PATTERNS OF STATIONARY REFLECTION

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Abstract. We consider the behavior of the stationary reflection property $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$ across the class of all cardinals and prove that, relative to the consistency of a supercompact cardinal, it has only trivial ZFC constraints.

In this paper we will examine the global behavior of stationary reflection. A stationary subset $S \subseteq \delta$ *reflects* if there is some ordinal $\alpha < \delta$ of uncountable cofinality such that $S \cap \alpha$ is a stationary subset of α. Stationary reflection is a basic notion of compactness that is studied widely in the set-theoretic literature because it serves to distinguish inner models like Gödel's Constructible Universe L from models containing very large cardinals. The question of whether a given cardinal δ has a non-reflecting stationary subset is known to be independent of ZFC. For example, if κ is weakly compact, then every stationary subset of κ reflects, but if κ is a successor cardinal in L, then every stationary subset of κ has a non-reflecting stationary subset.

Our project here is to prove an Easton-style result for stationary reflection across the class of all cardinals: we show that, given a fixed cofinality λ, the existence of a non-reflecting stationary subset of $\kappa \cap \mathrm{cof}(\lambda)$ does not depend on the existence of non-reflecting stationary subsets of $\mu \cap \mathrm{cof}(\lambda)$ for $\mu < \kappa$ as long as κ is regular. In other words, there is no Silver's Theorem for stationary reflection at a fixed cofinality. For the sake of exposition we prove our result for the fixed cofinalities $\aleph_n$ for $n < \omega$ because a result of Shelah allows us to handle the approachability ideal in a convenient manner. However, we believe that our methods will generalize to any fixed cofinality (see Conjecture 1 at the end of the paper).

The naive approach of simply forcing non-reflecting stationary sets wherever desired does not work, because we risk adding unintended non-reflecting sets. Hence, our work here will be of interest in part because of the methods we use in order to surmount this obstacle. First, the class forcing that we employ is a hybrid between iterated forcing and product forcing in the sense that the forcing at certain successors of singulars depends on the prior stages, while at the same time the forcing can be factored at successors of regular cardinals. Second, we make use of PCF-theoretic ideas by introducing *Easton scales* and exploiting the good points of these scales. (A similar hybrid notion appears in a paper of Cummings and Shelah [6], but our construction is quite different because of the scales that we use.) The good points allow us to restore stationary reflection when necessary. The techniques that we introduce with these scales should be applicable to further results.

Set theorists have long been interested in the global phenomena. The most notable example is Easton's result that the continuum function $\kappa \mapsto 2^\kappa$ is constrained only by monotonicity and König's Theorem (which implies that $\mathrm{cf}(2^\kappa) < \kappa$) when restricted to regular cardinals [7]. At first it was expected that this result would be

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extended to include singular cardinals, but singular cardinals turn out to be much more complicated: Silver proved that GCH cannot fail for the first time at a singular of uncountable cofinality [15]. Generally, singular cardinals present a challenge to global results.

Here we consider $\mathsf{SR}(S)$ where S is stationary in some κ, which states that every stationary subset $T \subseteq S$ reflects. Specifically, we examine $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$. This property winds up being compelling because we can precisely determine the ZFC constraints on its behavior. The constraints are fairly trivial:

- $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$ holds vacuously if $\kappa \leq \aleph_n$;
- $\mathsf{SR}(\aleph_{n+1} \cap \mathrm{cof}(\aleph_n))$ fails because no stationary subset of $\aleph_{n+1} \cap \mathrm{cof}(\aleph_n)$ can reflect;
- If λ is a singular cardinal, then λ has a non-reflecting stationary subset if and only if $\mathrm{cf}\,\lambda$ has a non-reflecting stationary subset.

One might object that the third bullet point makes singular cardinals uninteresting, but in our case it is the successors of singulars that are difficult to handle—so the difficult case nonetheless pertains to singular cardinals.

Our main theorem, which takes up the whole paper, is the following:

Theorem 1. *Suppose χ is a supercompact cardinal in V such that* GCH *holds above χ. Let F be a definable 2-valued function on the class of regular cardinals $\geq \chi$. Then there is a forcing extension $W \supset V$ in which $\chi = \aleph_{n+2}$, cofinalities $\geq \chi$ are preserved,* GCH *is preserved above χ, and for all regular $\kappa \in W$ such that $\kappa \geq \aleph_{n+2}$, there is a non-reflecting stationary subset of $\kappa \cap \mathrm{cof}(\aleph_n)$ if and only if $F(\kappa) = 1$.*

The hypothesis of the theorem follows from the consistency of a supercompact cardinal: If χ is supercompact in some model V_0, then there is a forcing extension V_1 in which the supercompactness of χ is indestructible under χ-directed closed forcing [11], and then GCH above χ can be forced using a product of Cohen posets.

It follows from our theorem that there is a lot of flexibility concerning the behavior of $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$ as κ varies:

Corollary 2. *Relative to the consistency of a supercompact cardinal, it is consistent that there is a model in which a given regular $\kappa \geq \aleph_{n+2}$ has a non-reflecting stationary subset of $\kappa \cap \mathrm{cof}(\aleph_n)$ precisely when:*

- *κ is the successor of a regular cardinal;*
- *κ is the successor of a singular cardinal;*
- *κ is inaccessible;*
- *κ is* not *inaccessible.*

We assume familiarity with basics of forcing and large cardinals [9].

1. Methods

1.1. **Large Cardinal Notions.**

We need large cardinals to obtain stationary reflection. In fact, stationary reflection at a successor of a singular cardinal λ^+ implies the failure of $\square_\lambda$, which has strong inner model-theoretic consequences [13]. Hence, some large cardinal hypothesis is necessary for our result.

Definition 3. A cardinal κ is λ-*supercompact* for $\lambda \geq \kappa$ if there is an elementary embedding $j : V \to M \subseteq V$ from the class of all sets to a proper subclass such that

$j \restriction V_\kappa = \mathrm{id} \restriction V_\kappa$, $j(\kappa) > \lambda$, and $M^\lambda \subseteq M$. A cardinal κ is *supercompact* if κ is λ-supercompact for all $\lambda \geq \kappa$.

If κ is supercompact, then it $\mathsf{SR}(\lambda \cap \mathrm{cof}(<\kappa))$ holds for all regular $\lambda \geq \kappa$. We plan to force over a model with a supercompact cardinal and then lift the supercompact embedding to the forcing extension in order to recover reflection. Our ability to lift this embedding without destroying stationarity subsets is the technical core of this paper.

We will need weakly compact cardinals for approachability, but their definition can be a black box [9]. And if we have a supercompact cardinal, then we have many weakly compact cardinals.

Fact 4. *If κ is supercompact, then there are κ-many weakly compact cardinals below κ.*

This is because a supercompact κ is measurable, and under the measurable embedding $j : V \to M$, $M \models$ "κ is weakly compact".

1.2. **Forcing Techniques.** We will consider partial orders that are not quite closed, so we will define a weakening of closure that appears frequently in the literature [3, 2].

Definition 5.

- Consider a poset $\mathbb{P}$ and an ordinal α. We describe $G^{\mathsf{II}}_\alpha(\mathbb{P})$, a two-player game of perfect information, as follows: Players I and II take turns building a sequence of conditions $\langle p_\beta : \beta < \alpha \rangle$. Player I plays at all odd successors (successors of the form $\xi + n$ where ξ is 0 or a limit and n is odd), while Player II plays at even successors and limits. If at some stage $\gamma < \alpha$ there is no lower bound of $\langle p_\beta : \beta < \gamma \rangle$, then Player I wins the game. If Player II does not lose at any $\beta < \alpha$, then Player II wins the game.
- We say that $\mathbb{P}$ is *κ-weakly strategically closed* if Player II has a winning strategy for $G^{\mathsf{II}}_\kappa(\mathbb{P})$.
- The game $G^{\mathsf{I}}_\alpha(\mathbb{P})$ is analogous to $G^{\mathsf{II}}_\alpha(\mathbb{P})$. The difference is that in $G^{\mathsf{I}}_\alpha(\mathbb{P})$, Player I plays at odd successors *and* limits, while Player II only plays at even successors.
- We say that $\mathbb{P}$ is *κ-strongly strategically closed* if Player II has a winning strategy for $G^{\mathsf{I}}_\kappa(\mathbb{P})$.

We need strategic closure because we will use it to get preservation of cardinals in our forcing extension. Note that κ-strong strategic closure implies κ-weak strategic closure.

Fact 6. *If κ is regular and $\mathbb{P}$ is κ-weakly strategically closed, then $\mathbb{P}$ is κ-distributive, meaning that it does not add new sequences of ordinals of length $< \kappa$.*

Now we introduce a poset for adding a non-reflecting stationary subset of $\kappa \cap \mathrm{cof}(\lambda)$. The poset that we will use in our construction will be akin to a product of instances of this poset—with some additional constraints.

Definition 7. If κ and λ are regular uncountable cardinals with $\lambda < \kappa$, then $\mathbb{S}(\kappa, \lambda)$ is the poset consisting of conditions p such that:

- p is a function from $\alpha + 1$ to $\{0, 1\}$ for some $\alpha < \kappa$;
- if $p(\beta) = 1$ for some $\beta \leq \alpha$, then $\mathrm{cf}(\beta) = \lambda$;

- for all $\beta \leq \max \operatorname{dom} p$ such that $\operatorname{cf} \beta > \omega$, there is a closed unbounded subset $c \subseteq \beta$ such that $c \cap \{\gamma \in \operatorname{dom} p : p(\gamma) = 1\} = \emptyset$.

If $p, q \in \mathbb{S}(\kappa, \lambda)$ then $p \leq q$ (i.e. p is stronger than q) precisely when $q \subseteq p$.

The non-reflecting stationary set added by $\mathbb{S}(\kappa, \lambda)$ is the set whose characteristic function is defined by the generic.

Fact 8. *If G is $\mathbb{S}(\kappa, \lambda)$-generic, then $\{\alpha < \kappa : \exists p \in G, p(\alpha) = 1\}$ is a non-reflecting stationary subset if $\kappa \cap \operatorname{cof}(\lambda)$.*

The poset $\mathbb{S}(\kappa, \lambda)$ has several nice properties.

Facts 9.

(1) $\mathbb{S}(\kappa, \lambda)$ is λ^+-closed.
(2) $\mathbb{S}(\kappa, \lambda)$ is κ-weakly strategically closed, and hence κ-distributive.
(3) If GCH holds, then $|\mathbb{S}(\kappa, \lambda)| = \kappa$ and therefore $\mathbb{S}(\kappa, \lambda)$ satisfies the κ^+-chain condition.

Hence, under GCH, the poset $\mathbb{S}(\kappa, \lambda)$ preserves cardinals and cofinalities. However, it will take more work to show that one can add non-reflecting sets at class-many cardinals without affecting cardinals and cofinalities.

Because $\mathbb{S}(\kappa, \lambda)$ is not κ-closed, we need to define a poset that will represent the quotient of $\mathbb{S}(\kappa, \lambda)$ inside a κ-closed poset. This will help us lift the supercompact embedding at the end of our construction.

Definition 10. If G is $\mathbb{S}(\kappa, \lambda)$-generic and $S = \{\alpha < \kappa : \exists p \in G, p(\alpha) = 1\}$, then in the extension by G we define $\mathbb{T}(\kappa, \lambda)$ to be the set of closed bounded sets $t \subseteq \kappa$ such that $t \cap S = \emptyset$. The conditions are ordered by end-extension, i.e. $t \leq s$ if and only if $t \cap (\max s + 1) = s$.

The forcing $\mathbb{T}(\kappa, \lambda)$ kills the stationarity of of S.

Fact 11. *$\mathbb{S}(\kappa, \lambda) * \mathbb{T}(\kappa, \lambda)$ has a κ-directed closed dense subset, and therefore $\mathbb{T}(\kappa, \lambda)$ is κ-distributive.*

Again, it will take more work to show that we can utilize this stationary-killing forcing for class-many cardinals. However, we will make use of the limited closure properties of this poset.

Fact 12. *$\mathbb{T}(\kappa, \lambda)$ is λ-closed.*

The last forcing concept that we will crucially use pertains to the support of the (quasi) product that we employ for our construction.

Definition 13. Suppose I is a set of cardinals, and consider the product of posets $\mathbb{P} = \prod_{\kappa \in I} \mathbb{P}_\kappa$. Given $p \in \mathbb{P}$, the *support* of p, denoted $\operatorname{sprt}(p)$, is the set of $\kappa \in I$ such that $p(\kappa) \neq 1_{\mathbb{P}_\kappa}$. We say that a subset $\mathbb{Q}$ of the full-support product $\mathbb{P}$ is an *Easton product*—or has *Easton support*—if $\mathbb{Q}$ consists of those $p \in \mathbb{P}$ such that for all regular μ, $|\operatorname{sprt}(p) \cap \mu| < \mu$. This definition also applies when I is a proper class.

1.3. **Approachability and the H_Θ Technique.** In this paper we will make ample use of a technique developed by Shelah and others that is based on H_Θ, the set of all sets x with transitive closure of size strictly less than Θ. The general idea begins by listing the parameters $a_1, \ldots, a_k$ relevant to an argument and choosing a regular cardinal Θ which is "large enough" in the sense that H_Θ will correctly witness all steps of the ensuing argument. Then we choose a well-order $<_\Theta$ on H_Θ, and we consider the model $\mathcal{H} := H_\Theta(\in, <_\Theta, a_1, \dots, a_k)$. Finally, we run the argument in the model $\mathcal{H}$. The point of this technique is that we want to use a club $\langle M \cap \mu : M \prec \mathcal{H}\rangle$ for some cardinal μ, but writing down the actual definition of such a club directly is infeasible. There are several good sources for further reading [8, 1].

Occasionally the H_Θ technique requires the notion of approachability. When we consider models $M \prec H_\Theta$, the models are not necessarily closed, so approachability is a way for these models to be in some sense closed enough.

