ON COMPACTNESS OF WEAK SQUARE AT SINGULARS OF UNCOUNTABLE COFINALITY

MAXWELL LEVINE

Abstract. Cummings, Foreman, and Magidor proved that Jensen's square principle is non-compact at $\aleph_\omega$, meaning that it is consistent that $\square_{\aleph_n}$ holds for all $n < \omega$ while $\square_{\aleph_\omega}$ fails. We investigate the natural question of whether this phenomenon generalizes to singulars of uncountable cofinality. Surprisingly, we show that under some mild hypotheses, the weak square principle $\square^*_\kappa$ is in fact compact at singulars of uncountable cofinality, and that an even stronger version of these hypotheses is not enough for compactness of weak square at $\aleph_\omega$.

1. Introduction and Background

The properties of singulars of uncountable cofinality are notoriously different from those of countable cofinality. A prime example is Silver's theorem that GCH cannot fail for the first time at a singular of uncountable cofinality. In contrast, Magidor showed that GCH can fail for the first time at $\aleph_\omega$. There is therefore a natural question of whether this phenomenon generalizes to more complex structures.[1]

Here we focus on the combinatorial properties of inner models, notably square principles. Jensen originally distilled the principle $\square_\kappa$ (where κ is some given cardinal) to study the properties of Gödel's Constructible Universe L [10]. Many variations of $\square_\kappa$ have been studied since then. (Precise definitions will be given below, but Cummings-Foreman-Magidor [3] is the canonical reference for this area.) There is in general a tension between square principles and large cardinals, one instance of which is that $\square_\kappa$ fails if κ is larger than a supercompact cardinal. Moreover, the failure of $\square_\kappa$ at a singular cardinal κ requires considerable consistency strength from large cardinals [16]. The models of interest in this area realize some compatibility of both square principles and the compactness properties exhibited by large cardinals.

In this paper we will address the compactness of square principles themselves: whether or not $\square_\kappa$ necessarily holds for some cardinal κ if $\square_\delta$ holds for sufficiently many cardinals $\delta < \kappa$. Cummings, Foreman, and Magidor proved that it is consistent that $\square_{\aleph_n}$ holds for $1 \leq n < \omega$ but that $\square_{\aleph_\omega}$ fails [4]. Later, Krueger improved the result by obtaining a bad scale on $\aleph_\omega$ in a similar model [11]. But such results can also go in the other direction: Cummings, Foreman, and Magidor also proved that if $\square_{\aleph_n}$ holds for all $n < \omega$, there is an object that to some extent resembles a $\square_{\aleph_\omega}$-sequence but with a weaker coherence property [5]. The main result of this paper is along these lines:

[1] See [12] and [13] for recent examples.

Theorem 1.1. *Suppose that κ is a singular strong limit of cofinality $\lambda > \omega$ such that for some stationary set $S \subseteq \kappa$, $\square^*_\delta$ holds for all $\delta \in S$ and $\prod_{\delta \in S} \delta^+$ carries a good scale. Then $\square^*_\kappa$ holds.*

This represents some progress on a question raised by Golshani online regarding a supposed Silver's Theorem for special Aronszajn trees [9]: at any cardinal δ, $\square^*_\delta$ is equivalent to the existence of a special δ^+-Aronszajn tree [1].

The difference between the results of Cummings-Foreman-Magidor and Theorem 1.1 is that the resulting sequence is fully coherent—not just coherent at points of uncountable cofinality. In other words, we are able to obtain some compactness for a canonical object by obtaining exactly that canonical object in the end. We nonetheless depend on the goodness of scales, as do Cummings-Foreman-Magidor.

Note that the use of stationarity in Theorem 1.1 is necessary. Starting from $V \models$ "κ supercompact", we could work in $V[\mathrm{Col}(\aleph_1, <\kappa)]$ and force with product of square-adding posets $\prod_{\alpha<\omega_1} \mathbb{S}_{\aleph_{\alpha+1}}$ to get to a model W. This model would have a bad scale carried by $\aleph_{\omega_1}$ for the following reason: the added squares could be threaded by a product $\prod_{\alpha<\omega_1} \mathbb{T}_{\aleph_{\alpha+2},\aleph_{\alpha+1}}$ (where the threads added to the squares originally of length $\aleph_{\alpha+2}$ have length $\aleph_{\alpha+1}$). This will preserve regularity of $\aleph^W_{\omega_1}$ using the fact that if τ is a regular cardinal such that $\mathbb{P}$ has size $\leq \tau$ and $\mathbb{Q}$ is τ^+-distributive, then $\Vdash_{\mathbb{P}}$ "$\mathbb{Q}$ is τ^+-distributive". Standard lifting arguments then show that there is a bad scale on $\aleph^W_{\omega_1}$ in the extension by the product of threads, but this implies that there is already a bad scale in W, and hence that $\square^*_{\aleph_{\omega_1}}$ fails.

Theorem 1.1 also contrasts with the following supporting result:

Theorem 1.2. *Assuming the consistency of a supercompact cardinal, there is a model in which $\aleph_\omega$ is a strong limit, there is a good scale on $\aleph_\omega$, $\square_{\aleph_n}$ holds for all $n < \omega$, and $\square^*_{\aleph_\omega}$ fails—specifically, there is also a bad scale on $\aleph_\omega$.*

This shows that the uncountable cofinality of κ in Theorem 1.1 distinguishes it from the countable case in a pronounced way. The proof of this theorem will use techniques similar to those used for the non-compactness results mentioned above (i.e. [4, 11]). We note that if $\kappa_0 = \aleph_0$ and $\langle \kappa_n : 1 < n < \omega \rangle$ is a sequence of supercompact cardinals in some ground model, then in an extension by $\prod_{n<\omega} \mathrm{Col}(\kappa_0, <\kappa_n)$, we have that all scales on $\aleph_\omega$ are good, $\square^*_{\aleph_\omega}$ fails, and $\square^*_{\aleph_n}$ holds for all $n < \omega$.[2] However, getting $\square_{\aleph_n}$ to hold for $n < \omega$ makes a stronger point as a contrast with Theorem 1.1.

For the remainder of the introduction, we will focus on definitions. In Section 2 we will prove Theorem 1.1, and in Section 3 we will prove Theorem 1.2.

1.1. **Definitions.** We define square sequences in terms of a hierarchy introduced by Schimmerling [17].

