ON DISJOINT STATIONARY SEQUENCES

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Abstract. We answer a question of Krueger by obtaining disjoint stationary sequences on successive cardinals. The main idea is an alternative presentation of a mixed support iteration, using it even more explicitly as a variant of Mitchell forcing. We also use a Mahlo cardinal to obtain a model in which $\aleph_2 \notin I[\aleph_2]$ and there is no disjoint stationary sequence on $\aleph_2$, answering a question of Gilton.

1. Introduction and Background

In order to develop a more vivid picture of the infinite cardinals, set theorists study a variety of objects that can potentially exist on these cardinals. The objects of interest for this paper are called *disjoint stationary sequences*. These were introduced by Krueger to answer a question of Abraham and Shelah about forcing clubs through stationary sets. Beginning in joint work with Friedman, Krueger wrote a series of papers in this area, connecting a wide range of concepts and answering seemingly unrelated questions of Foreman and Todorčević [4, 11, 12, 13, 14, 15]. Generally, the new arguments hinged on the behavior of two-step iterations of the form $\mathrm{Add}(\tau) * \mathbb{P}$.

In order to extend the application of these arguments as widely as possible, Krueger developed the notion of mixed support forcing [12, 15]. These forcings are to some extent an analog of the forcing that Mitchell used to obtain the tree property at double successors of regular cardinals. Their most notable feature is the appearance of quotients insofar as the forcings took the form $\mathbb{M} \simeq \bar{\mathbb{M}} * \mathrm{Add}(\tau) * \mathbb{Q}$ where $\bar{\mathbb{M}}$ is a partial mixed support iteration. The appearance of $\mathrm{Add}(\tau)$ after the initial component, together with the preservation properties of the quotient $\mathbb{Q}$, allowed Krueger's new arguments to go through various complicated constructions. Mixed support iterations have found several applications since [5], particularly in regard to guessing models [16].

The main idea in this paper is to use a version of Mitchell forcing to accomplish the task of a mixed support iteration. Specifically, this version of Mitchell forcing takes the form $\mathbb{M} \simeq \bar{\mathbb{M}} * \mathrm{Add}(\tau) * \mathbb{Q}$.[1] The trick used to obtain this structural property is reminiscent of the one usd by Cummings et al. in "The Eightfold Way" to demonstrate that subtle variations in the definitions of Mitchell forcing—up to merely shifting a Lévy collapse by a single coordinate—can substantially alter the properties of the forcing extension. The benefit of the forcing used here is that it comes with a projection analysis of the sort that Abraham used for Mitchell forcing [1]. Both the forcing itself and its quotients are projections of products of the form

[1]The extent to which all variations of these forcings are equivalent or not is left as a loose end. Here we only deal with the case where the two-step iteration $\mathrm{Add}(\tau) * \mathbb{P}$ takes the form $\mathrm{Add}(\tau) * \dot{\mathrm{Col}}(\mu, \delta)$.

$\mathbb{A} \times \mathbb{T}$ where $\mathbb{A}$ has a good chain condition and $\mathbb{T}$ has a good closure property. This allows us to obtain preservation properties conveniently, without having to delve into too many technical details. Abraham in fact used this projection analysis to extend Mitchell's result to successive cardinals. This is exactly what we do here for disjoint stationary sequences, answering the first component of a question of Krueger [15, Question 12.8]:

Theorem 1. *Suppose $\lambda_1 < \lambda_2$ are two Mahlo cardinals in V. Then there is a forcing extension in which there are disjoint stationary sequences on $\aleph_2$ and $\aleph_3$.*

We lay out the basic definition and concepts in the following subsections and then develop the proof in Section 2. We also achieve one of Krueger's separations for successive cardinals, which answers a component of another one of his questions [15, Question 12.9]:

Theorem 2. *Suppose $\lambda_1 < \lambda_2$ are two Mahlo cardinals in V. Then there is a forcing extension in which for $\mu \in \{\aleph_1, \aleph_2\}$, there are stationarily many $N \in [H(\mu^+)]^\mu$ that are internally stationary but not internally club.*

The last main result is motivated by work of Gilton and Krueger, who answered a question from "The Eightfold Way" by obtaining stationary reflection for subsets of $\aleph_2 \cap \mathrm{cof}(\omega)$ together with failure of approachability at $\aleph_2$ (i.e. $\aleph_2 \notin I[\aleph_2]$) using disjoint stationary sequences [5]. This result used the fact that the existence of a disjoint stationary sequence implies failure of approachability. Gilton asked for the exact consistency strength of the failure of approachability at $\aleph_2$ together with the nonexistence of a disjoint stationary sequence on $\aleph_2$ [7, Question 9.0.15]. (He pointed out that Cox found this separation using PFA [2].) It is known that the failure of approachability requires the consistency strength of a Mahlo cardinal, and in Section 3 we show that a Mahlo cardinal is sufficient for the separation:

Theorem 3. *Suppose that λ is Mahlo in V. Then there is a forcing extension in which $\aleph_2 \notin I[\aleph_2]$ and there is no disjoint stationary sequence on $\aleph_2$.*

Disjoint stationary sequences are known to be interpretable in terms of canonical structure (see Fact 6 below), and the main idea for Theorem 3 is a simple master condition argument that exploits this connection.

We note that all three of these theorems can be generalized to arbitrarily high cardinals.

1.1. **Basic Definitions.** We assume familiarity with the basics of forcing and large cardinals. We use the following conventions: If $\mathbb{P}$ is a forcing poset, then $p \le q$ for $p, q \in \mathbb{P}$ means that p is stronger than q. We say that $\mathbb{P}$ is κ-closed if for all $\le_{\mathbb{P}}$-decreasing sequences $\langle p_\xi : \xi < \tau \rangle$ with $\tau < \kappa$, there is a lower bound p, i.e. $p \le p_\xi$ for all $\xi < \tau$. We say that $\mathbb{P}$ has the κ-chain condition of all antichains $A \subseteq \mathbb{P}$ have cardinality strictly less than κ.

Now we give our main definitions:

Definition 4. Given a regular cardinal μ, a *disjoint stationary sequence* on μ^+ is a sequence $\langle \mathcal{S}_\alpha : \alpha \in S \rangle$ such that:

- $S \subseteq \mu^+ \cap \mathrm{cof}(\mu)$ is stationary,
- $\mathcal{S}_\alpha$ is a stationary subset of $P_\mu(\alpha)$ for all $\alpha \in S$,
- $\mathcal{S}_\alpha \cap \mathcal{S}_\beta = \emptyset$ if $\alpha \neq \beta$.

We write $\mathsf{DSS}(\mu^+)$ to say that there is a disjoint stationary sequence on μ^+.

Definition 5. Given a stationary $N \in [H(\Theta)]^\kappa$,[2] we say:

- N is *internally unbounded* if $\forall x \in P_\kappa(N), \exists M \in N, x \subseteq M$,
- N is *internally stationary* if $P_\kappa(N) \cap N$ is stationary in $P_\kappa(N)$,
- N is *internally club* if $P_\kappa(N) \cap N$ is club in $P_\kappa(N)$,
- N is *internally approachable* if there is an increasing and continuous continuous chain $\langle M_\xi : \xi < \kappa \rangle$ such that $|M_\xi| < \kappa$ and $\langle M_\eta : \eta < \xi \rangle \in M_{\xi+1}$ for all $\xi < \kappa$ such that $N = \bigcup_{\xi<\kappa} M_\xi$.

