Set Theory
Winter term 2024/2025
Exercises

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Exercise sheet 09 from 20.12.2024

Due at the beginning of the exercise session on 10.01.2025 at 12:00. Typically, any problem is worth 4 points. A total of at least 50% of all available points is required for the "Studienleistung".

This homework will be a bonus, meaning that it can count towards your grade, but it will not count towards the sum of the homework points.

1. (8 Points) Let M be a ctm and $\mathbb{P} \in M$ a poset. Assume that κ is an ordinal in M such that $M \models \kappa$ is a cardinal. Show that if $M \models |\mathbb{P}| < \kappa$ and G is a $\mathbb{P}$-generic filter over M, $M[G] \models \kappa$ is a cardinal.

Proof sketch: Assume otherwise. Then there is a $\mathbb{P}$-generic filter G and a $\mathbb{P}$-name τ such that in $M[G]$, τ^G is a surjection from some $\gamma < \kappa$ onto κ. By the forcing theorem, there is a condition $p \in G$ which forces that $\tau\colon \check{\gamma} \to \check{\kappa}$ is surjective.

For $\alpha < \gamma$, let A_α consist of all those $q \leq p$ such there is $\beta_q^\alpha \in \kappa$ with $q \Vdash \tau(\check{\alpha}) = \check{\beta}_q^\alpha$. What is the size of $\bigcup_{\alpha<\gamma}\{\beta_q^\alpha \mid q \in A_\alpha\}$? How does this contradict the assumption that p forces τ to be surjective?

2. (8 Points) Consider the poset $\mathbb{P}$ from exercise 1 on sheet 08. Show that $M \models |\mathbb{P}| = \aleph_1^M$ and deduce that $M[G] \models \aleph_2^M = \aleph_0^+$ whenever G is a $\mathbb{P}$-generic filter over M.