $s = \operatorname{stem} r'$, then this contradicts the fact that $t \in r'$ is bad. $\square$

Proposition 2.3. *Let $\dot{\mathbb{U}}$ be a $\mathbb{L}$-name for a forcing poset. Suppose $(p, \dot{c}) \in \mathbb{P} * \dot{\mathbb{U}}$ and suppose $\dot{x}$ is a name for an element of V. There is some $(q, \dot{d}) \leq_0 (p, \dot{c})$ and some h_n such that for all $t \in q$ with $|t| = h_n$, $(q \restriction t, \dot{d})$ decides a value for $\dot{x}$.*

Proof. We call a node $t \in p$ *bad* if it not the case that there is $(q, \dot{d}) \leq (p, \dot{c})$ with $q \leq_0 p \restriction t$ and $n < \omega$ such that every $s \in q$ with $|s| = n$ is such that $(q \restriction s, \dot{d})$ decides a value for $\dot{x}$.

We argue that if t is bad and I is its splitting ideal, then the set of $\alpha \in \operatorname{osucc}_p(t)$ such that $t^\frown\langle\alpha\rangle$ is bad is I-positive. Otherwise there is an I-positive $W \subseteq \operatorname{osucc}_p(t)$ such that for all $\alpha \in W$, there is $q_\alpha \leq_0 p \restriction t$, n_α, and $\dot{d}_\alpha$ such that for every $s \in q_\alpha$ with $|s| = n_\alpha$, $(q_\alpha \restriction s, \dot{d}_\alpha)$ decides $\dot{x}$. We find some I-positive $W' \subseteq W$ and some $n < \omega$ such that for all $\alpha \in W'$, $n_\alpha = n$. Then let $q = \bigcup_{\alpha \in W'} q_\alpha$ and let $\dot{d}$ be the gluing of the $\dot{d}_\alpha$'s below q. Then we can see that $q \leq_0 p \restriction t$, and hence that t cannot be bad.

Then we obtain an overall contradiction in a manner similar to the proof of Proposition 2.2. $\square$

This proof uses ideas from the result that the classical version of Namba forcing (from Jech's textbook [Jec03, Chapter 28]) consistently has the weak ω_1-approximation property [Lev23], some material from which is to some extent repeated here. The main changes are that we must alter the statement to suit our situation, and that we must apply it to a Laver-style Namba forcing.

Proof. Suppose for contradiction that $\dot{F}$ is an $\mathbb{L}$-name for a (necessarily) new cofinal subset of ν of order-type ω_1 all of whose initial segements are in V.

Let $\varphi(i,q,\dot{d})$ denote the formula

$$i<\omega_1 \wedge (q,\dot{d}) \in \mathbb{L} * \dot{\mathbb{U}} \wedge (q,\dot{d}) \wedge \exists \langle A_\alpha : \alpha \in \operatorname{osucc}_q(\operatorname{stem}(q))\rangle \text{ s.t.}$$
$$\forall \alpha \in \operatorname{osucc}_q(\operatorname{stem}(q)), (q \restriction (\operatorname{stem}(q)^\frown\langle\alpha\rangle), \dot{d}) \Vdash \text{“}\dot{F}\restriction i \in A_\alpha\text{”} \wedge$$
$$\forall \alpha,\beta \in \operatorname{osucc}_q(\operatorname{stem}(q)), \alpha \neq \beta \implies A_\alpha \cap A_\beta = \emptyset.$$

Claim 2.4. *$\forall j<\omega_1, (p,\dot{c}) \in \mathbb{L}$, there is some $i \in (j,\omega_1)$ and some $(q,\dot{d}) \leq_0 (p,\dot{c})$ such that $\varphi(i,p,\dot{c})$ holds.*

Proof. Let $W = \operatorname{osucc}_p(\operatorname{stem}(p))$. By induction on $\alpha \in W$ we will define a sequence of conditions, $\langle (q_\alpha,\dot{d}_\alpha) : \alpha \in W\rangle$, a sequence of natural numbers $\langle n_\alpha : \alpha \in W\rangle$, a sequence of countable ordinals $\langle i_\alpha : \alpha \in W\rangle$, and the sets $\langle A_\alpha : \alpha \in W\rangle$ of cardinality strictly less than ν. After these objects are defined, we will finalize a choice of $(q,\dot{d})$ and the A_α's.

If $\alpha = \min W$, then we can choose an arbitrary $i_\alpha \in (j,\omega_1)$. We apply Proposition 2.3 to find some $(q_\alpha,\dot{d}_\alpha) \leq_0 (p \restriction (\operatorname{stem} p^\frown\langle\alpha\rangle),\dot{c})$ and some $n_\alpha \in \omega$ such that for all $s \in q_\alpha$ with $|s| = n_\alpha$, $(q_\alpha,\dot{d}_\alpha)$ decides $\dot{F}\restriction i_\alpha$. Then we let $A_\alpha = \{a : \exists t \in \operatorname{lev}_{n_\alpha}(q_\alpha), (q_\alpha \restriction t, \dot{d}_\alpha) \Vdash \text{“}\dot{F}\restriction i_\alpha = a\text{”}\}$. (Establishing this case is just a formality.)

Now suppose that the members of our sequences have been defined for $\beta \in W \cap \alpha$. Let $B = \bigcup_{\beta \in \alpha \cap W} A_\beta$, which is in particular of cardinality strictly less than ν. Then observe that

$$(p \restriction t^\frown \langle\alpha\rangle, \dot{c}) \Vdash \text{“}\{\dot{F}\restriction i : i \in (j,\omega_1)\} \not\subseteq B\text{”}$$

since otherwise there would be some $(q,\dot{d}) \leq (p \restriction t^\frown\langle\alpha\rangle,\dot{c})$ such that $(q,\dot{d}) \Vdash \text{“}\{\dot{F}\restriction i : i \in (j,\omega_1)\} \subseteq B\text{”}$. We know that $\dot{F}\restriction i$ is forced to be bounded in ν: If $\nu = \aleph_{\omega+1}^V$ then this is by Fact 1.18, otherwise it follows using the chain condition and our assumption that $2^{\aleph_\omega} = \aleph_{\omega+1}$. It is therefore implied that $(q,\dot{d})$ forces $\dot{F}$ to be bounded in ν, and this is a contradiction of our assumptions.

This means that

$$(p \restriction t^\frown \langle\alpha\rangle, \dot{c}) \Vdash \text{“}\exists i \in (j,\omega_1), \dot{F}\restriction i \notin B\text{”}.$$

So we let $\dot{k}$ be the $\mathbb{P}$-name for the ordinal in (j,ω_1) witnessing this expression. By Proposition 2.2, there is some $(q'_\alpha,\dot{d}'_\alpha) \leq_0 (p \restriction t^\frown\langle\alpha\rangle,\dot{c})$ and some i_α such that $(q'_\alpha,\dot{d}'_\alpha) \Vdash \text{“}\dot{k} = i_\alpha\text{”}$. Apply Proposition 2.3 to find some $n_\alpha \in \omega$ and some $(q_\alpha,\dot{d}_\alpha) \leq_0 (q'_\alpha,\dot{d}'_\alpha)$ such that for any $t \in q_\alpha$ of height n_α, $(q_\alpha \restriction t,\dot{d}_\alpha)$ decides $\dot{F}\restriction i_\alpha$. Then (as in the base case) we let $A_\alpha = \{a : \exists t \in (q_\alpha)_{n_\alpha}, (q_\alpha \restriction t,\dot{d}_\alpha) \Vdash \text{“}\dot{F}\restriction i_\alpha = a\text{”}\}$. Observe that $A_\alpha \cap A_\beta = \emptyset$ for all $\beta < \alpha$.

Let I be ideal in the sequence of $\mathcal{I}$ such that $\mathbb{L}$ is defined to have I-positive splitting for $\operatorname{osucc}_p(t)$. Finally, we choose some $W' \subseteq W$ that is I-positive and such that there is some i such that for all $\alpha \in W'$, $i_\alpha = i$. Then let $q = \bigcup_{\alpha \in W'} q_\alpha$. Let $\dot{d}$ be the gluing of the $\dot{d}_\alpha$'s below q_α. □

We plan to build a fusion sequence using Claim 2.4. For this purpose we define a game $\mathcal{G}_k$ for $k < \omega_1$.

Suppose round n of the game is being played where $n = 0$ is the first round. If $n = 0$ then let $(q_*, \dot{d}_*)$ be the starting condition $(p_{-1}, \dot{d}_{-1})$ where $|\operatorname{stem}(\bar{p})| = m$ and let $i_* = 0$. Otherwise if $n > 0$ let $(q_*, \dot{d}_*, i_*)$ be (q_n, i_n). First Player I chooses a subset $Z_n \subseteq \operatorname{osucc}_{q_*}(\operatorname{stem}(q_*))$ with $Z_n \in I_{m+n}$ and some $\delta_n < k$. Then Player II chooses some $\alpha \in \operatorname{osucc}_{q_*}(\operatorname{stem}(q_*)) \setminus Z_n$ and some condition $(q_n, \dot{d}_n) \leq_0 (q_* \restriction \operatorname{stem} q_* {}^\frown \langle \alpha \rangle, \dot{d}_*)$ and some $i_n \in (\delta_n, k)$ such that $\varphi(q_n, \dot{d}_n, i_n)$ holds. Hence we have the following diagram:

Player I	Z_0, δ_0		Z_1, δ_1		Z_2, δ_2		$\dots$
Player II		$q_0, \dot{d}_0, i_0$		$q_1, \dot{d}_1, i_1$		$q_2, \dot{d}_2, i_2$	$\dots$

Player II loses at stage n if they cannot an appropriate pair $(q_n, \dot{d}_n, i_n)$ witnessing $\varphi(i_n, q_n, \dot{d}_n)$ for some $i_n < k$, i.e. if they cannot in particular find such $i_n \in (\delta_n, k)$. Otherwise, if Player II does not lose at any finite stage, then Player II wins.

The following can be proved using standard arguments for Namba-style games (see [Nam71],[Lev23, Claim 10],[CM11, Fact 5], [Kru13, Proposition 3.4]).

Claim 2.5. *For some $k < \omega_1$, Player II has a winning strategy in $\mathcal{G}_k$.*

Sketch of Proof. The essential idea is the following: By the Gale-Stewart Theorem, the failure of the claim implies that for all $i < \omega_1$ is a winning strategy σ_i for Player I in $\mathcal{G}_i$. Then take an elementary submodel $M \prec H(\theta)$ with $\langle \sigma_i : i < \omega_1 \rangle \in M$. Then it is possible to construct a run of the game $\mathcal{G}_k$ such that Player I uses the strategy σ_k but nonetheless loses the game, and this is done by ensuring that Player II's moves are all in M even though $\sigma_k \notin M$. The crux of the argument is that it is possible to take the union of $\aleph_1$-many sets in the relevant ideal to obtain a set in that ideal. □

Now we will build a condition $q \in \mathbb{L}$ by a fusion process in such a way that any stronger condition deciding $\dot{F} \restriction k$ will also code the generic sequence for $\mathbb{L}$.

Fix a sequence $\langle \delta_n : n < \omega \rangle$ converging to k. Let $p_0 = \bar{p}$ be the starting point where $|\operatorname{stem}(\bar{p})| = m$. We will define a fusion sequence $\langle p_n : n < \omega \rangle$ and a sequence $\langle \dot{c}_n : n < \omega \rangle$ by induction on $n < \omega$ in such a way that $p_{n+1} \Vdash$ "$\dot{c}_{n+1} \leq \dot{c}_n$" and such that:

> For all $n < \omega$, then for all $t \in p_n$ with $|t| = m + n$, the following is the case: Let $s_0 \sqsubseteq s_1 \sqsubseteq \ldots \sqsubseteq s_n = t$ be the sequence of all nodes up to and including t. Then there is a sequence $Z_0^t, \ldots, Z_n^t$ such that
>
> $$(Z_0^t, \delta_0), (p_0 \restriction s_0, \dot{c}_0, i_0), \ldots, (Z_n^t, \delta_n), (p_n \restriction s_n, \dot{c}_n, i_n)$$
>
> is a run of the game $\mathcal{G}_k$ in which Player II's moves are determined by the winning strategy obtained in Claim 2.5.

Note that the third point implies the following: For all positive $n < \omega$, for all $t \in p_n$ with $|t| = m + n$, there is $i_t \in (\delta_n, k)$ and a sequence $\langle A_s : s \in \operatorname{succ}_{p_n}(t) \rangle$ witnessing that $\varphi(i_t, p_n \restriction t, \dot{c}_n)$ holds.

We construct the fusion sequence as follows: Start with stage -1 for convenience and let $p_{-1} = p$. Now assume we have defined p_{n-1}, and we are considering $t \in p_{n-1}$ with $|t| = m + n$. Let $s_0 \sqsubseteq s_1 \sqsubseteq \ldots \sqsubseteq s_{n-1} = t$ be the sequence of splitting nodes up to and including t. Let S_t be the set of $\alpha \in \operatorname{osucc}_{p_{n-1}}(t)$ such that for some Z^α_n, the winning strategy for Player II applied to the sequence

$$(Z^t_0, \delta_0), (p_0 \restriction s_0, \dot{c}_0, i_0), \ldots, (Z^t_{n-1}, \delta_{n-1}), (p_{n-1} \restriction s_{n-1}, , \dot{c}_{n-1}, i_{n-1}), (Z^\alpha_n, \delta_n)$$

produces some $(q_n, , \dot{d}_n, i_n)$ where $(q_n, \dot{d}_n) \leq_0 (p_{n-1} \restriction t^\frown \langle \alpha \rangle, \dot{c}_{n-1})$. We claim that $|S_t| \notin I_{m+n}$. Otherwise Player I would have a winning move for the sequence

$$(Z^t_0, \delta_0), (p_0 \restriction s_0, \dot{c}_0, i_0), \ldots, (Z^t_{n-1}, \delta_{n-1}), (p_{n-1} \restriction s_{n-1}, \dot{c}_{n-1}, i_{n-1})$$

by playing S_t as the I_{m+n}-component of their move. For each such t and $\alpha \in S_t$, choose $q_{t,\alpha}$ to be produced by the winning strategy for Player II as the $\mathbb{L}$-component of their move. Now let $p_n = \bigcup \{q_{t,\alpha} : |t| = m+n, \alpha \in S_t\}$.