Although disjoint stationary sequences may seem unrelated the separation of variants of internal approachability, there are deep connections here, for example:

Fact 6 (Krueger, [15])**.** *If μ is regular and $2^\mu = \mu^+$, then $\mathsf{DSS}(\mu^+)$ is equivalent to the existence of a stationary set $U \subseteq [H(\mu^+)]^\mu$ such that every $N \in U$ is internally internally unbounded but not internally club.*

1.2. **Projections and Preservation Lemmas.** Technically speaking, our main goal is to show that certain forcing quotients behave nicely. We will make an effort to demonstrate the preservation properties of these quotients directly. These quotients will be defined in terms of projections:

Definition 7. If $\mathbb{P}_1$ and $\mathbb{P}_2$ are posets, a *projection* is an onto map $\pi : \mathbb{P}_1 \to \mathbb{P}_2$ such that:

- $p \leq q$ implies that $\pi(p) \leq \pi(q)$,
- if $r \leq \pi(p)$, then there is some $q \leq p$ such that $\pi(q) \leq r$.

A projection is *trivial* if $\pi(p) = \pi(q)$ implies that p and q are compatible.

Trivial projections are basically ismorphisms:

Fact 8. *If $\pi : \mathbb{P}_1 \to \mathbb{P}_2$ is a trivial projection, then $\mathbb{P}_1 \simeq \mathbb{P}_2$.*

For our purposes, we are interested in the preservation of stationary sets. The chain condition gives us preservation fairly straightforwardly. The following fact is implicit in parts of the literature, and a version of it can be found in this paper in the form of Proposition 26.

Fact 9. *If $\mathbb{P}$ has the μ-chain condition and $S \subset P_\mu(X)$ is stationary, then $\mathbb{P}$ forces that S is stationary in $P_\mu(X)$.*

However, we must place demands on our stationary sets in order for them to be preserved by closed forcings.

Definition 10. A stationary set $S \subset P_\mu(H(\Theta))$ is *internally approachable of length* τ if for all $N \in S$ with $N \prec H(\Theta)$, there is a continuous chain of elementary submodels $\langle M_i : i < \tau \rangle$ such that $N = \bigcup_{i<\tau} M_i$ and for all $i < \tau$, $\langle M_i : i < j \rangle \in M_{j+1}$. In this case we write $S \subseteq \mathcal{IA}(\tau)$.

Fact 11. *If $S \subset P_\mu(H(\Theta)) \cap \mathcal{IA}(\tau)$ is an internally approachable stationary set, $\tau < \mu$, and $\mathbb{P}$ is μ-closed, then $\mathbb{P}$ forces that S is stationary in $P_\mu(H(\Theta)^V)$.*

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[2]See Jech for details on stationary sets [10].

1.3. **Costationarity of the Ground Model.** The notion of ground model costationarity is a key ingredient in arguments pertaining to disjoint stationary sequences. It will specifically give us the disjointness, since we will be picking stationary sets that are not added by initial segments of these forcings.

Gitik obtained the classical result:

Fact 12 (Gitik [8]). *If $V \subset W$ are models of ZFC with the same ordinals, $W \setminus V$ contains a real, and κ is a regular cardinal in W such that $(\kappa^+)^W \leq \lambda$, then $P_\kappa^W(\lambda) \setminus V$ is stationary.*

Because we will need Fact 11, we will actually use Krueger's refinement of Gitik's theorem:

Fact 13 (Krueger [15]). *Suppose $V \subset W$ are models of ZFC with the same ordinals, $W \setminus V$ contains a real, μ is a regular cardinal in W, and $X \in V$ is such that $(\mu^+)^W \subseteq X$, and that in W, Θ is a regular cardinal such that $X \subset H(\Theta)$. Then in W the set $\{N \in P_\mu(H(\Theta)) \cap \mathcal{IA}(\omega) : N \cap X \notin V\}$ is stationary.*

2. The New Mitchell Forcing

2.1. **Defining the Forcing.** In this subsection we will illustrate the basic idea of this paper by using our new take on Mitchell forcing to prove a known result:

Theorem 14 (Krueger [15]). *If λ is a Mahlo cardinal and $\mu < \lambda$ are regular cardinals, there is a forcing extension in which $2^\omega = \mu^+ = \lambda$ and there is a disjoint stationary sequence on λ.*

Specifically, we will define a forcing $\mathbb{M}^+(\tau, \mu, \lambda)$ such that the model W in Theorem 14 can be realized as an extension by $\mathbb{M}^+(\omega, \mu, \lambda)$.

For standard technical reasons, we define a poset ismorphic to $\text{Add}(\tau, \lambda)$:

Definition 15. Given a regular τ and a set of ordinals Y, we let $\text{Add}^*(\tau, Y)$ be the poset consisting of partial functions $p : \{\delta \in Y : \delta \text{ is inaccessible}\} \times \tau \to \{0, 1\}$ where $|\operatorname{dom} p| < \tau$. We let $p \leq_{\text{Add}^*(\tau,Y)} q$ if and only if $p \supseteq q$.

Note: In later subsections we will conflate $\text{Add}(\tau, \lambda)$ and $\text{Add}^*(\tau, \lambda)$ to simplify notation.

Definition 16. Let λ be inaccessible and let $\tau < \mu < \lambda$ be regular cardinals such that $\tau^{<\tau} = \tau$. We define a forcing $\mathbb{M}^+(\tau, \mu, \lambda)$ that consists of pairs (p, q) such that:

(1) $p \in \text{Add}^*(\tau, \lambda)$,
(2) q is a function such that:
  (a) $\operatorname{dom} q \subset \{\delta < \lambda : \delta \text{ is inaccessible}\}$,
  (b) $|\operatorname{dom} q| < \mu$,
  (c) $\forall \delta \in \operatorname{dom}(q), p \upharpoonright ((\delta + 1) \times \tau) \Vdash_{\text{Add}^*(\tau,\delta+1)}$ "$q(\delta) \in \dot{\text{Col}}(\mu, \delta)$".

We let $(p, q) \leq (p', q')$ if and only if:

(i) $p \leq_{\text{Add}^*(\tau,\lambda)} p'$,
(ii) $\operatorname{dom} q \supseteq \operatorname{dom} q'$,
(iii) for all $\delta \in \operatorname{dom} q', p \upharpoonright ((\delta + 1) \times \tau) \Vdash_{\text{Add}^*(\tau,\delta+1)}$ "$q(\delta) \leq_{\dot{\text{Col}}(\mu,\delta)} q'(\delta)$"

First we go through the more routine properties that one would expect of this forcing.

Proposition 17. *$\mathbb{M}^+(\tau,\mu,\lambda)$ is τ-closed and λ-Knaster.*

Proof. Closure uses the facts that $\mathrm{Add}^*(\tau,\lambda)$ is τ-closed and $\Vdash_{\mathrm{Add}^*(\tau,\delta+1)}$ "$\dot{\mathrm{Col}}(\mu,\delta)$ is μ-closed" for all δ. Knasterness uses a standard application of the Delta System Lemma. □

Crucially, we get a nice termspace:

Definition 18. Let $\mathbb{T} = \mathbb{T}(\mathbb{M}^+(\tau,\mu,\lambda))$ be the poset consisting of conditions q such that:

(1) $\operatorname{dom} q \subset \lambda \cap \{\delta < \lambda : \delta \text{ is inaccessible}\}$,
(2) $|\operatorname{dom} q| < \mu$,
(3) $\forall \delta \in \operatorname{dom} q, \Vdash_{\mathrm{Add}^*(\tau,\delta+1)} q(\delta) \in \dot{\mathrm{Col}}(\mu,\delta)$".

Most importantly, we let $q \le q'$ if and only if:

(i) $\operatorname{dom} q \supseteq \operatorname{dom} q'$,
(ii) for all $\delta \in \operatorname{dom} q$, $\Vdash_{\mathrm{Add}^*(\tau,\delta+1)}$ "$q(\delta) \le q'(\delta)$".