Definition 14. Suppose μ is regular with $\mu^{<\mu} = \mu$ and $\vec{a} = \langle a_\alpha : \alpha < \mu\rangle$ is an enumeration of $[\mu]^{<\mu}$. Then a point $\alpha < \mu$ is *approachable* with respect to $\vec{a}$ if and only if there is an unbounded set $A \subset \alpha$ with $\operatorname{ot} A = \operatorname{cf} \alpha$ such that for all $\beta < \alpha$ there is some $\gamma < \alpha$ such that $A \cap \beta = a_\gamma$. We write $S \in I[\mu]$ if there is a club $C \subset \mu$ such that every point in $S \cap C$ is approachable with respect to $\vec{a}$.

Observe that $I[\mu]$ does not depend on the specific enumeration $\vec{a}$ of $[\mu]^{<\mu}$.

1.4. **PCF Theory on Easton Products.** The purpose of this section is to define Easton scales and to give conditions by which these scales have good points. These good points will in turn be used for the interleaving argument that will be the crux of our strategic closure and stationary preservation lemmas.

Definitions 15.

(1) If λ is a singular cardinal and $L \subseteq \lambda$ is a set of regular cardinals unbounded in λ, let $f \in \prod L$ mean that $\operatorname{dom} f$ is an Easton subset of L, i.e. $|\operatorname{dom} f \cap \kappa| < \kappa$ for *every* regular $\kappa < \lambda$, and that for all $\kappa \in \operatorname{dom} f$, $f(\kappa) < \kappa$. We say that $\prod L$ is a product of *width* λ.

(2) If $\prod L$ is a product of width λ and $D, E \subseteq L$ are Easton sets (for all regular $\kappa < \lambda$, $|D \cap \kappa|, |E \cap \kappa| < \kappa$), then we write $D \subseteq^* E$ if there is some $\tau < \lambda$ such that $D \cap (\tau, \lambda) \subset E$, and we say that this relation is *witnessed* by τ. We write that $f <^* g$ if there is some $\tau < \lambda$ witnessing that $\operatorname{dom} f \subseteq^* \operatorname{dom} g$ and such that for all $\kappa \in \operatorname{dom} f \cap (\tau, \lambda)$, $f(\kappa) < g(\kappa)$. The corresponding notions for equality and for non-strict inequality are denoted $f =^* g$ and $f \leq^* g$ respectively.

(3) If $\prod L$ is a product of width λ, then we say that two $<^*$-increasing sequences $\vec{f} = \langle f_\beta : \beta < \alpha\rangle$ and $\vec{g} = \langle g_\delta : \delta < \gamma\rangle$ *cofinally interleave each other* if for all $\beta_0 < \alpha$, there is some $\delta_0 < \gamma$ such that $f_{\beta_0} <^* g_{\delta_0}$, and for all $\delta_0 < \gamma$, there is $\beta_0 < \alpha$ such that $g_{\delta_0} <^* f_{\beta_0}$.

(4) Given a product $\prod L$ of width λ, an *Easton scale* is a sequence $\langle f_\alpha : \alpha < \lambda^+\rangle$ of functions such that:
   (a) $\forall \alpha < \lambda^+$, $f_\alpha \in \prod L$;
   (b) $\forall \alpha < \beta < \lambda^+$, $f_\alpha <^* f_\beta$;
   (c) $\forall g \in \prod L$, there is some $\alpha < \lambda^+$ such that $g <^* f_\alpha$.

Note that these resemble the definitions of products and scales as used in standard PCF theory, but we are generally considering more than $\operatorname{cf} \lambda$-many cardinals below each singular λ.

Proposition 16. *If $2^\lambda = \lambda^+$, and $\prod L$ is a product of width λ, then there is an Easton scale on L.*

Proof. Using the fact that $|\prod L| = 2^\lambda = \lambda^+$, let $\langle g_\alpha : \alpha < \lambda^+\rangle$ be an enumeration of $\prod L$. Build a sequence $\langle f_\alpha : \alpha < \lambda^+\rangle \subseteq \prod L$ by induction. If $\alpha = \beta + 1$, let $\operatorname{dom} f_\alpha = \operatorname{dom} f_\beta \cup \operatorname{dom} g_\beta$ and let $f_\alpha(\kappa) = \max\{f_\beta(\kappa), g_\beta(\kappa)\} + 1$ for all $\kappa \in \operatorname{dom} f_\alpha$. If α is a limit, pick $A \subseteq \alpha$ unbounded of order-type $\operatorname{cf}\alpha$. Then let $\operatorname{dom} f_\alpha = (\operatorname{cf}\alpha, \lambda) \cap \bigcup_{\beta\in A} \operatorname{dom} f_\beta$, noting in particular that if κ is regular and $\kappa \in (\operatorname{cf}\alpha, \lambda)$, then $\bigcup_{\beta\in A} \operatorname{dom} f_\beta \cap \kappa$ has cardinality less than κ. And for $\kappa \in \operatorname{dom} f_\alpha$, let $f_\alpha(\kappa) = \sup_{\beta\in A} f_\beta(\kappa)$, so $f_\alpha(\kappa) < \kappa$ for the same reason. Then $f_\gamma <^* f_\alpha$ for all $\gamma < \alpha$ because $<^*$ is transitive: if $\gamma < \alpha$ and $\beta \in A \cap (\gamma, \alpha)$, then by induction $f_\gamma <^* f_\beta <^* f_\alpha$. □

So that we do not repeat ourselves, we fix a product $\prod L$ of width λ and an Easton scale $\vec{f}$ on this product for the remainder of the section.

Definition 17. A point $\alpha < \lambda^+$ such that $\operatorname{cf}\alpha \neq \operatorname{cf}\lambda$ is *good* if there is some unbounded $A \subset \alpha$ of order-type $\operatorname{cf}\alpha$ and some $\tau < \lambda$ such that for all $\beta, \gamma \in A$ with $\beta < \gamma$, τ witnesses that $\operatorname{dom} f_\beta \subseteq^* \operatorname{dom} f_\gamma$ and for all $\kappa \in \operatorname{dom} f_\beta \cap (\tau, \lambda)$, $f_\beta(\kappa) < f_\gamma(\kappa)$.

Observe that this definition of goodness is analogous to the definition of goodness for scales on products of length $\operatorname{cf}\lambda$, and that it is also the case with Easton scales that every point α such that $\operatorname{cf}\alpha < \operatorname{cf}\lambda$ is good.

Proposition 18. *The following are equivalent for α such that $\operatorname{cf}\alpha > \operatorname{cf}\lambda$:*

(1) α is a good point for $\vec{f}$.
(2) There exists a sequence $\langle h_\xi : \xi < \operatorname{cf}\alpha\rangle \subset \prod L$ such that:
   (a) for some $\tau < \lambda$ and every $\xi, \eta < \operatorname{cf}\alpha$ with $\xi < \eta$, τ witnesses that $\operatorname{dom} h_\xi \subseteq^* \operatorname{dom} h_\eta$, and for all $\kappa \in \operatorname{dom} h_\xi \cap (\tau, \lambda)$, $h_\xi(\kappa) < h_\eta(\kappa)$;
   (b) $\langle h_\xi : \xi < \operatorname{cf}\alpha\rangle$ and $\langle f_\beta : \beta < \alpha\rangle$ cofinally interleave each other.

Proof. First suppose α is a good point for $\vec{f}$. If $A := \langle \beta_\xi : \xi < \operatorname{cf}\alpha\rangle$ and τ witness goodness at α, then let $h_\xi := f_{\beta_\xi}$.

Now we prove the converse. Let $\langle \lambda_i : i < \operatorname{cf}\lambda\rangle \subset L$ converge to λ. We use the so-called Sandwich Argument. Pick $A := \langle \beta_\xi : \xi < \operatorname{cf}\alpha\rangle$ such that for all $\xi < \operatorname{cf}\alpha$, $h_\xi <^* f_{\beta_\xi} \leq^* h_{\xi+1}$ (by thinning out the enumeration of h_ξ's if necessary). For each $\xi \in A$, let $i(\xi)$ be such that $\lambda_{i(\xi)} \geq \tau$ witnesses both $h_\xi <^* f_{\beta_\xi}$ and $f_{\beta_\xi} \leq^* h_{\xi+1}$. By the Pigeonhole Principle, there is some unbounded $X \subseteq \operatorname{cf}\alpha$ and some $j < \operatorname{cf}\lambda$ such that $i(\xi) = j$ for all $\xi \in X$. Let $A' = \langle \beta_\xi : \xi \in X\rangle$. Then if $\xi, \eta \in A'$ and $\xi < \eta$, then λ_i witnesses that

$$\operatorname{dom} f_{\beta_\xi} \subseteq^* \operatorname{dom} h_{\xi+1} \subseteq^* \operatorname{dom} h_\eta \subseteq^* \operatorname{dom} f_{\beta_\eta},$$

and moreover if $\kappa \in \operatorname{dom} f_{\beta_\xi} \cap (\lambda_j, \lambda)$, then it follows that,

$$f_{\beta_\xi}(\kappa) \leq h_{\xi+1}(\kappa) \leq h_\eta(\kappa) < f_{\beta_\eta}(\kappa).$$

This shows that A' and λ_j witness goodness of α. □

This equivalent definition of goodness is also analogous to one for scales on products of length λ. However, in the latter case there is a third definition for goodness in terms of the existence of an exact upper bound taking values of cofinality equal to $\operatorname{cf}\alpha$. This definition does not appear to work for Easton scales because even though the candidate for the exact upper bound can be defined analogously, the natural proof of exactness, in which one must take a supremum of less than $\operatorname{cf}\alpha$-many ordinals, fails due to the possibility that there are more than $\operatorname{cf}\alpha$-many points below λ being considered.

Using Proposition 18 we can show that approachable sets give us good points.

Lemma 19. *If $S \in I[\lambda^+]$, then there is some club $C \subset \lambda^+$ such that every $\alpha \in S \cap C$ of cofinality greater than $\operatorname{cf}\lambda$ is a good point for $\vec{f}$.*

Proof. Let D be the club such that all points in $D \cap S$ are approachable with respect to some enumeration $\vec{a} = \langle a_\alpha : \alpha < \lambda^+ \rangle$. We work with the structure $\mathcal{H} := H_\Theta(\in, <_\Theta, D, \vec{f}, \prod L, \vec{a})$ where Θ is large enough.

Let $\langle M_\xi : \xi < \lambda^+ \rangle$ be a continuous and $\in$-increasing sequence of elementary submodels of $\mathcal{H}$ of cardinality λ. Then if $\delta_\xi := M_\xi \cap \lambda^+$ for $\xi < \lambda^+$, it follows that $\vec{\delta} := \langle \delta_\xi : \xi < \lambda^+ \rangle$ is a club in λ^+. Moreover, $\vec{\delta} \subseteq D$. We will argue that $\vec{\delta}$ consists of good points for $\vec{f}$.

Let $\delta = \delta_\xi$ and $M = M_\xi$ for some $\xi < \lambda^+$. We will use the second definition of good points from Proposition 18. By approachability, there is some unbounded $A \subset \delta$ with $\operatorname{ot} A = \operatorname{cf}\delta$ such that for all $\beta < \delta$, $A \cap \beta = a_\gamma$ for some $\gamma < \delta$. In particular, this means that all initial segments of A are in the model M by elementarity because $M \cap \lambda^+ = \delta$.

Now we can work in M. Let $\langle \beta_i : i < \operatorname{cf}\delta \rangle$ enumerate A. We will define a sequence $\langle h_i : i < \operatorname{cf}\delta \rangle$ of functions in $\prod L$ and we will ensure that for all $\kappa > \operatorname{cf}\delta$, $\langle h_i(\kappa) : i \in X \rangle$ is strictly increasing. Namely, within M, let $\operatorname{dom} h_i = \bigcup_{j<i} \operatorname{dom} h_j$, and for $\kappa \in \operatorname{dom} h_i \cap (\operatorname{cf}\delta, \lambda)$, let $h_i(\kappa) := \max\{f_{\beta_i}(\kappa), \sup_{j<i} h_j(\kappa)\} + 1$. We know that h_i is definable in M for limits i because $\langle \beta_j : j < i \rangle \in M$. Furthermore, by elementarity and the fact that $M \cap \lambda^+ = \delta$, for every h_i there is some $\beta_k < \delta$ such that $h_i <^* f_{\beta_k}$. □

Then we use good points to show that certain interleaving arguments define unique (up to $=^*$) functions in $\prod L$.

Lemma 20. *Suppose α is a good point and $\operatorname{cf}\alpha \neq \operatorname{cf}\lambda$. Suppose also that $A \subset \alpha$ is unbounded in α and $\operatorname{ot} A = \operatorname{cf}\alpha$, and that $\langle g_\xi : \xi < \operatorname{cf}\alpha \rangle$ cofinally interleaves $\langle f_\beta : \beta < \alpha \rangle$. Define $\operatorname{dom}\bar{f} = (\operatorname{cf}\alpha, \lambda) \cap \bigcup_{\beta \in A} \operatorname{dom} f_\beta$, and for $\kappa \in \operatorname{dom}\bar{f}$, define $\bar{f}(\kappa) = \sup_{\beta \in A} f_\beta(\kappa)$. Similarly, define $\operatorname{dom}\bar{g} = (\operatorname{cf}\alpha, \lambda) \cap \bigcup_{\xi < \operatorname{cf}\alpha} \operatorname{dom} g_\xi$ and for $\kappa \in \operatorname{dom}\bar{g}$ define $\bar{g}(\kappa) = \sup_{\xi < \operatorname{cf}\alpha} g_\xi(\kappa)$.*

Then $\bar{f} =^* \bar{g}$. In particular, this works if $g_\xi := f_{\gamma_\xi}$ where $\langle \gamma_\xi : \xi < \operatorname{cf}\alpha \rangle$ enumerates an arbitrary cofinal sequence in α.

Proof. Enumerate A as $\langle \beta_\xi : \xi < \operatorname{cf}\alpha \rangle$.

Suppose first that $\operatorname{cf}\alpha < \operatorname{cf}\lambda$. For each $\{\xi, \eta\} \in [\operatorname{cf}\alpha]^2$, let $\tau_{\xi,\eta}$ witness that $f_\xi <^* f_\eta$ or vice versa, depending on whether ξ or η is bigger. Let $\tau := \sup\{\tau_{\xi,\eta} : \{\xi, \eta\} \in [\operatorname{cf}\alpha]^2\} < \lambda$. Then τ witnesses $\operatorname{dom}\bar{f} =^* \operatorname{dom}\bar{g}$ and for all $\kappa \in \operatorname{dom}\bar{f} \cap (\tau, \lambda)$, we have $\bar{f}(\kappa) = \bar{g}(\kappa)$.