Now let q be the fusion limit of $\langle p_n : n < \omega \rangle$ and let $\dot{d}$ be the $\mathbb{L}$-name for the lower bound of $\langle \dot{c}_n : n < \omega \rangle$. Then $(q, \dot{d})$ forces that the generic sequence for $\mathbb{P}$ can be recovered from $\dot{F} \restriction k$ as follows: Let $(r, \dot{e}) \leq (q, \dot{d})$ force $\dot{F} \restriction k = g \in V$. We can inductively choose a cofinal branch $b \subset r$ such for all $t \in b$, for some $i_t < k$, $g \restriction i_t = a_t$. Specifically, we construct b by defining a sequence $\langle s_n : n < \omega \rangle$ of splitting nodes as follows: Let $s_0 = \operatorname{stem} q$. Given s_n, let $s^*_{n+1} \sqsupseteq s_n$ be the next splitting node. Then since $\varphi(i_t, r \restriction s^*_{n+1}, \dot{e})$ holds for some $i_t \in (\delta_n, k)$, there is some $\alpha \in \operatorname{osucc}_r(s^*_{n+1})$ such that $(r \restriction s^*_{n+1}{}^\frown \langle \alpha \rangle, \dot{e}) \Vdash$ "$g \restriction i_t \in A_t$". Then let $s_{n+1} = s^*_{n+1}{}^\frown \langle \alpha \rangle$. Then let $b = \{t \in r : \exists n < \omega, t \sqsubseteq s_n\}$. This implies that $(r, \dot{e})$ forces that the generic object is equal to b, i.e. that $\bigcap \Gamma(\mathbb{P}) = b \in V$, but this is not possible.

Hence $(q, \dot{d}) \Vdash$ "$\dot{F} \restriction k \notin V$" lest we obtain the contradiction from the previous paragraph. This contradicts the premise from the beginning of the proof that initial segments of $\dot{F}$ are in V. □

Remark 2.6. The work in this section essentially resolves a question from the arXiv version of one of the first author's preprints pertaining to the Laver version of Namba forcing [Lev23, Question 2]. The necessary modification of the argument occurs in Claim 2.4.

2.2. **Some Exactness of Upper Bounds.** Because the forcing we use is meant to provide a master condition for the forcings adding the $\square_{\aleph_n}$'s (which is not needed in Cummings-Magidor [CM11]), we must make some adjustments to their arguments.

We want to examine the interaction of this repeating version with scales from the ground model.

Definition 2.7. We isolate two particular names.

(1) Let $\dot{b}_{\mathsf{full}}$ be a $\mathbb{L}$-name for the *generic branch*, meaning if G is $\mathbb{L}$-generic, then $\dot{b}_{\mathsf{full}}$ evaluates to

$$b_{\mathsf{full}} = \bigcup \{\operatorname{stem}(p) \ : \ p \in G\}.$$

(2) Recall the function d from the definition of $\mathbb{L}$. We write $d^{-1}_{\min}(m)$ for the minimal n with $d(n) = m$.

(3) Let $\dot{b}_{\mathsf{prod}}$ be a $\mathbb{L}$-name evaluating to the function

$$b_{\mathsf{prod}} = \{\langle (m, \operatorname{stem}(p)(d^{-1}_{\min}(m))), p \rangle \ : \ p \in G, m \in \omega, \operatorname{dom}(p) > d^{-1}_{\min}(m)\}.$$

Proposition 2.8. *Let $p \in \mathbb{L}$. Then there is some $q \leq_0 p$ such that*

$$q \Vdash \text{“}\dot{b}_{\mathsf{prod}} \restriction [|\operatorname{stem}(q)|, \omega) \text{ is strictly increasing”}.$$

Proof. Equivalently, we are trying to prove the existence of some $q \leq_0 p$ such that

$$q \Vdash \text{“}\dot{b}_{\mathsf{full}} \restriction [\mathrm{stem}(q), \omega) \cap \{n < \omega : \exists m, d_{\min}^{-1}(m) = n\} \text{ is strictly increasing”}.$$

We define a sequence $\langle p_n : n < \omega \rangle$ below p with $p_0 = p$. Suppose we have defined p_n such that $b_n \Vdash$ “$\dot{b}_{\mathsf{full}} \restriction [|\mathrm{stem}(q)|, |\mathrm{stem}(q)| + n] \cap \{n < \omega : d(n) = k\}$”. For all $t \in p_n$ with $|t| = |\mathrm{stem}(q)| + n$, choose $q_t \leq_0 p_n \restriction t$ such that if $d(n) = k$ then $\mathrm{osucc}_{q_t}(t) \cap \max\{q(\ell) : d(\ell) = k, \ell < n\} = \emptyset$. Then let $p_{n+1} = \bigcup\{q_t : t \in p_n, |t| = |\mathrm{stem}(q)| + n\}$. Then let $q = \bigcap_{n<\omega} p_n$ be the fusion limit, so q witnesses the proposition. □

Lemma 2.9. *Let $p \in \mathbb{L}$ with $|\mathrm{stem}(p)| = d_{\min}^{-1}(n)$ and suppose $\dot{\gamma}$ is a name for an ordinal that is forced by p to be below $\dot{b}_{\mathsf{full}}(n)$. Then there is some $q \leq_0 p$ and some $\delta < \aleph_{d(n)}$ such that $q \Vdash$ “$\dot{\gamma} \leq \delta$”.*

Proof. Let $N := |\mathrm{stem}(p)|$. By Proposition 2.8, we can assume that $q \Vdash$ “$\dot{b}_{\mathsf{prod}} \restriction [|\mathrm{stem}(q)|, \omega)$ is strictly increasing”. Suppose for contradiction that the lemma fails. For each splitting node $t \in p$, we say that t is *bad* if the lemma also fails with respect to $p \restriction t$, i.e. if there is no $q \leq_0 p \restriction t$ such that for some $\delta < \aleph_{d(n)}$, $q \Vdash$ “$\dot{\gamma} \leq \delta$”.

Now we will build an extension $r \leq_0 p$ such that for all $t \in r$, t is bad. This will give a contradiction because if $r' \leq r$ is any condition deciding $\dot{\gamma}$ and $t = \mathrm{stem}(r')$, then $r' \leq_0 \restriction p \restriction t$, contradicting badness of t.

We will build $r \leq_0 p$ using a fusion sequence $\langle p_n : n < \omega \rangle$ where $p_0 = p$ and where all $t \in p_n$ with $|t| \leq |\mathrm{stem}(p)| + n$ are bad. This is fulfilled by p_0 by assumption. Suppose then that we have p_n. Let $t \in p_n$ be such that $|t| = |\mathrm{stem}(p)| + n$.

Claim 2.10. $S_t := \{\alpha \in \mathrm{osucc}_t(p_n) : t^\frown\langle\alpha\rangle \text{ is bad}\}$ *is a stationary subset of $\aleph_{d(|t|)} \cap \mathrm{cof}(\omega_1)$.*

Proof of Claim. Otherwise, there is a stationary subset $S \subseteq \aleph_{d(|t|)} \cap \mathrm{cof}(\omega_1)$ such that for all $\alpha \in S$, there is some $q_\alpha \leq_0 p_n \restriction (t^\frown\langle\alpha\rangle)$ deciding a bound $\delta_\alpha < \aleph_{d(N)}$ for $\dot{\gamma}$.

We consider three cases.

$d(|t|) < d(N)$: Let $\delta = \sup\{\delta_\alpha : \alpha \in S\}$. Then $\delta < \aleph_{d(N)}$. If $q = \bigcup\{q_\alpha : \alpha \in S\}$, then $q \Vdash$ “$\dot{\gamma} < \delta$” and $q \leq_0 p_n \restriction t$, contradicting the fact that t is bad.

$d(|t|) > d(N)$: Then there is a stationary subset $S' \subseteq S$ and some δ such that for all $\alpha \in S$, $\delta_\alpha = \delta$. Since $\delta_\alpha < \aleph_{d(N)}$ for all α, this is of course the case for δ. Then we can find a contradiction analogous to the previous case.

$d(|t|) = d(N)$: This is essentially like the previous case, but now we use Fodor's Lemma in the "strong" sense: In this case we have $\delta_\alpha < \alpha$. We are also using the increasing statement from Proposition 2.8. □

Since S_t is a stationary subset of $\aleph_{d(|t|)} \cap \mathrm{cof}(\omega_1)$ for all such t, we let $p_{n+1} \cup \{p \restriction (t^\frown\langle\alpha\rangle) : \alpha \in S_t\}$. Having defined p_n for $n < \omega$, we let $r = \bigcap_{n<\omega} p_n$. Then r is the condition described in the second paragraph such that t is bad for all $t \in r$, hence we have finished the proof. □

Note that the proof of Lemma 2.9 uses the fact that $\mathbb{L}$ is defined to split into stationary sets by invoking Fodor's Lemma. The same is true of the next lemma.

Now we are able to prove a bounding lemma analogous to one obtained by Cummings and Magidor [CM11, Fact 4]. This lemma embodies the reason that we are using Laver-style Namba forcings in this paper.

Lemma 2.11. *Let $\vec{f} = \langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$ be a scale on $\prod_{n<\omega} \aleph_n$. Then $\mathbb{L}$ forces that $\dot{b}_{\mathsf{prod}}$ is an exact upper bound of $\vec{f}$.*

Proof. First we need to argue that $\mathbb{L}$ forces $\dot{b}_{\mathsf{prod}}$ to be an upper bound of $\vec{f}$. This follows from a relatively simple argument: If $\alpha < \aleph_{\omega+1}$ and $p \in \mathbb{L}$, construct $q \leq_0 p$ such that for all $k = d^{-1}_{\min}(n) \geq |\operatorname{stem} q|$ and $t \in q$ with $|t| = k$, $\operatorname{osucc}_q(t) \setminus f_\alpha(k) = \emptyset$.

Now we prove the more complicated assertion, which is that if $p \Vdash$ "$\dot{h} < \dot{b}_{\mathsf{prod}}$" where $\dot{h}$ is taken to be arbitrary, then we can find $q \leq p$ such that for some $\xi < \aleph_{\omega+1}$, $q \Vdash$ "$\dot{h} <^* f_\xi$".

Let $\{k_n : n < \omega\}$ enumerate $\{d^{-1}_{\min}(n) : n < \omega\}$. Let N be such that $p \Vdash$ "$\forall n \geq N, \dot{h}(n) < \dot{b}_{\mathsf{prod}}(n)$".

We will define a fusion sequence $\langle p_n : n < \omega\rangle$ and values $g(n)$ of a function g by induction on n such that $p_n \Vdash$ "$\dot{h}(n) < g(n)$" for $n \geq N$. Let $p_n = p$ for $n \leq N$. Suppose we have defined p_{n-1} and $n \geq N$. Let $n = d(k_n)$. For all $t \in p_{n-1}$ with $|t| = k_n$ and all $\alpha \in \operatorname{osucc}_{p_{n-1}}(t)$, apply Lemma 2.9 to find $q'_{t^\frown\langle\alpha\rangle} \leq p_{n-1} \restriction t$ and some $\delta_{t^\frown\langle\alpha\rangle}$ such that $q'_{t^\frown\langle\alpha\rangle} \Vdash$ "$\dot{h}(n) \leq \delta_{t^\frown\langle\alpha\rangle}$". By Fodor's Lemma, there is a stationary $S_t \subseteq \operatorname{osucc}_{p_{n-1}}(t)$ and a value δ_t such that for all $\alpha \in S_t$, $q_{t^\frown\langle\alpha\rangle} \Vdash$ "$\dot{h}(n) \leq \delta_t$". Let $q_t = \bigcup_{\alpha \in S_t} q'_{t^\frown\langle\alpha\rangle}$.

Then let $p_n = \bigcup\{q_t : t \in p_{n-1}, |t| = k_n\}$. Let $g(n) = \sup\{\delta_t : t \in p_{n-1}, |t| = k_n\}$. Since $d(n') < m$ for all $n' < k_n$, it follows that $|\{t \in p_n : |t| = k_n\}| < \aleph_n$, and so $g(n) < \aleph_n$.

We finally let $q = \bigcap_{n<\omega} p_n$ be the fusion limit. Observe that $q \Vdash$ "$\dot{h}(n) \leq g(n)$ for $n \geq N$. If $\xi < \aleph_{\omega+1}$ is large enough that $g <^* f_\xi$ then we are done. □

2.3. **A Poset for Adding a Good Scale.** Here we will develop a poset that forces that there is a good scale.

Fix a singular λ of cofinality κ and let $\langle \lambda_i : i < \kappa\rangle$ be a strictly increasing sequence of regular cardinals converging to λ. We say $f <_j g$ if for all $i \geq j$, $f(i) < g(i)$. Hence $f <^* g$ if $f <_j g$ for some $j < \kappa$.

We define a poset for forcing a good scale.

Definition 2.12. Given some $\vec{\lambda} = \langle \lambda_i : i < \kappa\rangle$, let $\mathbb{G}(\vec{\lambda})$ be a partial order whose conditions have the form $\langle f_\beta : \beta \leq \alpha\rangle$ for some $\alpha < \lambda^+$ such that for all $\beta \leq \alpha$:

(1) $f_\beta \in \prod_{i<\kappa} \lambda_i$;
(2) for all $\gamma < \beta$, $f_\gamma <^* f_\beta$;
(3) if $\operatorname{cf}(\beta) > \kappa$, then β is a good point with respect to $\langle f_\gamma : \gamma < \beta\rangle$.