Proposition 19. *There is a projection* $\mathrm{Add}^*(\tau,\lambda) \times \mathbb{T}(\mathbb{M}^+(\tau,\mu,\lambda)) \twoheadrightarrow \mathbb{M}^+(\tau,\mu,\lambda)$.

Proof. We let π be the projection with the definition $\pi(p,q) = (p,q)$. This is automatically order-preserving because the ordering $\le_{\mathrm{Add}^*(\tau,\lambda)\times\mathbb{T}}$ is coarser than the ordering $\le_{\mathbb{M}^+(\tau,\mu,\lambda)}$. For obtaining the density condition, suppose $(r,s) \le_{\mathbb{M}^+(\tau,\mu,\lambda)} (p_0,q_0)$. We want to find some (p_1,q_1) such that $(p_1,q_1) \le_{\mathrm{Add}^*(\tau,\lambda)\times\mathbb{T}} (p_0,q_0)$ and $(p_1,q_1) \le_{\mathbb{M}^+(\tau,\mu,\lambda)} (r,s)$. To do this, we first let $p_1 = r$, and then we define q_1 with $\operatorname{dom} q_1 = \operatorname{dom} r$ such that at each coordinate $\delta \in \operatorname{dom} q_1$, we use standard arguments on names to show that we can get both $p_0 \restriction ((\delta+1)\times\tau) \Vdash_{\mathrm{Add}^*(\tau,\lambda)}$ "$q_1(\delta) \le s(\delta)$" as well as $1_{\mathrm{Add}^*(\tau,\lambda)} \Vdash_{\mathrm{Add}^*(\tau,\lambda)}$ "$q_1(\delta) \le q_0(\delta)$". □

Proposition 20. *$\mathbb{T}$ is μ-closed.*

Proof. This as an application of the Mixing Principle. Given a $\le_{\mathbb{T}}$-decreasing sequence $\langle q_i : i < \tau \rangle$ with $\tau < \mu$ we let $d = \bigcup_{i<\tau} \operatorname{dom} q_i$. Then we define a lower bound $\bar{q}$ with domain d such that for all $\delta \in d$, $q(\delta)$ is a canonically-defined name for a lower bound of the $q_i(\delta)$'s (where i is large enough that $\delta \in \operatorname{dom} q_i$). □

Then we get the standard consequences of the termspace analysis:

Proposition 21. *The following are true in any extension by $\mathbb{M}^+(\tau,\mu,\lambda)$:*

(1) V-cardinals up to and including μ are cardinals.
(2) For all $\alpha < \lambda$, $|\alpha| = \mu$.
(3) $\lambda = \mu^+$.
(4) $2^\tau = \lambda$.

Proof. *(1)* follows from the projection analysis and the fact that $\mathbb{T}$ is μ-closed and $\mathrm{Add}^*(\tau,\lambda)$ is τ^+-cc, and from τ-closure of $\mathbb{M}^+(\tau,\mu,\lambda)$. *(2)* follows from the fact that for all inaccessible $\delta < \lambda$, $\mathbb{M}^+(\tau,\mu,\lambda)$ projects onto $\mathrm{Col}(\mu,\delta)$. *(3)* follows from *(1)* and *(2)* plus λ-Knasterness. *(4)* follows from the fact that $\mathbb{M}^+(\mu,\lambda)$ projects onto $\mathrm{Add}^*(\tau,\lambda)$, so it forces that $2^\tau \ge \lambda$. Since the poset has size λ, it also forces that $2^\tau \le \lambda$. □

The following lemma is the crux of the new idea.

Lemma 22. *If $\delta_0 < \lambda$ is inaccessible, then there is a forcing equivalence*

$$\mathbb{M}^+(\tau, \mu, \lambda) \simeq \mathbb{M}^+(\tau, \mu, \delta_0) * \mathrm{Add}(\tau) * \mathcal{Q}$$

where $\mathbb{M}^+(\tau, \mu, \delta_0) * \mathrm{Add}(\tau)$ forces that $\mathcal{Q}$ is a projection of a product of a μ-closed forcing and a τ^+-cc forcing.

Proof. More precisely, we will show that there is a forcing equivalence $\mathbb{M}^+(\tau, \mu, \lambda) \simeq \mathbb{M}^+(\tau, \mu, \delta_0) * \mathrm{Add}(\tau) * (\mathcal{P} \times \mathcal{R})$ where the following hold in the extension by $\mathbb{M}^+(\tau, \mu, \delta_0) * \mathrm{Add}(\tau)$:

- $\mathcal{R}$ is a projection of a product of a μ-closed forcing and $\mathrm{Add}^*(\tau, \lambda)$, and
- $V[\mathbb{M}^+(\tau, \mu, \delta_0)][\mathrm{Add}(\tau, 1)] \models$ "$\mathcal{P}$ is μ-closed".

The statement of the lemma can then be obtained by merging $\mathcal{P}$ with the closed component of the product that projects onto $\mathcal{R}$.

First we describe $\mathcal{P}$ and $\mathcal{R}$. To do this, we fix some notation. Given $Y \subseteq \lambda$, we let π^Y_{Add} denote the projection $(p, q) \to p \restriction (Y \times \tau)$ from $\mathbb{M}^+(\tau, \mu, \lambda)$ onto $\mathrm{Add}^*(\tau, Y)$. For any poset $\mathbb{P}$, we employ the convention that $\Gamma(\mathbb{P})$ denotes a canonical name for a $\mathbb{P}$-generic. If $X \subset \mathbb{P}$, then we use the notation $\uparrow X := \{q \in \mathbb{P} : \exists p \in X, p \leq q\}$.

We will let

$$\mathcal{P} := \mathrm{Col}(\mu, \delta_0)^{V[(\uparrow(\pi^{\delta_0}_{\mathrm{Add}}{}''\Gamma(\mathbb{M}^+(\tau,\mu,\delta_0)))) \times \Gamma(\mathrm{Add}(\tau))]}$$

if we are working in an extension by $\mathbb{M}^+(\tau, \mu, \delta_0) * \mathrm{Add}(\tau)$. (In other words, the poset $\mathcal{P}$ will be the version of $\mathrm{Col}(\mu, \delta_0)$ as interpreted in the extension of V by $\mathrm{Add}^*(\tau, \delta + 1)$ where the initial coordinates come from $\mathbb{M}^+(\tau, \mu, \delta_0)$ and the last coordinate comes from the additional copy of $\mathrm{Add}(\tau)$.)

Still working in an extension by $\mathbb{M}^+(\tau, \mu, \delta_0) * \mathrm{Add}(\tau)$, the poset $\mathcal{R}$ consists of pairs (p, q) such that the following hold:

(1) $p \in \mathrm{Add}^*(\tau, (\delta_0, \lambda))$,
(2) q is a function such that
   (a) $\mathrm{dom}\, q \subset \{\delta \in (\delta_0, \lambda) : \delta \text{ is inaccessible}\}$,
   (b) $|\mathrm{dom}\, q| < \mu$,
   (c) $\forall \delta \in \mathrm{dom}(q), p \restriction ((\delta_0, (\delta + 1)) \times \tau) \Vdash_{\mathrm{Add}^*(\tau, (\delta_0, \delta+1))}$ "$q(\delta) \in \dot{\mathrm{Col}}(\mu, \delta)$".

The ordering is the one analogous to that of $\mathbb{M}^+(\tau, \mu, \lambda)$. An easy adaptation of the arguments for the projection analysis for $\mathbb{M}^+(\tau, \mu, \lambda)$ will then give a projection analysis for $\mathcal{R}$.

The rest of the proof of the lemma consists of verifying the more substantial claims.