Now suppose that $\operatorname{cf}\alpha > \operatorname{cf}\lambda$. This uses the same idea as the Sandwich Argument. By picking subsequences, we can assume without loss of generality that for all $\xi < \operatorname{cf}\alpha$, $g_\xi <^* f_{\beta_{\xi+1}}$ and $f_{\beta_\xi} <^* g_{\xi+1}$. Furthermore, let τ be large enough to witness goodness of α with respect to A. Pick a sequence $\langle \lambda_i : i < \operatorname{cf}\lambda \rangle$ converging to λ. Let $\lambda_{\xi(i)} \geq \tau$ witness both $g_\xi <^* f_{\beta_{\xi+1}}$ and $f_{\beta_\xi} <^* g_{\xi+1}$. Then there is some unbounded $X \subset \operatorname{cf}\alpha$ and some $j < \operatorname{cf}\lambda$ such that for all $\xi \in X$, $i(\xi) = j$. It follows that for all $\xi, \eta \in X$ with $\xi < \eta$, λ_j witnesses that

$$\operatorname{dom} g_\xi \subseteq^* \operatorname{dom} f_{\beta_{\xi+1}} \subseteq^* \operatorname{dom} f_{\beta_\eta} \subseteq^* \operatorname{dom} g_{\eta+1},$$

so it follows that λ_i witnesses $\operatorname{dom} \bar{f} =^* \operatorname{dom} \bar{g}$. Furthermore, for any $\xi, \eta \in X$ with $\xi < \eta$ and any $\kappa \in \operatorname{dom} \bar{g} \cap (\lambda_i, \lambda)$,

$$g_\xi(\kappa) < f_{\beta_{\xi+1}}(\kappa) \leq f_{\beta_\eta}(\kappa) < g_{\eta+1}(\kappa).$$

It follows that for such κ, $\bar{f}(\kappa) = \bar{g}(\kappa)$. Hence $\bar{f} =^* \bar{g}$. □

Definition 21. If α is a good point for $\vec{f}$, we say that $\vec{f}$ is *continuous* at α if there is some unbounded $A \subset \alpha$ of order-type $\operatorname{cf} \alpha$ and some $\tau < \lambda$ such that $(\tau, \lambda) \cap \operatorname{dom} f_\alpha = (\tau, \lambda) \cap \bigcup_{\beta \in A} \operatorname{dom} f_\beta$, and such that for all $\kappa \in (\tau, \lambda)$, $f_\alpha(\kappa) = \sup_{\beta \in A} f_\beta(\kappa)$.

Hence, Lemma 19 gives us:

Corollary 22. *If $S \in I[\lambda^+]$, then for any product $\prod L$ of width λ, there is a club $C \subset \lambda^+$ and a scale $\vec{f} = \langle f_\alpha : \alpha < \lambda^+ \rangle$ that is continuous at all points in $S \cap C$.*

As a restatement of Lemma 20, we have the proposition that will later on (in subsection 2.3, subsection 2.5, and subsection 2.7) vindicate our discussion of good points:

Proposition 23. *Suppose α is a good point and $\vec{f}$ is continuous at α. Suppose also that $\langle g_\xi : \xi < \operatorname{cf} \alpha \rangle$ cofinally interleaves $\langle f_\beta : \beta < \alpha \rangle$, and that the function g is defined such that $\operatorname{dom} g = (\operatorname{cf} \alpha, \lambda) \cap \bigcup_{\xi < \operatorname{cf} \alpha} \operatorname{dom} g_\xi$, and such that for all $\kappa \in \operatorname{dom} g$, $g(\kappa) = \sup_{\xi < \operatorname{cf} \alpha} g_\xi(\kappa)$. Then $g =^* f_\alpha$.*

2. Constructing the Model

Now we will commence with the proof of Theorem 1.

2.1. Preparation of the Ground Model.

Let χ be a supercompact cardinal and assume GCH above χ. Our present goal is to define a suitable forcing extension $V[G]$ in which $\chi = \aleph_{n+2}$. We will let $W := V[G]$ and work in W in subsection 2.2 through subsection 2.7 below. Then in subsection 2.8, we will refer back to V.

There are two cases that we consider. If we are trying to prove the result about stationary subsets of $\kappa \cap \operatorname{cof}(\aleph_0)$, then we do not actually need to deal with approachability, so we may simply take the Lévy Collapse $\mathbb{C} = \operatorname{Col}(\aleph_1, < \chi)$ and let G be $\mathbb{C}$-generic, so that $V[G] \models \chi = \aleph_2$.

If we are trying to prove the result for $\aleph_n$, $n > 0$, then we need to make arrangements so that we will have enough approachability when we need it. We use the following result of Shelah (which appears in a stronger form as Fact 2.10 in the paper about forcing approachability with set-sized forcing [14]):

Fact 24. *If λ is a singular strong limit and $\nu = \lambda^+$ in V, and W is a forcing extension of V in which λ is still a singular strong limit and ν is still its successor, then:*

$$W \models \{\alpha < \nu : V \models \text{“}\operatorname{cf} \alpha \textit{ is weakly compact”}\} \in I[\nu].$$

Because χ is supercompact, there are χ-many weakly compact cardinals below it. We pick any weakly compact $\psi < \chi$. Let $\mathbb{C}_1 = \mathrm{Col}(\aleph_{n-1}, <\psi)$, let $\mathbb{C}_2 = \mathrm{Col}(\psi^+, <\chi)$, and let $\mathbb{C} = \mathbb{C}_1 \times \mathbb{C}_2$. Then $V[G]$ will be our W, where $W \models \chi = \aleph_{n+2}$. The point is that enough approachability will persist in mild forcing extensions of W.

Proposition 25. *If λ is a singular strong limit and ν is the successor of λ in $V[G]$ and $n > 0$, then $\nu \cap \mathrm{cof}(\aleph_n) \in I[\nu]$ in any $\aleph_n$-distributive forcing extension of $V[G]$ in which λ is still a strong limit and ν is still its successor.*

Proof. If $(\mathrm{cf}\,\alpha)^{V[G]} = \aleph_n$, then the cofinality of α is preserved in any $\aleph_n$-distributive forcing extension of $V[G]$. Moreover, any such extension will preserve the conditions of Fact 24. $\square$

Observe that we cannot use the Lévy Collapse to get ψ equal to $\aleph_{\omega+1}$ (or any successor of a singular). This is precisely why we restrict ourselves to $\mathsf{SR}(\kappa\cap\mathrm{cof}(\aleph_n))$ for fixed n.

2.2. **Defining the Forcing.** Now we work in W. For ease of use, we classify regular cardinals as follows:

(A) If $F(\kappa) = 1$, then $\kappa \in \mathcal{C}_A$.
(B) If $F(\kappa) = 0$ and κ is inaccessible, a successor of a regular cardinal, a successor of a singular cardinal of cofinality equal to $\aleph_n$, or a successor of a singular cardinal λ of cofinality not equal to $\aleph_n$ such that $\{\kappa < \lambda : F(\kappa) = 1\}$ is bounded in λ, then $\kappa \in \mathcal{C}_B$.
(C) If $F(\lambda^+) = 0$ and λ is a singular of cofinality not equal to $\aleph_n$ such that $\mathcal{C}_A$ is unbounded in λ, then $\lambda^+ \in \mathcal{C}_C$.

So $\mathcal{C}_A$ is the class of cardinals κ that will have non-reflecting subsets of $\kappa\cap\mathrm{cof}(\aleph_n)$, $\mathcal{C}_B$ is the class of cardinals κ where $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$ will hold but where nothing special needs to be done with the forcing, and $\mathcal{C}_C$ will be those cardinals where in order to make $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$ hold, we will need to force an extra club through κ.

We define a class partial order $\mathbb{S}$ for adding nonreflecting stationary subsets to cardinals κ where $F(\kappa) = 1$. Since we have committed to $\aleph_n$, let $\mathbb{S}(\kappa) = \mathbb{S}(\kappa, \aleph_n)$ to simplify the notation.

Definition 26. For every $\lambda^+ \in \mathcal{C}_C$, we fix an Easton scale $\vec{f_\lambda} := \langle f^\lambda_\alpha : \alpha < \lambda^+\rangle$ on $\prod(\mathcal{C}_A \cap \lambda)$ that is continuous at every $\alpha \in \lim D^*_\lambda \cap \mathrm{cof}(\aleph_n)$ for some club $D^*_\lambda \subseteq \lambda^+$. We can do this because of Corollary 22.

We define $\mathbb{S}$ to consist of conditions p such that:

(1) p has Easton support: $\mathrm{sprt}\, p$ is a set of regular cardinals $\geq \chi$ such that if κ is inaccessible, then $|\mathrm{sprt}(p) \cap \kappa| < \kappa$.
(2) If $\kappa \in \mathcal{C}_A$, then $p(\kappa)$ is a condition in $\mathbb{S}(\kappa)$.
(3) If $\kappa \in \mathcal{C}_B$, then $p(\kappa)$ is the trivial forcing.
(4) If $\lambda^+ \in \mathcal{C}_C$, then $p(\lambda^+)$ is a closed bounded subset $c \subseteq \lambda^+$ such that if $\alpha \in \lim c \cap \mathrm{cof}(\aleph_n) \cap D^*_\lambda$, then the following condition holds:

   There is some $\tau < \lambda$ such that $\mathrm{dom}\, f^\lambda_\alpha \cap (\tau, \lambda) \subseteq \mathrm{sprt}\, p$ and such that for all $\kappa \in \mathrm{dom}\, f^\lambda_\alpha \cap (\tau, \lambda)$, $f^\lambda_\alpha(\kappa) \in \mathrm{dom}\, p(\kappa)$ and $p(\kappa)(f^\lambda_\alpha(\kappa)) = 0$.

   If this condition holds for α, we say that α has the *Annulment Property.*

If $p, q \in \mathbb{S}$, then $p \leq q$ if:

(a) $\mathrm{sprt}\, q \subseteq \mathrm{sprt}\, p$;

(b) for all $\kappa \in \operatorname{sprt} q \cap \mathcal{C}_A$, $p(\kappa) \upharpoonright (\max \operatorname{dom} q(\kappa) + 1) = q(\kappa)$;
(c) for all $\lambda^+ \in \operatorname{sprt} q \cap \mathcal{C}_C$, $p(\lambda^+) \cap (\max q(\lambda^+) + 1) = q(\lambda^+)$.

We have considerable freedom to extend conditions.

Proposition 27. *Suppose:*

- $p \in \mathbb{S}$, *X is a set of regular cardinals $\geq \chi$ and is such that $|\kappa \cap X| < \kappa$ for all inaccessible κ and* $\operatorname{sprt} p \cap X = \emptyset$*;*
- $\gamma_\kappa \in (\max \operatorname{dom} p(\kappa), \kappa)$ *for all* $\kappa \in \operatorname{sprt} p \cap \mathcal{C}_A$*, and* $\delta_{\lambda^+} \in (\max p(\lambda^+), \lambda^+)$ *for all* $\lambda^+ \in \operatorname{sprt} p \cap \mathcal{C}_C$*;*
- $\gamma_\kappa \in \kappa$ *for all* $\kappa \in X \cap \mathcal{C}_A$ *and* $\delta_{\lambda^+} < \lambda^+$ *for all* $\lambda^+ \in X \cap \mathcal{C}_C$*.*

Consider the function q with support $\operatorname{sprt} p \cup X$ *defined such that:*

- $q(\kappa)$ *is any extension of* $p(\kappa)$ *such that* $\max \operatorname{dom} q(\kappa) = \gamma_\kappa$ *for* $\kappa \in \operatorname{sprt} p \cap \mathcal{C}_A$*;*
- $q(\lambda^+) = p(\lambda^+) \cup \{\delta_{\lambda^+}\}$ *for* $\lambda^+ \in \operatorname{sprt} p \cap \mathcal{C}_C$*;*
- $q(\kappa)$ *is any condition in* $\mathbb{S}(\kappa)$ *such that* $\max \operatorname{dom} q(\kappa) = \gamma_\kappa$ *for* $\kappa \in X \cap \mathcal{C}_A$*;*
- $q(\lambda^+) = \{\delta_{\lambda^+}\}$ *for* $\lambda^+ \in X \cap \mathcal{C}_C$*.*

Then $q \in \mathbb{S}$.

Proof. We need to show that we have not violated that Annulment Property for points in $p(\lambda^+)$, $\lambda^+ \in \operatorname{sprt} p \cap \mathcal{C}_C$. To this end, there are two observations to be made. The δ_{λ^+}'s that we chose are not limit points of $q(\lambda^+)$, so the Annulment Property is dealt with vacuously. As for the old limit points $\alpha \in D^*_\lambda$ of cofinality $\aleph_n$ from $p(\lambda^+)$, the Annulment Property is already verified from some $\tau < \lambda$ and cardinals from $\operatorname{dom} f^\lambda_\alpha \cap (\tau, \lambda) \subseteq \operatorname{sprt} p$, so expanding the domain of p does not violate the Annulment Property for these points. □

As we pointed out in the introduction, the question of whether $\mathsf{SR}(\kappa \cap \operatorname{cof}(\aleph_n))$ holds is trivially settled for $\kappa \leq \aleph_{n+1}$, so we only concern ourselves with cardinals $\kappa \geq (\aleph_{n+2})^W$. Hence χ (which is supercompact in V but is equal to $\aleph_{n+2}$ in W) is the smallest cardinal where $\mathbb{S}$ can possibly be nontrivial.

We adopt the convention whereby $\mathbb{S}[\mu, \nu]$ and $\mathbb{S}[\mu, \nu)$ refer to $\mathbb{S}$ restricted to intervals.