Definition 1.3. We say that $\langle \mathcal{C}_\alpha \mid \alpha \in \lim(\kappa^+) \rangle$ is a $\square_{\kappa,\lambda}$-*sequence* if for all limit $\alpha < \kappa^+$:

(1) each $C \in \mathcal{C}_\alpha$ is a club subset of α with $\mathrm{ot}(C) \leq \kappa$;
(2) for every $C \in \mathcal{C}_\alpha$, if $\beta \in \lim(C)$, then $C \cap \beta \in \mathcal{C}_\beta$;
(3) $1 \leq |\mathcal{C}_\alpha| \leq \lambda$.

[2] The reasons these properties hold follow: Magidor and Shelah showed that all scales on $\aleph_\omega$ are good in this model [14], $\square^*_{\aleph_\omega}$ fails because the strong reflection property holds (see Section 4 of [3]), and $\square^*_{\aleph_n}$ holds for all $n < \omega$ because of GCH by a theorem of Specker.

The principle $\square_{\kappa,1}$ is the original $\square_\kappa$, and $\square_{\kappa,\kappa}$ is the weak square, denoted $\square^*_\kappa$.

Definition 1.4. If μ is a cardinal and $S \subset \lim(\mu^+)$ is stationary, then we say that $\langle C_\alpha : \alpha \in S\rangle$ is a *partial square sequence* if for all $\alpha \in S$:

(1) C_α is closed and unbounded in α;
(2) $\mathrm{ot}(C_\alpha) \leq \mu$;
(3) if $\beta \in S$ and $\gamma \in \lim C_\alpha \cap \lim C_\beta$, then $C_\alpha \cap \gamma = C_\beta \cap \gamma$.

Definition 1.5.

(1) If τ is a cardinal and $f, g : \tau \to \mathrm{ON}$, then $f <^* g$ if there is some $j < \tau$ such that $f(i) < g(i)$ for all $i \geq j$. The analogous definitions hold for $>^*$ and $=^*$.
(2) Given a singular cardinal κ, we say that a strictly increasing sequence $\vec{\kappa} = \langle \mu_i : i < \mathrm{cf}\,\kappa\rangle$ of regular cardinals converging to κ is a *product* when we regard $\prod_{i<\mathrm{cf}\,\kappa} \mu_i$ as a space.
(3) Given a product $\vec{\kappa} = \prod_{i<\mathrm{cf}\,\kappa} \mu_i$, a sequence $\langle f_\alpha : \alpha < \kappa^+\rangle$ is a *scale* on $\vec{\kappa}$ if:
    (a) for all $\alpha < \kappa^+$, $f_\alpha \in \vec{\kappa}$, i.e. $f_\alpha(i) < \mu_i$ for all $i < \mathrm{cf}\,\kappa$;
    (b) for all $\beta < \alpha < \kappa^+$, $f_\alpha <^* f_\beta$;
    (c) for all $g \in \vec{\kappa}$, there is some $\alpha < \kappa^+$ such that $g <^* f_\alpha$ (i.e. $\langle f_\alpha : \alpha < \kappa^+\rangle$ is cofinal in the product $\vec{\kappa}$).

    We also say that the product $\vec{\kappa}$ *carries* $\vec{f}$.
(4) We will use the term *pseudo-scale* for an object resembling a scale that is not necessarily cofinal in its product $\vec{\kappa}$, i.e. it satisfies (a) and (b) of the previous item.
(5) Given a scale (or pseudo-scale) $\vec{f} = \langle f_\alpha : \alpha < \kappa^+\rangle$, $\alpha < \kappa^+$ is *good* if there is some unbounded $A \subset \alpha$ with $\mathrm{ot}\,A = \mathrm{cf}\,\alpha$ and some $j < \mathrm{cf}\,\kappa$ such that for all $i \geq j$, $\langle f_\beta(i) : \beta \in A\rangle$ is strictly increasing.
(6) If there is a club $D \subset \kappa^+$ such that every $\alpha \in D$ with $\mathrm{cf}\,\alpha > \mathrm{cf}\,\kappa$ is a good point of $\vec{f}$, then $\vec{f}$ is a *good scale.* An analogous definition applies for good pseudo-scales.

The reason for defining pseudo-scales is that the cofinality clause of the definition of a scale will be largely irrelevant for our purposes. The next fact is what we use to obtain failure of $\square^*_\kappa$ in Theorem 1.2.

Fact 1.6. *If κ is singular, then $\square^*_\kappa$ implies that all pseudo-scales on κ are good.*[3]

2. ZFC Results

In this section we will prove the main results of the paper. We clarify notions of continuity in Subsection 2.1, then we prove Theorem 1.1 in Subsection 2.2, and then we sketch an analogous theorem for partial squares in Subsection 2.3.

2.1. **Continuity.** Our goal in this section is to obtain a strong concept of the continuity used by Cummings, Foreman, and Magidor for scales on a singular cardinal κ of cofinality λ. The material concerning points α such that $\mathrm{cf}(\alpha) > \lambda$ is the same as theirs, but we want to consider some issues that arise when $\mathrm{cf}(\alpha) \leq \lambda$. Specifically, continuity is trivial if $\mathrm{cf}(\alpha) < \lambda$, and we would like to modify the concept of continuity for the situation where $\mathrm{cf}(\alpha) = \lambda$ so that the square sequences we define are coherent.

[3] This is in Cummings' survey [1], but without the distinction involving pseudo-scales.

Fix a singular κ of cofinality $\lambda > \omega$. We will consider some fixed stationary $S \subseteq \lambda$ and a product $\vec{\kappa} = \prod_{i \in S} \mu_i$. This formulation will be important when we are considering $\alpha \in \kappa^+ \cap \operatorname{cof}(\lambda)$. Fix a pseudo-scale $\vec{f}$ on $\vec{\kappa}$.

Proposition 2.1. *If* $\operatorname{cf} \alpha > \operatorname{cf} \kappa$ *and* α *is a good point, then for any cofinal* $B \subset \alpha$ *with* $\operatorname{ot} B = \operatorname{cf} \alpha$*, there is some* $B^* \subseteq B$ *such that* B^* *witnesses goodness of* α*.*

This follows from what is known as "The Sandwich Argument."