Ordering is by end-extension: if $p, q \in \mathbb{G}(\vec{\lambda})$, then $p \leq q$ if and only if $p \restriction \operatorname{dom} q = q$. We drop the notation for $\vec{\lambda}$ when the context is clear.

Proposition 2.13. *$\mathbb{G}(\vec{\lambda})$ is κ^+-directed closed.*

Proof. $\mathbb{G}(\vec{\lambda})$ is *tree-like*, meaning that $p, q \in \mathbb{G}$ are compatible if and only if $p \leq q$ or $q \leq p$. Therefore it is enough to show that $\mathbb{G}(\vec{\lambda})$ is κ^+-closed. This follows from the facts that points β with $\operatorname{cf}(\beta) < \kappa$ are automatically good and that we do not require points β with $\operatorname{cf}(\beta) = \kappa$ to be good. □

Proposition 2.14. *$\mathbb{G}(\vec{\lambda})$ is $(\lambda+1)$-strategically closed.*[4]

[4]For the definition of strategic closure, see [Cum10, Definition 5.15].

Proof. The play will take the form of a decreasing sequence $\langle p_\xi : \xi \leq \lambda \rangle \subseteq \mathbb{G}(\vec{\lambda})$ in which $\gamma_\xi = \max \operatorname{dom} p_\xi$, i.e. each p_ξ will formally have the presentation $p_\xi = \langle f^\xi_\zeta : \zeta \leq \gamma_\xi \rangle$, but because we have $p_\xi \leq p_{\xi'}$ for $\xi > \xi'$, we can write $p_\xi = \langle f_\zeta : \zeta \leq \gamma_\xi \rangle$. Player II will play so that if $\xi < \xi' < \lambda$ are such that ξ, ξ' are even and j is minimal such that $\xi < \xi' < \lambda_j$, then $f_\xi <_j f_{\xi'}$.

Suppose $\xi < \lambda$ is an even successor with j minimal such that $\xi < \lambda_j$ and $\xi = \eta+2$. Then Player II will choose h such that $p_{\eta+1}(\gamma_{\eta+1}) <^* h$ and $p_\eta(\gamma_\eta) <_j h$ and will play $p_\xi := p_{\eta+1}{}^\frown\langle \gamma_{\eta+1}+1, h\rangle$.

Suppose $\xi < \lambda$ is a limit and j is minimal such that $\xi < \lambda_j$. Then let $\gamma_\xi = \sup_{\eta<\xi} \gamma_\eta$ and let $h(i) = \sup\{f_{\gamma_\eta}(i) : \eta < \xi, \eta \text{ even}\}$ for $i \geq j$ and $h(i) = 0$ otherwise. If $\operatorname{cf}(\xi) \leq \kappa$, there is no consideration with regard to goodness. If $\operatorname{cf}(\xi) > \kappa$, then if we let $f^*_{\gamma_\eta}(i) = f_{\gamma_\eta}(i)$ for $i \geq \lambda_j$ and $f^*_{\gamma_\eta}(i) = 0$ otherwise, then it follows by construction that $\langle f^*_{\gamma_\eta} : \eta < \xi, \eta \text{ even}\rangle$ is $<_j$-increasing and cofinally interleaved with $\langle f_{\gamma_\eta} : \eta < \xi \rangle$. This is one of the equivalent definitions of goodness, so if we define p_ξ such that $\operatorname{dom} p_\xi = \gamma_\xi + 1$, $p_\xi \leq p_\eta$ for $\eta < \xi$, and $p_\xi(\gamma_\xi) = h$ where h is an exact upper bound of $\langle f_{\gamma_\eta} : \eta < \xi \rangle$, then p_ξ is a condition.

If $\xi = \lambda$, then we can find a lower bound by Proposition 2.13. $\square$

By distributivity we have cardinal preservation.

Proposition 2.15. *If $2^\lambda = \lambda^+$ then $\mathbb{G}(\vec{\lambda})$ preserves cardinals and cofinalities.*

Proposition 2.16. *$\mathbb{G}(\vec{\lambda})$ adds a good scale to $\vec{\lambda}$.*

Proof. If $\dot{h}$ is a $\mathbb{G}(\vec{\lambda})$-name for a function in the product as forced by some condition p, then choose $p' \leq p$ such that $p' \Vdash$ "$\dot{h} = g$". Then choose $p'' \leq p'$ such that $p''(\max \operatorname{dom} p'')$ dominates g. $\square$

3. Proving the Main Theorem

This section constitutes the proof of Theorem 1.2.

3.1. **Defining the Iteration and Establishing Basic Properties of the Target Model.**

Now we define the model that witnesses the main theorem.

First we establish some notation. For this section, and some model V_0, let $\mathbb{L}^{V_0}$ refer $\mathbb{L}$ is defined in the model V_0.

If τ is a cardinal, we recall Jensen's forcing $\mathbb{S}_\tau$ notion for adding a $\square_\tau$-sequence: Conditions are functions s such that:

(1) $\operatorname{dom} s \in \tau^+$,
(2) $\forall \alpha \in \operatorname{dom} s$, $s(\alpha)$ is a closed unbounded subset of α, of order-type $\leq \tau$,
(3) $\forall \alpha, \beta \in \operatorname{dom} s$, if β is a limit point of $s(\alpha)$, then $s(\alpha) \cap \beta = s(\beta)$.

We of course want:

Fact 3.1. *(See* [CFM01, Section 6]*.) The forcing $\mathbb{S}_\tau$:*

(1) is $(\tau+1)$-strategically closed and
(2) adds a $\square_\tau$-sequence.

Now let $\mathbb{S} = \prod_{n<\omega} \mathbb{S}_{\aleph_n}$. Also, let $\mathbb{G}^{V_0} = \mathbb{G}(\prod_{n<\omega} \aleph_n^{V_0})$, the poset defined in Subsection 2.3.

We make a standard definition explicit for clarity:

Definition 3.2. A *wide Aronszajn tree* of height $\aleph_1$ is a tree T of height $\aleph_1$ and *any* width that has no cofinal branches.

So, wide Aronszajn trees are not required to have countable levels.

We will also employ Baumgartner's specializing forcing:

Fact 3.3 (Baumgartner)**.** *Suppose that T is a wide Aronszajn tree of height $\aleph_1$. Then there is a forcing $\mathbb{B}(T)$ such that $\mathbb{B}(T)$ has the countable chain condition and such that $\mathbb{B}(T)$ adds a function $b : T \to \omega$ such that if $t \sqsubseteq t'$ are elements of T, then $b(t) = b(t')$ implies $t = t'$ (i.e. $\mathbb{B}(T)$ forces T to be special). (See* [Jec03, Chapter 16]*.)*

We need to describe a tree that will be used in this construction: Let g be $\mathbb{S} * \dot{\mathbb{G}}$-generic and let k be $\dot{\mathbb{L}}[g]$-generic over $V[g]$. Let $D \subseteq \nu$ be a club of order-type ω_1 in $V[g][k]$. Let

$$T_{\mathrm{IA}} = \{\langle Z_i : i < j\rangle \in H(\nu)^{V[g]} \cap ((H(\nu)^{V[g]})^{<\omega_1}) : \sup \bigcup_{i<j} (Z_i \cap \nu) \in D\}$$

where the tree order is determined by end extension, i.e. $\langle Z_i^0 : i < j_0\rangle \leq_{T_{\mathrm{IA}}} \langle Z_i^1 : i < j_1\rangle$ if and only if $j_1 \geq j_0$ and $Z_i^0 = Z_i^1$ for all $i < j_0$.

Claim 3.4. *Let g be $\mathbb{S} * \dot{\mathbb{G}}$-generic, let k be $\dot{\mathbb{L}}[g]$-generic over $V[g]$. Let $\dot{\mathbb{F}}$ be the $\mathbb{S} * \dot{\mathbb{G}} * \dot{\mathbb{L}}$ name for $\mathbb{B}(T_{\mathrm{IA}})$ if T_{IA} has height $\geq \omega_1$ and let $\dot{\mathbb{F}}$ be the name for the trivial forcing of T_{IA} has height $< \omega_1$. Let f be $\dot{\mathbb{F}}$-generic over $V[g][k]$. Then if $W \supseteq V[g][k][f]$ is an outer model preserving ω_1 (and hence $\mathrm{cf}(\nu) = \omega_1$), then there is no continuous sequence $\langle M_i : i < \omega_1\rangle$ such that $\langle M_i : i < j\rangle \in H(\nu)^{V[g]}$ for all $j < \omega_1$ and $H(\nu)^{V[g]} = \bigcup_{i<\omega_1} M_i$.*

There is also a specific case of a theorem of Cummings that we will use for clarity [Cum97, Theorem 2]:

Fact 3.5 (Cummings)**.** *Let $W \supseteq V$ be an extension such that $W \models$ "$|\aleph_n^V| = \aleph_1$" for all $n < \omega$. if $V \models$ "There is a good scale on $\aleph_\omega$", then $W \models$ "$|\aleph_{\omega+1}^V| = \aleph_1$".*

Proof of Claim 3.4. First we make some observations. Consider the tree T consisting of elements of the form

$$\langle Z_i : i < j\rangle \in H(\nu)^{V[g]} \cap (H(\nu)^{V[g]})^{<\omega_1}$$

defined without the restriction to D. By Theorem 2.1, we know that $V[g][k] \models$ "$\mathrm{cf}(\nu) = |\nu| = |({}^{<\omega_1}\nu)^{V[g]}| = \aleph_1$", so T has cardinality $\aleph_1$ in $V[g][k]$. Also by Theorem 2.1, it follows that under the direct extension ordering, there are no branches b of T such that for all $\delta < \nu$, there is some $\langle M_i : i < j\rangle \in b$ with $\bigcup_{i<j} \sup(M \cap \nu) \geq \delta$. It is immediate from $\mathrm{ot}(D) = \omega_1$ that T_{IA} has height no greater than ω_1.

Now we consider two cases. The first is that the height of T_{IA} is equal to some $\gamma < \omega_1$ in $V[g][k]$. Suppose for contradiction that $\langle M_i : i < \omega_1\rangle$ is a sequence as in the statement of the claim. Let $E := \langle \beta_i : i < \omega_1\rangle$ enumerate $\{\bigcup_{j<i} \sup(M_j \cap \nu) : i < \omega_1\}$. By continuity, E is a club, so $D \cap E$ is a club in ν. Choose some $\delta \in D \cap E$ such that $\mathrm{ot}(D \cap E \cap \delta) > \gamma$. If $\delta = \beta_i$, and j^* is such that $\bigcup_{i<j^*} (M_i \cap \nu) \in D$. Then $\langle M_i : i < j^*\rangle$ has at least δ-many predecessors in T_{IA}, which is a contradiction.

Now suppose that T_{IA} has height ω_1 in $V[g][k]$. We argue that in $V[g][k]$, T_{IA} is a wide Aronszajn tree of cardinality and height ω_1, in particular that T_{IA} has no cofinal branches in $V[g][k]$. By the observation stated in the first paragraph about branches that are unbounded in ν, it is sufficient to consider the possibility of a branch b of height ω_1 such that

$$\exists\beta < \nu, \langle M_i : i < j\rangle \in b, \bigcup_{i<j}(M_i \cap \nu) < \beta.$$

Let $\langle \alpha_i : i < \omega_1\rangle$ be an increasing enumeration of D and let i^* be minimal such that $\alpha_{i^*} \geq \beta$. But for all $\langle M_i : i < j\rangle, \langle M_i : i < j+1\rangle \in b$, there must be an element in D which is in $M_{j+1} \setminus M_j$. Therefore this is impossible since there are countably many elements of D below α_{i^*}.

To finish the claim, suppose for contradiction $\langle M_i : i < \omega_1\rangle$ is a sequence as in the statement. Let E be as defined in the first case. Let $\langle \gamma_i : i < \omega_1\rangle$ enumerate $D \cap E$ and let $\vec{M}_i = \langle M_j : j < \xi(i)\rangle$ be the corresponding elements of T_{IA}. Then each $\vec{M}_i$ belongs to a level of height $\geq i$ within T_{IA}. This contradicts the facts ω_1 is preserved and that the generic function added by f can only take countably many values. □

We start in a ground model V in which κ is a $\kappa^{+\omega+1}$-supercompact cardinal. The preparation is defined as follows: Fix a Laver supercompact guessing function $\ell : \kappa \to V_\kappa$ such that for every x and $\lambda \geq |\operatorname{tc}(x)|$ up to $\kappa^{+\omega+1}$, there is a λ-supercompact embedding $j : V \to M$ with critical point κ_0 such that $j(\ell)(\kappa) = x$ [Lav78].

We define a revised countable support (see [FMS89]) iteration $\mathbb{I} = \langle \mathbb{I}_\alpha, \dot{\mathbb{J}}_\alpha : \alpha < \kappa\rangle$ as follows:

(1) Suppose α is inaccessible and that $\ell(\alpha)$ is an $\mathbb{I}_\alpha$-name for a poset of the form
$$\dot{\mathbb{S}} * \dot{\mathbb{G}} * \dot{\mathbb{L}} * \dot{\mathbb{B}}(T_{\mathrm{IA}}) * \dot{\mathrm{Col}}(\aleph_1, \chi)$$
where T_{IA} indicates the wide Aronszajn tree discussed above and χ is some regular cardinal. If T_{IA} has height ω_1, then let $\dot{\mathbb{J}}_\alpha = \ell(\alpha)$. If T_{IA} has height less than ω_1, let $\dot{\mathbb{J}}_\alpha = \dot{\mathbb{S}} * \dot{\mathbb{G}} * \dot{\mathbb{L}} * \dot{\mathrm{Col}}(\aleph_1, \chi)$.

(2) Suppose α is inaccessible and $\ell(\alpha)$ is an $\mathbb{I}_\alpha$-name for a poset of the form
$$\mathbb{S} * \dot{\mathbb{G}} * \dot{\mathrm{Add}}(\alpha^{+\omega+1}) * \dot{\mathbb{L}}.$$
Then let $\dot{\mathbb{J}}_\alpha = \ell(\alpha)$.

