The Semi-Weak Square Principle

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Abstract

Cummings, Foreman, and Magidor proved that for any $\lambda < \kappa$, $\square_{\kappa,\lambda}$ implies that κ carries a PCF-theoretic object called a very good scale, but that $\square^*_\kappa$ (which one could write as $\square_{\kappa,\kappa}$) is consistent with the absence of a very good scale at κ. They asked whether $\square_{\kappa,<\kappa}$ is enough to imply the existence of a very good scale for κ, and we resolve this question in the negative. Furthermore, we sharpen a theorem of Cummings and Schimmerling and show that $\square_{\kappa,<\kappa}$ implies the failure of simultaneous stationary reflection at κ^+ for any singular κ. This implies as a corollary that if Martin's Maximum holds, then $\square_{\kappa,<\kappa}$ fails for singular cardinals κ of cofinality ω_1.

Keywords: Set Theory, Forcing, Large Cardinals

1. Background

Singular cardinals occupy a curious place in set theory. They are amenable to the forcing technique devised by Cohen to establish the independence of the continuum hypothesis, but many of their properties can be proved outright from ZFC using the PCF theory developed by Shelah. The properties of singular cardinals in a given model depend on a fundamental tension between the extent to which the model supports large cardinals, and the extent to which it resembles Gödel's constructible universe L.

The square principle, denoted $\square_\kappa$ if it holds at a cardinal κ, was distilled by Jensen in order to find Suslin trees at successors of singular cardinals in L [8], and it embodies many of the constructions that can be carried out in L. It can hold in larger models, but it fails above supercompact cardinals. Jensen also defined a weak square principle, denoted $\square^*_\kappa$, and later on Schimmerling introduced a hierarchy of intermediate principles $\square_{\kappa,\lambda}$ for $1 \leq \lambda \leq \kappa$ in order to study a wider range of inner models [12]. In Schimmerling's notation, $\square_\kappa$ is equivalent to $\square_{\kappa,1}$ and $\square^*_\kappa$ is equivalent to $\square_{\kappa,\kappa}$.

In their paper, "Squares, Scales, and Stationary Reflection," Cummings, Foreman, and Magidor demonstrated a complex interplay between generalized square principles, the reflection properties of large cardinals, and the scales used in PCF theory [3]. They proved that for a singular κ and $\lambda < \kappa$, $\square_{\kappa,\lambda}$ implies the existence of a so-called very good scale, and they showed that a very good scale implies the failure of simultaneous stationary reflection at κ^+. On the other hand, they constructed a model (using countably many supercompact cardinals) in which both $\square^*_\kappa$ and simultaneous reflection at κ^+ hold—hence there is no very good scale at κ in that model. They asked whether the semi-weak square, $\square_{\kappa,<\kappa}$, implies the existence of a very good scale, and we will prove here that it does not—assuming the existence of a supercompact cardinal. Furthermore, we show that $\square_{\kappa,<\kappa}$ does nonetheless imply the failure of simultaneous stationary reflection.

Large cardinals are necessary for our first result. If κ is a singular cardinal that does not carry a very good scale, then $\square_\kappa$ fails, and hence $0^\#$ exists by Jensen's Covering Lemma [6]. Although it is possible for $\square_\kappa$ to hold in models that satisfy certain large cardinals [13], the failure of $\square_\kappa$ has strong consequences for inner models [11]. An exact lower bound for the non-existence of a very good scale is unknown.

The conceptual context for this paper is contained entirely in Cummings, Foreman, and Magidor's original study [3]. We aim to give a reasonably self-contained exposition for anyone with a working knowledge of forcing theory and supercompact cardinals.

The essential definitions for this paper are as follows:

Definition 1. If $2 < \mu \leq \kappa$, then we say that $\square_{\kappa,<\mu}$ *holds* if there is a $\square_{\kappa,<\mu}$-*sequence*, which is a sequence $\langle \mathcal{C}_\alpha : \alpha \in \lim(\kappa^+) \rangle$ such that for all $\alpha \in \lim(\kappa^+)$:

- $\mathcal{C}_\alpha$ consists of clubs $C \subset \alpha$ of order-type less than or equal to κ;
- $\forall C \in C_\alpha, \forall \beta \in \lim C$, $C \cap \beta \in \mathcal{C}_\beta$;
- $1 \leq |\mathcal{C}_\alpha| < \mu$.

We refer to $\square_{\kappa,<\kappa}$ as the *semi-weak square principle*.

The principles $\square_{\kappa,\mu}$ are defined similarly, but where the second inequality in the third bullet point is not strict. The semi-weak square principle is weaker than $\square_\kappa$, in which the $\mathcal{C}_\alpha$ sets are singletons, but stronger than $\square^*_\kappa$, in which the $\mathcal{C}_\alpha$ sets can have size κ. The author previously constructed a model in which $\square_{\kappa,<\kappa}$ holds while $\square_{\kappa,\lambda}$ fails for $\lambda < \kappa$ [9]. Since $\square_{\kappa,\lambda}$ for $\lambda < \kappa$ implies the existence of a very good scale at singular κ (a notion that we will define momentarily), we automatically know that $\square_{\kappa,\lambda}$ fails for all $\lambda < \kappa$ in the model that we construct in this paper.