Proof. Suppose $A \subset \alpha$ witnesses goodness. Let $\tau = \operatorname{cf} \alpha$ and enumerate $A' := \langle \alpha_\xi : \xi < \tau \rangle \subset A$ and $B' := \langle \beta_\xi : \xi < \tau \rangle \subset B$ in such a way that for all $\xi < \tau$, $f_{\alpha_\xi} \leq^* f_{\beta_\xi} <^* f_{\alpha_{\xi+1}}$. Observe that A' also witnesses goodness of α with respect to some j'. For each $\xi < \tau$, let $j_\xi \geq j'$ be such that $i \geq j_\xi$ implies $f_{\alpha_\xi}(i) \leq f_{\beta_\xi}(i) < f_{\alpha_{\xi+1}}(i)$. Then there is some unbounded $X \subset \tau$ and some $j < \lambda$ such that for all $\xi < \tau$, $j_\xi = j$. Since j also witnesses goodness with respect to A', this means that if $\xi, \eta \in X$ and $\xi < \eta$, then for all $i \geq j$, we have $f_{\beta_\xi}(i) < f_{\alpha_{\xi+1}}(i) \leq f_{\alpha_\eta}(i) \leq f_{\beta_\eta}(i)$. We have proved the proposition with $B^* = \langle \beta_\xi : \xi \in X \rangle$. $\square$

Modulo a short argument, this implies:

Proposition 2.2. *If a product* $\vec{\kappa}$ *carries a good scale* $\vec{f}$*, then there is a scale* $\vec{g}$ *such that* every α *with* $\operatorname{cf} \alpha > \operatorname{cf} \kappa$ *is a good point of* $\vec{g}$*.*

Definition 2.3. Suppose $\vec{f} = \langle f_\alpha : \beta < \alpha \rangle$ is a $<^*$-increasing sequence on the product $\vec{\kappa} = \prod_{i \in S} \mu_i$, and that $A \subset \alpha$ is unbounded for some $\alpha < \kappa^+$ with $\operatorname{ot} A = \operatorname{cf} \alpha$.

- $\vec{f}_A$ denotes the function $i \mapsto \sup_{\beta \in A} f_\beta(i)$;
- if $\operatorname{cf} \alpha = \operatorname{cf} \kappa$ and $A = \langle \beta_i : i < \operatorname{cf} \kappa \rangle$, $\vec{f}_A^{\Delta}$ denotes the function $i \mapsto \sup_{j<i} f_{\beta_j}(i)$.

Definition 2.4. If f and g are functions on a product $\vec{\kappa}$, we write $f =^*_\Delta g$ if there is a club $C \subseteq \lambda$ such that for all $i \in C \cap S$, $f(i) = g(i)$. The definition for $f <^*_\Delta g$ is analogous.

Definition 2.5. A scale $\vec{f} = \langle f_\alpha : \alpha < \kappa^+ \rangle$ is *totally continuous* if the following hold:

- if $\operatorname{cf} \alpha < \operatorname{cf} \kappa$, then for all cofinal $A \subset \alpha$ with $\operatorname{ot} A = \operatorname{cf} \alpha$, $(\vec{f} \restriction \alpha)_A =^* f_\alpha$;
- if $\operatorname{cf} \alpha = \operatorname{cf} \kappa$, then for all *clubs* $A \subset \alpha$ such that $\operatorname{ot} A = \operatorname{cf} \alpha$, we have $f_\alpha =^*_\Delta (\vec{f} \restriction \alpha)^\Delta_A$;
- if $\operatorname{cf} \alpha > \operatorname{cf} \kappa$, then α is a good point, f_α is an exact upper bound of $\langle f_\beta : \beta < \alpha \rangle$, and for all cofinal $A \subset \alpha$ witnessing goodness of α, we have $(\vec{f} \restriction \alpha)_A =^* f_\alpha$.

Even though these cases are different, we will say *by continuity* if we invoke any of them.

Now we work towards:

Lemma 2.6. *If* $\operatorname{cf} \kappa = \lambda > \omega$*,* $S \subset \lambda$ *is stationary, and* $\vec{\kappa} = \prod_{i \in S} \mu_i$ *is a product of regular cardinals on* κ *that carries a good scale, then it carries a totally continuous good scale.*

Fix a $<^*$-increasing sequence $\vec{f} = \langle f_\alpha : \beta < \alpha \rangle$ on a product $\prod_{i<\mathrm{cf}\,\kappa} \mu_i$. The following is straightforward:

Proposition 2.7. *Suppose $\alpha < \kappa^+$, $\mathrm{cf}\,\alpha < \mathrm{cf}\,\kappa$, $A, B \subset \alpha$ are unbounded and $\mathrm{ot}\,A = \mathrm{ot}\,B = \mathrm{cf}\,\alpha$. Then $\vec{f}_A =^* \vec{f}_B$.*

Proposition 2.8. *If $\mathrm{cf}\,\alpha > \mathrm{cf}\,\kappa$ and $A \subset \alpha$ witnesses goodness, then $\vec{f}_A$ is an exact upper bound of $\langle f_\beta : \beta \in A \rangle$.*

Proof. It is straightforward that $\vec{f}_A$ is an upper bound. For exactness, suppose that $g <^* \vec{f}_A$. Let $j < \lambda$ witness goodness with respect to A as well as $g <^* \vec{f}_A$, and for all i with $j \leq i < \lambda$, let $\beta_i \in A$ be such that $g(i) < f_{\beta_i}(i)$. If $\beta = \sup_{j \leq i < \lambda} \beta_i$, then by goodness we have $g <^* f_\beta$. □

Remark. If $\mathrm{cf}\,\alpha \leq \mathrm{cf}\,\kappa$, then $\langle f_\beta : \beta < \alpha \rangle$ has no exact upper bound: Let $\langle \beta_\xi : \xi < \mathrm{cf}\,\alpha \rangle$ be increasing and cofinal in α and let $\langle S_\xi : \xi < \mathrm{cf}\,\alpha \rangle$ be a partition of $\mathrm{cf}\,\kappa$ into disjoint unbounded sets. Define g such that $g(i) = f_{\beta_\xi}(i)$ if and only if $i \in S_\xi$. Then $g <^* f_\alpha$, but there is no $\beta < \alpha$ such that $g <^* f_\beta$.

Proposition 2.9. *If $\mathrm{cf}\,\alpha > \mathrm{cf}\,\kappa$, $A \subset \alpha$ witnesses goodness of α, and $A' \subset A$ is unbounded in α, then $\vec{f}_A =^* \vec{f}_{A'}$.*

Proof. It is immediate that $\vec{f}_{A'} \leq^* \vec{f}_A$. Suppose for contradiction that $\vec{f}_{A'} <^* \vec{f}_A$ as witnessed by $j < \lambda$. Assume that j is also large enough to witnesses goodness with respect to A, which implies that it witnesses goodness with respect to A' as well. Then for all i with $j \leq i < \lambda$, there is some $\beta_i \in A$ such that $\vec{f}_{A'}(i) < f_{\beta_i}(i) < \vec{f}_A(i)$. Let β be an element of A' greater or equal to $\sup_{j \leq i < \lambda} \beta_i < \alpha$. By goodness of A', $i \geq j$ implies that $f_{\beta_i}(i) \leq f_\beta(i)$, and so we have $f_\beta(i) \leq \vec{f}_{A'}(i) < f_\beta(i)$, a contradiction. □

