Set Theory
Winter term 2024/2025
Exercises

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Exercise sheet 10 from 10.01.2025

Due at the beginning of the exercise session on 17.01.2025 at 12:05. Typically, any problem is worth 4 points. A total of at least 50% of all available points is required for the “Studienleistung”.

1. (3+3 Points) Let M be a ctm, let $\mathbb{P}, \mathbb{Q} \in M$ be partial orders and $\pi\colon \mathbb{P} \to \mathbb{Q}$ an isomorphism with $\pi \in M$.

(a) Show that if G is $\mathbb{P}$-generic over M, then $\pi[G]$ is $\mathbb{Q}$-generic over M and $M[G] = M[\pi[G]]$.

(b) Let π_* be the lift of π to the class of $\mathbb{P}$-names. Show that whenever $\phi(v_1, \ldots, v_n)$ is a formula, $\dot{x}_1, \ldots, \dot{x}_n$ are $\mathbb{P}$-names and $p \in \mathbb{P}$, then

$$p \Vdash \phi(\dot{x}_1, \ldots, \dot{x}_n) \;\Rightarrow\; \pi(p) \Vdash \phi(\pi_*(\dot{x}_1), \ldots, \pi_*(\dot{x}_n))$$

Hint: Please use the Forcing Theorem.

2. (3+3 Points) Let $\mathbb{P}$ be a partial order. We say that $\mathbb{P}$ is *almost homogeneous* if for any $p, q \in \mathbb{P}$ there exists an automorphism π of $\mathbb{P}$ such that $\pi(p)$ and q are compatible.

(a) Show that $\mathrm{Add}(\omega, \omega)$ is almost homogeneous.

(b) Suppose that M is a ctm, $\mathbb{P} \in M$ is almost homogeneous, $x_1, \ldots, x_n \in M$ and $p \in \mathbb{P}$. Show that if $p \Vdash \phi(\check{x}_1, \ldots, \check{x}_n)$, then $1 \Vdash \phi(\check{x}_1, \ldots, \check{x}_n)$. Deduce that for any formula ϕ, either $1 \Vdash \phi(\check{x}_1, \ldots, \check{x}_n)$ or $1 \Vdash \neg\phi(\check{x}_1, \ldots, \check{x}_n)$.

3. (4 Points) We say that a poset $\mathbb{P}$ is *κ-closed* (elsewhere this might be called *$<\kappa$-closed*) if for any descending sequence $(p_\alpha)_{\alpha<\mu}$ (where $\mu < \kappa$) there is $p \in \mathbb{P}$ such that $p \leq p_\alpha$ for any $\alpha < \mu$.

Suppose κ is a singular cardinal and $\mathbb{P}$ is a κ-closed poset. Show that $\mathbb{P}$ is κ^+-closed.