Claim 23. $\mathbb{M}^+(\tau, \mu, \lambda) \simeq \mathbb{M}^+(\tau, \mu, \delta_0) * \mathrm{Add}(\tau, 1) * (\mathcal{P} \times \mathcal{R})$.

Proof. We identify $\mathbb{M}^+(\tau, \mu, \delta_0) * \mathrm{Add}(\tau, 1) * (\mathcal{P} \times \mathcal{R})$ with the dense subset of conditions $((r, s), t, u, (r, \dot{s}'))$ such that $\dot{s}'$ is forced to have a specific domain in V. The fact that this subset is dense follows from the fact that $\mathbb{M}^+(\tau, \mu, \lambda) * \mathrm{Add}(\tau)$ has the μ-covering property.

We will argue that there is a trivial projection defined by

$$\pi : (p, q) \mapsto (\underbrace{(p \restriction (\delta_0 \times \tau), q \restriction \delta_0)}_{\mathbb{M}^+(\mu, \delta_0)}, \underbrace{p \restriction (\{\delta_0\} \times \tau)}_{\mathrm{Add}(\tau)}, \underbrace{q^*(\delta_0)}_{\mathcal{P}}, \underbrace{(\bar{p}, \bar{q})}_{\mathcal{R}})$$

such that

- $\bar{p} := p \restriction ((\delta_0, \lambda) \times \tau)$;

- $q^*(\delta_0)$ is obtained by changing $q(\delta_0)$ from an $\mathrm{Add}^*(\tau, \delta_0+1)$-name to an $\mathrm{Add}(\tau)$-name as interpreted in the extension by the relevant generic, namely $(\uparrow (\pi^{\delta_0}_{\mathrm{Add}}``\Gamma(\mathbb{M}^+(\tau,\mu,\delta_0))))$;
- $\bar{q}$ has domain (δ_0,λ), and for each $\delta \in (\delta_0,\lambda)$, $\bar{q}(\delta)$ has changes analogous to the changes made to $q^*(\delta_0)$.

It is clear that π is order-preserving. We also want to show that if

$$((r,s),t,u,(r',\dot{s}')) \leq_{\mathbb{M}^+(\tau,\mu,\delta_0)*\mathrm{Add}(\tau)*(\mathcal{P}\times\mathcal{R})} \pi(p_0,q_0)$$

then there is some $(p_1,q_1) \leq_{\mathbb{M}^+(\mu,\lambda)} (p_0,q_0)$ such that $\pi(p_1,q_1) \leq ((r,s),t,u,(r',s'))$. This can be done by taking:

- $p_1 = r \cup \tilde{t} \cup r'$ where $\tilde{t}$ writes t as as a partial function $\{\delta\}\times\tau \to \{0,1\}$,
- $q_1 = s \cup \tilde{u} \cup \tilde{s}'$ where $\tilde{u}$ reinterprets u as a $\mathrm{Add}^*(\delta_0+1)$-name and for each $\delta \in \mathrm{dom}(\dot{s}')$, $\tilde{s}'$ reinterprets $\dot{s}'(\delta)$ as a $\mathrm{Add}^*(\delta+1)$-name.

Last, we argue that $\pi(p_0,q_0) = \pi(p_1,q_1)$ implies that (p_0,q_0) and (p_1,q_1) are compatible. Suppose that (p_0,q_0) and (p_1,q_1) are incompatible. If p_0 and p_1 are incompatible as elements of $\mathrm{Add}^*(\tau,\lambda)$, then one of $p_i \restriction (\delta_0 \times \tau)$, $p_i \restriction (\{\delta_0\}\times\tau)$, and $p_i \restriction ((\delta_0,\lambda)\times\tau)$ must be distinct for $i=0$ and $i=1$. Otherwise, there is some $p' \leq p_0, p_1$ and some $\delta \in \mathrm{dom}\, q_0 \cap \mathrm{dom}\, q_1$ inaccessible such that $p' \Vdash$ "$q_0(\delta) \perp q_1(\delta)$", which implies that $q_0(\delta) \neq q_1(\delta)$. Therefore, one of $q_i \restriction \delta_0$, $q_i(\delta_0)$, or $q_i \restriction (\delta_0,\lambda)$ is distinct for $i \in \{0,1\}$. □

Claim 24. *$V[\mathbb{M}^+(\tau,\mu,\delta_0)][\mathrm{Add}(\tau,1)] \models$ "$\mathcal{P}$ is μ-closed".*

Proof. In fact, our argument will also show that $V[\mathbb{M}^+(\tau,\mu,\delta_0)][\mathrm{Add}(\tau,1)] \models$ "$\mathcal{P} = \mathrm{Col}(\mu,\delta_0)$". We fix some arbitrary generics:

- G is $\mathbb{M}^+(\tau,\mu,\delta_0)$-generic over V,
- r is $\mathrm{Add}(\tau)$-generic over $V[G]$,
- H is the $\mathrm{Add}^*(\tau,\delta_0)$-generic induced from G by $\pi^{\delta_0}_{\mathrm{Add}}$,
- K is the generic for the quotient of $\mathbb{M}^+(\tau,\mu,\delta_0)$ by $\mathrm{Add}^*(\tau,\delta_0)$, i.e. the generic such that $V[H][K] = V[G]$,
- T is the generic for the termspace forcing $\mathbb{T}(\mathbb{M}^+(\tau,\mu,\delta_0))$, so that $V[G] \subset V[T][H]$.

It is enough to argue that $V[G][r] \models$ "$\mathcal{P}$ is μ-closed" knowing that $V[H][r] \models$ "$\mathcal{P}$ is μ-closed". Because adjoining G does not change the definition of $\mathrm{Add}(\tau)$, and because K is defined in terms of the subsets of τ adjoined by the filter H, we have $V[G][r] = V[H][K][r] = V[H][r][K]$. Therefore, it is enough to show that K does not add $<\mu$-sequences over $V[H][r]$, so that $V[H][r]$'s version of $\mathrm{Col}(\mu,\delta_0)$ remains μ-closed in $V[G][r]$. We have

$$V[H][r] \subset V[H][r][K] = V[H][K][r] = V[G][r] \subset V[T][H][r] = V[H][r][T],$$

and Easton's Lemma implies that T does not add new $<\mu$-sequences over $V[H][r]$, so therefore K does not add new $<\mu$-sequences over $V[H][r]$ since it is an intermediate factor of the extension. □

This completes the proof of the lemma. □

Now we have an application for the case where $\tau = \omega$.

Proposition 25. *If λ is Mahlo then $V[\mathbb{M}^+(\omega,\mu,\lambda)] \models \mathsf{DSS}(\lambda)$.*

This basically repeats Krueger's argument for [15, Theorem 9.1].