Proposition 2.10. *Suppose $\alpha < \kappa^+$, $\mathrm{cf}\,\alpha > \mathrm{cf}\,\kappa$ and $A, B \subset \alpha$ both witness goodness of α. Then $\vec{f}_A =^* \vec{f}_B$.*

Proof. Assume that j is large enough to witness goodness with respect to both A and B. Use the Sandwich Argument from Proposition 2.1 to find $A' \subset A$ and $B' \subset B$ such that $\vec{f}_{A'} =^* \vec{f}_{B'}$. Our result then follows from Proposition 2.9. □

Proposition 2.11. *Suppose $\alpha < \kappa^+$, $\mathrm{cf}\,\alpha = \mathrm{cf}\,\kappa$, and C, D are both clubs in α such that $\mathrm{ot}\,C = \mathrm{ot}\,D = \mathrm{cf}\,\alpha$. Then $f_C^\Delta =^*_\Delta f_D^\Delta$.*

Proof. Suppose otherwise. Enumerate $C = \langle \beta_i : i < \mathrm{cf}\,\kappa \rangle$ and $D = \langle \gamma_i : i < \mathrm{cf}\,\kappa \rangle$. Then without loss of generality, $\{i < \mathrm{cf}\,\kappa : \vec{f}_C^\Delta(i) < \vec{f}_D^\Delta(i)\}$ is stationary in $\mathrm{cf}\,\kappa$. Let E be the club $\{i < \mathrm{cf}\,\kappa : \forall j_1, j_2 < i, \exists j^* < i \text{ witnessing } f_{\gamma_{j_1}} <^* f_{\gamma_{j_2}}\}$. Observe that if $i \in \lim E$, then $\langle f_{\gamma_j}(i) : j < i \rangle$ is strictly increasing, so for all $\delta < \sup_{j<i} f_{\gamma_j}(i)$, there is some $j' < i$ such that $\delta < f_{\gamma_{j'}}(i)$. Let $S := \lim E \cap \{i < \mathrm{cf}\,\kappa : \vec{f}_C^\Delta(i) < \vec{f}_D^\Delta(i)\}$.

Then for all $i \in S$, there is some $j < i$ such that $\vec{f}_C^\Delta(i) < f_{\gamma_j}(i)$. By Fodor's Lemma, there is a stationary $T \subset S$ and some $k < \mathrm{cf}\,\kappa$ such that for all $i \in T$, $\vec{f}_C^\Delta(i) < f_{\gamma_k}(i)$. If ℓ is large enough that $\gamma_k < \beta_\ell$, then there is some m such that for all $i \geq m$, $f_{\gamma_k}(i) < f_{\beta_\ell}(i)$. If $i > m, \ell$, then $f_{\gamma_k}(i) < f_{\beta_\ell}(i) \leq \vec{f}_C^\Delta(i)$. But T is of course unbounded, so this implies that we can find an i such that $f_{\gamma_k}(i) < \vec{f}_C^\Delta(i) < f_{\gamma_k}(i)$, a contradiction. □

Proof of Lemma 2.6. We are working with a product $\vec{\kappa} := \prod_{i<\lambda} \mu_i$. Let $\vec{g} = \langle g_\alpha : \alpha < \kappa^+ \rangle$ be a good scale on this product. Then we define a totally continuous scale $\vec{f} = \langle f_\alpha : \alpha < \kappa^+ \rangle$ by induction as follows using the propositions from this section: If $\alpha = \beta + 1$, choose $\gamma < \kappa^+$ large enough that $f_\beta <^* g_\gamma$. Then let f_α be such that $g_\gamma <^* f_\alpha$. If α is a limit and $\operatorname{cf} \alpha < \lambda$, choose any A, a cofinal subset of α of order-type $\operatorname{cf} \alpha$. Then let $f_\alpha := \vec{f}_A$. (Proposition 2.7.) If α is a limit and $\operatorname{cf} \alpha = \lambda$, choose A to be any club subset of α of order-type $\operatorname{cf} \alpha$. Then let $f_\alpha := \vec{f}_A^\Delta$. (Proposition 2.11.) Lastly, suppose α is a limit and $\operatorname{cf} \alpha > \lambda$. Then α is a good point in terms of $\langle f_\beta : \beta < \alpha \rangle$ because it is cofinally interleaved with $\langle g_\beta : \beta < \alpha \rangle$. Hence we can choose any cofinal $A \subset \alpha$ and let $f_\alpha := \vec{f}_A$. (Proposition 2.8 and Proposition 2.10.) $\square$

2.2. **The Construction for Weak Square.** Commencing with the proof of Theorem 1.1, fix a singular κ with cofinality $\lambda > \omega$ such that $S^* := \{\delta < \kappa : \square_\delta^* \text{ holds}\}$ is stationary (and of order-type λ). It will be sufficient to assume that for all $\tau < \kappa$, $\tau^\lambda < \kappa$, and to assume that $\prod_{\delta \in S^*} \delta^+$ carries a good pseudo-scale.

Proposition 2.12. *There is a club $E \subset \kappa$ consisting of singular cardinals.*

Proof. If $E \subset \kappa$ is any club of order-type λ, then all ordinals in $E' := \lim(E) \setminus (\lambda+1)$ are greater than λ and have cofinality less than λ, so they are singular. Moreover, we can argue that there is a club $E'' \subset E'$ of cardinals. Otherwise, there is a stationary $T \subset E'$ and a regressive function $\delta \mapsto |\delta| < \delta$ on T. This function is constant with value υ on a stationary subset $T' \subset T$, but this contradicts that fact that T' is unbounded in κ. $\square$

Using Proposition 2.12, let $\langle \kappa_i : i < \lambda \rangle$ be a continuous, cofinal, and strictly increasing sequence of singular cardinals in κ. It follows that $S := \{i < \lambda : \kappa_i \in \lim(S^*)\}$ is stationary in λ. Note that $\prod_{i \in S} \kappa_i^+$ also carries a good pseudo-scale, so we can use Lemma 2.6 to find a totally continuous pseudo-scale $\vec{f} = \langle f_\alpha : \alpha < \kappa^+ \rangle$ on the same product.