Proposition 28. *For all regular μ,* $\mathbb{S} \cong \mathbb{S}[\chi, \mu] \times \mathbb{S}[\mu^+, \mathrm{ON})$*, and more generally,* $\mathbb{S}[\kappa, \nu) \cong \mathbb{S}[\kappa, \mu] \times \mathbb{S}[\mu^+, \nu)$ *for regular $\mu < \nu$.*

Proof. The map $p \mapsto (p \upharpoonright [\chi, \mu], p \upharpoonright [\mu^+, \mathrm{ON}))$ maps $\mathbb{S}$ into the product $\mathbb{S}[\chi, \mu] \times \mathbb{S}[\mu^+, \mathrm{ON})$. The Annulment Property is the only nontrivial point, and initial segments of cardinals do not affect whether it holds. □

It is important to note that the above proposition fails if $\mu^+ \in \mathcal{C}_C$. For this reason, when we write $\mathbb{S}[\mu^+, \lambda)$, we will assume that μ is regular.

2.3. **Preservation Properties of the Forcing.** Because of the use of Easton support, a simple counting argument yields an upper bound on the cardinality of $\mathbb{S}[\chi, \mu]$ for regular μ:

Proposition 29. *For all regular μ, $\mathbb{S}[\chi, \mu]$ has size $\leq \mu$. Hence $\mathbb{S}[\chi, \mu]$ has the μ^+-chain condition.*

The analysis of $p \in \mathbb{S}$ requires us to take careful consideration of functions on $\operatorname{sprt} p \cap \lambda$ for singular cardinals λ such that $\lambda^+ \in \mathcal{C}_C$. This requires us to commit to some notation.

Definition 30. Given $p \in \mathbb{S}$ and $\lambda^+ \in \mathcal{C}_C \cap \operatorname{dom} p$,

- let f^λ_p be f^λ_α where $\alpha = \max p(\lambda^+)$;
- and let g^λ_p be the function on $\operatorname{sprt} p \cap \mathcal{C}_A \cap \lambda$ such that $g^\lambda_p(\kappa) = \max \operatorname{dom} p(\kappa)$ if $\kappa \in \operatorname{dom} g^\lambda_p$.

Lemma 31. *Given $q \in \mathbb{S}$, there is some $r \leq q$ such that for all $\lambda^+ \in \operatorname{sprt} q \cap \mathcal{C}_C$, $f^\lambda_q \leq^* g^\lambda_r$. This also applies to the restrictions $\mathbb{S}[\mu^+, \nu)$, $\mathbb{S}[\mu^+, \mathrm{ON})$.*

Notice the crucial use of Easton support in the proof.

Proof. We will argue for $\mathbb{S}$ because the argument for its restrictions is the same. Let $q \in \mathbb{S}$ and assume without loss of generality that $\operatorname{sprt} q \cap \mathcal{C}_C$ has maximal element Λ^+. We will define a sequence of functions $h_\lambda \in \prod(\mathcal{C}_A \cap \lambda)$ by induction on $\lambda^+ \in \operatorname{sprt} q \cap \mathcal{C}_C$ such that $h_\mu \leq^* h_\lambda \restriction \mu$ for all $\mu < \lambda$ and such that $f^\lambda_q \leq^* h_\lambda$ for all $\lambda \leq \Lambda$ such that $\lambda^+ \in \operatorname{sprt} q \cap \mathcal{C}_C$. The base case $\lambda^+ = \min\{\operatorname{sprt} q \cap \mathcal{C}_C\}$ works easily by letting $h_\lambda = f^\lambda_q$. For the successor case, there is a greatest element μ of $\operatorname{sprt} q \cap \mathcal{C}_C$ below λ, so we let $h_\lambda(\kappa) = h_\mu(\kappa)$ for $\kappa \in (\operatorname{sprt} q) \cap \mu$ and $h_\lambda(\kappa) = f^\lambda_q(\kappa)$ for $\kappa \in \operatorname{sprt} q \cap (\mu, \lambda)$.

For the limit case, suppose we have defined h_μ for all elements of $\operatorname{sprt} q \cap \mathcal{C}_C$ below λ. Let $\delta := \sup(\operatorname{sprt} q \cap \mathcal{C}_C \cap \lambda)$, noting that $\delta \leq \lambda$, possibly strictly. We know that δ is singular because otherwise it would be inaccessible (we are working in a model of GCH) and we would have $\operatorname{sprt} q$ unbounded in an inaccessible cardinal, but we chose to define $\mathbb{S}$ with Easton support. Hence, we let $\langle \lambda_i : i < \operatorname{cf} \delta \rangle$ be a sequence in $\operatorname{sprt} q \cap \mathcal{C}_C$ converging to δ such that $\operatorname{cf} \delta < \lambda_0$. Let h_λ be a function with the domain $\operatorname{dom} h_{\lambda_0} \cup \big([\lambda_0, \delta) \cap \bigcup_{i < \operatorname{cf} \delta} \operatorname{dom} h_{\lambda_i}\big) \cup \big([\delta, \lambda) \cap \operatorname{dom} f^\lambda_q\big)$ such that:

$$h_\lambda(\kappa) = \begin{cases} h_{\lambda_0}(\kappa) & \kappa < \lambda_0 \\ \max\{\sup_{i < \operatorname{cf} \lambda} h_{\lambda_i}(\kappa), f^\lambda_q(\kappa)\} & \kappa \in [\lambda_0, \delta) \\ f_q(\kappa) & \kappa \in [\delta, \lambda) \end{cases}$$

It is immediately apparent that $f^\lambda_q \leq^* h_\lambda$. If $\mu < \lambda$ and $\mu^+ \in \operatorname{sprt} q \cap \mathcal{C}_C$, then there is some i such that $\mu < \lambda_i$, and so $f^\mu_q \leq^* h_{\lambda_i} \restriction \mu \leq^* h_\lambda \restriction \mu$.

At the end of this process, we obtain h_Λ. Let $r \leq q$ be a condition such that $\operatorname{dom} h_\Lambda \subseteq \operatorname{sprt} r$ and such that $\max \operatorname{dom} r(\kappa) \geq \max\{\max \operatorname{dom} q(\kappa), h_\Lambda(\kappa)\}$ for all $\kappa \in \operatorname{sprt} r \cap \mathcal{C}_A$. □

Now we are in a position to get distributivity through strategic closure.

Lemma 32. *For all regular μ, $\mathbb{S}[\mu^+, \mathrm{ON})$ is μ^+-weakly strategically closed. The same holds for $\mathbb{S}[\mu^+, \nu)$ for any $\nu > \mu^+$.*

Proof. We will do the proof for $\mathbb{S}[\mu^+, \mathrm{ON})$ because the argument for $\mathbb{S}[\mu^+, \nu)$ is the same. Suppose Players I and II are constructing a descending sequence $\langle p_\xi : \xi < \mu^+ \rangle$ in $\mathbb{S}[\mu^+, \mathrm{ON})$. We will demonstrate a strategy for Player II such that the play can continue at any $\xi < \mu^+$.

For even successors $\xi = \eta + 1$, let p_ξ be a condition such that for all $\kappa \in \operatorname{dom} p_\eta \cap \mathcal{C}_A$, $\gamma^\kappa_\xi := \max \operatorname{dom} p_\xi(\kappa) > \max \operatorname{dom} p_\eta(\kappa)$ and $p_\xi(\gamma^\kappa_\xi) = 0$, and furthermore, that for all $\lambda^+ \in \operatorname{dom} p_\eta \cap \mathcal{C}_C$, $\max p_\xi(\lambda^+) > \max p_\eta(\lambda^+)$. We also want to employ an interleaving argument, so we consider two sub-cases. If ξ is of the form $\xi' + 4k$ where ξ' is a limit and $k < \omega$, then Player II will in addition make sure that $\max p_\xi(\lambda^+)$ is large enough so that $g^\lambda_{p_\eta} \leq^* f^\lambda_{p_\xi}$ using Proposition 27. If ξ is of the form $\xi' + 4k + 2$

where ξ' is a limit and $k < \omega$, Player II will apply Lemma 31 to find p_ξ such that for all $\lambda^+ \in \operatorname{sprt} p_\eta \cap \mathcal{C}_C$, $f^\lambda_{p_\eta} \leq^* g^\lambda_{p_\xi}$.

At limits ξ, Player II will choose p_ξ as follows: First, let $\operatorname{sprt} p_\xi = \bigcup_{\eta<\xi} \operatorname{sprt} p_\eta$. If $\lambda^+ \in \operatorname{dom} p_\xi \cap \mathcal{C}_C$, then $p_\xi(\lambda^+) = \bigcup_{\eta' \leq \eta < \xi} p_\eta(\lambda^+) \cup \{\sup_{\eta' \leq \eta < \xi} \max p_\eta(\lambda^+)\}$ for big enough η'. If $\kappa \in \operatorname{dom} p_\eta \cap \mathcal{C}_A$, then let $p_\xi(\kappa)$ have a domain with maximum $\gamma^\kappa_\xi := \sup_{\eta<\xi} \max \operatorname{dom} p_\eta(\kappa)$ such that $p_\xi(\kappa) \restriction \operatorname{dom} p_\eta(\kappa) = p_\eta(\kappa)$ for $\eta < \xi$ and $p_\xi(\kappa)(\gamma^\kappa_\xi) = 0$.

This produces a valid condition: If $\kappa \in \operatorname{sprt} p_\xi \cap \mathcal{C}_A$, then $\langle \gamma^\kappa_\eta : \eta < \xi \rangle$ is a club avoiding $\{\alpha < \kappa : p_\xi(\alpha) = 1\}$. If $\lambda^+ \in \operatorname{sprt} p_\xi \cap \mathcal{C}_C$, then consider $\delta := \max p_\xi(\lambda^+)$. If ξ has cofinality not equal to $\aleph_n$ then the same is true of δ, and so the Annulment Property is satisfied vacuously. The Annulment Property is also vacuously satisfied if $\max p_\xi(\lambda^+)$ is not in D^*_λ, the club such that points of cofinality $\aleph_n$ are continuity points of $\vec{f}_\lambda$. The serious case is if ξ, and hence δ, has cofinality equal to $\aleph_n$ and $\delta \in D^*_\lambda$. Then for all $\lambda^+ \in \operatorname{sprt} p_\xi \cap \mathcal{C}_C$, $\langle f^\lambda_\gamma : \gamma < \delta \rangle$ and $\langle g^\lambda_{p_\eta} : \eta < \xi \rangle$ cofinally interleave each other (because of Player II's choices at successor steps), so it follows from Proposition 23 that $f^\lambda_\delta =^* g^\lambda_{p_\xi}$. Because we have guaranteed that $p(\kappa)(\gamma^\kappa_\xi) = 0$ for all $\kappa \in \operatorname{sprt} p_\xi \cap \mathcal{C}_A$, the Annulment Property is satisfied. □

We can use a weak version of Easton's Lemma:

Fact 33. *If $\mathbb{P}$ and $\mathbb{Q}$ are forcing posets where $|\mathbb{P}| < \mu$ and $\mathbb{Q}$ is μ-distributive, then $\Vdash_{\mathbb{P}}$ "$\mathbb{Q}$ is μ-distributive.*

Fact 33 works because Easton's Lemma is proved in two basic steps: First, we show that $\Vdash_{\mathbb{Q}}$ "$\mathbb{P}$ has the μ-chain condition", which follows immediately from the hypotheses of Fact 33. The second step, in which we prove that $\Vdash_{\mathbb{P}}$ "$\mathbb{Q}$ is μ-distributive, proceeds the same way as the original version of Easton's Lemma.

In turn, Fact 33 is used to prove that $\mathbb{S}$ preserves cardinals and cofinalities in the same manner as the original Easton construction.

Proposition 34. *$\mathbb{S}$ preserves cardinals, cofinalities, and* GCH*. In particular, it preserves inaccessible cardinals.*

The acute reader may observe that we have not guaranteed that the Easton scales $\vec{f}_\lambda$ are still Easton scales in $W^{\mathbb{S}}$. In other words, we do not know that in $W^{\mathbb{S}}$, that $\vec{f}_\lambda$ is still cofinal in $\mathcal{C}_A \cap \lambda$. This is one preservation property that we lack. However, it will turn out in the key moment that this does not matter.

2.4. **Adding Non-Reflecting Stationary Sets.** Most of this construction will focus on showing that $\mathsf{SR}(\kappa \cap \operatorname{cof}(\aleph_n))$ holds for κ such that $F(\kappa) = 0$, but let us show that we add the non-reflecting sets that we intended to add. An important point to keep in mind is that $\mathbb{S}$ is $\aleph_{n+1}$-strategically closed.

Lemma 35. *For all $\kappa \in \mathcal{C}_A$, $\Vdash_{\mathbb{S}}$ "$\kappa \cap \operatorname{cof}(\aleph_n)$ has a non-reflecting stationary set".*

Proof. We claim that for all κ such that $F(\kappa) = 1$, if H is $\mathbb{S}$-generic over W, then the set $S_\kappa = \bigcup_{p \in H} p(\kappa)$ is a non-reflecting stationary set. This set is non-reflecting at any given $\gamma < \kappa$ as witnessed by any $p \in H$ with $\max \operatorname{dom} p(\kappa) \geq \gamma$, so we must give a reason why it is stationary. Work in W and suppose $\Vdash_{\mathbb{S}}$ "$\dot{C}$ is club in κ".

Use strategic closure to build a decreasing sequence $\langle p_\xi : \xi < \aleph_n \rangle$ of conditions in $\mathbb{S}$ (we will suppress the distinction of even successors for the sake of readability) and a continuous and increasing sequence $\langle \alpha_\xi : \xi < \aleph_n \rangle$ in κ as follows: Given

p_ξ, α_ξ, let $\alpha_{\xi+1} \in (\alpha_\xi, \kappa)$ be such that there is some $q \le p_\xi$ with $q \Vdash \alpha_{\xi+1} \in \dot{C}$. Then let $p_{\xi+1} \le q$ be such that $\max \operatorname{dom} p_{\xi+1}(\kappa) > \alpha_{\xi+1}$. If ξ is a limit, let p_η be a lower bound of $\langle p_\xi : \xi < \eta \rangle$ and let $\alpha_\xi = \sup_{\eta<\xi} \alpha_\eta$, noting that $p_\eta \Vdash \alpha_\xi \in \dot{C}$.