Definition 2. Let κ be singular and let $\langle \kappa_i : i < \operatorname{cf} \kappa \rangle$ be a sequence of regular cardinals converging to κ. Consider the ordering of eventual domination, where $f <^* g$ if there is some $j < \operatorname{cf} \kappa$ such that for all $i \geq j$, $f(i) < g(i)$. A *scale* at κ is a sequence $\langle f_\alpha : \alpha < \kappa^+ \rangle$ of functions with domain $\operatorname{cf} \kappa$ such that:

- $\forall \alpha, \forall i < \operatorname{cf} \kappa$, $f_\alpha(i) < \kappa_i$;

- $\forall \alpha < \beta < \kappa^+$, $f_\alpha <^* f_\beta$;
- $\forall g : \mathrm{cf}\,\kappa \to \mathrm{ON}$ such that $\forall i < \mathrm{cf}\,\kappa, g(i) < \kappa_i$, $\exists \alpha < \kappa^+$, $g <^* f_\alpha$.

In other words, a scale is a sequence of functions of length κ^+ in a given product $\prod_{i<\mathrm{cf}\,\kappa} \kappa_i$ that is increasing and cofinal in the ordering of eventual domination.

Definition 3. For a singular cardinal κ and a scale $\vec{f} = \langle f_\alpha : \alpha < \kappa^+ \rangle$ in a product $\prod_{i<\mathrm{cf}\,\kappa} \kappa_i$, a point $\alpha < \kappa^+$ is *very good* if $\mathrm{cf}\,\alpha > \mathrm{cf}\,\kappa$ and there is a club $C \subset \alpha$ and an index $j < \mathrm{cf}\,\kappa$ such that for all $i \geq j$, $\langle f_\beta(i) : \beta \in C \rangle$ is strictly increasing. A scale $\vec{f} = \langle f_\alpha : \alpha < \kappa^+ \rangle$ is *very good* if all points α such that $\mathrm{cf}\,\alpha > \mathrm{cf}\,\kappa$ are very good.

Note that a very good scale exists at κ if and only there is a scale $\vec{f} = \langle f_\alpha : \alpha < \kappa^+ \rangle$ and a club $D \subset \kappa^+$ such that every point $\alpha \in D$ of cofinality greater than $\mathrm{cf}\,\kappa$ is very good. There are also good scales and better scales, both of which are weaker notions than very good scales, but they fall outside the scope of this paper.

Definition 4. Consider a regular cardinal τ and a stationary subset $S \subset \tau$. S *reflects* at $\alpha < \tau$ if $\mathrm{cf}\,\alpha > \omega$ and $S \cap \alpha$ is stationary as a subset of α. If $\langle S_i : i < \gamma \rangle$ is a sequence of stationary subsets of τ, then $\langle S_i : i < \gamma \rangle$ *reflects simultaneously* at $\alpha < \tau$ if $S_i \cap \alpha$ is stationary for all $i < \gamma$.

If κ is singular, we say that *simultaneous reflection holds at* κ^+ if for every regular $\mu < \kappa$ and every sequence $\langle S_i : i < \mathrm{cf}\,\kappa \rangle$ of stationary subsets of $\kappa^+ \cap \mathrm{cof}(< \mu)$, there is some $\alpha < \kappa^+$ at which $\langle S_i : i < \mathrm{cf}\,\kappa \rangle$ reflects simultaneously.

Observe that if $\langle \kappa_i : i < \mathrm{cf}\,\kappa \rangle$ is a sequence of regular cardinals converging to κ, then the sequence of stationary sets $\langle \kappa^+ \cap \mathrm{cof}(\kappa_i) : i < \mathrm{cf}\,\kappa \rangle$ cannot reflect simultaneously. To say that simultaneous reflection holds at κ^+ means that every sequence of stationary sets that can plausibly reflect simultaneously will do so. Our definition here is used for the sake of simplicity, but other treatments use a potentially stronger definition of simultaneous reflection. There are models in which, for all $n, m < \omega$, every sequence of $\aleph_n$-many stationary sets of $\aleph_{\omega+1} \cap \mathrm{cof}(< \aleph_m)$ reflect simultaneously [10] [3].

Now that we have introduced the objects of study, we introduce the tools we will use to obtain our results. Supercompact cardinals will allow us to construct a model where singular cardinals can fail to carry very good scales.

Definition 5. A cardinal κ is *supercompact* if for every $\lambda \geq \kappa$ there is an elementary embedding $j : V \to M \subset V$ with critical point κ such that $j(\kappa) > \lambda$ and $M^\lambda \subset M$.

The main forcing poset we need comes from a family of posets defined by Jensen for adding generalized square sequences by initial segments. Cummings, Foreman, and Magidor cover the class of $\square_{\kappa,<\mu}$-adding posets, but we will provide a treatment here for completeness.