Let $\vec{\mathcal{C}}_i = \langle \mathcal{C}_\xi^i : \xi < \kappa_i^+ \rangle$ witness $\square_{\kappa_i}^*$ for all $i \in S$. Since κ_i is a limit cardinal for all i, we can assume that for all such i these $\square_{\kappa_i}^*$-sequences have the property that $\operatorname{ot} C < \kappa_i$ for all $C \in \mathcal{C}_\xi^i$, $\xi < \kappa_i^+$ (see [1]). If $\alpha < \kappa^+$, we define $\mathcal{F}_\alpha$ as follows:

- If $\operatorname{cf}(\alpha) \neq \lambda$, we let $\mathcal{F}_\alpha$ be the set of functions F such that $\operatorname{dom} F = S$ and such that $\forall i \in S$, $F(i) \in \mathcal{C}_{f_\alpha(i)}^i$.
- If $\operatorname{cf}(\alpha) = \lambda$, we let $\mathcal{F}_\alpha$ be the set of functions F such that $\operatorname{dom} F = S$ and such that *for some* $h =_\Delta^* f_\alpha$, $\forall i \in S$, $F(i) \in \mathcal{C}_{h(i)}^i$.

Regardless of whether or not $\operatorname{cf}(\alpha) = \lambda$, we will say that some $h \in \prod_{i \in S} \kappa_i^+$ *witnesses* $F \in \mathcal{F}_\alpha$ if for all $i \in S$, $F(i) \in \mathcal{C}_{h(i)}^i$.

For each $\alpha < \kappa^+$ and $F \in \mathcal{F}_\alpha$, we define $C_F \subset \alpha$ as follows:

- If $\beta < \alpha$ and $\operatorname{cf} \beta \neq \lambda$, then $\beta \in C_F$ if and only if there is some $j < \lambda$ such that for all $i \in S \setminus j$, $f_\beta(i) \in \lim F(i)$.
- If $\beta < \alpha$ and $\operatorname{cf} \beta = \lambda$, then $\beta \in C_F$ if and only if there the set of limit ordinals $\gamma \in C_F$ with $\operatorname{cf}(\gamma) < \lambda$ is unbounded in β.

Now we define our $\square_\kappa^*$-sequence at α depending on the cofinality:

- If $\operatorname{cf} \alpha < \lambda$, then $\mathcal{C}_\alpha := \{C_F : F \in \mathcal{F}_\alpha \text{ and } C_F \text{ is unbounded in } \alpha\} \cup \{C \subset \alpha : C \text{ is a club in } \alpha \text{ and } \operatorname{ot} C < \lambda\}$.

- If $\operatorname{cf}\alpha = \lambda$, choose a club $C \subset \alpha$ such that $\operatorname{ot} C = \lambda$ and let $\mathcal{C}_\alpha := \{C_F : F \in \mathcal{F}_\alpha \text{ and } C_F \text{ is unbounded in } \alpha\} \cup \{C\}$.
- If $\operatorname{cf}\alpha > \lambda$, let $\mathcal{C}_\alpha := \{C_F : F \in \mathcal{F}_\alpha\}$.

Lemma 2.13. *For all $\alpha \in \lim(\kappa^+)$ and $C \in \mathcal{C}_\alpha$, C is closed.*

Proof. It is enough to show that for all $\alpha \in \lim(\kappa^+)$ and $F \in \mathcal{F}_\alpha$, C_F is closed. The proof of this lemma does not depend on whether or not $\operatorname{cf}(\alpha) = \lambda$; that is, it does not depend on whether $F \in \mathcal{F}_\alpha$ is witnessed specifically by f_α or some $h =^*_\Delta f_\alpha$. Let $\langle \beta_\xi : \xi < \tau \rangle \subseteq C_F$ be a strictly increasing sequence with supremum $\beta < \alpha$ where τ is regular. For each $\xi < \tau$, let j_ξ witness that $\beta_\xi \in C_F$, i.e. for all $i \geq j$, $f_{\beta_\xi}(i) \in \lim F(i)$.

Case 1: $\tau < \lambda$. If $j' = \sup_{\xi<\tau} j_\xi$, then for all $i \geq j'$, we have $\sup_{\xi<\tau} f_{\beta_\xi}(i) \in \lim F(i)$. By continuity, there is also some j'' such that for all $i \geq j''$, $f_\beta(i) = \sup_{\xi<\tau} f_{\beta_\xi}(i)$. Hence, if j is larger than j' and j'', then j witnesses that $\beta \in C_F$ by closure of $F(i)$ for $i \in S$.

Case 2: $\tau > \lambda$. By the Pigeonhole Principle there is some unbounded $Z \subset \tau$ and some $j' < \lambda$ such that $j_\xi = j'$ for all $\xi \in Z$. By Proposition 2.1, there is some j'' and some $Z' \subset Z$ such that $\{\beta_\xi : \xi \in Z'\}$ and j'' witness goodness. It then follows by continuity that for all $i \geq j''$, $f_\beta(i) = \sup_{\xi \in Z'} f_{\beta_\xi}(i)$. If $j \geq j', j''$, then j witnesses that $\beta \in C_F$ as in the previous case.

Case 3: $\tau = \lambda$. By Case 2, we can assume that $\operatorname{cf}(\beta_\xi) < \lambda$ for all $\xi < \lambda$. Then closure follows by definition. $\square$

Lemma 2.14. *For all $\alpha \in \lim(\kappa^+)$, $C \in \mathcal{C}_\alpha$ is unbounded in α.*

Proof. It is sufficient to show that if $\operatorname{cf}\alpha > \operatorname{cf}\kappa$, then C_F is unbounded in α for an arbitrary $F \in \mathcal{F}_\alpha$. We will use the fact that $F \in \mathcal{F}_\alpha$ can only be witnessed by f_α. Consider some $\bar{\alpha} < \alpha$. We will find an element of C_F larger than $\bar{\alpha}$. By induction we define a sequence of ordinals $\langle \alpha_n : n < \omega \rangle$ in the interval $(\bar{\alpha}, \alpha)$, a $<^*$-increasing sequence of functions $\langle g_n : n < \omega \rangle$ in $\prod_{i \in S} \kappa_i^+$, and an undirected list of ordinals $\langle j_n : n < \omega \rangle$ in λ.

Suppose that α_n and g_n are defined. Let g_{n+1} be defined so that for all $i < \operatorname{cf}\kappa$, $g_{n+1}(i)$ an element of $F(i)$ larger than $f_{\alpha_n}(i)$. Using the facts that $g_{n+1} <^* f_\alpha$ and that f_α is an exact upper bound of $\langle f_\beta : \beta < \alpha \rangle$, find α_{n+1} so that $g_{n+1} <^* f_{\alpha_{n+1}}$, and let $j_{n+1} < \lambda$ witness this.

