Exercise sheet 08 from 13.12.2024

Due at the beginning of the exercise session on 20.12.2024 at 12:00. Typically, any problem is worth 4 points. A total of at least 50% of all available points is required for the "Studienleistung".

1. (8 Points) Let M be a ctm. Denote by $\aleph_1^M$ the unique ordinal in M such that $M \models \aleph_1^M = \aleph_0^+$. We will show that there exists a generic extension of M in which $\aleph_1^M$ is countable. In M, let $\mathbb{P}$ be the partial order consisting of functions $p\colon A \to B$ where $A \subseteq \aleph_0$ and $B \subseteq \aleph_1^M$ are finite. We order $\mathbb{P}$ by $p \leq q$ if and only if $p \supseteq q$.

For any G which is a $\mathbb{P}$-generic filter over M we define $f_G := \bigcup_{p \in G} p$.

Show the following:

(a) Whenever G is a $\mathbb{P}$-generic filter over M, f_G is an element of $M[G]$.

(b) There exists a $\mathbb{P}$-name τ such that for any $\mathbb{P}$-generic filter G over M we have $\mathrm{val}(\tau, G) = f_G$.

Now fix a $\mathbb{P}$-generic filter G over M. Show the following (by constructing suitable dense sets):

(c) $M[G] \models f_G$ is a function.

(d) $M[G] \models \mathrm{dom}(f_G) = \aleph_0$.

(e) $M[G] \models \mathrm{rge}(f_G) = \aleph_1^M$.

(f) Since $M[G]$ is a model of enough ZFC there will be an ordinal $\aleph_1^{M[G]}$ in $M[G]$ such that $M[G] \models \aleph_1^{M[G]} = \aleph_0^+$. By proposition 5.2.12, $\aleph_1^{M[G]} \in M$. Is $\aleph_1^{M[G]} > \aleph_1^M$, $\aleph_1^{M[G]} = \aleph_1^M$ or $\aleph_1^{M[G]} < \aleph_1^M$? Can the last case occur in any generic extension?

2. (8 Points) Let $\mathbb{P}$ be a partial order.

(a) Assume that $\mathbb{P}$ does not contain an atom. Show that $\mathbb{P}$ contains an infinite antichain.

(b) Assume that for any natural number n, $\mathbb{P}$ has an antichain of size n. Show that $\mathbb{P}$ contains an infinite antichain.

Hint: ~~Assuming that $\mathbb{P}$ has arbitrarily large finite antichains, can the set of atoms be dense in $\mathbb{P}$?~~ Check two cases, depending on if the set of atoms is dense in $\mathbb{P}$ or not.

(c) (hard) Let M be a ctm and $\mathbb{P}$ a partial order in M that does not contain any atoms. Show that (in the ambient universe V) there are $2^{\aleph_0}$ many $\mathbb{P}$-generic filters over M.

Hint: Combine the technique of item (a) with the proof of Proposition 5.1.3 in the lecture notes.