Definition 6. $\mathbb{S}_{\kappa,<\mu}$ is the poset of bounded initial segments of $\square_{\kappa,<\mu}$ sequences with closed domain, ordered by direct extension. More precisely, $\mathbb{S}_{\kappa,<\mu}$ is the set of conditions p such that:

- $\operatorname{dom} p = \{\alpha \leq \delta : \alpha \text{ a limit}\}$ for some limit $\delta < \kappa^+$;
- $\forall \alpha \in \operatorname{dom} p$, $p(\alpha)$ is a set of clubs $C \subset \alpha$ of order type less than or equal to κ;
- $\forall \alpha \in \operatorname{dom} p, \forall C \in p(\alpha), \forall \beta \in \lim C$, $C \cap \beta \in p(\beta)$;
- $\forall \alpha \in \operatorname{dom} p$, $1 \leq |p(\alpha)| < \mu$.

For the ordering, $p \leq q$ if p end-extends q, meaning that $\max p \geq \max q$ and $p \restriction (\max \operatorname{dom} q + 1) = q$.

Proposition 1. *$\mathbb{S}_{\kappa,<\mu}$ is $(\kappa+1)$-strategically closed.*

Proof. We show that any descending sequence $\langle p_\xi : \xi < \delta \rangle$ for $\delta \leq \kappa$ has a lower bound as long as the p_ξ's were selected at limits and even successors (ordinals of the form $\xi = \eta + n$ where n is even and η is a limit) with the following strategy: Given p_ξ with $\max \operatorname{dom} p_\xi = \gamma$, if $\xi + 1$ is an even successor, let $p_{\xi+1} = p_\xi ^\frown \langle \gamma + \omega, \{\langle \gamma + n : n < \omega \rangle\}\rangle$. For the limit case, suppose $\langle p_\xi : \xi < \delta \rangle$ with $\delta \leq \kappa$ is a decreasing sequence such that $\gamma_\xi := \max \operatorname{dom} p_\xi$. Then if $\gamma_\delta = \sup_{\xi < \delta} \gamma_\xi$, this sequence has the lower bound p_δ where $\max \operatorname{dom} p_\delta = \gamma_\delta$ and $p_\delta(\gamma_\delta) = \{\langle \gamma_\alpha : \alpha < \delta \rangle\}$, provided that the previous p_ξ's were chosen similarly for limits ξ, and provided that the $p_{\xi+1}$'s were chosen as described for even successors. $\square$

The strategy described in the proof will be used in a construction later on. The strategic closure also gives us preservation of cardinals.

Proposition 2. *$\mathbb{S}_{\kappa,<\mu}$ is κ^+-distributive and therefore preserves cardinals up to and including κ^+.*

If $2^\kappa = \kappa^+$ holds, then $|\mathbb{S}_{\kappa,<\mu}| = \kappa^+$ and $\mathbb{S}_{\kappa,<\mu}$ has the κ^{++}-chain condition, and so $\mathbb{S}_{\kappa,<\mu}$ preserves all cardinals. But we will avoid assumptions about the continuum function because our construction only concerns cardinals up to κ^+. The most important property of $\mathbb{S}_{\kappa,<\mu}$ follows from strategic closure as well:

Proposition 3. *If H is $\mathbb{S}_{\kappa,<\mu}$-generic, then $V[H] \models \square_{\kappa,<\mu}$.*

Proof. One can see that if H is $\mathbb{S}_{\kappa,<\mu}$-generic, then $\bigcup H$ is a $\square_{\kappa,<\mu}$-sequence provided that all limit ordinals below κ^+ are in the domain of some condition $p \in H$. This follows from the assertion that for all $\gamma \in \kappa^+$, $D_\gamma := \{p \in \mathbb{S}_{\kappa,<\mu} : \gamma \in \operatorname{dom} p\}$ is dense in $\mathbb{S}_{\kappa,<\mu}$, which we can prove by induction on γ. Successor cases are handled as in the successor case of the strategy for $\mathbb{S}_{\kappa,<\mu}$. Suppose γ is a limit and $\langle \gamma_\xi : \xi < \operatorname{cf} \gamma \rangle$ (where $\operatorname{cf} \gamma \leq \kappa$ of course) is strictly increasing and cofinal in γ. Then construct a sequence $\langle p_\xi : \xi < \operatorname{cf} \gamma \rangle$ by choosing some $p_{\xi+1} \leq p_\xi$ with $\gamma_\xi \in p_{\xi+1}$ at odd successors and choosing p_ξ with the strategy for all other ξ. Then there is a lower bound p for this sequence such that $\gamma \in \operatorname{dom} p$. $\square$

In order to make use of supercompact cardinals, we need to supplement the $\mathbb{S}_{\kappa,<\mu}$ poset with the threading poset.