(3) If α is inaccessible and $\ell(\alpha)$ is an $\mathbb{I}_\alpha$-name for $\mathrm{Col}(\aleph_1, \chi)$ for some $\chi < \kappa$, then let $\dot{\mathbb{J}}_\alpha$ be a name for $\mathrm{Col}(\aleph_1, \chi)$.

(4) Otherwise $\dot{\mathbb{J}}_\alpha$ is a name for the trivial poset.

Proposition 3.6. *If G is $\mathbb{I}$-generic over V, then $V[G] \models$ "$\kappa = \aleph_2$ and all posets of cardinality $\leq \kappa^{+\omega+1}$ that preserve stationary subsets of ω_1 are semiproper".*

Proof. By standard arguments, $\mathfrak{I}$ has the κ-chain condition and thus preserves regularity of κ. Because of the iterands for Case (2), there are surjections from $\aleph_1$ to χ for all $\chi < \kappa$, hence κ cannot be any larger than $\aleph_2$ in the extension. The collapse iterands in (1) ensure the statement about semiproperness by a lemma of Foreman, Magidor, and Shelah [FMS89, Lemma 3]. □

Let $G_{\mathbb{I}}$ be $\mathbb{I}$-generic over V, let $G_{\mathbb{S}}$ be $\mathbb{S}$-generic over $V[G_{\mathbb{I}}]$, and let $G_{\mathbb{G}}$ be $\mathbb{G}$-generic over $V[G_{\mathbb{I}}][G_{\mathbb{S}}]$. Then $V[G] := V[G_{\mathbb{I}}][G_{\mathbb{S}}][G_{\mathbb{G}}]$ will be the model witnessing Theorem 1.2.

Proposition 3.7. *The following are true in $V[G]$:*

(1) $\forall \tau > \aleph_1$, $2^\tau = \tau^+$,
(2) *For all* $n < \omega$, $\square_{\aleph_n}$ *holds,*
(3) *All scales on* $\aleph_\omega$ *are good.*

Proof. Without loss of generality, we can assume that GCH holds above κ. Then this is preserved by $\mathbb{I}$ since it has cardinality κ. Proposition 2.16 gives us the third point because if there is a good scale on the full product $\prod_{n<\omega} \aleph_n$, then restrictions to sub-products are automatically good. □

3.2. **The Lifting Argument.** Most of the work here consists of the following:

Lemma 3.8. *In* $V[G]$*, there are stationarily-many* $N \prec H(\aleph_{\omega+1})$ *of cardinality* $\aleph_1$ *that are not sup-internally approachable.*

We will in fact phrase the result in terms of guessing models.

Definition 3.9. Let N be a set such that $\aleph_1 \subseteq N$. Assume τ and λ are regular uncountable cardinals. Then we say that N is *weakly* (τ,λ)*-guessing* if whenever $f : \tau \to \mathrm{ON}$ and $f \restriction i \in N$ for cofinally many $i < \sup(N \cap \tau)$ and f is unbounded in $\sup(N \cap \lambda)$, then there is some $g \in N$ such that $f \restriction N = g$.

Observe that if N is (ω_1,λ)-weakly guessing for some λ, then we can take $f \in N$. Also, if N is sufficiently elementary, the definition implies that there can be no such f.

Proposition 3.10. *Suppose* $\lambda > \aleph_1$ *and that* N *with* $\aleph_1 = |N| = \mathrm{cf}(\aleph_1)$ *is weakly* (ω_1,λ)*-guessing. Then* N *is not sup-internally approachable.*

Proof. Suppose that N is sup-internally approachable via $\langle M_i : i < \omega_1\rangle$. Let $\delta = \sup(N \cap \lambda)$. Let $A = \langle \sup(M_i \cap \lambda) : i < \omega_1\rangle$. Then N has all initial segments of A. Since A is unbounded in δ, it follows that N cannot be weakly (ω_1,κ)-guessing. □

Lemma 3.11. *In* $V[G]$*, for all* $\theta \geq \aleph_{\omega+1}$*, there are stationarily-many* $N \prec H(\theta)$ *of cardinality* $\aleph_1$ *such that:*

(1) $\mathrm{cf}(N \cap \aleph_{\omega+1}) = \aleph_1$,
(2) N *is weakly* $(\omega_1, \aleph_{\omega+1})$*-guessing.*

Proof of Lemma 3.11. Let $\nu = \kappa^{+\omega+1} = \aleph_{\omega+1}^{V[G]}$.

Use the properties of the Laver function to find an embedding $j : V \to M$ such that $j(\kappa) > \nu$, $M^\nu \subset M$, and $j(\ell)(\kappa)$ is the $\mathbb{I}$-name for

$$\mathbb{S} * \dot{\mathbb{G}} * \dot{\mathbb{L}} * \dot{\mathbb{B}}(T_{\mathrm{IA}}) * \dot{\mathrm{Col}}(\aleph_1, \chi).$$

We let $\rho := \sup j[\nu]$.

First we will show that we can obtain a lift of $j : V \to M$ in a well-behaved forcing extension:

Claim 3.12. *There is a forcing extension* $V[G][K] \supset V[G]$ *such that:*

(1) $j : V \to M$ *can be lifted to* $j : V[G] \to M[j(G)] = M[G][K]$*,*
(2) $V[G][K] \models$ "$|\aleph_{\omega+1}^{V[G]}| = |\aleph_1^{V[G]}| = \aleph_1$",
(3) *In* $M[G][K]$*, if* $f : \omega_1 \to \mathrm{ON}$ *is not bounded in* ν*, then there is some* $i < \omega_1$ *such that* $f \restriction i \notin V[G]$*.*

This gives us the material to obtain the models:

Claim 3.13. *If $N_0 = H(\nu)^{V[G]}$, then in $M[G][K]$, the following hold:*

(1) $|j[N_0]| = \aleph_1 \subseteq j[N_0]$,
(2) $j[N_0] \prec j(N_0)$,
(3) $\sup(j[N_0] \cap j(\nu)) = \omega_1$,
(4) $j[N_0]$ *is weakly* $(\omega_1, \aleph_{\omega+1})$*-guessing.*

Now we can prove the claims.

Proof of Claim 3.12. Let j be as fixed at the beginning of the proof of Lemma 3.11. We will lift j in a series of steps in which we extend the range of j to the next generic. We will establish conditions *(1)* and *(2)* in the first step and argue that it is preserved as long as $\aleph_1$ is preserved in the following steps.

Lifting to domain $V[G_{\mathbb{I}}]$*:* By elementarity and Laver guessing and the fact that the iterands preserve stationary subsets of ω_1 (Fact 1.19 for the case of $\mathbb{L}$) we have that

$$\dot{\mathbb{S}} * \dot{\mathbb{G}} * \dot{\mathbb{L}} * \dot{\mathbb{B}}(T_{\mathrm{IA}}) * \dot{\mathrm{Col}}(\aleph_1, \chi)$$

is forced by $\mathbb{I}$ to preserve stationary subsets of ω_1.

Therefore, by the definition of the iteration and the output of the guessing function, we have

$$j(\mathbb{I}) = \mathbb{I} * \dot{\mathbb{S}} * \dot{\mathbb{G}} * \dot{\mathbb{L}} * \dot{\mathbb{B}}(T_{\mathrm{IA}}) * \dot{\mathbb{E}}$$

where $\dot{\mathbb{E}}$ is a remainder term that includes the Lévy collapse component. (This is where we use $\kappa^{+\omega+1} = \nu$-supercompactness of the embedding.)

Let $K_{\mathbb{L}}$ be $\mathbb{L}$-generic over $V[G]$, let $K_{\mathbb{B}}$ be $\mathbb{B}(T_{\mathrm{IA}})$-generic over $V[G][K_{\mathbb{L}}]$,and let $K_{\mathbb{E}}$ be $\mathbb{E}$-generic over $V[G][K_{\mathbb{L}}][K_{\mathbb{B}}]$.

Then since $j"\mathbb{I} \subseteq \mathbb{I}$, Silver's classical lifting argument (see [Cum10]) gives us an embedding $j : V[G_{\mathbb{I}}] \to M[G_{\mathbb{I}} * G_{\mathbb{S}} * G_{\mathbb{G}} * K_{\mathbb{L}} * K_{\mathbb{B}} * K_{\mathbb{E}}] = M[j(G_{\mathbb{I}})]$.

The argument that Condition *(2)* holds works as follows: The Lévy collapse iterand collapses ν to have cardinality and cofinality $\aleph_1$. Hence it is sufficient for Condition *(2)* to demonstrate preservation of $\aleph_1$ in the next steps.

In $V[G][K_{\mathbb{L}}][K_{\mathbb{B}}]$, T_{IA} is special, and in particular still a wide $\aleph_1$-Aronszajn tree. We can assume that its height is ω_1 since otherwise Claim 3.4 indicates a trivial case. By Claim 3.4, a function with the properties included in Condition *(3)* can be used to construct a chain that makes up a cofinal branch of T_{IA} (see e.g. [Eis10, Theorem 3.11]). Such a chain would collapse $\aleph_1$. Therefore, to ensure Condition *(3)* it is enough to show that $\aleph_1$ is preserved as computed in terms of extensions of M.

Lifting to domain $V[G_{\mathbb{I}}][G_{\mathbb{S}}]$: Let $\gamma_n := \sup j[\aleph_n^{V[G_{\mathbb{I}}]}]$ for all $n < \omega$. Then $\mathrm{cf}(\gamma_n) = \omega$ in $V[j(G_{\mathbb{I}})]$. Now consider $\bigcup_{s \in G_{\mathbb{S}}} j(s)$. Taking a cofinal ω-sequence $r_n \subseteq \gamma_n$ for each $n < \omega$, if we take $s^*(n) = \bigcup_{p \in G_{\mathbb{S}}} j(s(n))$, then we can inductively argue that $\bar{s} := \langle s^*(n) \cup \langle \gamma_n, r_n \rangle : n < \omega \rangle$ is a master condition for $j[G_{\mathbb{S}}]$ in $j(G_{\mathbb{S}})$ since coherence for the condition at γ_n is trivial because there are no limit points. (See [CFM03] for more detail.)

Then $\bar{s} \in j(\mathbb{S})$ is a master condition for $j[G_{\mathbb{S}}]$. Then we choose $K_{\mathbb{S}}$ to be a $j(\mathbb{S})$-generic containing p.

Lifting to domain $V[G_{\mathbb{I}}][G_{\mathbb{S}}][G_{\mathbb{G}}]$*:* We use another master condition argument. Let $\rho = \sup j[\mu]$. Let $\bar{p} = \bigcup_{p \in G_{\mathbb{G}}} j[p] = j[\vec{f}]$. Let $j(\vec{f}) = \langle f^*_\alpha : \alpha < j(\mu) \rangle$.

We now argue that $\bar{p}$ can be extended to a condition in $j(\mathbb{G})$. It is sufficient to argue that $M[j(G_{\mathbb{I}}) * j(G_{\mathbb{S}})]$ models that $\bar{p}$ can be extended to a condition in $j(\mathbb{G})$ since $j(\mathbb{G}) \in M[j(G_{\mathbb{I}}) * j(G_{\mathbb{S}})]$. It is enough to argue from the perspective of

$M[j(G_{\mathbb{I}} * G_{\mathbb{S}})]$. Specifically, we need to argue that ρ is a good point of $j(\vec{f})$, which then implies that $\bar{p}^\frown\langle\rho, h\rangle$ is a condition where h is any exact upper bound of $\bar{p}$. Let b_{prod} be the restriction of the generic function added by $\mathbb{L}$ denoted using the notation from Definition 2.7. Enumerate b_{prod} as $\langle b_n : n < \omega\rangle$. Abusing notation slightly, let $j(b_{\mathsf{prod}})$ denote the function $n \mapsto j(b_n)$. Note that even though b_{prod} is not in the domain of j, $M[j(G_{\mathbb{I}} * G_{\mathbb{S}})]$ contains $j \restriction \kappa^{+n}$ for all $n < \omega$, and $b_{\mathsf{prod}} \in M[j(G_{\mathbb{I}} * G_{\mathbb{S}})]$, so $j(b_{\mathsf{prod}}) \in M[j(G_{\mathbb{I}} * G_{\mathbb{S}})]$. We will argue that the function $j(b_{\mathsf{prod}})$ is an exact upper bound of $\bar{p}$. This will be sufficient since the range of this function consists points of cofinality $\aleph_1$ by the third point of Definition 1.13. Therefore we can apply the definition of goodness from Fact 1.7.

Claim 3.14. *Suppose that $h \in V[j(G_{\mathbb{I}} * G_{\mathbb{S}})]$ and that $h <^* j(b_{\mathsf{prod}})$. Then there is some $\alpha < \nu$ such that $M[j(G_{\mathbb{I}} * G_{\mathbb{S}})] \models$ "$h <^* f^*_{j(\alpha)}$" .*

Proof. We use the fact that $\mathrm{cf}(b_{\mathsf{prod}}(n)) = \aleph_1$ in $V[G_{\mathbb{I}} * G_{\mathbb{S}}]$. Fixing any $n < \omega$:, the image of j is unbounded in $\mathrm{cf}(j(b_{\mathsf{prod}})(n))$, so we can assume that $h(n)$ is in the image of j for all $n < \omega$. Let h' be the function such that $j(h'(n)) = h(n)$ for all $n < \omega$. We have that $j(G_{\mathbb{I}} * G_{\mathbb{S}}) = G * K_{\mathbb{L}} * K'$ where K' is a generic for a semiproper forcing in $V[G * K_{\mathbb{L}}]$. In particular, semiproper forcings are strictly bounding for functions $\omega \to \omega_1$. The space $\{f : f < b_{\mathsf{prod}}\}$ is cofinally interleaved with an embedding of ${}^\omega\omega_1$. Therefore there is some $h'' \in V[G][K_{\mathbb{L}}]$ such that $h' < h''$. By Lemma 2.11, there is some $\alpha < \aleph_{\omega+1}^{V[G]}$ such that $h'' <^* f_\alpha$. Suppose this is witnessed by $m < \omega$. Then for all $n \geq m$, we have that $j(f_\alpha(n)) = f^*_{j(\alpha)}(n) > j(h'(n)) = h(n)$. Hence $M[j(G_{\mathbb{I}} * G_{\mathbb{S}})] \models$ "$h <^* f^*_{j(\alpha)}$". □

Applying Claim 3.14, we have that where $j(\alpha) < \rho$. Hence $\bar{p}^\frown\langle\rho, h\rangle$ is a master condition for j"$G_{\mathbb{G}}$. Let $K_{\mathbb{G}}$ be any generic for $j(\mathbb{G})$ containing $\bar{p}$. Since $V[G_{\mathbb{I}}][G_{\mathbb{S}}] \models$ "$\mathbb{G}$ is countably closed", $M[j(G_{\mathbb{I}}][j(G_{\mathbb{S}})] \models$ "$j(\mathbb{G})$ is countably closed" and therefore preserves $\aleph_1$, this completes the lifting argument.