Then $\sup_{\xi<\aleph_n} \alpha_\xi = \sup_{\xi<\aleph_n} \max \operatorname{dom} p_\xi(\kappa)$, and we can denote this ordinal by γ. Let $\bar{p}$ be a condition below each of the p_ξ's such that $\bar{p}(\kappa)(\gamma) = 1$. Then $\bar{p} \Vdash \gamma \in \dot{C} \cap \dot{S}_\kappa$. □

2.5. **Defining the Quotient Poset.** The purpose of this section is to define a poset that will be used to lift supercompact embeddings, and to prove that it has some reasonable properties.

Recall that $\mathbb{T}(\kappa, \lambda)$ is the poset that shoots a club of order-type κ through the complement of the stationary set added by $\mathbb{S}(\kappa, \lambda)$. Hence we let $\mathbb{T}(\kappa) = \mathbb{T}(\kappa, \aleph_n)$.

Definition 36. Let $p \in \mathbb{S}[\mu^+, \nu)$.

- We say that $T \in E(p)$ if T is function with domain $\operatorname{sprt} p \cap \mathcal{C}_A$ such that for all $\kappa \in \operatorname{sprt} p \cap \mathcal{C}_A$:
  (1) $p(\kappa) \Vdash_{\mathbb{S}(\kappa)} T(\kappa) \in \mathbb{T}(\kappa)$;
  (2) there is some $d \in W$ such that $p(\kappa) \Vdash T(\kappa) = \check{d}$;
  (3) $\max \operatorname{dom} p(\kappa) = \max T(\kappa)$.
- If $T \in E(p)$, $p^\frown T$ is a condition such that $\operatorname{sprt}(p^\frown T) = \operatorname{sprt} p$, $(p^\frown T)(\kappa) = (p(\kappa), T(\kappa))$ if $\kappa \in \mathcal{C}_A$, and $(p^\frown T)(\kappa) = p(\kappa)$ if $\kappa \in \mathcal{C}_C$.
- $\mathbb{D}[\mu^+, \nu)$ is the poset of conditions of the form $p^\frown T$ for $p \in \mathbb{S}[\mu^+, \nu)$, $T \in E(p)$. If $p, q \in \mathbb{D}[\mu^+, \nu)$, then $p \le q$ if $\operatorname{sprt} q \subseteq \operatorname{sprt} p$ and $p(\kappa)$ is coordinate-wise stronger than q, i.e. $p(\kappa) \le q(\kappa)$ for all $\kappa \in \operatorname{sprt} q$.

Lemma 37. *$\mathbb{D}[\mu^+, \nu)$ is μ^+-strongly strategically closed.*

Proof. We describe a decreasing sequence of conditions $\langle r_\xi : \xi < \mu^+ \rangle$ in $\mathbb{Q}[\mu^+, \nu)$, and we describe a strategy for Player II that allows play to continue at any $\xi < \mu^+$. To do this, we will describe conditions $\langle p_\xi : \xi < \mu^+ \rangle$ in $\mathbb{S}[\mu^+, \nu)$ and extensions $\langle T_\xi : \xi < \mu^+ \rangle$ such that $r_\xi = p_\xi^\frown T_\xi$. We will also use a sequence $\langle d_\xi^\kappa : \eta_\kappa \le \xi < \mu^+, \xi \text{ an even successor} \rangle$ where η_κ is such that $\xi \ge \eta_\kappa$ implies $\kappa \in \operatorname{sprt} r_\xi$ and d_ξ^κ is a closed bounded subset of κ that is an element of W.

Player II only plays at even successors $\xi = \eta + 1$. Then let $s \le p_\eta$ be a condition with $\operatorname{sprt} s = \operatorname{sprt} p_\eta$ such that for all $\kappa \in \operatorname{sprt} p_\eta \cap \mathcal{C}_A$, there is some closed bounded $c^\kappa \subseteq \kappa$ such that $s(\kappa) \Vdash T_\eta(\kappa) = c^\kappa$. Choose $s' \le s$ such that $\max \operatorname{dom} s'(\kappa) > c^\kappa$ for all $\kappa \in \operatorname{sprt} p_\eta \cap \mathcal{C}_A$ and $\max s'(\lambda^+) > \max p_\eta(\lambda^+)$ for all $\lambda^+ \in \operatorname{sprt} p_\eta \cap \mathcal{C}_C$. Then let $p_\xi = s'$, and let d_ξ^κ be defined for all $\kappa \in \operatorname{sprt} p_\xi$ such that $d_\xi^\kappa = c^\kappa \cup \{\max \operatorname{dom} q_\xi(\kappa)\}$. As in the weak strategic closure of $\mathbb{S}[\mu^+, \mathrm{ON})$, we consider two sub-cases for the purpose of an interleaving argument: If ξ is of the form $\xi' + 4k$ where ξ' is a limit and $k < \omega$, then Player II will additionally guarantee that $\max p_\xi(\lambda^+)$ is large enough so that $g_{p_\eta}^\lambda \le^* f_{p_\xi}^\lambda$, and if ξ is of the form $\xi' + 4k + 2$ where ξ' is a limit and $k < \omega$, then Player II will use Lemma 31 to guarantee that $f_{p_\eta}^\lambda \le^* g_{p_\xi}^\lambda$.

It remains to argue that play can continue at any limit stage ξ. Specifically, we claim that if $\langle r_\eta : \eta < \xi \rangle$ have already been defined—and hence $\langle p_\eta : \eta < \xi \rangle$ and $\langle T_\eta : \eta < \xi \rangle$ has already been defined—then we can found a lower bound regardless of whether Player I chooses to play that specific lower bound. Let p_ξ be defined so that $\operatorname{sprt} p_\xi = \bigcup_{\eta<\xi} \operatorname{sprt} p_\eta$, such that $p_\xi(\kappa) := \bigcup_{\eta' \le \eta < \xi} p_\eta(\kappa) \cup$

$\langle \sup_{\eta' \leq \eta < \xi} \max \operatorname{dom} p_\eta(\kappa), 0 \rangle$ for large enough η' and $\kappa \in \operatorname{sprt} p_\xi \cap \mathcal{C}_A$, and such that $p_\xi(\lambda^+) := \bigcup_{\eta' \leq \eta < \xi} p_\eta(\lambda^+) \cup \{\sup_{\eta' \leq \eta < \xi} \max p_\eta(\lambda^+)\}$ for $\lambda^+ \in \operatorname{sprt} p_\xi(\lambda^+)$.

Then p_ξ is a lower bound for $\langle p_\eta : \eta < \xi \rangle$. First, $\bigcup_{\eta < \xi} d^\kappa_\eta$ is a club avoiding $p_\xi(\kappa)$ for all $\kappa \in \operatorname{sprt} p_\xi \cap \mathcal{C}_A$. Furthermore, we can argue that all points of cofinality $\aleph_n$ from $p_\xi(\lambda^+)$ for $\lambda^+ \in \operatorname{sprt} p_\xi \cap \mathcal{C}_C$ satisfy the Annulment Property, in which case the discussion proceeds exactly as in the proof of strategic closure of $\mathbb{S}[\mu^+, \mathrm{ON})$, the key point being Proposition 23. Finally, for $\kappa \in \operatorname{sprt} r_\xi \cap \mathcal{C}_A$, let $T(\kappa) := \bigcup_{\eta' \leq \eta < \xi, \eta \text{ ev. succ.}} d^\kappa_\eta \cup \{\sup_{\eta' \leq \eta < \xi, \eta \text{ ev. succ.}} \max d^\kappa_\eta\}$. Then we can see that $r_\xi := p_\xi{}^\frown T$ is a lower bound for $\langle r_\eta : \eta < \xi \rangle$. □

Proposition 38. *There is a complete embedding ι from $\mathbb{S}[\mu^+, \nu)$ to $\mathbb{D}[\mu^+, \nu)$.*

Proof. The map ι sends p to $p^\frown T$ where $T(\kappa) = 1_{\mathbb{T}(\kappa)}$ for each $\kappa \in \operatorname{sprt} p \cap \mathcal{C}_A$. This map evidently preserves $\leq$ and $\perp$, and it is a complete embedding because given $r \in \mathbb{D}[\mu^+, \nu)$, if $p \in \mathbb{S}[\mu^+, \nu)$ is the version of r without the "T-part," then p is the reduction of r modulo the embedding. □

It follows that $\mathbb{D}[\mu^+, \nu)$ can be factored as $\mathbb{S}[\mu^+, \nu)$ and a quotient.

Definition 39. Let $\mathbb{Q}[\chi, \nu)$ be $\mathbb{D}[\chi, \nu)/\iota(H)$ where H is $\mathbb{S}[\chi, \nu)$-generic. More generally, $\mathbb{Q}[\mu^+, \nu)$ is defined as $\mathbb{D}[\mu^+, \nu)/\iota(H')$ where H' is $\mathbb{S}[\mu^+, \nu)$-generic.

In particular, $\mathbb{S}[\chi, \nu) * \mathbb{Q}[\chi, \nu)$ is forcing-equivalent to $\mathbb{D}[\chi, \nu)$.

Observation 40. *The poset $\mathbb{Q}[\chi, \nu)$ is a subset of W.*

This observation will be vital to our interleaving argument for stationary preservation.

Proposition 41. *For regular μ:*

- $\Vdash_{\mathbb{S}}$ "*$\mathbb{Q}[\chi, \mu]$ has size $\leq \mu$*".
- $\Vdash_{\mathbb{S}}$ "*$\mathbb{Q}[\mu^+, \nu)$ is μ^+-distributive*".

Proof. The first point follows from a counting argument. The second follows from the fact that $\mathbb{Q}[\mu^+, \nu)$ is a factor of the μ^+-strongly strategically closed, and hence μ^+-distributive, poset $\mathbb{D}[\mu^+, \nu)$. □

Proposition 42. *If μ is a regular cardinal, then $\mathbb{Q}[\chi, \mu] = \mathbb{Q}[\chi, \mu^+)$ preserves stationary subsets of μ^+.*

Despite these ostensibly nice properties, $\mathbb{Q}[\chi, \mu]$ is not $\aleph_{n+1}$-closed, which means that we need to be innovative to prove stationary preservation in general.

2.6. **Freezing Arguments for Stationary Preservation.** Our immediate goal is to prove that $\mathbb{Q}[\chi, \nu]$ preserves stationary subsets of $\nu \cap \operatorname{cof}(\aleph_n)$ for $\nu \in \mathcal{C}_B$. First we establish a general lemma:

Lemma 43. *(Freezing Lemma) Let μ be a regular cardinal, and $\mathbb{P}_1$ and $\mathbb{P}_2$ be posets such that $\Vdash_{\mathbb{P}_2}$ "μ is regular" and $\Vdash_{\mathbb{P}_2}$ "$\mathbb{P}_1$ is μ-c.c.". (In particular, we can suppose that $\mathbb{P}_1$ is a poset of size $< \mu$ and $\mathbb{P}_2$ is μ-distributive.) If $(p, q) \in \mathbb{P}_1 \times \mathbb{P}_2$ and $(p, q) \Vdash$ "$\dot{C} \subseteq \nu$ is a club" for some regular $\nu \geq \mu$, then for all $\beta < \nu$, there is some $q' \leq q$ and some $\alpha \in (\beta, \nu)$ such that $(p, q') \Vdash$ "$\alpha \in \dot{C}$".*

Proof. Let G_2 be $\mathbb{P}_2$-generic over V. In $V[G_2]$ there is a $\mathbb{P}_1$-name $\ddot{C}$ such that for any $\mathbb{P}_1$-generic G_1 over $V[G_2]$, $\dot{C}_{G_1 \times G_2} = \ddot{C}_{G_1}$. This is because, without loss of generality, $\dot{C}$ is a nice name. Hence we let $\langle \check{\alpha}, r \rangle \in \ddot{C}$ if and only if $\langle \check{\alpha}, (r,s) \rangle \in \dot{C}$ for some $s \in G_2$ (bearing in mind that there is an abuse of notation when referring to $\check{\alpha}$ because the definition of $\check{\alpha}$ depends on the poset being used). So if $(p,q) \Vdash$ "$\dot{C}$ is a club in ν", then $p \Vdash$ "$\ddot{C}$ is a club in ν".

Working in $V[G_2]$, the μ-c.c. of $\mathbb{P}_1$ implies that $\langle \alpha < \nu : p \Vdash \alpha \in \ddot{C} \rangle$ is a club in ν. Hence there is some $\alpha \in (\beta, \nu)$ such that $p \Vdash$ "$\alpha \in \ddot{C}$". Let $q' \leq q$, $q' \in G_2$ witness this. Then $(p, q') \Vdash$ "$\alpha \in \dot{C}$". □

There are two notable sub-cases for cardinals in $\mathcal{C}_B$. First we consider certain inaccessible cardinals.

Lemma 44. *Suppose H is $\mathbb{S}[\chi,\mu)$-generic over W. If μ is an inaccessible cardinal and $S \subseteq \mu \cap \mathrm{cof}(\aleph_n)$ is stationary in $W[H]$, then the stationarity of S is preserved by $\mathbb{Q}[\chi,\mu)$.*

Proof. Work in $W[H]$ and fix a stationary set $S \subseteq \mu \cap \mathrm{cof}(\aleph_n)$. Let $q_0 \in \mathbb{Q}[\chi,\mu)$, $q_0 \Vdash$ "$\dot{C} \subseteq \mu$" is a club. Let Θ be large enough for the following discussion, and consider the structure $\mathcal{H} := H_\Theta(\in, <_\Theta, \mathbb{Q}[\chi,\mu), q_0, \dot{C}, \mu)$. Using the stationarity of S, we can find an elementary submodel $M \prec \mathcal{H}$ such that $\delta := M \cap \mu = \sup(M \cap \mu) \in S$. Moreover, since μ is inaccessible (and we are working with GCH), we can find such an M that is closed under sequences of length less than $\aleph_n$. Fix a sequence $\langle \delta_\xi : \xi < \aleph_n \rangle$ converging to δ.