Definition 7. Suppose we are given an $\mathbb{S}_{\kappa,<\mu}$-generic filter H and the $\square_{\kappa<\mu}$-sequence $\vec{\mathcal{C}} = \bigcup H$. If δ is an uncountable regular cardinal less than κ, let $\mathbb{T}_\delta$ be the poset of closed bounded sets $c \subset \kappa^+$ of order-type less than δ such that $\forall \alpha \in \lim c$, $c \cap \alpha \in \mathcal{C}_\alpha$. The ordering on $\mathbb{T}_\delta$ is end-extension: $d \leq c$ if $\max d \geq \max c$ and $d \cap (\max c + 1) = c$.

The significance of the threading poset lies in the following:

Proposition 4. *The subset* $D(\mathbb{S}_{\kappa,<\mu} * \mathbb{T}_\delta) := \{(p, \dot{c}) : \exists d, p \Vdash \dot{c} = \check{d}, \max \operatorname{dom} p = \max c\} \subset \mathbb{S}_{\kappa,<\mu} * \mathbb{T}_\delta$ *is δ-closed and dense.*

Proof. For density: given any $(p, \dot{c})$, p can be extended to p' forcing $\dot{c} = \check{d}$ for some d using κ^+-distibutivity. Then we extend p' to q with $\max \operatorname{dom} q > \max d$, and so we find $(q, d \cup \{\max \operatorname{dom} q\}) \in D$. For closure: If $\langle (p_\xi, d_\xi) : \xi < \eta \rangle$ is a sequence in $D(\mathbb{S}_{\kappa,<\mu} * \mathbb{T}_\delta)$ with $\eta < \delta$, then it has a lower bound (p^*, d^*) where $\max \operatorname{dom} p^* = \delta^* := \sup_{\xi<\eta} p_\xi$, $p(\delta^*) = \{\bigcup_{\xi<\eta} d_\xi\}$, and $d^* = \bigcup_{\xi<\eta} d_\xi \cup \{\sup_{\xi<\eta} \max d_\xi\}$. $\square$

It follows that the iteration $\mathbb{S}_{\kappa,<\mu} * \mathbb{T}_\delta$ is equivalent to a δ-closed forcing.

2. Semi-weak square does not imply existence of a very good scale

The threading poset is poorly behaved in general—it is not even countably closed—so our pivotal lemma shows that, for the poset $\mathbb{S}_{\kappa,<\kappa}$, the iteration with the threading poset can be made to preserve stationarity in some cases. Specifically, the threading poset preserves certain stationary sets that were obtained by applications of the pigeonhole principle.

Lemma 1. *Let H be $\mathbb{S}_{\kappa,<\kappa}$-generic and let $\nu = (\kappa^+)^V$. If $f : \nu \to \mu$ is a partition in $V[H]$ for some $\mu < \kappa$ and τ, δ are regular cardinals with $\tau < \delta$, then there is some $i < \mu$ such that $\Vdash_{\mathbb{T}_\delta}$ "$f^{-1}(i) \cap \operatorname{cof}(\tau)$ is stationary in ν".*

Proof. Let $\mathbb{S}$ denote $\mathbb{S}_{\kappa,<\kappa}$. Work in V, and suppose that $p \Vdash_{\mathbb{S}}$ "$\dot{f} : \nu \to \mu$". We can rewrite $\dot{f}$ as a $\mathbb{S} * \mathbb{T}_\delta$-name by inductively substituting instances of $\langle \sigma, p \rangle \in \dot{f}$ with $\langle \sigma, \langle p, \emptyset \rangle \rangle$ and instead consider: $(p, \emptyset) \Vdash_{\mathbb{S}*\mathbb{T}_\delta}$ "$\dot{f} : \nu \to \mu$ and $\dot{f} \in V^{\mathbb{S}}$". We want to show that there is some $p^* \leq p$ and some $i < \mu$ such that $(p^*, \emptyset) \Vdash_{\mathbb{S}*\mathbb{T}_\delta}$ "$\dot{f}^{-1}(i) \cap \operatorname{cof}(\tau)$ is stationary". We will inductively define a decreasing sequence $\langle p_\xi : \xi \leq \tau \rangle$ in $\mathbb{S}$ below p, decreasing sequences $\langle t^i_\xi : \xi \leq \tau \rangle$ for all $i < \mu$ such that $(p_\xi, t^i_\xi) \in D(\mathbb{S} * \mathbb{T}_\delta)$, and an increasing sequence of ordinals $\langle \alpha_\xi : 0 < \xi \leq \tau \rangle$.

Zero Step: We will define a decreasing sequence of conditions $\langle q_i : 0 < i \leq \mu \rangle \subset \mathbb{S}$ below p and a sequence $\langle \gamma_i : i \leq \mu \rangle$ such that $\gamma_i := \max \operatorname{dom} q_i$. We can assume that $(p, \emptyset) \nVdash$ "$\dot{f}^{-1}(0) \cap \operatorname{cof}(\tau)$ is stationary" since otherwise we would be done, and so we find some $(q_1, t^0_0) \leq p$ and a $\mathbb{S} * \mathbb{T}_\delta$-name $\dot{C}_0$ such that $(q_1, t^0_0) \Vdash$ "$\dot{C}_0$ is a club in ν and $\dot{C}_0 \cap \dot{f}^{-1}(0) \cap \operatorname{cof}(\tau) = \emptyset$". Similarly, if $i+1$ is an odd successor, then we may assume that there is some $q_{i+1} \leq q_i$ and t_0^i, t_0^{i+1} such that for $j = i, i+1$, $(q_{i+1}, t_0^j) \Vdash$ "$\dot{C}_j$ is a club in ν and $\dot{C}_j \cap \dot{f}^{-1}(j) \cap \mathrm{cof}(\tau) = \emptyset$". For even successors and limits, choose q_i using the strategy for $\mathbb{S}$. Finally, let $p_0 = q_\mu$.

