Set Theory	Lecturer: Maxwell Levine
Winter term 2024/2025	Assistant: Hannes Jakob
Exercises	

Exercise sheet 12 from 24.01.2025

Due at the beginning of the exercise session on 31.01.2025 at 12:05. Typically, any problem is worth 4 points. A total of at least 50% of all available points is required for the “Studienleistung”. This is the last sheet.

1. Let M be a ctm and $\mathbb{P}, \mathbb{Q} \in M$ posets. Let $\iota \colon \mathbb{P} \to \mathbb{Q}$ be a *dense embedding* in M, i.e. ι is injective, $p' \leq p$ if and only if $\iota(p') \leq \iota(p)$ and $\iota[\mathbb{P}]$ is a dense subset of $\mathbb{Q}$.

Show that whenever G is $\mathbb{P}$-generic over M, the set $\uparrow(\iota[G]) := \{q \in \mathbb{Q} \mid \exists r \in \iota[G]\ r \leq q\}$ is $\mathbb{Q}$-generic over M and whenever H is $\mathbb{Q}$-generic over M, the set $\iota^{-1}[H]$ is $\mathbb{P}$-generic over M. Conclude that any forcing extension by $\mathbb{P}$ is also a forcing extension by $\mathbb{Q}$ and vice versa.

Hint: A filter is generic if and only if it intersects every open dense set.

2. Let M be a ctm and $\mathbb{P} \in M$ a poset. Assume $\mathbb{P}$ contains elements p and q which are incompatible. Show that the collection of all $\tau \in M$ such that $p \Vdash \tau = \emptyset$ is not a set in M.

3. Let M be a ctm and $\mathbb{P} \in M$ a poset. Assume $\dot{\mathbb{Q}} \in M$ is a $\mathbb{P}$-name for a poset and κ is a cardinal.

(a) Show that if $\mathbb{P}$ is κ-Knaster and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is κ-Knaster, then $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-Knaster.

(b) Show that if $\mathbb{P}$ is $<\kappa$-distributive and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $<\kappa$-distributive, then $\mathbb{P} * \dot{\mathbb{Q}}$ is $<\kappa$-distributive.

4. Let M be a ctm and $\mathbb{P}, \mathbb{Q} \in M$ posets. such that $\mathbb{P}$ and $\mathbb{Q}$ are κ-cc. Show that $\mathbb{P} \times \mathbb{Q}$ is κ-cc if and only if $\Vdash_{\mathbb{P}} \check{\mathbb{Q}}$ is κ-cc.