We will define a decreasing sequence of conditions $\langle q_\xi : \xi < \aleph_n \rangle \subseteq M$ in $\mathbb{Q}[\chi,\mu)$, an increasing sequence of ordinals $\langle \alpha_\xi : \xi < \aleph_n \rangle$, and an increasing sequence of cardinals $\langle \kappa_\xi : \xi < \aleph_n \rangle$ such that $\kappa_0 = \chi$, if ξ is a successor then κ_ξ is regular, and $\operatorname{sprt} q_\xi \subseteq \kappa_\xi$. For successor stages, we are given q_ξ, α_ξ, and we view $\mathbb{Q}[\chi,\mu)$ as $\mathbb{Q}[\chi,\kappa_\xi] \times \mathbb{Q}[\kappa_\xi^+,\mu)$ and use the Freezing Lemma to find $\alpha_{\xi+1} \in (\max\{\alpha_\xi, \delta_\xi\}, \delta)$ and $q_{\xi+1} \leq q_\xi$ such that $q_{\xi+1} \restriction [\chi,\kappa_\xi] = q_\xi$ and $q_{\xi+1} \Vdash \alpha_{\xi+1} \in \dot{C}$. And of course, $\kappa_{\xi+1}$ is a regular cardinal large enough that $\operatorname{sprt} q_{\xi+1} \subseteq \kappa_{\xi+1}$. If ξ is a limit, then let $\alpha_\xi = \sup_{\eta<\xi} \alpha_\eta$ and let $\kappa_\xi = \sup_{\eta<\xi} \kappa_\eta$. And let q_ξ be defined so that $\operatorname{sprt} q_\xi = \sup_{\eta<\xi} \operatorname{sprt} q_\eta$ and such that $q_\xi \restriction [\kappa_\eta, \kappa_{\eta+1}] = q_\eta$ for $\eta < \xi$. Then q_ξ will be an element of $\mathbb{Q}[\chi,\mu)$ and it will be the case that $q_\xi \Vdash \alpha_\xi \in \dot{C}$. Furthermore, we can see that $\alpha_\xi, \kappa_\xi, q_\xi$ are in M because they are defined with regard to the parameters from $\mathcal{H}$ and the sequence $\langle \delta_\eta : \eta < \xi \rangle$, which is in M.

Once we have $\langle q_\xi : \xi < \aleph_n \rangle$, let $\overline{q}$ be defined such $\operatorname{sprt} \overline{q} = \bigcup_{\xi<\aleph_n} \operatorname{sprt} q_\xi$ (note that $\operatorname{sprt} \overline{q} \subseteq \sup_{\xi<\aleph_n} \kappa_\xi < \mu$) and such that for all $\xi < \aleph_n$, $\overline{q} \restriction [\chi,\kappa_\xi] = q_\xi$. Then $\overline{q}$ is a lower bound of $\langle q_\xi : \xi < \aleph_n \rangle$, so $\overline{q} \Vdash \delta = \sup_{\xi<\aleph_n} \alpha_\xi \in \dot{C}$, and thus $\overline{q} \Vdash \dot{C} \cap S \neq \emptyset$. □

Now we consider the non-trivial case for singular cardinals in $\mathcal{C}_B$.

Lemma 45. *Suppose H is $\mathbb{S}[\chi,\lambda)$-generic over W. If λ is singular cardinal of cofinality $\aleph_n$ and $S \subseteq \lambda^+ \cap \mathrm{cof}(\aleph_n)$ is stationary in $W[H]$, then the stationarity of S is preserved by $\mathbb{Q}[\chi,\lambda)$.*

Proof. Work in $W[H]$ and fix a stationary set $S \subseteq \mu \cap \mathrm{cof}(\aleph_n)$. Let $q_0 \in \mathbb{Q}[\chi,\lambda)$, $q_0 \Vdash$ "$\dot{C} \subseteq \lambda^+$" is a club. Fix a sequence of regular cardinals $\vec{\lambda} = \langle \lambda_n : n < \omega \rangle$ converging to λ, let Θ be a large enough regular cardinal, and consider the structure

$\mathcal{H} := H_\Theta(\in, <_\Theta, \mathbb{Q}[\chi, \lambda), q_0, \dot{C}, \lambda^+, \vec{\lambda})$. Using the stationarity of S, we can find an elementary submodel $M \prec \mathcal{H}$ of size λ such that if $\delta = M \cap \lambda^+$, then $\delta \in S$. If $n = 0$, then we pick any sequence $\langle \delta_n : n < \omega \rangle$ converging to δ. If $n > 0$, then we will appeal to the fact that $\lambda^+ \cap \mathrm{cof}(\aleph_n) \in I[\lambda^+]$ (i.e. there is enough approachability) to choose M (and hence δ) so that there is a sequence $\langle \delta_\xi : \xi < \aleph_n \rangle$ converging to δ such that $\langle \delta_\xi : \xi < \eta \rangle \in M$ for all $\eta < \aleph_n$.

Fix an increasing and continuous sequence of cardinals $\langle \lambda_\xi : \xi < \aleph_n \rangle$ converging to λ such that $\lambda_0 = \chi$, and such that if ξ is a successor ordinal then λ_ξ is regular. We will define a decreasing sequence $\langle q_\xi : \xi < \aleph_n \rangle \subseteq M$ of conditions from $\mathbb{Q}[\chi, \lambda)$ and an increasing sequence of ordinals $\langle \alpha_\xi : \xi < \aleph_n \rangle$ as follows: If q_ξ and α_ξ have been defined, view $\mathbb{Q}[\chi, \lambda)$ as the product $\mathbb{Q}[\chi, \lambda_\xi] \times \mathbb{Q}[\lambda_\xi^+, \lambda)$ and use the Freezing Lemma to find $q_{\xi+1}$ and $\alpha_{\xi+1} \in (\max\{\alpha_\xi, \delta_\xi\}, \delta)$ such that $q_{\xi+1} \Vdash$ "$\alpha_{\xi+1} \in \dot{C}$" and $q_{\xi+1}(\kappa) = q_\xi(\kappa)$ for all $\kappa < \lambda_\xi$. If ξ is a limit, then let $\alpha_\xi = \sup_{\eta < \xi} \alpha_\eta$ and let q_ξ be a condition such that $q_\xi \restriction [\lambda_\eta, \lambda_{\eta+1}] = q_\eta \restriction [\lambda_\eta, \lambda_{\eta+1}]$ for all $\eta < \xi$, and such that if $\kappa \in [\lambda_\xi, \lambda)$, then then $q_\xi(\kappa)$ is a lower bound of $\langle q_\eta(\kappa) : \eta < \xi \rangle$ using the $\aleph_n$-closure of $\mathbb{T}(\kappa, \aleph_n)$. So q_ξ is an actual condition in $\mathbb{Q}[\chi, \lambda)$ and q_ξ forces that $\alpha_\xi \in C$. Furthermore, q_ξ is in M because it was defined from parameters in $\mathcal{H}$ and the sequence $\langle \delta_\eta : \eta < \xi \rangle \in M$.

Finally, define $\bar{q}$ such that $q \restriction [\lambda_\xi, \lambda_{\xi+1}] = q_\xi \cap [\lambda_\xi, \lambda_{\xi+1}]$ for all $\xi < \aleph_n$. Then $\bar{q}$ is a lower bound of $\langle q_\xi : \xi < \aleph_n \rangle$, so q forces that $\sup_{\xi < \aleph_n} \alpha_\xi$, i.e. δ, is in $\dot{C}$. Thus $\bar{q} \Vdash$ "$\dot{C} \cap S \neq \emptyset$". $\square$

2.7. **The Interleaving Argument for Stationary Preservation.** Now we turn our attention to stationary preservation for the cardinals in $\mathcal{C}_C$. We make use of the following very important property:

Proposition 46. *Fix $\lambda^+ \in \mathcal{C}_C$ and let $\bar{H}$ be $\mathbb{S}[\chi, \lambda^+]$-generic over W. There is a club $D_\lambda \subseteq \lambda^+$ in $W[\bar{H}]$ such that for all $\alpha \in D_\lambda \cap \mathrm{cof}(\aleph_n)$, there is some $\tau < \lambda$ such that for all $\kappa \in \mathrm{dom}\, f^\lambda_\alpha \cap (\tau, \lambda)$, $f^\lambda_\alpha(\kappa) \notin S_\kappa$.*

Proof. The club D_λ is defined to be $D^*_\lambda \cap \bigcup_{p \in H} p(\lambda^+)$ (recall that D^*_λ was defined as a club such that all $\alpha \in \lim D^*_\lambda \cap \mathrm{cof}(\aleph_n)$ are points of continuity for $\vec{f_\lambda}$). The fact that it is closed follows immediately from the definition, and the fact that it is unbounded follows from our ability, given any $\beta < \lambda^+$ to extend any $p \in \mathbb{S}[\chi, \lambda^+]$ to $q \leq p$ such that $\max q(\lambda^+) > \beta$. And if $q \in \bar{H}$ is such that $\alpha < \max \mathrm{dom}\, q(\lambda^+)$, then the Annulment Property of α witnesses the fact that there is some $\tau < \lambda$ such that $f^\lambda_\alpha(\kappa) \notin S_\kappa$ for all $\kappa \in \mathrm{dom}\, f^\lambda_\alpha \cap (\tau, \lambda)$. $\square$

The crux of this construction uses the fact that continuous points of the Easton scales $\vec{f_\lambda}$ are defined uniquely up to interleaving for large $\kappa < \lambda$. We need a lemma that shows that we can decide elements of a club added in the extension in such a way that allows us to throw away initial segments of the domains of the conditions while assuring that we are still making the correct decisions about the club.

We need to introduce some notation for the following discussion. Recall that elements of $\mathbb{Q}[\chi, \nu]$ formally belong to $\mathbb{D}[\chi, \nu]$ and so we can refer to their support.

Definition 47. If $q \in \mathbb{Q}[\chi, \nu]$, then $q[\mu, \nu]$ is a condition such that $q[\mu, \nu](\kappa)$ is the trivial condition for $\kappa \notin [\mu, \nu]$ and $q[\mu, \nu](\kappa) = q(\kappa)$ for $\kappa \in [\mu, \nu]$.

Lemma 48. *Suppose λ is a singular cardinal such that $\lambda^+ \in \mathcal{C}_C$ and let $\langle \lambda_\xi : \xi < \mathrm{cf}\, \lambda \rangle$ be a sequence of regular cardinals converging to λ. Suppose that $\Vdash_{\mathbb{Q}[\chi, \lambda^+]} \Vdash$*

"$\dot{C} \subseteq \lambda^+$ is a club" (i.e. this is forced by the empty condition). Then for all $\beta < \lambda^+$ and $q \in \mathbb{Q}[\chi, \lambda^+]$, there is some $\alpha \in (\beta, \lambda^+)$ and some $q' \le q$ such that for all $\xi < \operatorname{cf} \lambda$, $q'[\lambda^+_\xi, \lambda^+] \Vdash \alpha \in \dot{C}$.

Of course, $\mathbb{Q}[\chi, \lambda^+]$ is trivial at λ^+, but this is the interval we will be considering when we apply this lemma.

Proof. Starting by working in W, considering conditions in $\mathbb{D}[\chi, \lambda^+]$. Let $q = T$ and consider $p^\frown T \in \mathbb{D}[\chi, \lambda^+]$. Let K be a $\mathbb{D}[\chi, \operatorname{cf} \lambda]$-generic containing $(p^\frown T)[\chi, \operatorname{cf} \lambda]$. Note that the cofinality of λ^+ is preserved in $W[K]$.

Now we work in $W[K]$. For each $i < \omega$ we will define a decreasing sequence $\langle p_{i,\xi}{}^\frown T_{i,\xi} : \xi < \operatorname{cf} \lambda \rangle$ of conditions in $\mathbb{D}[(\operatorname{cf} \lambda)^+, \lambda^+]$ below $(p^\frown T)[(\operatorname{cf} \lambda)^+, \lambda^+]$ and a (not necessarily increasing) sequence $\langle \alpha_{i,\xi} : i < \omega, \xi < \operatorname{cf} \lambda \rangle$ of ordinals in the interval (β, λ^+). Furthermore, we will define $\alpha^*_i := \sup_{\xi < \lambda} \alpha_i$ as we proceed. We will use strong strategic closure of $\mathbb{D}[(\operatorname{cf} \lambda)^+, \lambda^+]$ to keep the construction going, but we will suppress the distinction between even and odd successor ordinals, and we will not repeat the specifics of the strategy.

At successor stages we are given $p_{i,\xi}{}^\frown T_{i,\xi} \in \mathbb{D}[(\operatorname{cf} \lambda)^+, \lambda^+]$ for some $\xi < \lambda$ and $i < \omega$. Since $(p_{i,\xi}{}^\frown T_{i,\xi})[\lambda^+_\xi, \lambda^+] \Vdash$ "$\dot{C} \subset \lambda^+$ is a club", we can use the Freezing Lemma to find $r \le p_{i,\xi}{}^\frown T_{i,\xi}$ and $\alpha_{i,\xi} \in (\alpha^*_i, \lambda^+)$ such that $r[\lambda^+_\xi, \lambda^+] \Vdash \alpha_{i,\xi} \in \dot{C}$. Then let $p_{i,\xi+1}{}^\frown T_{i,\xi+1} \le r$ be chosen according to the strategy. Lower bounds can be chosen at limit stages as in the proof of strong strategic closure of $\mathbb{D}[(\operatorname{cf} \lambda)^+, \lambda^+]$ and $\alpha_{i,\xi}$ for a limit ξ can be chosen as in the successor stage of this argument. The case we were have defined $p_{i,\xi}{}^\frown T_{i,\xi}$ for all $\xi < \lambda$ and need to define $p_{i+1,0}{}^\frown T_{i+1,0}$ is another limit case.

Let $\overline{p}^\frown \overline{T}$ be a lower bound for the whole sequence. Let $\alpha = \sup_{i<\omega} \alpha_i$ and let q be the "T-part" of $\bar{p}^\frown \bar{T}$. Then for any $\mathbb{S}[(\operatorname{cf} \lambda)^+, \lambda^+]$-generic K' containing $\overline{p}$, we have arranged so that $q[\lambda^+_\xi, \lambda^+]$ forces that $\dot{C} \cap [\alpha_i, \alpha_{i+1}) \neq \emptyset$ for all $i < \omega$, and hence $q[\lambda^+_\xi, \lambda^+]$ forces that $\alpha \in \dot{C}$. Hence $s^\frown(\overline{p}^\frown \overline{T})$ witnesses the lemma. □

Lemma 49. *If λ is a singular cardinal such that $\lambda^+ \in \mathcal{C}_C$, then $\mathbb{Q}[\chi, \lambda^+]$ preserves stationary subsets of $\lambda^+ \cap \operatorname{cof}(\aleph_n)$.*

Recall that given $p \in \mathbb{S}$, we defined g^λ_p, which has domain $\operatorname{sprt} p \cap \mathcal{C}_A \cap \lambda$ and maps κ to $\max \operatorname{dom} p(\kappa)$.