Successor Step: At this point we are doing a proof by contradiction. Suppose that $\langle p_\xi : \xi \leq \eta \rangle$, $\langle t_\xi^i : 0 < \xi \leq \eta \rangle$ for $i < \mu$, and $\langle \alpha_\xi : 0 < \xi \leq \eta \rangle$ have been defined already. We will define a sequence of conditions $\langle q_i : i \leq \mu \rangle \subset \mathbb{S}$ below p_η; a set of closed bounded sets $\{s_i : i < \mu\}$ of order-type less than δ; and a collection of ordinals $\langle \beta_i : i < \mu \rangle$ above α_η such that $(q_i, s_j) \Vdash \beta_j \in \dot{C}_j$.

If $i+1$ is an odd successor and q_i has been defined, find q_{i+1}, s_i, s_{i+1}, β_i, and β_{i+1} such that for $j = i, i+1$, $(q_{i+1}, s_j) \leq (q_i, t_\eta^j)$, $\beta_j > \alpha_\eta$, and such that $(q_{i+1}, s_j) \Vdash$ "$\beta_j \in \dot{C}_j$". This works because $\dot{C}_j$ is forced to be a club for $j = i, i+1$. If i is a limit or an even successor, use the strategy for $\mathbb{S}$ to find q_i.

Now let $p_{\eta+1} = q_\mu$. Also let $t_{\eta+1}^i = s_i \cup \{\gamma^*\}$ where $\gamma^* = \sup_{i<\mu} \max \mathrm{dom}\, q_i$, and let $\alpha_{\eta+1} = \sup_{i<\mu} \beta_i$. Observe that $(p_{\eta+1}, t_{\eta+1}^i) \Vdash$ "$\dot{C}_i \cap [\alpha_\eta, \alpha_{\eta+1}) \neq \emptyset$" for all $i < \mu$.

Limit Step: For $i < \mu$, let $t_\eta^i = \bigcup_{\xi<\eta} t_\xi^i \cup \{\sup_{\xi<\eta} \max t_\xi^i\}$, and let $\alpha_\eta = \sup_{\xi<\eta} \alpha_\xi$. Let p_η be the condition such that $p \restriction \mathrm{dom}\, p_\xi = p_\xi$ for $\xi < \eta$, $\max \mathrm{dom}\, p_\eta$ is defined as $\gamma^* := \sup_{\xi<\eta} \max \mathrm{dom}\, p_\xi$, and $p_\eta(\gamma^*) = \{t_\eta^i : i < \mu\}$. Note that p_η is in fact a condition in $\mathbb{S}$ because $\mu < \kappa$. Since $\eta \leq \tau < \delta$, (p_η, t_η^i) is a condition in $\mathbb{S} * \mathbb{T}_\delta$. Also, since $(p_\eta, t_\eta^i) \Vdash$ "$\dot{C}_i \cap [\alpha_\xi, \alpha_{\xi+1}) \neq \emptyset$" for all $i < \mu$, it follows that $(p_\eta, t_\eta^i) \Vdash$ "$\alpha_\eta \in \dot{C}_i$" for all $i < \mu$.

The completes the construction. Choose $p^* \leq p_\tau$ deciding a value for $\dot{f}(\alpha_\tau)$. If $p^* \Vdash$ "$\dot{f}(\alpha_\tau) = i$" (which must be the case for some i), this contradicts the fact that $(p^*, t_\tau^i) \Vdash$ "$\alpha_\tau \in \dot{C}_i$ and $\dot{C}_i \cap \dot{f}^{-1}(i) \cap \mathrm{cof}(\tau) = \emptyset$". □

We must employ some technical theorems for the construction of a model with $\square_{\kappa,<\kappa}$ and no very good scale at κ. They are as follows:

Fact 1. *[2] Let $j : V \to M$ be an elementary embedding and let $\mathbb{P}$ be a forcing poset. If G is a $\mathbb{P}$-generic filter over V and H is a $j(\mathbb{P})$-generic filter over M such that $j[G] \subset H$, then j can be extended to an elementary embedding $j^* : V[G] \to M[H]$ given by $j^*(\dot{x}_G) := j(\dot{x})_H$ where $j^*(G) = H$.*