This completes the steps of the lifting argument. Finally, we let $K = K_{\mathbb{L}} * K_{\mathbb{B}} * K_{\mathbb{E}} * K_{\mathbb{S}} * K_{\mathbb{G}}$. □

Before starting with the proof of Claim 3.13, we establish some claims about the main object of interest generated by the lift. This effort allows us to speak of elementary submodels of $H(\aleph_{\omega+1})$ in particular.

Lemma 3.15. *Suppose that $\mathbb{P} \subseteq H(\aleph_{\omega+1})$, that $\mathbb{P}$ is countably closed, and that for arbtirarily high $n < \omega$, $\mathbb{P} \cong \mathbb{P}_A \times \mathbb{P}_B$ where $\mathbb{P}_A$ has cardinality $\aleph_n$ and $\mathbb{P}_B$ is $(\aleph_n+1)$-strategically closed. Then if g is $\mathbb{P}$-generic, then $H(\aleph_{\omega+1})[g] = H(\aleph_{\omega+1})^{V[g]}$.*

Proof. Note that if $\aleph_{\omega+1}$ is not preserved by $\mathbb{P}$, then $V[g] \models$ "$\mathrm{cf}(\aleph^V_{\omega+1}) = \aleph_n$". The factorization property of $\mathbb{P}$ therefore implies, using a variant of Easton's Lemma (see [Cum10, Remark 5.17]), that $\aleph_{\omega+1}$ is preserved.

First we argue that $H(\aleph_{\omega+1})[g] \subseteq H(\aleph_{\omega+1})^{V[g]}$ by $\in$-induction. Suppose that $\dot{X} \in H(\lambda)$ is a $\mathbb{P}$-name. Since any element of $\dot{X}$ is forced to be equal to some $\dot{z}$ such that $\langle\dot{z}, p\rangle \in \dot{X}$, and the $\dot{z}$'s are by induction forced to have transitive closure of cardinality $\leq \aleph_\omega$, the rest follows by $\in$-induction.

Now we argue that $H(\aleph_{\omega+1})^{V[g]} \subseteq H(\aleph_{\omega+1})[g]$. Again we can argue by $\in$-induction. Suppose that $\dot{X}$ is forced to have transitive closure of cardinality strictly less than $\aleph_{\omega+1}$. Thus, we suppose that $r \in \mathbb{P}$ and $r \Vdash$ "$\langle\dot{y}_\xi : \xi < \tau\rangle$ enumerates $\dot{X}$" for some $\tau \leq \aleph_\omega$ and moreover $r \Vdash$ "$\forall\xi < \tau, \exists\dot{w} \in H(\aleph_{\omega+1})^V, \dot{y}_\xi = \dot{w}$". Then it is sufficient to argue that there is some $r' \leq r$ and some $\langle \dot{z}_\xi : \xi < \tau \rangle \subseteq H(\aleph_{\omega+1})^V$ such that $r' \Vdash$ "$\dot{X} = \langle \dot{z}_\xi : \xi < \tau \rangle$".

First suppose that $\tau < \aleph_\omega$. Let $\mathbb{P} \cong \mathbb{P}_A \times \mathbb{P}_B$ be such that $\mathbb{P}_A$ has cardinality $\aleph_n$ and $\mathbb{P}_B$ is $(\aleph_n + 1)$-strategically closed where $\aleph_n > \tau$. Let r be identified with $(\bar{p}, \bar{q})$ under the factorization. (We are suppressing the details of the isomorphism $\mathbb{P} \cong \mathbb{P}_A \times \mathbb{P}_B$.) For each $\xi < \tau$, let $D_\xi \subseteq \mathbb{P}_B$ be the set of conditions q such that there is a maximal antichain $A \subseteq \mathbb{P}_A$ below $\bar{p}$ and a matrix $\{\dot{z}^\xi_{p,q} : p \in A\}$ such that $\dot{z}^\xi_{p,q} \in H(\aleph_{\omega+1})$ and $(p, q) \Vdash$ "$\dot{y}_\xi = \dot{z}^\xi_{p,q}$". We argue presently that the D_ξ's are dense below $\bar{q}$: Suppose $q' \leq \bar{q}$. Use the $(\aleph_n + 1)$-strategic closure to build a sequence of conditions $\langle q_i : i \leq \aleph_n \rangle$ below q' so that for each $i < \aleph_n$, there is some $p_i \in \mathbb{P}_A$ and $\dot{z}^\xi_{p_i,q_i} \in H(\aleph_{\omega+1})^V$ such that $(p_i, q_i) \Vdash$ "$\dot{y}_\xi = \dot{z}^\xi_{p_i,q_i}$", and do so in such a way that the p_i's are incompatible. Since $|\mathbb{P}_A| = \aleph_n$, this leads to the construction of a maximal antichain. Then $q_{\aleph_n} \in D_\xi$.

Having argued for the density of the D_ξ's, and noting that they are therefore open dense, use the strategic closure of $\mathbb{P}_B$ to find some q^* in the intersection $\bigcap_{\xi<\tau} D_\xi$. For each $\xi < \tau$, let A_ξ and $\{\dot{z}^\xi_{p,q^*} : p \in A_\xi\}$ witness that $q^* \in D_\xi$. Then let $\dot{z}_\xi$ be the name that glues the $\dot{z}^\xi_{p,q^*}$'s together along the antichain A_ξ. Since $\mathbb{P} \subseteq H(\aleph_{\omega+1})$, it follows that $\dot{z}_\xi \in H(\aleph_{\omega+1})$. Then $(\bar{p}, q^*)$ forces that $\dot{X}$ is enumerated by $\langle \dot{z}_\xi : \xi < \tau \rangle$, which is an element of $H(\aleph_{\omega+1})$.

Now suppose that $\tau = \aleph_\omega$. Let $\langle \dot{y}_\xi : \xi < \aleph_\omega \rangle$ be forced to be an enumeration of $\dot{X}$. Then by the case for $\tau < \aleph_\omega$, we can build a $\leq_\mathbb{P}$ decreasing sequence $\langle p_n : n < \omega \rangle$ in $\mathbb{P}$ such that p_n forces $\langle \dot{y}_\xi : \aleph_{n-1} \leq \xi < \aleph_n \rangle = \langle \dot{z}_\xi : \aleph_{n-1} \leq \xi < \aleph_n \rangle$ (setting $\aleph_{-1} = 0$) where the $\dot{z}_\xi$'s are elements of $H(\aleph_{\omega+1})$. Then the lower bound of the p_n's forces that $\dot{X}$ is enumerated by $\langle \dot{z}_\xi : \xi < \aleph_\omega \rangle$, which is an element of $H(\aleph_{\omega+1})$. □

Claim 3.16. *We have*

$$H(\nu)[G] := \{\dot{a}^G : \dot{a} \in H(\nu)\} = H(\nu)^{V[G]}.$$

Proof. It is a classical fact (the proof of which is roughly contained in the proof of Lemma 3.15) that if $\mathbb{P}$ is λ-cc for regular λ and $\mathbb{P} \subseteq H(\lambda)$, then $H(\lambda)[G] = H(\lambda)^{V[G]}$. It therefore follows that $H(\nu)[G_\mathbb{I}] = H(\nu)^{V[G_\mathbb{I}]}$. Lemma 3.15 implies that if $H' = H(\nu)^V[G_\mathbb{I}]$, then $H'[G_\mathbb{S}] = (H')^{V[G_\mathbb{I}][G_\mathbb{S}]}$. Finally, observe that if $H'' = H(\nu)^{V[G_\mathbb{I}][G_\mathbb{S}]}$, then we easily have $H''[G_\mathbb{G}] = (H'')^{V[G]}$ from the fact that $\dot{\mathbb{G}}$ is forced to be $\aleph_{\omega+1}$-distributive. □

Claim 3.17. *If $j : V[G] \to M[G][K]$ is defined as in Claim 3.12, then $j[N_0] \in M[G][K]$.*

Proof. By Claim 3.16, we have that $H(\nu)^V[G] = H(\nu)^{V[G]}$. Since $M^\nu \subset M$, we have both $H(\nu)^V \subseteq M$ and $j[H(\nu)^V] \in M$. Since $G * K \in M[G * K]$, it follows that

$$j[N_0] = j[H(\nu)^V[G]] = \{j(\dot{a}^G) : \dot{a} \in H(\nu)^V\} = \{j(\dot{a})^{G*K} : \dot{a} \in H(\nu)^V\},$$

is in $M[G][K]$. □

Proof of Claim 3.13. We will prove the requirements for $N := j[N_0]$ one by one.

1, Cardinality and Containment of $\aleph_1$*:* This is immediate from the second point of Claim 3.12 and the fact that $j(\aleph_1^{V[G]}) = j(\aleph_1^V) = \aleph_1$.

2, Elementarity: We use the Tarski-Vaught test in $M[G][K]$. Suppose we have $\bar{a} \in j[N_0]$ and $M[G][K] \models (\exists v \varphi(v, \bar{a}))^{H(j(\nu))}$. Let $\bar{b} \in N_0$ be such that $\bar{a} = j(\bar{b})$. So we are saying $M[G][K] \models \exists \varphi(v, j(\bar{b}))^{H(j(\nu))}$, so by elementarity, we have that in

$V[G]$, $H(\nu) \models \exists v \varphi(v, \bar{a})$. Let c be a witness, i.e. $V[G] \models \varphi(c, \bar{a})^{H(\nu)}$ for $c \in N_0$. Therefore $j(H(\nu)) \models \varphi(j(c), j(\bar{b}))$ where $j(c) \in j[N_0] = N$.

3, Uniformity for $\aleph_{\omega+1}$*:* We want to show that in $M[G][K]$, $\mathrm{cf}(j[N_0] \cap j(\nu)) = \aleph_1$. It is sufficient to observe that in $M[G][K]$, $\mathrm{cf}(j(\nu)) = \aleph_1$ since $\mathbb{L}$ forces $\mathrm{cf}(\nu) \geq \aleph_1$ and this is preserved by the rest of the iteration.

4, Guessing: Suppose for contradiction that in $M[G][K]$, there is an unbounded function $g : \omega_1 \to j(\nu)$ such that for all $i < \omega_1$, $g \restriction i \in j[N_0]$. Then for all $i < \omega_1$, there is some Y_i such that $j(Y_i) = g \restriction i$. Let f be the function $\bigcup_{i<\omega_1} Y_i$. We argue that f is unbounded in ν. If $\gamma < \nu$, then there is some $i < \omega_1$ such that $g(i) \in (j(\gamma), j(\nu))$. Then it follows by elementarity that $f(i) \in (\gamma, \nu)$. This therefore contradicts Condition *(3)* from Claim 3.13. □

Suppose $V[G] \models$ "$C \subseteq [H(\nu)]^{\aleph_1}$" is a club. Then given, lift $j : V[G] \to M[G][K]$ from Claim 3.12. we observe that $j[C]$ is a direct subset of $j[N_0]$ because if $j(x), j(y) \in j[C]$, then we can find $z \in C$ such that $z \supseteq x \cup y$ and then $j(z) \supseteq j(x \cup y)$. Furthermore, for all $a \in j[N_0]$, there is some $x \in C$ such that $a \in j(x)$. Since $M[G][K] \models$ "$j(C)$ is a club in $j(H(\nu)^{V[G]})$", it follows that $j[N_0] \in j(C)$. The statement of Lemma 3.11 therefore follows by elementarity given the properties that we proved for $j[N_0]$. □

3.3. **Failure of Weak Square.** Now we prove:

Lemma 3.18. $V[G] \models \neg \square^*_{\aleph_\omega}$.

We will use:

Fact 3.19 (Fuchs-Rinot [FR18])**.** *Suppose λ is a singular cardinal such that $\mu^{\mathrm{cf}(\lambda)} < \lambda$ for all $\mu < \lambda$, and suppose $\square^*_\lambda$ holds. Then in a generic extension by* $\mathrm{Add}(\lambda^+)$*, there is a non-reflecting stationary subset of $\lambda^+ \cap \mathrm{cof}(\lambda)$.*

Therefore it will be enough to show that if $\tilde{G}$ is $\mathrm{Add}(\aleph_{\omega+1})$-generic over $V[G]$, then $V[G][\tilde{G}] \models \mathsf{Refl}(\aleph_{\omega+1} \cap \mathrm{cof}(\omega))$, i.e. that all stationary subsets of $\aleph_{\omega+1} \cap \mathrm{cof}(\omega)$ reflect.

We have an analogous pair of claims to deal with.