Proof. Let G be $\mathbb{M}^+(\omega,\mu,\lambda)$-generic over V. The set of V-inaccessibles in λ will form the stationary set $S \subset \mu^+ \cap \mathrm{cof}(\mu)$ carrying the disjoint stationary sequence in the extension by $\mathbb{M}^+(\omega,\mu,\lambda)$. For every such $\delta \in S$, let $\bar{G}$ be the generic on $\mathbb{M}^+(\omega,\mu,\delta)$ induced by G and let r be the $\mathrm{Add}(\omega)$-generic induced by G via $\pi^{\{\delta\}}_{\mathrm{Add}}$. We use Fact 13 to obtain a stationary set $\mathcal{S}^*_\delta \subset P_\mu(H(\delta))^{V[\bar{G}][r]}$ such that for all $N \in \mathcal{S}^*_\delta$, $N \cap \delta \notin V[\bar{G}]$ and such that $\mathcal{S}^*_\delta$ is also internally approachable by a ω-sequence. Therefore we can apply Lemma 22 with Fact 11 and then Fact 9 to find that $\mathcal{S}^*_\delta$ is stationary in $V[G]$. We then let $\mathcal{S}_\delta = \{N \cap \delta : N \in \mathcal{S}^*_\delta\}$, and we see that $\langle \mathcal{S}_\delta : \delta \in S\rangle$ is a disjoint stationary sequence. □

2.2. **Proving the Main Theorems.** Now we will apply the new version of Mitchell forcing to answer Krueger's questions. Theorem 1 follows quickly:

Proof of Theorem 1. Begin with a ground model V in which $\lambda_1 < \lambda_2$ and the λ's are Mahlo. Let $\mathbb{M}_1 = \mathbb{M}^+(\omega,\aleph_1,\lambda_1)$. (Any λ_1-sized forcing that turns λ_1 into $\aleph_2$ and adds a disjoint stationary sequence on $\aleph_2$ would work, so we could also use a more standard mixed support iteration.) Then let $\dot{\mathbb{M}}_2$ be an $\mathbb{M}_1$-name for $\mathbb{M}^+(\omega,\lambda_1,\lambda_2)$. We argue that if G_1 is $\mathbb{M}_1$-generic over V and G_2 is $\dot{\mathbb{M}}_2[G_1]$-generic over $V[G_1]$, then $V[G_1][G_2] \models$ "$\mathsf{DSS}(\lambda_1) \wedge \mathsf{DSS}(\lambda_2)$". We get $\mathsf{DSS}(\lambda_2)$ from the fact that λ_2 remains Mahlo in $V[G_1]$ together with Proposition 25, so we only need to argue that the disjoint stationary sequence $\vec{\mathcal{S}} := \langle \mathcal{S}_\alpha : \alpha \in S\rangle \in V[G_1]$ remains a disjoint stationary sequence in $V[G_1][G_2]$.

Working in $V[G_1]$, preservation of $\vec{\mathcal{S}}$ follows from the projection analysis: Let H_1 and H_2 be chosen so that H_1 is $\mathbb{T} := \mathbb{T}(\mathbb{M}_2)$-generic over $V[G_1]$, H_2 is $\mathrm{Add}(\omega,\lambda_2)^{V[G_1]}$-generic over $V[G_1][H_1]$, and $V[G_1][G_2] \subseteq V[G_1][H_1][H_2]$. Since $\mathbb{T}$ is λ_1-closed, it preserves stationarity of S and the $\mathcal{S}_\alpha$'s, and $\mathrm{Add}(\omega,\lambda_2)^{V[G_1]}$ still has the countable chain condition in $V[G_1][H_1]$. It follows that the stationarity of S is preserved in $V[G_1][H_1][H_2]$, as well as the stationarity of the $\mathcal{S}_\alpha$'s (by Fact 9). Therefore $\vec{\mathcal{S}}$ is a disjoint stationary sequence on λ_1 in $V[G_1][G_2]$. □

It will take a bit more work to show that Theorem 2 holds in the same model given for Theorem 1. Note that we cannot just apply Fact 6 because $2^\omega = \aleph_3$ in the model for Theorem 1, plus it is consistent that there can be a stationary set which is internally unbounded but not internally stationary [13].

We will give some facts on preservation of the distinction between stationary sets that are internally stationary but not internally club:

Proposition 26. *Suppose $\mathbb{P}$ is ν-closed and $S \subseteq P_\delta(X)$ is a stationary set such that $|X|^{<\delta} \le \nu$ and $\delta \le \nu$. Then $\Vdash_{\mathbb{P}}$ "S is stationary in $P_\delta(X)$".*

Proof. Let $\dot{C}$ be a $\mathbb{P}$-name for a club in $P_\delta(X)$. Let $\vec{x} = \langle x_\xi : \xi \le \bar{\nu}\rangle$ be an enumeration of $P_\delta(X)$ (where $\bar{\nu} \le \nu$). We construct a sequence $\vec{z} = \langle z_\xi : \xi \le \bar{\nu}\rangle \subseteq P_\delta(X)$ and a $\le_{\mathbb{P}}$-descending sequence $\langle p_\xi : \xi \le \bar{\nu}\rangle$ such that for all ξ, $p_\xi \Vdash$ "$x_\xi \subseteq z_\xi \in \dot{C}$". Let D be the set of unions $\bigcup_{i<\bar{\delta}} z_{\xi_i}$ for all increasing chains $\langle z_{\xi_i} : i < \bar{\delta}\rangle \subset \vec{z}$ (where $\bar{\delta} < \delta$). Since D is a club in $P_\delta(X)$ defined in V, there is some $w \in D \cap S$. Let $\langle z_{\xi_i} : i < \bar{\delta}\rangle$ be an $\subseteq$-increasing chain with $\bar{\delta} < \delta$ such that $\bigcup_{i<\bar{\delta}} z_{\xi_i} = w$ and let $\xi^* < \bar{\nu}$ be such that $\xi^* > \sup_{i<\bar{\delta}} \xi_i$. Then $p_{\xi^*} \Vdash$ "$w \in \dot{C} \cap S$". □

Proposition 27. *Let $\mathbb{P}_1$ have the δ-chain condition, let $\mathbb{P}_2$ be ν-closed, and let X be a set such that $|X|^\delta \le \nu$ with $\delta^+ \le \nu$. If $S \subseteq [X]^\delta$ is stationary and internally* *stationary but not internally club, then* $\mathbb{P}_1 \times \mathbb{P}_2$ *forces that* S *is stationary and internally stationary but not internally club.*

Proof. First, S remains stationary in the extension by $\mathbb{P}_2$ by Proposition 26, and it remains stationary in the further extension by $\mathbb{P}_1$ by the fact that $\mathbb{P}_1$ still has the δ-chain condition together with Fact 9. If $N \in S$, then $N \cap P_\delta(N)$ is stationary, so its stationarity is preserved by the same reasoning, using the fact that we still have the appropriate chain condition. The fact that N is *not* internally club is preserved in the extension by $\mathbb{P}_2$ because of ν-closure and the fact that $\delta \leq \nu$, and then it is preserved in the further extension by $\mathbb{P}_1$ because the proof of Fact 9 shows that added clubs contain ground model clubs. □

We use a concept from Harrington and Shelah to handle Mahlo cardinals:

Definition 28. [9] Let $\mathcal{N}$ be a model of some fragment of ZFC. We say that $\mathcal{M} \prec \mathcal{N}$ is *rich* if the following hold:

(1) $\lambda \in \mathcal{M}$;
(2) $\bar{\lambda} := \mathcal{M} \cap \lambda \in \lambda$;
(3) $\bar{\lambda}$ is an inaccessible cardinal in $\mathcal{N}$;
(4) The size of $\mathcal{M}$ is $\bar{\lambda}$;
(5) $\mathcal{M}$ is closed under $<\bar{\lambda}$-sequences and $\bar{\lambda} < \lambda$.

Lemma 29. *If* λ *is Mahlo, then* $\mathbb{M}^+(\omega, \mu, \lambda)$ *forces that there are stationarily many* $Z \in [\mu^+]^\mu$ *which are internally stationary but not internally club.*

This follows Krueger's proof of [15, Theorem 10.1], making necessary changes for Mahlo cardinals, and including enough details to show that we can get the necessary preservation of stationarity simply from the projection analysis. We do not need guessing functions (which are used in Krueger's argument) because we are only obtaining one instance of separation per large cardinal.