Fact 2. *[2] Suppose κ is an inaccessible cardinal, $\lambda < \kappa$ is regular, and $\mathbb{P}$ is a λ-closed separative poset such that $|\mathbb{P}| < \kappa$. Then there is a complete embedding $i : \mathbb{P} \to Col(\lambda, <\kappa)$ such that forcing with $Col(\lambda, <\kappa)$ is equivalent to forcing with $Col(\lambda, <\kappa)/i(\mathbb{P})$. Moreover, if $Col(\lambda, A)$ where $\sup A = \kappa$ is used in place of $Col(\lambda, <\kappa)$, then the conclusion still holds.*

Fact 3. *[1] If τ is a regular uncountable cardinal, $S \subset \tau \cap \mathrm{cof}(\omega)$ is a stationary subset of τ, and $\mathbb{P}$ is a countably closed poset, then S remains stationary in any forcing extension by $\mathbb{P}$.*

Now we are in a position to prove our main result.

Theorem 1. *If κ is supercompact in V, then there is a forcing extension in which there is a singular cardinal λ such that $\square_{\lambda,<\lambda}$ holds and λ does not carry a very good scale. Moreover, for any $n < \omega$ we can arrange that $\lambda = \aleph_{\omega_n}$ in this forcing extension.*

Specifically, we can get a model where $\square_{\aleph_\omega,<\aleph_\omega}$ holds and $\aleph_\omega$ does not carry a very good scale, and this can generalize to singulars of uncountable cofinality. We are not necessarily limited to cardinals of the form $\aleph_{\omega_n}$ for $n < \omega$ either. The reader may note that collapsing cardinals is not required for obtaining the consistency result at λ. For this it is enough to work in a model where the supercompactness of κ is indestuctible under κ-directed closed forcing, to choose λ such that $\operatorname{cf}\lambda < \kappa < \lambda$, and to skip the Lévy Collapse in the following proof.

Proof. Let $\lambda = \kappa^{+\omega_n}$, and let G be $\operatorname{Col}(\omega_{n+1}, <\kappa)$-generic over V, so that $V[G] \models$ "$\lambda = \aleph_{\omega_n}$". Let $\mathbb{S}$ be $\mathbb{S}_{\lambda,<\lambda}$ as defined in $V[G]$ and let H be an $\mathbb{S}$-generic filter over $V[G]$. Then $V[G * H]$ is our intended model.

Suppose $\vec{f} = \langle f_\alpha : \alpha < \lambda^+\rangle$ is a scale at λ in some product $\prod_{i<\operatorname{cf}\lambda} \lambda_i$ and is contained in the model $V[G * H]$. Consider the maps $g_i : \alpha \mapsto f_\alpha(i) < \lambda_i$ for each $i < \operatorname{cf}\lambda$. Apply Lemma 1 to get stationary sets $S_i \subset \lambda^+ \cap \operatorname{cof}(\omega)$ for each i such that g_i is constant on S_i, and moreover such that $\Vdash^{V[G*H]}_{\mathbb{T}_{\omega_{n+1}}}$ "S_i is stationary in ν" where $\nu = (\lambda^+)^V$.

Now let $\tau := \max\{\nu, |\mathbb{S}|\}$ and take a τ-supercompact embedding $j : V \to M$ with critical point κ. We will make repeated use of the fact that if $\rho := \sup j[\nu]$, then the facts that $M \models$ "$j(\nu)$ is regular" and $M^\nu \subset M$ imply that $\rho < j(\nu)$.

Claim 1. *There exists an extension $V[G * H * L]$ in which S_i is stationary in ν for all $i < \operatorname{cf}\lambda$ and where j can be extended to an elementary embedding $j^+ : V[G * H] \to M[j(G * H)]$.*

Claim 2. *Using the embedding from Claim 1, we can show that the S_i's reflect simultaneously in $V[G * H * L]$, which means that they reflect simultaneously in $V[G * H]$.*

If $\gamma < \nu$ is the point of reflection from Claim 2, then γ cannot be a very good point for the scale $\vec{f}$. If it were, it would be witnessed by a club $C \subset \gamma$ and an index $j < \operatorname{cf}\lambda$ such that $\langle f_\beta(i) : \beta \in C\rangle$ is increasing for $i \geq j$. However, $\beta \mapsto f_\beta(j)$ is constant on $S_j \cap \gamma$. Since $C \cap S_j$ is stationary, and thus unbounded, this is not possible.

Proof of Claim 1. Let I be $\mathbb{T}_{\omega_{n+1}}$-generic over $V[G * H]$. We get preservation of stationarity of the S_i's due to the way they were chosen with Lemma 1.