Let $\beta = \sup_{n<\omega} \alpha_n$, which in particular is larger than $\bar{\alpha}$. We claim that $\beta \in C_F$ as witnessed by $j := \sup_{n<\omega} j_n < \lambda$. For each $i < \lambda$ such that $i \geq j$, $\langle g_n(i) : i < \omega \rangle$ and $\langle f_{\alpha_n}(i) : n < \omega \rangle$ interleave each other, so $\sup_{n<\omega} f_{\alpha_n}(i) \in \lim F(i)$ for such i. For sufficiently large i, $f_\beta(i) = \sup_{n<\omega} f_{\alpha_n}(i)$ by continuity, so this completes the proof. $\square$

Lemma 2.15. *For all $\alpha \in \lim(\kappa^+)$ and $C \in \mathcal{C}_\alpha$, if $\beta \in \lim C$, then $C \cap \beta \in \mathcal{C}_\beta$.*

Proof. The lemma is only substantial if $C = C_F$ for some $F \in \mathcal{F}_\alpha$, and it does not depend on whether $\operatorname{cf}(\alpha) = \lambda$. By assumption C_F is unbounded in β, so Lemma 2.13 implies that $\beta \in C_F$.

Case 1: $\operatorname{cf}\beta \neq \lambda$: Let $j < \lambda$ witness $\beta \in C_F$, meaning that if $i \geq j$ then $f_\beta(i) \in \lim F(i)$. By the coherence of $\vec{\mathcal{C}}^i$ for $i \in S$, it follows that $F(i) \cap f_\beta(i) \in \mathcal{C}^i_{f_\beta(i)}$ for such i. Let F' be a function with domain S such that $F'(i) \in \mathcal{C}^i_{f_\beta(i)}$ for all $i \in S$ and such that $F'(i) = F(i) \cap f_\beta(i)$ for $i \geq j$ in particular. Then $F' \in \mathcal{F}_\beta$ and $C_{F'}$

is unbounded in β, so $C_{F'} \in \mathcal{C}_\beta$. If $\gamma < \beta$, let $j' < \lambda$ witness $f_\gamma <^* f_\beta$. Then if $i \geq j, j'$, it follows that $f_\gamma(i) \in F(i)$ if and only if $f_\gamma(i) \in F'(i)$. We conclude that $C_F \cap \beta = C_{F'}$.

Case 2: $\operatorname{cf} \beta = \lambda$: Choose a sequence $\langle \beta_i : i < \lambda \rangle \subset C_F \cap \beta$; by closure (Lemma 2.13, Case 1) we can assume that $\langle \beta_i : i < \lambda \rangle$ is closed and unbounded in λ, and that $\operatorname{cf}(\beta_i) < \lambda$ for all $i < \lambda$. By Proposition 2.11, we can also assume that $f_\beta =^*_\Delta (\vec{f} \restriction \beta)^\Delta_{\langle \beta_i : i < \lambda \rangle}$, i.e. that there is a club $E \subset \lambda$ such that for all $i \in E$, $f_\beta(i) = \sup_{j<i} f_{\beta_j}(i)$. Let D be a club such that $D \subseteq E$ and such that for all $i \in D, j < i$, there is some $j' < i$ witnessing that $\beta_j \in C_F$, and moreoever such that for all $i \in D, j_1, j_2 < i$, there is some $j < i$ witnessing that $f_{\beta_{j_1}} <^* f_{\beta_{j_2}}$. It follows that for all $i \in D$ and $j < i$, $f_{\beta_j}(i) \in \lim F(i)$, and therefore that for all $i \in D$, $f_\beta(i) \in \lim F(i)$. Then let F' be defined so that $F'(i) = F(i) \cap f_\beta(i)$ for $i \in D \cap S$ and $F'(i) = F(i)$ for $i \in S \setminus D$. Then it follows that $C_F \cap \beta = C_{F'}$: in particular, if $\gamma \in C_{F'}$, then f_γ is dominated by f_β on a club, so it must be the case that $\gamma < \beta$. Hence we find that $F' \in \mathcal{F}_\beta$ is witnessed by h such that $h(i) = f_\beta(i)$ for $i \in D$ and $h(i) = h'(i)$ for the h' witnessing $F \in \mathcal{F}_\alpha$ (hence $h =^*_\Delta f_\beta$). Therefore we have shown that $C_F \cap \beta \in \mathcal{C}_\beta$. □

Lemma 2.16. *For all $\alpha \in \lim(\kappa^+)$ and $C \in \mathcal{C}_\alpha$, $\operatorname{ot} C < \kappa$.*

Proof. It is sufficent to show that $\operatorname{ot} C_F < \kappa$ for all $F \in \mathcal{F}_\alpha$ and all $\alpha < \kappa^+$ (independently of whether $\operatorname{cf}(\lambda) = \alpha$). Recall that we assumed that the $\square^*_{\kappa_i}$-sequences $\langle \mathcal{C}^i_\xi : i < \kappa_i^+ \rangle$ were defined so that for all for all $i < \lambda, \xi < \kappa_i^+, C \in \mathcal{C}^i_\xi$, $\operatorname{ot} C < \kappa_i$.

Fix $\alpha < \kappa^+$. For every $i \in S$, there is some $j < i$ such that $\operatorname{ot} F(i) < \kappa_j$. This means that there is a stationary $T \subseteq S$ and some k such that for all $i \in T$, $\operatorname{ot} F(i) < \kappa_k$. If $\beta \in C_F$ and $i \in T$, let $g_\beta(i) = \operatorname{ot}(F(i) \cap f_\beta(i))$ for all i such that $f_\beta(i) \in F(i)$ and 0 otherwise. The set $\{g_\beta : \beta \in C_F\}$ has size $\kappa_k^\lambda < \kappa$ (we assumed this bit of cardinal arithmetic), so it is enough to observe that if $\beta, \beta' \in C_F$ and $\beta < \beta'$, then g_β and $g_{\beta'}$ are distinct. □

Lemma 2.17. *For all $\alpha \in \lim(\kappa^+)$, $|\mathcal{C}_\alpha| \leq \kappa$.*

Proof. Our assumption that $\tau^\lambda < \kappa$ for all $\tau < \kappa$ implies that $|\{C \subset \alpha : \operatorname{ot} C < \lambda\}| = \kappa$, so it is enough to show that $|\{C_F : F \in \mathcal{F}_\alpha\}| \leq \kappa$ for all $\alpha \in \lim(\kappa^+)$.