Proof. As in the freezing arguments, we work in $W[H]$. By the mixing of names, it is enough to consider a $\mathbb{Q}[\chi, \lambda^+]$-name $\dot{C}$ where $\Vdash_{\mathbb{Q}[\chi,\lambda^+]}$ "$\dot{C} \subseteq \lambda^+$" is a club. Let Θ be a large enough regular cardinal and consider the structure $\mathcal{H} := H_\Theta(\in, <_\Theta, \mathbb{Q}[\chi, \lambda^+], \dot{C}, \lambda^+)$. Using the stationarity of S, we can find an elementary submodel $M \prec \mathcal{H}$ of size λ such that if $\delta = M \cap \lambda^+ = \sup(M \cap \lambda^+)$, then—and this is the crux of the whole construction—we have $\delta \in S \cap D_\lambda$. If $n > 0$, we can use approachability to select M and δ such that there is a sequence $\langle \delta_\xi : \xi < \aleph_n \rangle$ such that $\langle \delta_\xi : \xi < \eta \rangle \in M$ for all $\eta < \xi$. Otherwise, if $n = 0$, let $\langle \delta_\xi : \xi < \aleph_0 \rangle$ be any sequence converging to δ

We will define a decreasing sequence $\langle q_\xi : \xi < \aleph_n \rangle \subseteq M$ of conditions in $\mathbb{Q}[\chi, \lambda^+]$ and an increasing and continuous sequence of ordinals $\langle \alpha_\xi : \xi < \aleph_n \rangle$. We will also make use of the function $g^\lambda_{q_\xi}$ where $\operatorname{dom} g = \operatorname{sprt} q_\xi \cap \lambda$ and $g(\kappa) = \max q_\xi(\kappa)$. For the successor case, suppose q_ξ is already defined. We can choose $q_{\xi+1} \le q_\xi$ such that $f^\lambda_{\delta_\xi} <^* g^\lambda_{q_{\xi+1}}$, and moreover by Lemma 48 we can choose $q_{\xi+1}$ such that there is some $\alpha_\xi \in (\delta_\xi, \delta)$ such that $q_{\xi+1}[\tau, \lambda^+] \Vdash \alpha_\xi \in \dot{C}$ for cofinally many $\tau < \lambda$. If ξ is a limit, let q_ξ be a lower bound of $\langle q_\eta : \eta < \xi \rangle$ using the $\aleph_n$-closure of $\mathbb{T}(\kappa, \aleph_n)$ for each κ, and let $\alpha_\xi = \sup_{\eta<\xi} \alpha_\eta$. These are contained in M because they are defined with the parameters from $\mathcal{H}$ and the sequence $\langle \delta_\eta : \eta < \xi \rangle \in M$.

We claim that $\langle g^\lambda_{q_\xi} : \xi < \aleph_n \rangle$ and $\langle f^\lambda_{\delta_\xi} : \xi < \aleph_n \rangle$ cofinally interleave each other. By the construction, we know that $f^\lambda_{\delta_\xi} <^* g^\lambda_{q_{\xi+1}}$. By elementarity of M, and the fact that $q_\xi \in W$ for any $\xi < \aleph_n$, there is some $\eta < \aleph_n$ such that $g^\lambda_{q_\xi} <^* f^\lambda_{\delta_\eta}$. Hence, if we define g such that $\operatorname{dom} g = (\aleph_n, \lambda) \cap \bigcup_{\xi<\aleph_n} \operatorname{dom} g^\lambda_{q_\xi}$, and $g(\kappa) = \sup_{\xi<\aleph_n} g^\lambda_{q_\xi}(\kappa)$, then we see that $g =^* f^\lambda_\delta$ by Proposition 23. In other words, there is some τ be such that $\operatorname{dom} g \cap (\tau, \lambda) = \operatorname{dom} f_\delta \cap (\tau, \lambda)$ and such that $\kappa \in \operatorname{dom} g \cap (\tau, \lambda)$ implies $g(\kappa) = f^\lambda_\delta(\kappa)$, and thus $g(\kappa) \notin S_\kappa$. It follows that there is a lower bound $\overline{q}$ of $\langle q_\xi[\tau^+, \lambda^+] : \xi < \aleph_n \rangle$ and that $\overline{q} \Vdash \delta \in \dot{C} \cap S$. □

2.8. **Lifting the Embeddings.** Since subsection 2.2 we have been working in the model W. Recall that χ is supercompact in V. Recall also that $W = V[G]$ where G is $\mathbb{C}$-generic over V and $\mathbb{C}$ is defined in one of two ways. Either $n = 0$ and $\mathbb{C} = \operatorname{Col}(\aleph_1, < \chi)$ or else $n > 0$ and $\mathbb{C} = \operatorname{Col}(\aleph_{n-1}, < \psi) \times \operatorname{Col}(\psi^+, < \chi)$ where ψ is a weakly compact cardinal below χ. In particular, $V[G] \models$ "$\chi = \aleph_{n+2}$". We will also continue referring to cardinals $\geq \chi$ as belonging to one of $\mathcal{C}_A, \mathcal{C}_B$, or $\mathcal{C}_C$. In this section we will finally use the supercompactness of χ.

Our main tool will be a lifting argument that is due to Silver.

Fact 50. [2] *If $j : V \to M$ is an embedding, G is $\mathbb{P}$-generic over V, H is $j(\mathbb{P})$-generic over M, and $j[G] \subseteq H$, then j can be lifted to $j' : V[G] \to M[j(G)]$.*

Another key fact is in its original form due to Solovay, and it will allow us to set up more stationary preservation. Its purpose it to make ugly quotients behave nicely.

Fact 51. *(Absorption Theorem) Suppose κ is a regular cardinal and that $\mathbb{P}$ is a separative and κ-strongly strategically closed poset such that $|\mathbb{P}| < \lambda$. Then there is a complete embedding $\iota : \mathbb{P} \to \operatorname{Col}(\kappa, < \lambda)$ such that if G is $\mathbb{P}$-generic over V, then $\operatorname{Col}(\kappa, < \lambda)$ is forcing-equivalent to $\operatorname{Col}(\kappa, < \lambda)/\iota(G)$. Moreover, this works if $\operatorname{Col}(\kappa, < \lambda)$ is replaced by $\operatorname{Col}(\kappa, A)$ where $\sup A = \lambda$.*

We have two remarks on this version of the Absorption Theorem, which appears in a few other guises (the best source is Cummings' chapter in the Handbook [2]). First, the statement occasionally includes the hypothesis that λ is inaccessible, but this is not necessary—it implies that $\operatorname{Col}(\kappa, < \lambda)$ has the λ-chain condition, which is circumstantially helpful but not required. Second, the statement of the Absorption Theorem usually includes a hypothesis about the closure of $\mathbb{P}$, but here we are using strong strategic closure. This is in fact enough: the core of the proof of the Absorption Theorem is the fact that, $\mathbb{P}$ is forcing-equivalent to $\operatorname{Col}(\kappa, \lambda)$ if it is separative, κ-closed, has cardinality λ, and collapses λ to have size κ [9]. The reader can verify that it is enough to assume that $\mathbb{P}$ is κ-strongly strategically closed. Also, it is worth noting that the strongly strategically closed version of the Absorption Theorem has been used elsewhere [10].

We need another stationary-preservation fact for another component of our lifting argument.

Fact 52. *If $\mathbb{P}$ is κ^+-weakly strategically closed, $\mu\cap\mathrm{cof}(\kappa) \in I[\mu]$, and $S \subseteq \mu\cap\mathrm{cof}(\kappa)$ is stationary, then forcing with $\mathbb{P}$ preserves the stationarity of S.*

The proof of this fact is very similar to the proof of the fact that κ^+-closed posets preserve stationary subsets of $\mu \cap \mathrm{cof}(\kappa)$ if $\mu \cap \mathrm{cof}(\kappa) \in I[\mu]$, which can be found in several good sources [8] [1].

Finally, we will need to apply Fact 52 to a two-step iteration, so we need one more item.

Proposition 53. *If $\mathbb{P}$ is κ-closed and $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is κ-strategically closed, then $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-strategically closed.*

We include the proof for skeptical readers:

Proof. Consider a play $\langle (p_\xi, \dot{q}_\xi) : \xi < \kappa\rangle$ and let $\dot{\sigma}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\dot{\sigma}$ is a strategy for $\dot{\mathbb{Q}}$". We describe the strategy of Player II as follows: If $\xi = \eta + 1$ is an even successor, choose $p_\xi \le p_\eta$ and $\dot{q}_\xi$ such that $p_\xi \Vdash$ "$\dot{q}_\xi = \dot{\sigma}(\langle \dot{q}_\zeta : \zeta \le \eta\rangle)$". If ξ is a limit, then let p_* be a lower bound of $\langle p_\eta : \eta < \xi\rangle$. Then p_* forces that for all $\eta < \xi$ where Player II chooses conditions, $\dot{q}_\eta = \dot{\sigma}(\langle q_\zeta : \zeta < \eta\rangle)$, hence there is some $p_\xi \le p_*$ and some $\dot{q}_\xi$ such that $p_\xi \Vdash$ "$\forall \eta < \xi, \dot{q}_\xi \le \dot{q}_\eta$". And it is evident that this strategy works. □

Now we can do the main work of this section.

Lemma 54. *Suppose that one of the following holds:*

(1) $\mu \in \mathcal{C}_B$, $\bar{H}$ is $\mathbb{S}[\chi, \mu]$-generic, and $S \in V[G * \bar{H}]$ is a stationary subset of $\mu \cap \mathrm{cof}(\aleph_n)$.
(2) $\lambda^+ \in \mathcal{C}_C$, $\bar{H}$ is $\mathbb{S}[\chi, \lambda^+]$-generic, and $S \in V[G * \bar{H}]$ is a stationary subset of $\lambda^+ \cap \mathrm{cof}(\aleph_n)$.

Then S reflects in $V[G * \bar{H}]$.

Proof. Let $\nu = \mu$ or $\nu = \lambda^+$, depending on the case we are considering. Most of this lemma consists in proving the following:

Claim. *Suppose $j : V \to M$ is a ν-supercompact embedding with critical point χ and suppose one of the following holds:*

(1) $\nu = \mu \in \mathcal{C}_B$, G is $\mathbb{C}$-generic over V, $\bar{H}$ is $\mathbb{S}[\chi, \nu)$-generic over $V[G]$, and $S \subseteq \mu \cap \mathrm{cof}(\aleph_n)$ is stationary and is in $V[G * \bar{H}]$.
(2) $\nu = \lambda^+ \in \mathcal{C}_C$, G is $\mathbb{C}$-generic over V, $\bar{H}$ is $\mathbb{S}[\chi, \nu]$-generic over $V[G]$, and $S \subseteq \nu \cap \mathrm{cof}(\aleph_n)$ is stationary and is in $V[G * \bar{H}]$.

Then there is an extension $V[G * \bar{H} * L]$ in which S is still stationary and j can be lifted to $j^+ : V[G * \bar{H}] \to M[j^+(G * \bar{H})]$.

Proof of Claim. We will perform the lift in several stages. We will also do the proof for the case where $n > 0$, and $\mathbb{C} = \mathrm{Col}(\aleph_{n-1}, <\psi) \times \mathrm{Col}(\psi^+, <\chi) = \mathbb{C}_0 \times \mathbb{C}_1$, because this is strictly more difficult than the $n = 0$ case.

If $n > 0$ then we let G factor as $G_0 \times G_1$ where G_0 is $\mathrm{Col}(\aleph_{n-1}, <\psi)$-generic. The first step of the lift comes from the fact that $\mathrm{Col}(\aleph_{n-1}, <\psi) \in V_\chi$, hence $j(\mathrm{Col}(\aleph_{n-1}, <\psi)) = \mathrm{Col}(\aleph_{n-1}, <\psi)$, and so we can apply Fact 50 to show that j lifts to $j^0 : V[G_0] \to M[j^0(G_0)] = M[G_0]$. The stationarity of S is preserved because $|\mathrm{Col}(\aleph_{n-1}, <\psi)| < \nu$.

The next task is to lift the embedding through the forcing $\mathrm{Col}(\aleph_{n+1}, <\chi)$, which is the interpretation of $\mathrm{Col}(\psi^+, <\chi)$ in $V[G_0]$ because if $n > 0$ then $\psi^{V[G_0]} = \aleph_n$.

We observe that $j^0(\mathbb{C}_1) = j^0(\mathrm{Col}(\aleph_{n+1}, < \chi)) = \mathrm{Col}(\aleph_{n+1}, < \chi) \times \mathbb{R} = \mathbb{C}_1 \times \mathbb{R}$ where $\mathbb{R} := \prod_{\alpha \in [\chi, j^0(\chi))} \mathrm{Col}(\aleph_{n+1}, \alpha)$.

Next we use the quotient forcing.

Case 1: $\nu = \mu \in \mathcal{C}_B$. Then we let I be $\mathbb{Q}[\chi, \mu]$-generic over $V[G * \bar{H}]$. Then $\mathbb{Q}[\chi, \mu] \cong \mathbb{Q}[\chi, \mu)$ since $F(\mu) = 0$. Hence the stationarity of S is preserved by: Proposition 42 if μ is a successor of a regular; Lemma 44 if μ is inaccessible; Lemma 45 if μ is the successor of a singular of cofinality $\aleph_n$; and by the fact that $|\mathbb{Q}[\chi, \mu)| < \mu$ if μ is the successor of a singular λ and $\{\kappa < \lambda : F(\kappa) = 1\}$ is bounded in μ.

Case 2: $\nu = \lambda^+ \in \mathcal{C}_C$. We let I be $\mathbb{Q}[\chi, \lambda^+]$-generic over $V[G * \bar{H}]$. Then the stationarity of S is preserved by Lemma 49.