Now consider $j(\operatorname{Col}(\omega_{n+1}, <\kappa)) = \operatorname{Col}(\omega_{n+1}, <j(\kappa)) = \operatorname{Col}(\omega_{n+1}, <\kappa) \times \mathbb{R}$ where $\mathbb{R}$ is the $\leq \omega_n$-support product $\prod_{\alpha \in [\kappa, j(\kappa))} \operatorname{Col}(\omega_{n+1}, \alpha)$. Since $\mathbb{S} * \mathbb{T}_{\omega_{n+1}}$ is equivalent to an ω_{n+1}-closed poset of size less than $j(\kappa)$, Fact 2 implies that there exists an embedding $i : \operatorname{Col}(\kappa, <\lambda) * \mathbb{S} * \mathbb{T}_{\omega_{n+1}} \to \mathbb{R}$ such that $\mathbb{R}/i(\operatorname{Col}(\kappa, <\lambda) * \mathbb{S} * \mathbb{T}_{\omega_{n+1}})$ is equivalent to $\mathbb{R}$. Let J be $\mathbb{R}/i(\operatorname{Col}(\kappa, <\lambda) * \mathbb{S} * \mathbb{T}_{\omega_{n+1}})$-generic over $V[G*H*I]$. Because $\mathbb{R}$ is countably closed and is equivalent to our quotient, it follows that the S_i's are stationary in $V[G*H*I*J]$ by Fact 3. We also have that $j[\mathrm{Col}(\omega_{n+1}, <\kappa)] \subset \mathrm{Col}(\omega_{n+1}, <\kappa) \times \mathbb{R}$, so we can apply the Fact 1 to get a partial lift $j' : V[G] \to M[G*H*I*J] = M[j'(G)]$.

Now consider $j'(\mathbb{S})$. We argue that $j'(\mathbb{S})$ is countably closed in $V[j'(G)]$. $\mathbb{S}$ is countably closed because any countable descending sequence of conditions $\langle p_n : n<\omega\rangle$ has a lower bound p such that $\delta = \max\operatorname{dom} p = \sup_{n<\omega} \max\operatorname{dom} p_n$ and $p(\delta) = \{A\}$ where A is any ω-sequence of ordinals cofinal in δ. Hence $M[j'(G)] \models$ "$j'(\mathbb{S})$ is countably closed" by elementarity. Moreover, since $j(\mathrm{Col}(\omega_{n+1}, <\kappa))$ is countably closed, and since $M^{<\omega_1} \subset M$, it follows that $M[j'(G)]$ is closed under countable sequences. (This uses the fact that for a model $N \subset V$, $N^\gamma \subset N$ if and only if $\mathrm{ON}^\gamma \subset N$ [2].) It follows that $V[j'(G)] \models$ "$j'(\mathbb{S})$ is countably closed". Therefore, the S_i's are again still stationary if we force with $j'(\mathbb{S})$ by Fact 3.

Let $\mathcal{C} = \bigcup H$, write $\mathcal{C} = \langle \mathcal{C}_\alpha : \alpha \in \lim(\nu)\rangle$, and write $j(\mathcal{C})$ as $\langle \mathcal{C}^*_\alpha : \alpha \in \lim(j'(\nu))\rangle$. Furthermore, let $T = \bigcup I$. Consider $\beta \in \lim j'[T]$, so that there is some $\alpha < \nu$ such that $j'(\alpha) = \beta$. Since j' is continuous on sequences of ordinals of countable cofinality and T has order-type ω_{n+1}, $j'[T] \cap \beta = j'(T \cap \alpha) \in j'(\mathcal{C}_\alpha) = \mathcal{C}^*_\beta$. It follows that $s := \langle \mathcal{C}^*_\alpha : \alpha \in \lim(\rho)\rangle^\frown\langle \rho, \{j'[T]\}\rangle$ is a master condition for $j'[\mathbb{S}]$, meaning that for all $p \in \mathbb{S}$, $s \le j'(p)$. Hence we can force with a $j'(\mathbb{S})$-generic K containing s to apply Fact 1 and get the lift $j^+ : V[G*H] \to M[j^+(G*H)]$.

We now have $L = I*J*K$, which gives us the claim. □

Proof of Claim 2. Work in $V[G*H*L]$, where we have defined j^+. Recall that $\mathrm{cf}\,\lambda < \kappa$, and so $j^+(\mathrm{cf}\,\lambda) = \mathrm{cf}\,\lambda$ because $j^+ \restriction \mathrm{ON} = j \restriction \mathrm{ON}$. We again use $\rho = \sup j[\nu]$, where $\rho < j(\nu)$.

We show that for every $i < \mathrm{cf}\,\lambda$, $j^+(S_i) \cap \rho$ is stationary in ρ. Consider a club $C \subset \rho$ in $M[j^+(G*H)]$, and let $E = \{\alpha < \nu : j^+(\alpha) \in D\}$. Then E will be unbounded in ν and closed under countable sequences: if $\langle \alpha_n : n<\omega\rangle \subset E$ is a sequence with supremum α^*, then $\sup_{n<\omega} j^+(\alpha_n) = j^+(\alpha^*) \in D$, so $\alpha^* \in E$; if $\beta < \nu$ then define $\langle \beta_n : n<\omega\rangle \subset \nu$ above β and $\langle \gamma_n : n<\omega\rangle \subset \rho$ so that for all n, $j^+(\beta_n) < \gamma_n < j^+(\beta_{n+1})$, so $\beta < \sup \beta_n \in E$. Therefore there will be some $\alpha \in E \cap S_i$, so $j^+(\alpha) \in D \cap j^+(S_i) \neq \emptyset$.