Fix $\alpha \in \lim(\kappa^+)$. We first argue for the case in which $\operatorname{cf}(\alpha) \neq \lambda$. For all $i \in S'$ enumerate $\mathcal{C}^i_{f_\alpha(i)} = \langle C^i_\zeta : \zeta < \kappa_i \rangle$. For stationary sets $T \subset S'$ and $\zeta < \kappa$, let

$$X^k_T = \{F \in \mathcal{F}_\alpha : \forall i \in T, \exists \zeta < \kappa_k \text{ such that } F(i) = C^i_\zeta \text{ and } \operatorname{ot}(C^i_\zeta) < \kappa_k\}.$$

We claim that for all $F \in \mathcal{F}_\alpha$, there are $T \subset S'$ and $k < \lambda$ such that $C_F \in X^k_T$. Let $F \in \mathcal{F}_\alpha$. For each F and $i \in S'$, there is some $j < i$ such that we have $F(i) = C^i_\zeta$ for some $\zeta < \kappa_j$ and $\operatorname{ot}(C^i_\zeta) < \kappa_j$ as well. It follows that there is a stationary $T \subset S'$ and $k < \lambda$ such that for all $i \in T$, $F(i) = C^i_\zeta$ and $\operatorname{ot}(C^i_\zeta) < \kappa_k$ for some $\zeta < \kappa_k$.

Because $2^\lambda = \lambda^\lambda < \kappa$, there are at most κ-many X^k_T's. Therefore it remains to show that for all such T, k, that $|\{C_F : F \in X^k_T\}| \leq \kappa$. Let G_F be the set of functions $g_\beta = f_\beta \restriction T$ for all $\beta \in C_F$. If $\beta \neq \beta'$, then $g_\beta \neq g_{\beta'}$, so if $F' \neq F$ then $G_F \neq G_{F'}$. Now, for $i \in T$, let $R^k_T(i) = \bigcup_{\zeta < \kappa_k} C^i_\zeta$. Then for all $F \in F^k_T$, $G_F \subseteq \prod_{i \in T} R^k_T(i)$. Moreover, $\prod_{i \in T} R^k_T(i)$ has cardinality $\kappa_k^\lambda < \kappa$. It follows that $|\{C_F : F \in X^k_T\}| \leq \kappa$.

Now we comment on the case in which $\mathrm{cf}(\alpha) = \lambda$. For all $F \in \mathcal{F}_\alpha$, there is some $S' \subset S$ be a stationary set such that for all $i \in S'$, $F(i) \in C^i_{f_\alpha(i)}$. The argument above can be done for all $F \in \mathcal{F}_\alpha$ such that there is an h witnessing $F \in \mathcal{F}_\alpha$ where $h \restriction S' = f_\alpha \restriction S'$. Since $2^\lambda < \kappa$, and we only need to consider 2^λ-many possible S', this is sufficient. $\square$

This finishes the proof of Theorem 1.1.

2.3. **Sketching the Construction for Partial Square.** We observe that a result similar to Theorem 1.1 holds for partial squares:

Theorem 2.18. *Let κ be a singular strong limit cardinal of cofinality $\lambda > \omega$. Suppose there is a stationary set $S \subset \kappa$ such that $\square_\delta$ holds for all $\delta \in S$ and such that $\prod_{\delta \in S} \delta^+$ carries a good scale. Then there is a partial square sequence on $\kappa^+ \cap \mathrm{cof}(> \lambda)$.*

This can be proved with the same techniques as the previous theorem, and the setup is basically the same: We fix a singular strong limit κ with cofinality $\lambda > \omega$ such that $\{\delta < \kappa : \square_\delta \text{ holds}\}$ is stationary (and of order-type λ). Let $\langle \kappa_i : i < \lambda \rangle$ be continuous, cofinal, and strictly increasing in κ. We find that $S := \{i < \lambda : \square_{\kappa_i} \text{ holds}\}$ is stationary in λ, and we can construct a totally continuous scale $\vec{f} = \langle g_\alpha : \alpha < \kappa^+ \rangle$ on $\prod_{i \in S} \kappa_i^+$. Let $\mathcal{C}_i = \langle C^i_\xi : \xi < \kappa_i^+ \rangle$ witness $\square_{\kappa_i}$ for all $i \in S$. By Proposition 2.12, we can again assume that $\mathrm{ot}\, C^i_\xi < \kappa_i$ for all $\xi < \kappa_i^+, i < \lambda$. Now we can define the clubs of which our square sequence will consist. For each $\alpha \in \kappa^+ \cap \mathrm{cof}(> \lambda)$, let:

$$X_\alpha := \langle \beta < \alpha : \{i < \lambda : f_\beta(i) \in \lim C^i_{f_\alpha(i)}\} \text{ is co-bounded in } S \rangle.$$

Then we have an analog of Lemma 2.13:

Lemma 2.19. *For all $\alpha \in \lim(\kappa^+)$, if $\langle \beta_\xi : \xi < \tau \rangle \subset X_\alpha$'s and $\tau \neq \lambda$, then $\sup_{\xi < \tau} \beta_\xi \in X_\alpha$.*

Then let C_α be the closure of X_α inside α. The partial square sequence will be the sequence $\langle C_\alpha : \alpha \in \kappa^+ \cap \mathrm{cof}(> \lambda) \rangle$. Proofs of the various lemmas are analogous. Coherence for the case $\mathrm{cf}(\beta) = \lambda$ is easier since no witness needs to be constructed.

3. The Consistency Result

In this section we prove Theorem 1.2 using techniques of Cummings, Foreman, and Magidor [4], and of Krueger [11]. The construction of the model begins with a preparation like the one used to force the consistency of Martin's Maximum. Here we force with a product to obtain $\square_{\aleph_n}$ for $n < \omega$ as well as a good scale carried by $\prod_{m \text{ even}} \aleph_m$. Then we use a supercompact embedding to show that there is a bad scale on $\prod_{m \text{ odd}} \aleph_m$. (This idea of using good and bad scales on different products sometimes appears in the context of Prikry extensions, most notably in Gitik-Sharon [8].) The preparation pays off in the way a Namba forcing is used to lift the embedding past the posets adding the $\square_{\aleph_n}$'s and the good scale.

First we define the Namba forcing used for the construction in Subsection 3.1 and handle the ways in which it needs to be distinct from the ones used in the analogous constructions. Then we define a simple poset for adding a good scale in Subsection 3.2 which will be necessary for our construction. Then we show that everything fits together in Subsection 3.3.

3.1. **The Namba Forcing.**

Definition 3.1. Fix a bijection $d : \omega \to \omega \setminus \{0, 1\}$ such that:

(1) For all $m \geq 2$, there are infinitely many n such that $d(n) = m$;
(2) If n is the least number such that $d(n) = m$, then for all $k < n$, $d(k) < m$.

Let P_{even} be the the set of $n < \omega$ such that n is minimal such that $d(n) = m$ and m is even. Let P_{odd} be the analogous set where m is odd.