We proceed to work in $V[G * \bar{H} * I]$ where $\bar{H}$ is $\mathbb{S}[\chi, \nu]$-generic and I is $\mathbb{Q}[\chi, \nu]$-generic.

Since $\mathbb{S}[\chi, \nu] * \mathbb{Q}[\chi, \nu]$ is forcing-equivalent to the $\aleph_{n+1}$-strongly strategically closed poset $\mathbb{D}[\chi, \nu]$, we can apply the Absorption Theorem to find a complete embedding $\iota : \mathbb{S}[\chi, \nu] * \mathbb{Q}[\chi, \nu] \to \mathbb{R}$ such that $\mathbb{R}/\iota(\bar{H} * I)$ is forcing-equivalent to $\mathbb{R}$. Then let J be $\mathbb{R}$-generic over $V[G * \bar{H} * I]$. Now the embedding j^0 can be lifted to j' in this model because $j^0[G_1] \subseteq G_1 * \bar{H} * I * J$ (where conditions in G_1 are sent to themselves in the first coordinate). Hence we get a lift $j' : V[G] \to M[j'(G)]$.

Now we are working in $V[G*\bar{H}*I*J]$. Recall that $V[G] \models$ "$\chi = \aleph_{n+2}$". We claim that $j'(\mathbb{S}[\chi, \nu])$ is $\aleph_{n+1}$-strategically closed in $V[j'(G)]$: We have established that $\mathbb{S}[\chi, \nu]$ is $\aleph_{n+1}$-strategically closed in $V[G]$, and so $M[j'(G)] = M[G * \bar{H} * I * J] \models$ "$j'(\mathbb{S}[\chi, \nu])$ is $\aleph_{n+1}$-strategically closed" by elementarity. Because $M^\nu \subseteq M$ and because $j(\mathbb{C}_1)$ is $\aleph_{n+1}$-closed (using the fact that $\aleph_n$ is below the critical point of the embedding), we have that $M[j'(G)]$ is closed under sequences of length $\aleph_n$, and so $V[G * \bar{H} * I * J] \models$ "$j'(\mathbb{S}[\chi, \nu])$ is $\aleph_{n+1}$-strategically closed" because the failure of this would be witnessed at some ordinal $\alpha < \aleph_{n+1}$.

The iteration of $\mathbb{R} * j'(\mathbb{S}[\chi, \nu])$ preserves the stationarity of S over $V[G*\bar{H}*I*J]$: The iteration is $\aleph_{n+1}$-strategicially closed by Proposition 53, $\nu \cap \mathrm{cof}(\aleph_n) \in I[\nu]$ in the model $V[G*\bar{H}*I]$ by Proposition 25 and the fact that $\mathbb{D}[\chi, \nu]$ is $\aleph_{n+1}$-distributive over $V[G]$. If ν is the successor of a singular cardinal, we can apply Fact 52, and if ν is inaccessible then we can use an argument in the vein of Lemma 44. Either way we conclude that S remains stationary in $V[G * \bar{H} * I * J * K]$ if K is $j'(\mathbb{S}[\chi, \nu])$-generic over $V[G * \bar{H} * I * J]$.

But it remains to prove that we can find a generic for $j'(\mathbb{S})$ that allows us to apply Fact 50. For this we use a master condition argument. Define p as follows:

- $\mathrm{sprt}\, p = \{j'(\kappa) : \kappa \in [\chi, \nu] \setminus \mathcal{C}_B\}$;
- for all $\kappa \in \mathcal{C}_A$, $\mathrm{dom}\, p(j'(\kappa)) = \sup j'[\kappa] + 1$ and for all $\alpha \leq \sup j'[S_\kappa]$, $p(j'(\kappa))(\alpha) = 1$ if and only if $\alpha \in j'[S_\kappa]$;
- for all $\lambda^+ \in \mathcal{C}_C$, $p(j'(\lambda^+))$ is the closure of $j[D_\lambda]$ where D_λ comes from Proposition 46.

Claim. *p is a condition in $j'(\mathbb{S})$.*

Proof of Claim. The domain of p has Easton support from the point of view of $M[j(G)]$ because for all regulars $\kappa \in [\chi, \nu]$, the fact that $M^\kappa \subseteq M$ implies that $\sup j[\kappa] < j(\kappa)$, and hence $\sup j'[\kappa] < j'(\kappa)$. For each $\kappa \in \mathrm{sprt}\, p \cap \mathcal{C}_A$, let T_κ be the club added by $\mathbb{Q}[\chi, \nu]$ that avoids S_κ. Since j' is continuous for sequences of ordinals of length $\leq \aleph_n$, and T_κ avoids S_κ, it follows that $j[T_\kappa]$ avoids $j(S_\kappa)$.

We are left to verify the Annulment Property for points of cofinality $\aleph_n$ in the closure C of $j[D_\lambda]$ for $\lambda^+ \in \mathcal{C}_C$. Observe that if $\delta := \sup j'[D_\lambda] = \sup j'[\lambda^+]$, then $(\operatorname{cf}\delta)^V = \lambda^+$ and $(\operatorname{cf}\delta)^{M[j'(G)]} > \aleph_n$ by the $\aleph_{n+1}$-distributivity of $\mathbb{C} * \mathbb{R}$. Hence, if $\alpha \in j'(D_\lambda)$ has cofinality $\aleph_n$, then $\alpha = j'(\beta)$ where $\beta \in D_\lambda \cap \operatorname{cof}(\aleph_n)$, again by continuity of j' for sequences of length $\leq \aleph_n$.

Let $\bar{p} \in \bar{H}$ be a condition such that $\beta \in p(\lambda^+)$. Then by elementarity, the following is true in $M[j(G)]$:

> There is some $\tau < j'(\lambda)$ such that $\operatorname{dom}(j'f)^{j'(\lambda)}_\alpha \cap (\tau, j'(\lambda)) \subseteq \operatorname{sprt} j'(\bar{p})$ and such that for all $\kappa \in \operatorname{dom}(j'f)^{j'(\lambda)}_\alpha \cap (\tau, j'(\lambda))$, $(j'f)^{j'(\lambda)}_\alpha(\kappa) \in \operatorname{dom} j'(\bar{p})(\kappa)$ and $j'(\bar{p})(\kappa)((j'f)^{j'(\lambda)}_\alpha(\kappa)) = 0$.

Since $j'(\bar{p})$ satisfies the Annulment Property, the master condition p satisfies the Annulment Property for α because it extends $\bar{p}$ as a function as in Proposition 27. That is, since the Annulment Property was verified with respect to $j'(\bar{p})$ as witnessed by the interval $\operatorname{dom}(j'f)^{j'(\lambda)}_\alpha \cap (\tau, j'(\lambda))$, it is does not matter that the support of $\bar{p}$ is larger. □

Now that we have a master condition, force with a $j'(\mathbb{S}[\chi,\nu))$-generic K that contains p and let $L = I * J * K$. This allows us to extend j' to $j^+ : V[G * \bar{H}] \to M[j^+(G * \bar{H})]$, and so we have proved the claim. □

We work in $V[G * \bar{H} * L]$ in which our stationary set $S \subseteq \nu \cap \operatorname{cof}(\aleph_n)$ remains stationary. We consider $j(S)$ and $\rho := \sup j^+[\nu]$, noting that $\rho < j^+(\nu)$. We can argue that $M^+[j(G * \bar{H})] \models$ "$j^+(S) \cap \rho$ is stationary in ρ" as follows: Suppose $C \subseteq \rho$ is a club belonging to $M[j^+(G * \bar{H})]$.

Claim. $\bar{C} := \{\alpha < \nu : j^+(\alpha) \in C\}$ *is unbounded in* ρ *and* $\aleph_{n+1}$*-closed.*

Proof of Claim. The facts that C is a club and that j^+ is continuous for sequences of length $< \aleph_{n+1}$ imply that $\bar{C}$ is $\aleph_{n+1}$-closed. For unboundedness, we define $\langle \alpha_n : n < \omega \rangle \subset C$ and $\langle \beta_n : n < \omega \rangle \subset \nu$ as follows: Given α_n, find $\beta_{n+1} < \nu$ such that $j^+(\beta_{n+1}) > \alpha_n$, and given β_n, find $\alpha_{n+1} \in C$ such that $j^+(\beta_n) < \alpha_{n+1}$. Let $\bar{\gamma} := \sup_{n<\omega} \beta_n$. Then $j^+(\bar{\gamma}) = \sup_{n<\omega} \alpha_n$ by interleaving, so $j^+(\bar{\gamma}) \in C$ and thus $\bar{\gamma} \in \bar{C}$. □

Therefore, $\bar{C}$ intersects S, C intersects $j^+(S) \cap \rho$ and $M[j^+(G * \bar{H})] \models$ "$\exists \rho < j^+(\nu), j^+(S) \cap \rho$ is stationary". By elementarity, $V[G * \bar{H}] \models$ "S reflects". □

2.9. **Finishing the Theorem.** Now we can tie everything together, keeping in mind that $V[G] \models$ "$\chi = \aleph_{n+2}$".

Proof of Theorem 1. We have demonstrated that $\mathbb{S}$ adds non-reflecting stationary sets where directed by F—that is, $\kappa \in [\aleph_{n+2}, \mathrm{ON})$ such that $F(\kappa) = 1$. If $F(\mu) = 0$, then we consider the factoring $\mathbb{S}[\aleph_{n+2}, \mathrm{ON}) = \mathbb{S}[\aleph_{n+2}, \mu] \times \mathbb{S}[\mu^+, \mathrm{ON})$. For any stationary subset S of $\mu \cap \operatorname{cof}(\aleph_n)$, the distributivity of $\mathbb{S}[\mu^+, \mathrm{ON})$ implies that S is already contained in $V[G * \bar{H}]$ where $\bar{H}$ is $\mathbb{S}[\aleph_{n+2}, \mu]$-generic. And since $F(\mu) = 0$, the previous section shows that S reflects. □

3. Further Questions

We believe that our result can be extended beyond the $\aleph_n$'s:

Conjecture 1. *Suppose that in V, χ is supercompact and F is a two-valued function on the class of regular cardinals $\geq \chi$. Then for every $\phi < \chi$, there is a forcing extension $W \supset V$ such that $W \models \chi = \phi^{++}$ and such that for all cardinals $\kappa \geq \chi$, $W \models \mathsf{SR}(\kappa \cap \mathrm{cof}(\phi))$ if and only if $F(\kappa) = 0$.*

We are asserting that, for example, an Easton-style result can be obtained for $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_{\omega+1}))$. The main task in verifying this conjecture would be to obtain enough approachability where it is needed, and it should be possible to force approachability at all successors of singulars without using the result of Shelah. The idea would be augment $\mathbb{S}$ by also shooting clubs through the set of approachable points at successors of singulars λ^+ where reflection should be preserved.

The next question asks whether our use of Easton scales and the Annulment Property are demonstrably necessary for our construction:

Question 1. *Let λ be a singular cardinal, let $\langle \lambda_i : i < \mathrm{cf}\,\lambda \rangle$ be a sequence of regular cardinals converging to λ, let $\tau \neq \mathrm{cf}\,\lambda$ be regular, and let $\mathbb{S}(\lambda_i, \tau)$ for $i < \mathrm{cf}\,\lambda$ be as in Definition 7. If $\mathbb{P} := \prod_{i<\mathrm{cf}\,\lambda} \mathbb{S}(\lambda_i, \tau)$ is a full-support product, then does $\mathbb{P}$ add a non-reflecting stationary subset of $\lambda^+ \cap \mathrm{cof}(\tau)$?*

We are interested in the answer to this question given a general setting, because if $V \models \lambda^+ \in I[\lambda^+]$, then we will have $V[\mathbb{P}] \models \neg\mathsf{SR}(\lambda^+ \cap \mathrm{cof}(\tau))$. Here is a sketch: Given a continuous scale $\vec{f} = \langle f_\alpha : \alpha < \lambda^+ \rangle$, the non-reflecting stationary set S is essentially the stationary subset of $\lambda^+ \cap \mathrm{cof}(\tau)$ that is being destroyed by the club-shooting in the fourth point of Definition 26 (the definition of the main poset). That S is stationary can be proved with a straightforward genericity argument. To show that S is non-reflecting at a given point $\rho < \lambda^+$, one must take a club C_ρ defined like S but in terms of the clubs in $f_\rho(i)$ for $i < \mathrm{cf}\,\lambda$ that witness non-reflection of the generic object added by $\mathbb{S}(\lambda_i, \tau)$. The tricky point is that the goodness of ρ appears necessary to show that C_ρ is in fact unbounded. Hence, even though there is a natural methodological approach for this question that tells us something about models that satisfy both $\mathsf{SR}(\lambda^+)$ and $\lambda^+ \in I[\lambda^+]$ (see [12]), it is difficult to settle it in full generality.

We also consider limitations on the extent to which our result can be generalized:

Question 2. *Can the results of this paper be extended to stationary sets concentrating on points of arbitrary (or un-fixed) cofinality? In other words, suppose that F is a function on a class of regular cardinals to itself such that $F(\kappa)^+ < \kappa$ for all $\kappa \in \mathrm{dom}(F)$. Is it possible to obtain a model such that $\mathsf{SR}(\kappa \cap \mathrm{cof}(\lambda))$ holds precisely when $F(\kappa) = \lambda$?*

We suspect that the answer to this question is negative, and that there is a Silver's Theorem for stationary reflection that is waiting to be discovered.

Lastly, we have:

Question 3. *Does ZFC put any restrictions on the global behavior of $\square_\kappa$? Suppose F is a two-valued function on the class of all cardinals. Is it possible to obtain a model such that $\square_\kappa$ holds precisely if $F(\kappa) = 1$? And if there are ZFC restrictions, what exactly are they?*

Some progress has been made for this question. Cummings, Foreman, and Magidor constructed a model in which $\square_{\aleph_n}$ holds for all $n < \omega$, but where $\square_{\aleph_\omega}$ fails [4]. However, it appears difficult to generalize this result to singulars of uncountable cofinality. Cummings et al. also showed that the existence of square sequences below a singular cardinal κ implies the existence of something resembling but distinct from a $\Box_\kappa$-sequence [5]. It may be possible to take their argument further.

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