We have demonstrated that,

$$M[j^+(G*H)] \models \text{“}\exists \alpha < j^+(\nu), \forall i < \mathrm{cf}\,\lambda, j^+(S_i) \cap \alpha \text{ is stationary in } \alpha\text{”}.$$

It follows by elementarity that,

$$V[G*H] \models \text{“}\exists \alpha < \nu, \forall i < \mathrm{cf}\,\lambda, S_i \cap \alpha \text{ is stationary”}.$$

□

This completes the proof of our theorem. □

3. Semi-weak square implies the failure of simultaneous reflection

Up to now, the extent that semi-weak square impacts stationary reflection was given by the following:

Fact 4 (Cummings, Schimmerling)**.** *[5] If κ is a singular strong limit cardinal and $\square_{\kappa,<\kappa}$ holds, then for every stationary set $S \subset \kappa^+$, there is a $\mu < \kappa$ and a sequence of stationary subsets $\langle S_i : i < \operatorname{cf}\kappa\rangle$ of κ^+ such that if $S_i \cap \alpha$ is stationary for all $i < \operatorname{cf}\kappa$, then $\operatorname{cf}\alpha < \mu$.*

It turns out that we can weaken the hypotheses and strengthen the conclusion. Even though semi-weak square does not imply the existence of a very good scale, its impact on stationary reflection is the same as that of a very good scale—namely, simultaneous stationary reflection fails.

Theorem 2. *If κ is singular and $\square_{\kappa,<\kappa}$ holds, then for every stationary $S \subset \kappa^+$, there is a sequence $\langle S_i : i < \operatorname{cf}\kappa\rangle$ of subsets of S that do not reflect simultaneously.*

Proof. Fix a stationary set $S \subset \kappa^+$ and a $\square_{\kappa,<\kappa}$-sequence $\langle \mathcal{C}_\alpha : \alpha \in \lim(\kappa^+)\rangle$. By κ^+-completeness of the club filter (or Fodor's Lemma), there is some stationary $S' \subset S$ and a cardinal $\mu < \kappa$ such that $|\mathcal{C}_\alpha| < \mu$ for all $\alpha \in S'$. Let $\lambda := \operatorname{cf}\kappa$ and choose a strictly increasing sequence $\langle \kappa_i : i < \lambda\rangle$ of regular cardinals converging to κ such that $\mu < \kappa_0$.

We define a sequence of functions $\langle f_\alpha : \alpha \in S'\rangle$ in the product $\prod_{i<\lambda} \kappa_i$ that is increasing with respect to evenual domination. If $\alpha = \min S'$, then let $f_\alpha(i) = 0$ for all $i < \lambda$. For all other $\alpha \in S'$, let j be the least ordinal such that there is some $C \in \mathcal{C}_\alpha$ with $\operatorname{ot} C < \kappa_j$. Then let $f_\alpha(i) = 0$ for $i < j$, and for $i \geq j$ let,

$$f_\alpha(i) = \sup\left\{ \sup_{\beta \in C \cap S'} f_\beta(i) : C \in \mathcal{C}_\alpha, \operatorname{ot} C < \kappa_i \right\} + 1.$$

For every $i < \lambda$ use κ^+-completeness of the club filter to find $S_i \subset S'$ and $\delta_i < \kappa_i$ such that for all $\alpha \in S_i$, $f_\alpha(i) = \delta_i$. We claim that the sequence $\langle S_i : i < \lambda\rangle$ does not simultaneously reflect.

Suppose for contradiction that $\alpha < \kappa^+$ is a point of simultaneous reflection for this sequence. Pick $C \in \mathcal{C}_\alpha$ (it does not matter whether $\alpha \in S'$) and choose i such that $\operatorname{ot} C < \kappa_i$. Using the assertion that $S_i \cap \alpha$ is stationary, pick $\beta, \gamma \in \lim C \cap S_i$ such that $\beta < \gamma$. Then $C \cap \gamma \in \mathcal{C}_\gamma$, $\operatorname{ot}(C \cap \gamma) < \operatorname{ot}(C) < \kappa_i$, and of course $\beta \in C \cap S'$, so it follows by construction that $f_\gamma(i) > f_\beta(i)$. This contradicts the fact that $f_\gamma(i) = \delta_i = f_\beta(i)$. $\square$

Using this theorem we can sharpen a result of Cummings and Magidor, who proved that if Martin's Maximum (usually abbreviated MM) holds, then $\square_{\kappa,\lambda}$ fails for $\lambda < \kappa$ if κ is singular and has cofinality ω_1 [4]. They appeal to the fact that if MM holds, then for every regular $\mu > \omega_1$, every sequence $\langle S_i : i < \omega_1\rangle$ of stationary subsets of $\mu \cap \operatorname{cof}(\omega)$ reflects simultaneously [7]. Together with Theorem 2, this implies the last result of this paper:

Corollary 3. *If* MM *holds and κ is singular of cofinality ω_1, then $\square_{\kappa,<\kappa}$ fails.*

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