Proof of Lemma 29. Denote $\mathbb{M} := \mathbb{M}^+(\omega, \mu, \lambda)$ and let $\dot{C}$ be an $\mathbb{M}$-name for a club in $([H(\mu^+)]^\mu)^{V[\mathbb{M}]}$. We want to find an $\mathbb{M}$-name $\dot{Z}$ for an element of $([H(\mu^+)]^\mu)^{V[\mathbb{M}]} \cap \dot{C}$ that is internally stationary but not internally club. Let $\dot{F}$ be an $\mathbb{M}$-name for a function $(H(\mu^+)^{V[\mathbb{M}]})^{<\omega} \to H(\mu^+)^{V[\mathbb{M}]}$ with the property that all of its closure points are in $\dot{C}$. Let Θ be as large as needed for the following discussion and let $\mathcal{N}$ be the structure $(H(\Theta), \in, <_\Theta, \mathbb{M}, \dot{F}, \lambda, \mu)$ where $<_\Theta$ is a well-ordering of $H(\Theta)$.

Since λ is Mahlo, we can find some $\mathcal{M} \prec \mathcal{N}$ with $\mu \subset \mathcal{M}$ that is a rich submodel of cardinality $\bar{\lambda}$. Now set G to be $\mathbb{M}$-generic over V. Note that $H(\lambda)^{V[G]} = H(\lambda)[G]$ because $\mathbb{M}$ has the λ-chain condition and $\mathbb{M} \subset H(\lambda)$. We will argue that $Z := \mathcal{M}[G] \cap H(\lambda)[G]$ is what we are looking for.

Claim 30. $Z \in C := \dot{C}[G]$.

Proof. We have $\bar{\lambda} \leq |Z| \leq |\mathcal{M}| \leq \bar{\lambda}$ and $\bar{\lambda}$ has cardinality μ in $\mathcal{N}[G]$, so $Z \in [H(\lambda)^{V[G]}]^\mu$. If $a_1, \ldots, a_n \in Z$, there are $\mathbb{M}$-names $\dot{b}_1, \ldots, \dot{b}_n \in \mathcal{M} \cap H(\lambda)$ such that $a_i = \dot{b}_i[G]$ for all $1 \leq i \leq n$. By elementarity, $\mathcal{M}$ contains the $<_\Theta$-least maximal antichain $A \subset \mathbb{M}$ of conditions deciding $\dot{F}(\dot{b}_1, \ldots, \dot{b}_n)$. Since $|A| < \lambda$, $|A| \in \mathcal{M} \cap \lambda = \bar{\lambda}$, so it will follow that $A \subset \mathcal{M}$. Therefore if $p \in G \cap A$, then $p \in M$ in particular, so $p \Vdash \dot{F}(\dot{b}_1, \ldots, \dot{b}_n) = \dot{b}_*$ for some $\dot{b}_* \in \mathcal{M} \cap H(\lambda)$ where we automatically get $\dot{b}_* \in H(\bar{\lambda})$, and therefore $F(a_1, \ldots, a_n) = a_* := \dot{b}_*[G] \in \mathcal{M}[G] \cap H(\lambda)[G] = Z$ (where of course $F := \dot{F}[G]$). □

For the rest of the proof let $\bar{G} := \pi_{\mathcal{M}}(G)$ where $\pi_{\mathcal{M}}$ is the Mostowski collapse relative to $\mathcal{M}$. Since $\pi_{\mathcal{M}}(\mathbb{M}) = \mathbb{M}^+(\omega, \mu, \bar{\lambda})$, there is an extension $\pi_{\mathcal{M}} : \mathcal{M}[G] \cong \pi_{\mathcal{M}}(\mathcal{M})[\bar{G}]$. We also denote $h := \pi_{\mathcal{M}}(H(\lambda)[G] \cap \mathcal{M}[G])$. Note that $h^{<\bar{\lambda}} \subset h$ by the facts that $\mathcal{M}$ is rich and $\pi_{\mathcal{M}}(\mathbb{M})$ has the $\bar{\lambda}$-chain condition.

Claim 31. *Z is internally stationary.*

Proof. First, we argue that $S := P_\mu(h)^{\mathcal{N}[\bar{G}]}$ is stationary in $\mathcal{N}[G]$. By Lemma 22, the quotient $\mathbb{M}/\bar{G}$ is a projection of a forcing of the form $\mathbb{A}_1 * (\dot{\mathbb{T}} \times \mathbb{A}_2)$ where $\mathbb{A}_1$ has the countable chain condition, $\dot{\mathbb{T}}$ is an $\mathbb{A}_1$-name for a μ-closed forcing, and $\mathbb{A}_2$ also has the countable chain condition. Let K_1, K_T, and K_2 be respective generics such that $V[G] \subseteq V[\bar{G}][K_1][K_T][K_2]$. Working in $\mathcal{N}[\bar{G}]$, note that $S' \cap \mathcal{IA}(\omega)$ is stationary, and therefore has its stationarity preserved in $V[\bar{G}][K_1]$ by Fact 9.

We must also show that the stationarity of S' will be preserved by countably closed forcings over $\mathcal{N}[\bar{G}][K_1]$. Suppose $\langle M_n : n < \omega \rangle$ witnesses internal approachability of some $N \in S'$ in $V[\bar{G}]$ with respect to the structure $H(\lambda^+)^{V[\bar{G}]}$, and let $M_\omega := \bigcup_{n<\omega} M_n$. Then we can see that $\langle M_n[K_1] : n < \omega \rangle$ is a chain of elementary submodels of $H(\lambda)[\bar{G}][K_1] = H(\lambda)^{V[\bar{G}][K_1]}$. We also have $M_n[K_1] \cap V[\bar{G}] = M$ and $M_\omega[K_1] \cap V[\bar{G}] = M_\omega \in S'$ with $M_\omega[K_1] \prec H(\lambda)^{V[\bar{G}][K_1]}$. If we choose the M_n's to be elementary substructures of $H(\lambda^+)^{V[\bar{G}]}(\in, <^*, \dot{C}, \ldots)$ where $<^*$ is a well-ordering and $\dot{C}$ is a $\mathbb{A}_1 * \dot{\mathbb{T}}$-name for a club, then an argument almost exactly like the one showing that internal approachability is preserved (i.e. the proof of Fact 11) will show that S' is stationary in $\mathcal{N}[\bar{G}][K_1][K_T]$.

Then the extension of $\mathcal{N}[\bar{G}][K_1][K_T][K_2]$ over $\mathcal{N}[\bar{G}][K_1][K_T]$ preserves the stationarity of S' by another application of Fact 9, so we get stationarity in $\mathcal{N}[G]$.

Now that we have established preservation of stationarity of S', we can finish the argument. Since $|h| = \mu$ in $\mathcal{N}[G]$, we can write $h = \bigcup_{i<\mu} x_i$ where $\langle x_i : i < \mu \rangle$ is a continuous and $\subset$-increasing chain of elements of $P_\mu(h)$. The chain is a club in h, so there is a stationary $X \subseteq \mu$ such that $\{x_i : i \in X\} \subseteq T$. For all $i < \mu$, the fact that $|x_i| < \mu$ implies that $x_i \in h$, and so $x_i = \pi_{\mathcal{M}}(y_i)$ for some $y_i \in Z$. Therefore $\langle y_i : i < \mu \rangle$ is $\subset$-increasing and continuous with union Z, and in particular $\langle y_i : i \in X \rangle$ is stationary in Z. □

Claim 32. *Z is* not *internally club.*

Proof. Suppose for contradiction that Z is internally club and hence that there is a $\subset$-increasing and continuous chain $\langle Z_i : i < \mu \rangle \in \mathcal{N}[G]$ with $|Z_i| < \mu$ for all $i < \mu$ and $\bigcup_{i<\mu} Z_i = Z$. So for all $i < \mu$, $Z_i \subset Z$, and so $\langle \pi_{\mathcal{M}}[Z_i] : i < \mu \rangle$ is an $\subset$-increasing and continuous chain with union h. If we let $W_i := \pi_{\mathcal{M}}[Z_i]$ for all $i < \mu$, then the fact that $|W_i| < \mu$ implies that $W_i = \pi_{\mathcal{M}}(Z_i)$. Therefore $\langle W_i : i < \mu \rangle$ is a continuous and $\subset$-increasing chain of sets in $P_\mu(h)$ with union h.