Claim 3.20. *There is a forcing extension $V[G][\tilde{G}][K] \supset V[G][\tilde{G}]$ such that:*

(1) $j : V \to M$ *can be lifted to* $j : V[G * \tilde{G}] \to M[j(G * H)] = M[G][\tilde{G}][K]$,
(2) Stationary subsets of $\aleph_{\omega+1} \cap \mathrm{cof}(\omega)$ *in* $V[G][\tilde{G}]$ *are stationary in* $V[G][\tilde{G}][K]$,
(3) $V[G][\tilde{G}][K] \models$ "$\mathrm{cf}(\nu) = \aleph_1$".

Claim 3.21. *Suppose we have the lift $j : V[G * \tilde{G}] \to M[j(G * \tilde{G})]$ with the properties declared in Claim 3.20. Assume also that $|\nu| = \aleph_1$ in $V[G][\tilde{G}]$. Then $V[G][\tilde{G}] \models$* $\mathsf{Refl}(\aleph_{\omega+1} \cap \mathrm{cof}(\omega))$.

Now we can prove the claims.

Proof of Claim 3.20. We choose an embedding j with $j(\ell)(\kappa) = \dot{\mathbb{S}} * \dot{\mathbb{G}} * \dot{\mathrm{Add}}(\nu) * \dot{\mathbb{L}}$. We will lift the embedding in a series of steps in which we extend the range of an embedding to the next generic. We will argue for point *(2)* in the first step.

Lifting to domain $V[G_{\mathbb{I}}]$: Because of Case (2) of the definition of $\mathbb{I}$, we have $V[G_{\mathbb{I}}] \models \mathsf{WRP}$. Therefore $\mathbb{S} * \dot{\mathbb{G}} * \dot{\mathrm{Add}}(\aleph_{\omega+1}) * \dot{\mathbb{L}}$ is semiproper in $V[G_{\mathbb{I}}]$. By the chain condition of $\mathbb{I}$, we have $M[G_{\mathbb{I}}]^\nu \subset M[G_{\mathbb{I}}]$, so $\mathbb{S}*\dot{\mathbb{G}}*\dot{\mathrm{Add}}(\nu)*\dot{\mathbb{L}}$ is also semiproper in $M[G_{\mathbb{I}}]$. Therefore, by elementarity, we have that

$$j(\mathbb{I}) = \mathbb{I} * \dot{\mathbb{S}} * \dot{\mathbb{G}} * \dot{\mathrm{Add}}(\aleph_{\omega+1}) * \dot{\mathbb{L}} * \dot{\mathbb{E}}.$$

Let $K_{\mathbb{L}}$ be $\mathbb{L}$-generic over $V[G][\tilde{G}]$ and let $K_{\mathbb{E}}$ be $\mathbb{E}$-generic over $V[G][K_{\mathbb{L}}]$. Then Silver's classical lifting argument (see [Cum10]) gives us an embedding $j : V[G_{\mathbb{I}}] \to M[G_{\mathbb{I}} * G_{\mathbb{S}} * G_{\mathbb{G}} * G_{\mathbb{L}} * \tilde{G} * K_{\mathbb{L}} * K_{\mathbb{E}}] = M[j(G_{\mathbb{I}})]$ which is defined in $V[j(G_{\mathbb{I}})]$. (We will drop the dots to refer to the evaluated forcings in $V[j(G_{\mathbb{I}})]$.)

Now we argue for stationary preservation: We know that $\mathbb{L}$ forcing preserves stationary subsets of $\aleph_{\omega+1} \cap \mathrm{cof}(\omega)$. We also know that that $\mathbb{L}$ collapses $\aleph_{\omega+1}^W$ to have cardinality $\aleph_1$ by Fact 3.5. It will be sufficient, therefore, to show that stationary subsets of ω_1 are preserved in the remaining steps.

Lifting to domain $V[G_{\mathbb{I}}][G_{\mathbb{S}}]$: Analogous to Claim 3.12 in terms of both obtaining the lift and showing that the extension preserves stationary subsets of ω_1.

Lifting to domain $V[G_{\mathbb{I}}][G_{\mathbb{S}}][G_{\mathbb{G}}]$: Analogous to Claim 3.12.

Lifting to domain $V[G_{\mathbb{I}}][G_{\mathbb{S}}][G_{\mathbb{G}}][\tilde{G}]$: Use a master condition argument. As in the previous steps, we have $j``\tilde{G} \in M[j(G_{\mathbb{I}})]$. Therefore $q = \bigcup j``\tilde{G} \in M[j(G_{\mathbb{I}})]$, so we let K_{Add} be any $j(\mathrm{Add}(\nu))$-generic containing q. As in the previous two steps, $j(\mathrm{Add}(\nu))$ is countably closed and therefore preserves the relevant stationary sets.

Then we let $K = G_{\mathbb{L}} * G_{\mathbb{E}} * G_{\mathbb{S}} * G_{\mathbb{G}}$, completing the proof of the claim. $\square$

Proof of Claim 3.21. (See [CFM03, Claim 7].) Let $S \in V[G]$ be a stationary subset of $\aleph_{\omega+1} \cap \mathrm{cof}(\omega)$. Consider the lift $j : V[G][\tilde{G}] \to M[G][\tilde{G}][K]$ from Claim 3.20. Again, let $\rho = \sup j[\nu]$ where $\nu = \kappa^{+\omega+1}$. It is straightforward to show that $M[G][\tilde{G}][K] \models$ "$S \cap \rho$ is stationary ", from which the claim follows by elementarity. $\square$

This finishes the lemma on the failure of $\square^*_{\aleph_\omega}$ and it completes the proof of Theorem 1.2.

3.4. **Non-Tightness of the Models from the Main Theorem.** We include a last observation on the model from Theorem 1.2. Recall Definition 1.11.

Proposition 3.22. *In $V[G]$, for all $\theta \geq \aleph_{\omega+1}$, there are stationarily many $N \prec H(\theta)$ of cardinality $\aleph_1$ that are not tight for $K = \{\aleph_n : 2 \leq n < \omega\}$.*

Proof. Let $j[N_0]$ be as in Claim 3.12 and Claim 3.13. It is enough to show that $j[N_0]$ is not tight. Without loss of generality, we can assume that a fixed club $C \subseteq [H(\nu)^{V[G]}]^{\aleph_1}$ is chosen such that $\forall x \in C$, if $\mathrm{cf}(\sup(\nu \cap x)) = \omega_1$, then $\sup(\nu \cap x)$ is a good point of the scale $\vec{f}$ that was added by $\mathbb{G}$. This uses a predicate $[\alpha \mapsto f_\alpha]$ for $\vec{f}$ in the structure $H(\nu)^{V[G]}$ (since we cannot literally have $\vec{f} \in H(\nu)^{V[G]}$ because $|\vec{f}| = \nu$).

Recall again that by point (3) of definition Definition 1.13 and the preservation of $\aleph_1$, $\mathrm{cf}(j(\vec{f}))_\rho)(k) = \aleph_1$ for sufficiently large $k < \omega$.

First, we argue that $j(b_{\mathsf{prod}})$ is an exact upper bound of $j(\vec{f}) \restriction \rho$ (using the same abuse of notation from earlier). This follows by the argument for Claim 3.14.

Since $j[N_0] \cap j(\nu) = \rho$, no element of $j[N_0]$ can dominate $j(b_{\mathsf{prod}})$: Suppose $g \in j[N_0] \cap j(\nu)$. Then there is $\alpha < \nu$, $g <^* f^*_{j(\alpha)} <^* j(b_{\mathsf{prod}})$.

We conclude by noting that $j(b_{\mathsf{prod}}) \in \prod_{n<\omega}(j(\aleph_n^{V[G]}) \cap j[N_0])$. $\square$

4. Further Considerations

4.1. **Another Failure of Square for the Earlier Models.** We will define a "Miller version" of our Namba forcing, which is used by Cummings, Foreman, and Magidor as well as Krueger.

Definition 4.1. Fix a bijection $d : \omega \to \omega \setminus \{0, 1\}$ as in Definition 1.13.

The poset $\mathbb{M}$ will consist of trees p such that the following hold:

(1) p is a tree consisting of finite sequences t.
(2) For all $t \in p$ and $n \in \mathrm{dom}(t)$, $t(n) \in \aleph_{d(n)}$.
(3) For all $t \in p$, there is some $t' \sqsupseteq t$ such that if $n = \mathrm{dom}(t')$, then $\{\eta : t^\frown\eta \in p\}$ has cardinality $\aleph_{d(n)}$.

The ordering on $\mathbb{M}$ is given by inclusion.

We will not use $\mathbb{M}$ in conjunction with good scales, so there is no need to employ stationary splitting.

We take an interest in a different type of square sequence:

Definition 4.2. If λ is a regular cardinal and $\kappa \leq \lambda$, then $\Box(\lambda, \kappa)$ holds if there is a sequence $\langle \mathcal{C}_\alpha : \alpha < \lambda \rangle$ such that:

(1) for all $C \in \mathcal{C}_\alpha$, C is a club in α,
(2) for all $C \in \mathcal{C}_\alpha$ and $\beta \in \lim C$, $C \cap \beta \in \mathcal{C}_\beta$,
(3) there is no club $D \subseteq \lambda$ such that for all $\alpha \in \lim D$, $D \cap \alpha \in \mathcal{C}_\alpha$.

In this subsection, we will obtain:

Theorem 4.3. *Assuming the consistency of a supercompact cardinal, there is a model in which the following hold:*

(1) $\aleph_\omega$ *is a strong limit,*
(2) $\Box_{\aleph_n}$ *holds for all* $n < \omega$,
(3) $\Box(\aleph_{\omega+1}, \aleph_1)$ *fails.*

The point we are making is that we obtain a failure of a variant of the square principle that probably cannot be extracted from the failure of simultaneous reflection. Work of Hayut and Lambie-Hanson shows that (in particular) $\Box(\aleph_{\omega+1}, \aleph_1)$ is consistent with simultaneous reflection for $\aleph_0$-many subsets of $\aleph_{\omega+1}$ [HL17, Theorem 4.11]. Moreover, we make use of an approximation property that was not employed in the earlier papers. One could also make $\Box^*_{\aleph_\omega}$ fail, as Krueger does [Kru13].

4.1.1. *More Approximation.*

Theorem 4.4. *Let* $\dot{\mathbb{U}}$ *be a* $\mathbb{M}$*-name for a countably closed forcing. Then if*

$$\Vdash_{\mathbb{M} * \dot{\mathbb{U}}} \text{"}\dot{X} \text{ is unbounded in } \aleph_{\omega+1}^V \text{ and } \dot{X} \notin V\text{"},$$

then $\mathbb{M} * \dot{\mathbb{U}}$ *forces that there is some* $\delta < \aleph_{\omega+1}^V$ *such that* $\dot{X} \cap \delta \notin V$.

Proof. Let $\nu = \aleph_{\omega+1}^V$. Suppose for contradiction that $\dot{X}$ is a $\mathbb{M}$-name for a new cofinal subset of ν, all of whose initial segements are in V.

Let $\varphi(\delta, q, \dot{d})$ denote the formula

$$\begin{aligned} \delta < \nu \wedge & (q, \dot{d}) \in \mathbb{M} * \dot{\mathbb{U}} \wedge \exists \langle a_\alpha : \alpha \in \mathrm{osucc}_q(\mathrm{stem}(q)) \rangle \text{ s.t.} \\ & \forall \alpha \in \mathrm{osucc}_q(\mathrm{stem}(q)), (q \restriction (\mathrm{stem}(q)^\frown \langle \alpha \rangle), \dot{d}) \Vdash \text{"}\dot{X} \cap \delta = a_\alpha\text{"} \wedge \\ & \forall \alpha, \beta \in \mathrm{osucc}_q(\mathrm{stem}(q)), \alpha \neq \beta \implies a_\alpha \neq a_\beta. \end{aligned}$$

Claim 4.5. *Let κ be a cardinal and let λ be a regular cardinal such that $\kappa^+ < \lambda$. Define*

$$T = \{t \in {}^{<\lambda}2 : \exists q \leq p, q \Vdash \text{“}\dot{X} \cap \mathrm{dom}(t) = t\text{”}\}.$$

Then for all $p \in \mathbb{M}$, there is a club $C \subseteq \lambda$ such that the level T_δ has width $> \kappa$ for all $\delta \in C \cap \mathrm{cof}(\kappa^+)$.

Proof. Let T be the tree defined in the statement of the proposition. Suppose contrapositively that there is a stationary subset $S \subseteq \lambda \cap \mathrm{cof}(\kappa^+)$ such that for all $\delta \in S$, $|T_\delta| \leq \kappa$. By the proof of a theorem of Kurepa [Tod84, Lemma 2.7], it must be the case that for all $\delta \in S$ and $t \in T_\delta$, there is some $t' \sqsubseteq t$ such that $\mathrm{dom}(t') < \mathrm{dom}(t)$ and such that t' has a unique descendant in T_δ.

Now define a regressive function $h_0 : S \to \lambda$ such that for all $\delta \in S$, there is some $t \in T_\delta$ and some $t' \sqsubseteq t$ with $\mathrm{dom}(t') = h_0(\delta) < \delta$. By Fodor's Lemma, there is some $S' \subseteq S$ and some γ such that for all $\delta \in S'$, there is some $t \in T_\delta$ with a predecessor on level γ with only t as a successor in T_δ. By considering limit points of S, we can assume that the predecessor is on a level γ with $\gamma \in S$, and hence that there are at most κ-many to choose from. Hence we can apply Fodor again to find some $S'' \subseteq S'$ and some $\bar{t}$ such that for all $\delta \in S''$, there is some $t \in T_\delta$ such that $\bar{t} \sqsubseteq t$ and t is the unique descendant of $\bar{t}$ in T_δ.

