Set Theory
Winter term 2024/2025
Exercises

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Exercise sheet 02 from 25.10.2024

Due in the next exercise session on the 08.11.2024 at 12:00. Typically, any problem is worth 4 points. A total of at least 50% of all available points is required for the "Studienleistung".

1. (8 points) Prove Proposition 1.36 from the lecture notes. Specifically, prove that the following holds for any ordinals α, β, γ:

(a) $\alpha \cdot 0 = 0$.

(b) $\alpha \cdot 1 = \alpha$.

(c) $\alpha \cdot S(\beta) = \alpha \cdot \beta + \alpha$.

(d) If β is a limit then $\alpha \cdot \beta = \sup\{\alpha \cdot \xi \mid \xi < \beta\}$.

(e) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.

(f) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.

Remark: Points (a), (c) and (d) can be used to instead define ordinal multiplication inductively instead of explicitly.

2. Prove the Cantor-Bernstein Theorem: Let A and B be sets and $f\colon A \to B$ and $g\colon B \to A$ injective functions. Then there is a bijection $h\colon B \to A$.

Hint: Assume without loss of generality that $A \subseteq B$ and $f(a) = a$ for all $a \in A$ (why is that possible?). Define $C := \{g^n(b) \mid n \in \omega, b \in B \setminus A\}$ and then $h(x) = g(x)$ for $x \in C$ as well as $h(y) = y$ for $y \in B \setminus C$. Show that $h\colon B \to A$ is as required.

3. In the lecture we have defined the ordinal exponentiation α^β by induction on β. In this exercise we will find an explicit definition of α^β. First let $F(\beta, \alpha)$ consist of all functions $f\colon \beta \to \alpha$ such that $f(i) = 0$ for all but finitely many $i \in \beta$. For $f, g \in F(\beta, \alpha)$, let $f < g$ if and only if there is $i \in \beta$ such that $f(i) < g(i)$ and $f(j) = g(j)$ for all $j > i$.

(a) Show that $<$ is a well-order on $F(\beta, \alpha)$.

(b) Show that the ordertype of $(F(\beta, \alpha), <)$ is equal to α^β for any ordinals α, β.
Hint: Use induction on β.