Next we define a set $U \in \mathcal{N}[\bar{G}][r]$ (where r is the generic induced by G from $\pi_{\text{Add}}^{\{\bar{\lambda}\}}$) as

$$\{A \in P_\mu(H(\chi)) \cap \mathcal{IA}(\omega) : A \cap h \notin \mathcal{N}[\bar{G}]\}.$$

We have a real in $\mathcal{N}[\bar{G}][r] \setminus \mathcal{N}[\bar{G}]$ and $(\mu^+)^{\mathcal{N}[\bar{G}][r]} = \lambda \subset H(\lambda)$. Hence we apply Fact 13 to see that U is stationary in $\mathcal{N}[\bar{G}][r]$, and it remains stationary in $\mathcal{N}[G]$ by the preservation properties of the quotient (i.e. Lemma 22 combined with Fact 11 and Fact 9). Therefore in $\mathcal{N}[G]$, $\{A \cap h : A \in U\}$ is stationary in $P_\mu(h)$. Since $\langle W_i : i < \mu \rangle$ is club in h, there is some $i < \mu$ such that $W_i = A \cap h$ for some $A \in U$.

But by definition, $A \cap h \notin \mathcal{N}[\bar{G}]$, and subsets of W_i of cardinality $<\bar{\lambda}$ are in $\mathcal{N}[\bar{G}]$, so this is a contradiction. □

This completes the proof of the lemma. □

Proof of Theorem 2. Let $\mathbb{M}_1$ be any λ_1-sized forcing that turns λ_1 into $\aleph_2$ and adds stationarily many $N \in [H(\aleph_2)]^{\aleph_1}$ that are internally stationary but not internally club. Let $\dot{\mathbb{M}}_2$ be an $\mathbb{M}_1$-name for $\mathbb{M}^+(\omega, \lambda_1, \lambda_2)$, let G_1 be $\mathbb{M}_1$-generic over V, and let G_2 be $\dot{\mathbb{M}}_2[G_1]$-generic over $V[G_1]$. Then we can see that the theorem holds in $V[G_1][G_2]$: the distinction between internally stationary and internally club on $[H(\aleph_2)]^{\aleph_1}$ is preserved in $V[G_1][G_2]$ by Proposition 27, and we get a distinction between internally stationary and internally club for $[H(\aleph_3)]^{\aleph_2}$ by Lemma 29. □

3. A Club Forcing and a Guessing Sequence

3.1. **A review of the tools.** The main idea of the proof of Theorem 3 is to force a club through the complement of a canonical stationary set, which is described as follows:

Fact 33 (Krueger,[15])**.** *Suppose μ is an uncountable regular cardinal and $\mu^{<\mu} \leq \mu^+$. Let $\underline{x} = \langle x_\alpha : \alpha < \mu^+ \rangle$ enumerate $[\mu^+]^{<\mu}$ and let*

$$S(\underline{x}) := \{\alpha \in \mu^+ \cap \operatorname{cof}(\mu) : P_\mu(\alpha) \setminus \langle x_\beta : \beta < \alpha \rangle \text{ is stationary}\}.$$

Then $\mathsf{DSS}(\mu^+)$ holds if and only if $S(\underline{x})$ is stationary.

The natural thing to do is to define the following:

Definition 34. Let μ be an uncountable regular cardinal such that $\mu^{<\mu} = \mu^+$ and let $\underline{x}$ and $S(\underline{x})$ be defined as in Fact 33. Then let $\mathbb{P}(\underline{x})$ be the set of closed bounded subsets p of μ^+ such that $p \cap S(\underline{x}) = \emptyset$. We let $p' \leq p$ if and only if $p' \cap (\max p + 1) = p$.

We will also crucially need a characterization of diamonds. This following appears in joint work with Gilton and Stejskalová [6].

Fact 35. *The following are equivalent:*

1. *λ is Mahlo and $\diamondsuit_\lambda(\mathrm{Reg})$ (where of course $\mathrm{Reg} = \{\tau < \lambda : \tau$ regular$\}$) holds.*
2. *There is a function $\ell : \lambda \to V_\lambda$ such that for every transitive structure $\mathcal{N}$ satisfying a rich fragment of ZFC that is closed under λ^+-sequences in V, the following holds: For every $A \in \mathcal{N}$ with $A \in H(\lambda^+)$ and any $a \subset \mathcal{H}$ with $|a| < \lambda$, there is a rich $\mathcal{M} \prec \mathcal{N}$ with $a \cup \{\ell\} \subset \mathcal{M}$ such that $\ell(\bar{\lambda}) = \pi_{\mathcal{M}}(A)$ (where $\bar{\lambda} = \mathcal{M} \cap \lambda$ and $\pi_{\mathcal{M}}$ is the Mostowski collapse).*[3]

We can always use such an ℓ assuming the consistency of a Mahlo cardinal: If λ is Mahlo in a model V, then it is Mahlo in Gödel's class L where $\diamondsuit_\lambda(S)$ holds for all regular λ and stationary $S \subset \lambda$.

We use a poset that appears in Gilton's thesis [7] and is discussed in the same paper with the guessing sequence [6]. We denote this poset $\mathbb{M}^G_\ell(\kappa, \lambda)$ and black-box its basic properties:

Fact 36. [3, 7] *The following hold for $\mathbb{M}^G_\ell(\kappa, \lambda)$:*

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[3]The original is stated with a different quantification—for all rich structures, there exists a function, not the other way around. However, the proof works with the quantification used here.

- $\mathbb{M}_\ell^G(\kappa,\lambda)$ *has the* λ*-chain condition;*
- $\mathbb{M}_\ell^G(\kappa,\lambda)$ *is* κ*-closed;*
- *If* $\ell(\delta) = \mathbb{P}$ *for some* κ^+*-closed forcing, then we have the forcing equivalence:*

$$\mathbb{M}_\ell^G(\kappa,\lambda) \simeq \mathbb{M}_\ell^G(\kappa,\delta) * (\mathbb{P} \times \mathrm{Add}(\kappa,\delta^\oplus)) * \mathbb{N}_{\delta^\oplus}$$

where:
  - $\alpha^\oplus$ *takes the least inaccessible larger than* α*, and*
  - $\mathbb{N}_{\delta^\oplus}$ *is a projection of a product of a square-*κ^+*-cc and a* κ^+*-closed forcing.*

3.2. **The proof.** Now we prove Theorem 3. Fix κ and λ as in the statement of the theorem and let $\mu = \kappa^+$. We can assume that $\diamondsuit_\lambda(\mathrm{Reg})$ holds, so let ℓ witness Fact 35 and let $\mathbb{M} = \mathbb{M}_\ell^G(\kappa,\lambda)$. We have $V[\mathbb{M}] \models \mu^{<\mu} \le \mu^+$, so we fix an $\mathbb{M}$-name $\underline{\dot{x}}$ of $[\mu^+]^{<\mu}$ in $V[\mathbb{M}]$ as well as a sequence of names $\langle \dot{x}_\alpha : \alpha < \mu^+\rangle$ that canonically represent the elements listed by $\underline{\dot{x}}$. Then let $\dot{\mathbb{P}}$ be an $\mathbb{M}$-name for $\mathbb{P}(\underline{\dot{x}})$. Let G be $\mathbb{M}$-generic over V and let H be $\mathbb{P} := \dot{\mathbb{P}}[G]$-generic over $V[G]$. Then the model in which the theorem is realized is $V[G][H]$.