Let $\bar{\delta} = \mathrm{dom}(\bar{t})$ and let $(q, \dot{d}) \leq (p, \dot{c})$ decide $\bar{t} = \dot{X} \cap \bar{\delta}$. Since $\bar{t} \in T$, $(q, \dot{d}) \Vdash$ “$\bar{t} = \dot{X} \cap \bar{\delta}$”. Let t_δ be the unique descendant of $\bar{t}$ in T_δ. For all $\gamma < \lambda$, the fact that $\dot{X}$ is forced to be unbounded unbounded implies that $(q, \dot{d}) \Vdash$ “$\exists \delta \in (\gamma, \lambda), \dot{X} \cap (\gamma, \lambda) \neq \emptyset$”. Therefore for all $\gamma < \lambda$, there is some $\delta \in (\gamma, \lambda)$ such that $\sup t_\delta \in (\gamma, \lambda)$ by uniqueness of t_δ. Otherwise, $(q, \dot{d})$ would force that $\dot{X}$ is bounded in λ. Moreover, for these values γ, δ we have $(q, \dot{d}) \Vdash$ “$\dot{X} \cap \gamma = t_\delta \cap \gamma$”. Hence $(q, \dot{d}) \Vdash$ “$\dot{X} = \bigcup_{\delta \in S''} t_\delta$”. This contradicts the fact that $\dot{X}$ is forced to be new. □

Claim 4.6. $\forall \gamma < \nu, (p, \dot{c}) \in \mathbb{M}$*, there is some $\delta \in (\gamma, \nu)$, and some $(q, \dot{d}) \leq (p, \dot{c})$ with* $\mathrm{stem}\, q = \mathrm{stem}\, p$ *such that $\varphi(\delta, p, \dot{c})$ holds.*

Proof. Let $W = \mathrm{osucc}_p(\mathrm{stem}(p))$.

For each $\alpha \in W$, let

$$T_\alpha = \{t \in {}^{<\lambda}2 : \exists (q, \dot{d}) \leq (p \restriction (\mathrm{stem}(p)^\frown \langle \alpha \rangle), \dot{c}), (q, \dot{d}) \Vdash \text{“}\dot{X} \cap \mathrm{dom}(t) = t\text{”}\}.$$

Let C_α be the set of δ such that $|T_\alpha| \geq \aleph_\omega$. Then C_α is a club by Claim 4.5. Let $C = \bigcap_{\alpha \in W} C_\alpha$. Fix some $\delta \in (C \cap \mathrm{cof}(|W|^+)) \setminus (\gamma + 1)$.

By induction on $\alpha \in W$ we will define a sequence of conditions, $\langle (q_\alpha, \dot{d}_\alpha) : \alpha \in W \rangle$ and the distinct sets $\langle a_\alpha : \alpha \in W \rangle$.

Choose $(q_\alpha, \dot{d}_\alpha) \leq (p \restriction (\mathrm{stem}\, p^\frown \langle \alpha \rangle), \dot{c})$ forcing “$\dot{X} \cap \delta = a_0$” for some a_0.

Now suppose that the members of our sequences have been defined for $\beta \in W \cap \alpha$. Let $B = \{a_\beta : \beta \in W \cap \alpha\}$, which is in particular of cardinality strictly less than $|W|$.

We do not have $(p \restriction t^\frown \langle \alpha \rangle, \dot{c}) \Vdash$ “$\dot{X} \cap \delta \in B$”, because this would contradict the fact that $\delta \in C_\alpha$. Therefore there is some $(q_\alpha, \dot{d}_\alpha) \leq (p, \dot{c})$ and some $a_\alpha \notin B$ such that $(q_\alpha, \dot{d}_\alpha) \Vdash$ “$\dot{X} \cap \delta = a_\alpha$”.

Then let $q = \bigcup_{\alpha \in W} q_\alpha$. Let $\dot{d}$ be the gluing of the $\dot{d}_\alpha$'s below q_α. □

Now we will build a condition $(q, \dot{d}) \in \mathbb{M} * \dot{\mathbb{U}}$ and an ordinal δ by a fusion process in such a way that any stronger condition deciding $\dot{X} \restriction \delta$ will also code the generic sequence for $\mathbb{M}$.

Define a sequence $\langle (p_n, \dot{c}_n) : n < \omega \rangle$ of conditions and a sequence $\langle \gamma_n : n < \omega \rangle$ of ordinals in $\aleph_{\omega+1}$.

The construction is defined as follows: Let $(p_0, \dot{c}_0)$ be obtained by applying Claim 4.6 to $(p, \dot{c})$.

In general that $(p_n, \dot{c}_n)$ and γ_n have been defined. Consider all nodes t of the n^{th} splitting level of p_n. Then apply Claim 4.6 to $(p_n, \dot{c}_n)$ and γ_n to obtain $(q_t, \dot{c}_t)$ and some γ_t. Let $p_{n+1} = \bigcup q_t$ and let $\dot{c}_{n+1}$ be the gluing of the $\dot{c}_t$'s. Let γ_{n+1} be above the supremum of the γ_t's.

At the end we let $q = \bigcap_{n<\omega} p_n$ be the fusion limit, we let $\dot{d}$ be the name forced to be a lower bound of the $\dot{c}_n$'s, and we let $\delta = \sup_{n<\omega} \gamma_n$.

Then $(q, \dot{d})$ forces that the generic sequence for $\mathbb{M}$ can be recovered from the forced value of $\dot{X} \restriction \delta$ as follows: At the 0^{th} level, we choose a node based on the forced value for γ_0. Choose a $\sqsubseteq$-increasing sequences nodes inductively so that at step n, one is at the n^{th} splitting level and checks the node corresponding to the forced value $\dot{X} \cap \gamma_n$.

Hence $(q, \dot{d}) \Vdash$ "$\dot{X} \restriction \delta \notin V$" or else we obtain the contradiction from the previous paragraph. This contradicts the premise from the beginning of the proof that initial segments of $\dot{X}$ are in V. $\square$

4.1.2. *Relationship to Squares.*

Definition 4.7. Let N be a set such that $\aleph_1 \subseteq N$ and assume that κ is a regular cardinal. We say that N is *cofinally weakly κ-guessing* if for all unbounded $X \subseteq N \cap \kappa$ such that $X \cap \gamma \in N$ for $\gamma \in N \cap \kappa$, it follows that there is some $Y \in N$ such that $Y \cap (\sup(N \cap \kappa)) = X$.

Proposition 4.8. *Suppose that λ is regular and $\theta > \lambda$. Suppose that there are stationarily-many $N \prec H(\theta)$ with $\mathrm{cf}(N \cap \lambda) = \aleph_1 = |N|$ that cofinally weakly λ-guessing. Then $\square(\lambda, \omega_1)$ fails.*

Proof. (See [Cox-Krueger, [CK18]].) Suppose for contradiction that we have $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha : \alpha < \lambda \rangle$, a $\square(\lambda, \omega_1)$-sequence. Consider the structure $\mathcal{A} = (H(\theta), \in, <_\theta, \vec{\mathcal{C}})$. Let $N \prec \mathcal{A}$ be as in the hypothesis of the proposition. Let $\delta = \sup(N \cap \lambda)$.

Take any $D \in \mathcal{C}_\delta$. By standard arguments, the fact that $\mathrm{cf}(N \cap \lambda)$ is uncountable implies that $N \cap \delta$ contains a club in δ. (See e.g. [CFM04, Lemma 4.3].) Therefore $D \cap N \cap \delta$ is a club. Take some $\gamma \in D \cap N \cap \delta$. Then $D \cap \gamma \in \mathcal{C}_\gamma$, and since $\aleph_1 \subseteq N$, we have $\mathcal{C}_\gamma \subseteq N$, so therefore $D \cap \gamma \in N$. By unboundedness it follows that this will be true for any $\gamma \in N \cap \lambda$.

By the fact that N is cofinally weakly λ-guessing, there is some $E \in N$ such that $E \cap \sup(N \cap \lambda) = D$. This implies that N thinks that E is a thread of the $\square(\lambda, \omega_1)$-sequence, which is a contradiction. $\square$

4.1.3. *Sketching the Rest of the Argument.* The construction for Theorem 4.3 is very similar to that of Theorem 1.2 with four prominent differences: First, since we are not attempting to get all scales on $\aleph_\omega$ to be good, we will not use $\mathbb{G}(\prod_{2 \le n < \omega} \aleph_n)$, which makes the construction strictly easier. Second, we want to refer to models in $H(\theta)$ for $\theta \geq \aleph_{\omega+2}$ so that they may contain unbounded subsets of $\aleph_{\omega+1}$ as elements. Third, for similar reasons we are using a different sort of guessing—cofinal weak guessing for $\aleph_{\omega+1}$ rather than weak $(\omega_1, \aleph_{\omega+1})$-guessing. Fourth, we no longer need worry about the branches of the analog of T_{IA}.

Let g be $\mathbb{S}$-generic and let k be $\dot{\mathbb{M}}[g]$-generic over $V[g]$. Let T_{IA^+} be a the analog of T_{IA} for $H(\nu)^{V[g]} \cap ((H(\nu)^{V[g]})^{<\aleph_{\omega+1}})$ where the tree order is determined by end extension.

We start in a ground model V in which κ is a supercompact cardinal and fix a Laver supercompact guessing function $\ell : \kappa \to V_\kappa$.

We define a revised countable support iteration $\mathbb{I} = \langle \mathbb{I}_\alpha, \dot{\mathbb{J}}_\alpha : \alpha < \kappa \rangle$ as follows:

(1) Suppose α is inaccessible and that $\ell(\alpha)$ is an $\mathbb{I}_\alpha$-name for a poset of the form
$$\dot{\mathbb{S}} * \dot{\mathbb{M}} * \dot{\mathbb{B}}(T_{\mathrm{IA}^+}) * \dot{\mathrm{Col}}(\aleph_1, \chi)$$
where T_{IA^+} indicates the wide Aronszajn tree discussed above and χ is some regular cardinal. Then let $\dot{\mathbb{J}}_\alpha = \dot{\mathbb{S}} * \dot{\mathbb{M}} * \dot{\mathrm{Col}}(\aleph_1, \chi)$.

(2) If α is inaccessible and $\ell(\alpha)$ is an $\mathbb{I}_\alpha$-name for $\mathrm{Col}(\aleph_1, \chi)$ for some $\chi < \kappa$, then let $\dot{\mathbb{J}}_\alpha$ be a name for $\mathrm{Col}(\aleph_1, \chi)$.

(3) Otherwise $\dot{\mathbb{J}}_\alpha$ is a name for the trivial poset.

As before we fix $\nu = \aleph_{\omega+1}^{V[G]}$.

The target model $V[G]$ will be a forcing extension by $\mathbb{I} * \dot{\mathbb{S}}$. To prove that $V[G] \models \neg\square(\aleph_{\omega+1}, \aleph_1)$, we would argue for:

Lemma 4.9. *In $V[G]$, for all $\theta \geq \aleph_{\omega+1}$, there are stationarily-many $N \prec H(\theta)$ such that:*

(1) $\mathrm{cf}(N \cap \aleph_{\omega+1}) = \aleph_1$,
(2) N is cofinally weakly $\aleph_{\omega+1}$-guessing.

This would be obtained from the following claims, where we note that the last point of each is changed for Theorem 4.3.

Claim 4.10. *There is a forcing extension $V[G][K] \supset V[G]$ such that:*

(1) $j : V \to M$ can be lifted to $j : V[G] \to M[j(G)] = M[G][K]$,
(2) $V[G][K] \models$ "$|\aleph_{\omega+1}^{V[G]}| = |\aleph_1^{V[G]}| = \aleph_1$",
(3) In $M[G][K]$, if $f : \nu \to \nu$ is unbounded in $\aleph_{\omega+1}^{V[G]}$ and $f \restriction \delta \in V[G]$ for all $\delta < \nu$, then $f \in V[G]$.

Claim 4.11. *If $N_0 = H(\nu)^{V[G]}$, then in $M[G][K]$, the following hold:*

(1) $|j[N_0]| = \aleph_1 \subseteq j[N_0]$,
(2) $j[N_0] \prec j(N_0)$,
(3) $\sup(j[N_0] \cap j(\nu)) = \omega_1$,
(4) $j[N_0]$ is cofinally weakly $\aleph_{\omega+1}$-guessing.

4.2. **Sup-Internal Approachability without Internal Unboundedness.** We will provide a theorem that fits the mold of Krueger's distinctions of variants of internal approachability.

Theorem 4.12. MM *implies that for all regular $\lambda \geq \aleph_2$ and all singular $\lambda > \aleph_2$ of cofinality ω, for sufficiently large $\theta > \lambda$, there are stationarily-many $N \prec H(\theta)$ of cardinality $\aleph_1$ that are sup-internally approachable at λ^+ but not internally unbounded.*

This theorem also demonstrates what might happen in a construction like that of Theorem 1.2, but without the use of Baumgartner's specializing forcing.

Definition 4.13. Consider some $\lambda \geq \aleph_2$. Let $\mathbb{U}$ consist of countable sequences $\langle M_i : i \leq j \rangle$ such that:

1. for all $i < j$, M_i is countable,
2. $\langle M_i : i < j \rangle \in H(\lambda)$,
3. $\langle M_i : i \leq j \rangle$ is continuous and $\subseteq$-increasing,
4. for all successors $j' + 1 \leq j$, $\langle M_i : i < j' \rangle \in M'_j$.

If $p, q \in \mathbb{U}$, we let $p \leq q$ if and only if $\operatorname{dom} p \supseteq \operatorname{dom} q$ and $p \restriction \operatorname{dom} q = q$.

It is easy to observe that:

Proposition 4.14. $\mathbb{U}$ *is countably closed.*

In fact, $\mathbb{U}$ is equivalent to the Lévy collapse $\mathrm{Col}(\aleph_1, |H(\lambda)|)$ under GCH.

