Set Theory
Winter term 2024/2025
Exercises

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Exercise sheet 03 from 08.11.2024

Due at the beginning of the exercise session on 15.11.2024 at 12:00. Typically, any problem is worth 4 points. A total of at least 50% of all available points is required for the "Studienleistung".

1. Let κ be a regular cardinal $> \omega$. Show that if $f\colon \kappa \to \kappa$, then $\{\alpha < \kappa : \forall \xi < \alpha, f(\xi) < \alpha\}$ is closed and unbounded in κ.

2. We define the following variants of the axiom of choice:

(a) AC_ω (the *axiom of countable choice*) states that whenever f is a function on ω such that $f(n)$ is a nonempty set for each $n \in \omega$, there is a function g on ω such that $g(n) \in f(n)$ for every $n \in \omega$.

(b) DC_ω (the *axiom of countable dependent choice*) states that whenever X is a set and $R \subseteq X \times X$ has the property that for any $x \in X$ there is $y \in X$ with $(x, y) \in R$, then there is a function f on ω such that $((f(n), f(n + 1)) \in R$ for every $n \in \omega$.

Answer (with proofs) the following questions (in ZF):

(a) Does the axiom of choice imply DC_ω?

(b) Does DC_ω imply AC_ω?
Hint: Given a function f on ω, let X consist of all pairs (n, x) where $x \in f(n)$. Define a relation $R \subseteq X \times X$ by letting $((n, x), (n + 1, y)) \in R$ (for all possible x, y). By DC_ω there is a function f with $(f(n), f(n + 1)) \in R$ for all $n \in \omega$. Is f already as required? If not, how can you modify f so that it is as required?

For two cardinals κ and λ we let $\lambda^{<\kappa}$ be the size of the set of all functions $f\colon \alpha \to \lambda$ where $\alpha < \kappa$.

3. Show that if κ is regular then $2^{<\kappa} = \kappa^{<\kappa}$.

4. Show that if GCH holds then $\kappa^{<\kappa} = \kappa$ if and only if κ is regular.