Note: If $\mathcal{M} \prec \mathcal{N}$ is rich and $\pi_\mathcal{M}$ is the Mostowski collapse relative to $\mathcal{M}$, we will typically denote $\pi_\mathcal{M}(a)$ as $\bar{a}$.

The following lemma is the crux of the proof:

Lemma 37. *Let* $\mathcal{M} \prec \mathcal{N}$ *be a rich model chosen to witness Fact 35 in the sense of having the properties that* $\mathcal{M} \cap \lambda = \bar{\lambda}$ *and* $\ell(\bar{\lambda}) = \pi_\mathcal{M}(\dot{\mathbb{P}}(\underline{\dot{x}}))$*. Suppose* $\bar{G}_0 * \bar{H}_0$ *is* $\bar{\mathbb{M}} * \bar{\mathbb{P}}$*-generic over* V*.*

Then there is a $G_0 * H_0$ *which is* $\mathbb{M} * \mathbb{P}(\underline{\dot{x}})$*-generic over* V *and a rich* $\mathcal{M} \prec \mathcal{N}$ *such that:*

1. *if* $j : \bar{\mathcal{M}} \to \mathcal{M} \subset \mathcal{N}$ *is the inverse of the Mostowski collapse, then there is a lift* $j : \bar{\mathcal{M}}[\bar{G}_0][\bar{H}_0] \to \mathcal{N}[G_0][H_0]$*;*
2. $\bar{\mathcal{M}}[\bar{G}_0][\bar{H}_0]^{<\bar{\lambda}} \subseteq \bar{\mathcal{M}}[\bar{G}_0]$*;*
3. $\mathcal{N}[G_0]$ *is an extension of* $\mathcal{N}[\bar{G}_0][\bar{H}_0]$ *by* $\mathrm{Add}(\kappa,(\mathcal{M}\cap\lambda)^\oplus) * \mathbb{N}_{(\mathcal{M}\cap\lambda)^\oplus}$*.*

Proof. We will lift the elementary embedding $j : \bar{\mathcal{M}} \to \mathcal{N}$ to $j : \bar{\mathcal{M}}[\bar{G}_0][\bar{H}_0] \to \mathcal{N}[G_0][H_0]$. We therefore fix the notation $\bar{\lambda} = \mathcal{M} \cap \lambda$, and we have an $\bar{\mathbb{M}}$-generic $\bar{G}_0$, so we let $\mathbb{P} = \dot{\mathbb{P}}(\underline{\dot{x}})[G_0]$.

To perform the lift, we need to show that we can absorb the generic $\bar{H}_0$. We can see that $\mathcal{N}[\bar{G}_0] \models$ "$\pi_\mathcal{M}(\mathbb{P})$ is $\bar{\lambda}$-closed", which follows from the fact that $\bar{\mathbb{M}}$ has the $\bar{\lambda}$-chain condition. By the guessing property of ℓ we have a forcing equivalence $\mathbb{M}/G_0 \simeq (\bar{\mathbb{P}} \times \mathrm{Add}(\kappa,\bar{\lambda}^\oplus)) * \dot{\mathbb{N}}_{\bar{\lambda}^\oplus}$, giving us *(3)*.

The first stage of the lift $j : \bar{\mathcal{M}}[\bar{G}_0] \to \mathcal{N}[G_0]$ works by choosing a generic G' over $\mathbb{M}/\bar{G}_0$ such that G' projects to $\bar{H}_0$. Then we let $G_0 = \bar{G}_0 \times G'$ and we see that $j``\bar{G}_0 \subseteq G_0$.

To lift the embedding further, we use a master condition argument. Specifically, we want to show that $\cup\bar{H}_0 \cup \{\bar{\lambda}\}$ is a condition in $\mathbb{P}$. This follows because $\bar{\lambda} \notin S(\underline{x})$ (as evaluated in $\mathcal{N}[G_0]$) because $\bar{\mathcal{M}}[\bar{G}_0]^{<\bar{\lambda}} \subset \bar{\mathcal{M}}[\bar{G}_0]$ and therefore $P_\mu(\bar{\lambda}) \setminus \langle x_\beta : \beta < \bar{\lambda}\rangle$ will be empty, so of course it will be nonstationary. Hence we choose H_0 to be a generic containing $\cup\bar{H}_0 \cup \{\bar{\lambda}\}$. It then follows that $\bar{\mathcal{M}}[\bar{G}_0][\bar{H}_0]^{<\bar{\lambda}} \subseteq \bar{\mathcal{M}}[\bar{G}_0]$, giving us *(2)*. ☐

Proposition 38. $\dot{\mathbb{P}}[G]$ *is* λ*-distributive over* $V[G]$*.*

Proof. Suppose there were $(m, \dot{p}) \in \mathbb{M} * \dot{\mathbb{P}}$ forcing that some $\dot{f}$ collapses λ over V. Then a suitably-chosen $\mathcal{N} := (H(\Theta), \in, <_\Theta, \mathbb{M} * \dot{\mathbb{P}}, (m, \dot{p}), \dot{f}, \ldots)$ would contain the $<_\Theta$-least such example, and so we can find a rich $\mathcal{M} \prec \mathcal{N}$ witnessing Fact 35 with $(m, \dot{p}) \in \mathcal{M}$ and such that $\ell(\bar{\lambda}) = \pi_{\mathcal{M}}(\dot{\mathbb{P}})$. Then *(2)* from Lemma 37 obtains a contradiction. ☐

Proposition 39. $V[G][H] \models \neg\mathsf{DSS}(\mu^+)$.

Proof. Since $\mathbb{P}$ is λ-distributive over $V[G]$, $\underline{x}$ remains an enumeration of $[\mu^+]^{<\mu}$ in $V[\mathbb{M}][\mathbb{P}]$. Moreover, $\mathbb{P}$ forces that $S(\underline{x})$ is nonstationary in $V[\mathbb{M}][\mathbb{P}]$, so we can apply Fact 33. ☐

Proposition 40. $V[G][H] \models \neg\mathsf{AP}(\mu^+)$.

Proof. This is exactly as in Lemma 5.9 [6], where we imitate the argument of "The Eightfold Way" and use property *(3)* of the lift, except that here $\mathbb{P}$ stands for a $\mathbb{P}(\underline{x})$ rather than the iteration $\mathbb{P}_\alpha$ used in [6]. The main point is that if we are using an embedding $j : \bar{\mathcal{M}}[\bar{G}][\bar{H}] \to \mathcal{N}[G][H]$, then the extension by $G * H$ over the extension by $\bar{G} * \bar{H}$ has the correct branch preservation properties (as given by the distributivity of $\dot{\mathbb{P}}[G]$ and the closure and square-chain condition of the posets projecting onto $\mathbb{N}_{\bar{\lambda}^\oplus}$). ☐

Now we are finished with the proof of Theorem 3.

4. Further directions

We propose some other considerations along the lines of the question: Why did we have to do more work to get Theorem 2 after obtaining Theorem 1? Or rather, is the assumption $2^\mu = \mu^+$ necessary for Fact 6?

Question 1. *Is it consistent for μ regular that exactly one of $\mathsf{DSS}(\mu^+)$ and "internally club and internally unbounded are distinct for $[H(\mu^+)]^\mu$" holds?*

On a similar note, the assumption that $2^\mu = |H(\mu^+)|$ is also used in a folklore result that assuming $2^\mu = \mu^+$, the distinction between internally unbounded and internally approachable for $[\mu^+]^\mu$ requires a Mahlo cardinal.

Question 2. *What is the exact equiconsistency strength of the separation of internally approachable and internally unbounded for $[H(\mu^+)]^\mu$ for regular μ?*

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