Proposition 4.15. *Suppose that $\mathbb{P}$ is semiproper in V and $\mathbb{A}$ is countably closed in V. Then the product $\mathbb{P} \times \mathbb{A}$ preserves stationary subsets of ω_1 over V.*

Proof. (See [CK17, Theorem 5.5].) Let $S \subseteq \omega_1$ be stationary. Let θ be large enough that $H(\theta)$ contains $\mathbb{P}$, $\mathbb{A}$, and any other relevant sets. Suppose $\dot{C}$ is a $\mathbb{P} \times \mathbb{A}$ name forced by some $(\bar{p}, \bar{q})$ to be a club subset of ω_1. By semiproperness of $\mathbb{P}$, there is a club $C^* \subseteq [H(\theta)]^\omega$ of elementary submodels $M \prec H(\theta)$ such that for all $p \in M \cap \mathbb{P}$, there is $p' \leq p$ such that p' is $(M, \mathbb{P})$-semigeneric. By taking a chain from C^*, we can build a club $D \subseteq \omega_1$ such that for all $\delta \in D$, $\delta = M \cap \omega_1$ for some $M \in C^*$ containing $\dot{C}$ and $(\bar{p}, \bar{q})$. Therefore we have some $M \in C^*$ with $M \cap \omega_1 = \delta \in S$.

Let $\langle \delta_n : n < \omega \rangle$ be a countable and strictly increasing sequence converging to δ. For each n let

$$D_n = \{q \in \mathbb{A} : \exists p \in \mathbb{P}, (p, q) \| \min(\dot{C} \setminus \delta_n)\}.$$

By the Product Lemma, D_n is open dense in $\mathbb{A}$. Choose a decreasing sequence $\langle q_n : n < \omega \rangle$ below $\bar{q}$ such that $q_n \in D_n \cap M$ for all $n < \omega$. Let q be a lower bound of the q_n's using closure of $\mathbb{A}$. Let $p \leq \bar{p}$ be $(M, \mathbb{P})$-semigeneric.

We can argue that $(p, q) \Vdash$ "$\delta \in \dot{C}$". Let

$$D'_n = \{p \in \mathbb{P} : (p, q_n) \| \min(\dot{C} \setminus \delta_n)\},$$

which is dense in $\mathbb{P}$, and let $\dot{\alpha}_n$ be the $\mathbb{P}$-name for the ordinal decided by an element of D'_n. By semigenericity, $p \Vdash$ "$\dot{\alpha}_n \in M$". Therefore, for all $n < \omega$, $(p, q) \Vdash$ "$\dot{C} \cap (\delta_n, \delta) \neq \emptyset$", therefore $(p, q) \Vdash$ "$\delta \in \dot{C}$". □

See Woodin for the following [Woo99, Theorem 2.53]:

Fact 4.16 (Woodin)**.** *Suppose that $\mathbb{P}$ is a forcing poset and that for all sequences $\mathcal{D} = \langle D_\alpha : \alpha < \omega_1 \rangle$ of dense subsets of $\mathbb{P}$, there is a $\mathcal{D}$-generic filter. Then for any regular cardinal θ such that $\mathbb{P} \in H(\theta)$, there are stationarily many $[H(\theta)]^{\aleph_1}$ with $\omega_1 \subseteq N$ for which there exists an $(N, \mathbb{P})$-generic filter.*

Proof of Theorem 4.12. Work in a model of MM. Let $\theta > (2^\lambda)^+$ where $\mathbb{U}$ is defined with respect to $H(\lambda)$ and such that $H(\theta)$ contains all relevant sets. Suppose λ is regular. Let $\mathbb{P}$ be the standard Miller-style Namba forcing of height ω and width λ. Or more precisely, $\mathbb{P}$ consists of trees $p \subseteq {}^{<\omega}\lambda$ such that:

1. for all $t \in p$, $|\operatorname{osucc}_p(t)| \in \{1, \lambda\}$,
2. for all $t \in p$, there is some $t' \sqsupseteq t$ such that $|\operatorname{osucc}_p(t)| = \lambda$,

and such that $p \leq q$ if and only if $p \subseteq q$.

If λ is singular of cofinality ω, then we can define $\mathbb{P}$ to be some singular Namba forcing like $\mathbb{L}$.

By Proposition 4.15, we can apply Fact 4.16 to obtain stationarily-many $N \in P_{\omega_2}(H(\theta))$ with $\omega_1 \subseteq N$ for which there exists an $(N, \mathbb{P} \times \mathbb{U})$-generic filter. Take an arbitrary such example N and let g denote the filter.

First we argue that N is not internally unbounded. Since $\mathbb{P}$ forces that λ is singularized to have cofinality ω, there is a name $\dot{s}$ for the generic sequence, which is not covered by any set in V. By genericity of g, there are conditions forcing that respective points belong to the sequence $\langle s_n : n < \omega \rangle$. If $x \in N$, and without loss of generality $x \subseteq \mathrm{ON}$, then there is a dense set D_x of conditions (p, u) such that there is some α such that $(p, u) \Vdash$ "$\alpha \in \dot{s} \setminus x$". We have $D_x \in N$ by definability. Let $(p, u) \in D_x \cap N \cap g$. Therefore x cannot cover $\langle s_n : n < \omega \rangle$. (See [CK18, Lemma 5.2].)

Now we argue that N is sup-internally approachable at λ^+. Let $\delta = \sup(N \cap \lambda^+)$. We have $\mathrm{cf}(\delta) = \omega_1$ so take a cofinal sequence $\langle \delta_i : i < \omega_1 \rangle$. Let D_i be the dense set of conditions $(p, u) \in \mathbb{P} \times \mathbb{U}$ such that $\sup_{i \in \mathrm{dom}\, u} \sup(u_i \cap \lambda^+) \geq \delta_i$ and $\mathrm{dom}\, u_i \geq i$. Let $(p_i, u_i) \in g \cap D_i$ and write $u_i = \langle M_\xi : \xi < j_i \rangle$. Then $\langle M_\xi : \xi < \omega_1 \rangle$ witnesses sup-internal approachability of N at λ^+. □

4.3. **A Comment on the Proper Forcing Axiom.** The development of $\mathbb{G}(\vec{\lambda})$ up to this point suffices for the main theorem. We will add an additional observation relating this material to the consideration of PFA.

Definition 4.17 (Yoshinobu)**.** (See [Yos17].) Given a poset $\mathbb{P}$, we let $G^*(\mathbb{P})$ denote a two-player game described here. Player I chooses an at most countable subset of $\mathbb{P}$ and Player II plays conditions in $\mathbb{P}$ as follows:

Player I	A_0		A_1		$\dots$		$A_{\omega+1}$
Player II		b_0		b_1	$\dots$	b_ω	

Player II wins the game if and only if she is able to make her ω_1'st move.

They obey the following rules, in which Player I is responsible for (1)-(3) and Player II is responsible for (4):

(1) $\langle A_\gamma : \gamma \in \omega_1 \setminus \mathrm{Lim} \rangle$ is $\subseteq$-increasing.
(2) For each $\gamma \in \omega_1 \setminus \mathrm{Lim}$. A_γ has a common extension in $\mathbb{P}$.
(3) For each $\gamma < \omega_1$, it holds that $\bigwedge A_{\gamma+1} \leq_{\mathcal{B}(\mathbb{P})} b_\gamma$.
(4) For each $\gamma < \omega_1$, b_γ is a common extension of A_γ (where for limits we define $A_\gamma = \bigcup \{A_\xi : \xi \in \gamma \setminus \mathrm{Lim}\}$).

Definition 4.18. A function from at most countable subsets of $\mathbb{P}$ to $\mathbb{P}$ is called a $(*)$-*tactic*. We say that Player II *plays according to a* $(*)$-*tactic* τ if for $\delta < \omega_1$, she chooses $\tau(A_\delta)$ as her δ^{th} move as long as it is legal. We say that a $(*)$-tactic τ is *winning* if Player II wins all games where she plays according to τ. We say that $\mathbb{P}$ is $(*)$-*tactically closed* if there is a winning $(*)$-tactic for $G^*(\mathbb{P})$.[5]

Fact 4.19 (Yoshinobu)**.** $(*)$-*tactically closed forcings preserve* PFA [Yos17].

Proposition 4.20. *If λ is a singular cardinal of countable cofinality and $\prod_{n<\omega} \lambda_n$ is a product on λ, then $\mathbb{G}(\prod_{n<\omega} \lambda_n)$ is $(*)$-tactically closed.*

[5]Reduced from the source material, in which Yoshinobu also defines $(*)$-operations.

Corollary 4.21. PFA *is compatible with the statement that for a singular of countable cofinality λ, all scales on λ are good.*

Proof of Proposition 4.20. We will define a play $\langle A_\xi, p_\xi : \xi < \omega_1\rangle$ in Yoshinobu's game where $\max\operatorname{dom} p_\xi = \gamma_\xi$.

We will show that Player II has a winning $(*)$-tactic. In particular, Player II will play in such a way such that if $\xi < \xi' < \omega_1$ are limit ordinals then $p_\xi(\gamma_\xi) < p_{\xi'}(\gamma_{\xi'})$ (where this indicates *everywhere* domination). We will also concurrently define a sequence of $h'_\xi s \in \prod_{n<\omega} \lambda_n$ where ξ's are steps where Player II plays.

If $\xi = \eta + 1$, let $\tau(A_\xi) = p_\xi = p_\eta \cup \langle \gamma_\eta, h_\xi\rangle$ where $h_\xi(i) = p_\eta(\gamma_\eta)(n) + 1$ for all $n < \omega$.

If ξ is a limit, consider A_ξ. Since the A_ξ's have a common extension, $\bigcup_{p\in A_\xi} p$ is a condition (modulo a maximum). Let $h_\xi(n) = \sup_{p\in A_\xi} p(\max\operatorname{dom} p)(n) + 1$. Let

$$p_\xi = \langle \sup_{p\in A_\xi}(\max\operatorname{dom} p), h_\xi\rangle \cup \bigcup_{p\in A_\xi} p.$$

Then p_ξ is a condition because $\operatorname{cf}(\max\operatorname{dom} p_\xi) = \omega$.

Now we have defined $\langle A_\xi, p_\xi : \xi < \omega_1\rangle$. Let

$$h_{\omega_1}(n) = \sup_{p\in\bigcup_{\xi<\omega_1} A_\xi} p(\max\operatorname{dom} p)(n)$$

and let

$$p_{\omega_1} = \left\langle \sup_{p\in\bigcup_{\xi<\omega_1} A_\xi}(\max\operatorname{dom} p), h_{\omega_1}\right\rangle \cup \bigcup_{p\in\bigcup_{\xi<\omega_1} A_\xi} p.$$

Then since the h_ξ's are everywhere increasing, it follows by Fact 1.7 that h_{ω_1} is an exact upper bound of the sequence of functions defined by the run of the game. Therefore p_{ω_1} can be played as Player II's ω_1'th move. $\square$

Remark 4.22. We could have redefined $\mathbb{G}(\prod_{n<\omega}\lambda_n)$ slightly to show that that PFA is consistent with the existence of a *very good scale* on the product $\prod_{n<\omega}\lambda_n$.

Comments on the Literature. We would like to respectfully comment on some unresolved issues in the literature that are relevant to our work.

Part of our motivation came from the introduction of the paper of Cummings, Foreman, and Magidor in which the obtained the quasi-compactness of squares for $\aleph_\omega$. They had set out to investigate, as they put it, "the problem of the relationship between the sets of good and approachable points" [CFM04, Page 2]. Their framework was the notion of *canonical structure*, which refers to the infinitary objects whose definitions are essentially independent of any particular choices made in defining other objects [CFM04, CFM06]. (In connection with approachability, see also [FM97, BDJ+00].)

Some particulars of the model for Theorem 1.2 were fashioned after a statement of Cummings et al. from their first canonical structures paper. The statement essentially went as follows [CFM04, Example 6.7]: Assume that $\aleph_\omega$ is a strong limit and that $2^{\aleph_\omega} = \aleph_{\omega+1}$. Assume also that $\vec{f} = \langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$ is a continuous scale. Suppose that $S \in I[\aleph_{\omega+1}\cap\operatorname{cof}(\omega_1)]$. Then there is a club $C \subseteq \aleph_{\omega+1}$ such that if $N \prec H(\aleph_{\omega+2})$ has cardinality $\aleph_1$ and uniform cofinality ω_1, $\sup(N\cap\aleph_{\omega+1}) = \gamma \in C\cap S$, and $\chi_N =^* f_{\sup(N\cap\aleph_{\omega+1})}$, then N is internally approachable. The authors provided a sketch of the argument that did not appear require even internal unboundedness.

It seemed that the conclusion was supposed to indicate something like sup-internal approachability.

However, Hannes Jakob pointed out to us that if there are stationarily many N that are internally unbounded but not internally approachable (which is consistent [Kru07]), then this stands as a counterexample to Example 6.7, which therefore cannot be literally true. To see this, observe that internally unbounded models of cardinality $\aleph_1$ are tight in $\prod_{n<\omega} \aleph_n$ and $\aleph_1$-uniform. It is then implied that the weak approachability ideal for $\aleph_{\omega+1}$ (see e.g. [Eis10]) is distinct from the approachability ideal, which is a contradiction if $\aleph_\omega$ is a strong limit.

Particularly in this context of these considerations, it is also natural to ask whether we can obtain the conclusion of Theorem 1.2 together with CH. A reasonable approach would involve a version of the iteration presented above in Subsection 3.1 that does not add reals to models of CH. The problem of iterating Namba forcing without adding reals had been considered for a long time before being solved independently by Jensen, using the notion of subcomplete forcing [Jen14], and Shelah, using the notion of the $\mathbb{I}$-condition [She17, Chapter X]. The trouble is in the difference between Laver-style and Miller-style Namba forcings, which are to some extent incompatible in iterations, as is demonstrated by a theorem of Magidor and Shelah [She17, Claim 4.2, Chapter XI]. The $\mathbb{I}$-condition of Shelah applies to the Miller version, but our methods for Theorem 1.2 use the Laver version. Shelah did in fact announce results for the Laver version of the $\mathbb{I}$-condition ([She17, Remark XV.4.16A, Part 2], referencing Shelah # 311), but the paper never appeared.

Acknowledgments. We thank Hannes Jakob for providing the counterexample mentioned above, and we thank James Cummings for some helpful correspondence.

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