The poset $\mathbb{P}$ will consist of trees T such that the following hold:

(1) T is a tree consisting of finite sequences t.
(2) For all $t \in T$ and $n \in \text{dom}(t)$, $t(n) \in \aleph_{d(n)}$.
(3) Let $t \in T$ be the unique node maximal in the ordering of T such that for all $s \in T$, either $t \subseteq s$ or $s \subseteq t$. Then t is called the *stem* of T and is denoted $\text{stem}(T)$. The following hold for $t \in T$ with $t \supseteq \text{stem}(T)$:
    (a) If $n = \text{dom}(t)$ and $n \in P_{\text{even}}$. Then $\{\eta : t^\frown\eta \in T\}$ is a stationary subset of $\aleph_{d(n)}$.
    (b) If $n = \text{dom}(t)$ and n' is the largest element of ω such that $n' \in P_{\text{even}}$ and $n' \leq n$, then $\{\eta : t'^\frown\eta \in T\}$ has cardinality $\geq \min\{\aleph_{d(n')}, \aleph_{d(n)}\}$.

The ordering on $\mathbb{P}$ is given by inclusion: $T_1 \leq T_2$ (i.e. T_1 contains more information than T_2) if and only if $T_1 \subseteq T_2$.

If $T \in \mathbb{P}$, we write $n(T) := |\text{stem}(T)|$. If $S, T \in \mathbb{P}$ and $n < \omega$, we write $S \leq_n T$ if $S \leq T$, $\text{stem}(S) = \text{stem}(T)$, and for all t with $|t| \leq n(S) + n$, $t \in S$ if and only if $t \in T$.

The following is immediate from sub-item (b) of item 3 in Definition 3.1:

Proposition 3.2. *For all $m < \omega$ and $t \in T$, there are infinitely many n with $d(m) = n$ such that for some $t' \supseteq t$, $\{\eta : t'^\frown\eta \in T\}$ has cardinality $\aleph_m$.*

Because the forcing we use is meant to provide a master condition for the forcings adding the $\square_{\aleph_n}$'s (which is not needed in Cummings-Magidor [6]), we must make some adjustments to their arguments. They use the concept of badness mentioned here, but we need to stretch out the fusion sequences.

Lemma 3.3. *Let $T \in \mathbb{P}$ and suppose $n(T) = n$ where $n \in P_{\text{even}}$. If $\dot{\alpha}$ is a name for an ordinal below some δ with $\delta < \aleph_{d(n)}$, then there is some $T' \leq_0 T$ deciding $\dot{\alpha}$.*

Proof. Like Cummings and Magidor [6], we say that $T \in \mathbb{P}$ is *bad* if the lemma fails, meaning that for some unfixed n, we have $n(T) = n \in P_{\text{even}}$, yet there is no $T' \leq_0 T$ deciding $\dot{\alpha}$. We claim that for any bad T with $n(T) = n \in P_{\text{even}}$, the set X of $\eta < \aleph_{d(n)}$ such that $T \restriction t^\frown\eta$ is bad is stationary. Otherwise, Fodor's Lemma implies that there is some stationary $X' \subset X$ and some $\beta < \mu$ such that for all $\eta \in X$, there is some $T_\eta \restriction t^\frown\eta$ forcing "$\dot{\alpha} = \beta$". Then consider $T' = \bigcup_{\eta \in X'} T_\alpha$ where $T' \leq_0 T$, which contradicts the assumption of badness.

Supposing that some $T \in \mathbb{P}$ is bad, we work towards a contradiction by defining a fusion sequence $\langle T_n : n < \omega \rangle$ below T as follows: Let $T_0 = T$ and let $\langle k(n) : n < \omega \rangle$ enumerate $P_{\text{prod}} \setminus n(T)$. If T_n is defined, then for all $t \in T_n$ such that $\text{dom}\, t = k(n)$, we choose a collection Y_t of nodes $t' \supseteq t$ of length $< k(n+1)$ such that for all $t' \in Y_t$, t' has $\min\{\aleph_{d(k(n))}, \aleph_{\text{dom}(t')}\}$-many immediate successors in Y_t. Then for each $t' \in Y_t$ with $\text{dom}(t') = k(n+1)$, let $X_{t'}$ be the stationary set of points η such that $T_n \restriction (t'^\frown\eta)$ is bad. Then let $T_{n+1} = \{T_n \restriction u(t)^\frown\eta : t \in T, \text{dom}\, t =$

$k(n), t' \in Y_t, \eta \in X_{t'}\}$. Then let T' be the fusion limit of $\langle T_n : n < \omega \rangle$. Observe that sub-item (b) of item 3 in Definition 3.1 still holds for T'.

Then let $T'' \leq T'$ decide $\dot{\alpha}$. Let $t'' \supseteq \mathrm{stem}(T'')$ be such that $|t''| \in P_{\mathrm{even}}$. Then $T'' \restriction t'' \leq_0 T_n \restriction t'$ for some $T_n, t' \in Y_t$ from the construction, and $T'' \restriction t''$ decides $\dot{\alpha}$, which contradicts badness of $T_n \restriction t'$ in the construction. $\square$

Lemma 3.4. *Let $T \in \mathbb{P}$ and suppose $n(T) = n$ where $n \in P_{\mathrm{even}}$. If $\dot{\alpha}$ is* any *name for an ordinal, then there are $T' \leq_0 T$ and $i < \omega$ such that every i-step extension of T' decides a value for $\dot{\alpha}$.*

Proof. We argue again with analogy to Cummings-Magidor. For this lemma we say that T is *bad* if there is no $T' \leq_0 T$ and no $i < \omega$ such that every i-step extension decides $\dot{\alpha}$. We first claim that if T is bad and $n(T) = n$ with $n \in P_{\mathrm{even}}$, then there are non-stationarily many $\eta \in \aleph_{d(n)}$ such that $n(T) \restriction \mathrm{stem}(T)^\frown \eta$ is bad. The argument uses Fodor's Theorem as in Lemma 3.3. We then do a fusion argument where the only differences with the Cummings-Magidor argument are the use of the Y_t's as in Lemma 3.3 and the fact that at the end we need to find a contradiction using a stem of length $n \in P_{\mathrm{even}}$. $\square$

Now we can collect some properties of our Namba forcing for which existing arguments suffice without alteration.

Facts 3.5. *The following are true for $\mathbb{P}$ relative to a ground model V:*
1. *For all $n < \omega$, $\mathbb{P}$ forces that $\mathrm{cf}(\aleph_n^V) = \omega$.*
2. *$\mathbb{P}$ forces that $\aleph_{\omega+1}^V$ is an ordinal of cardinality and cofinality $\geq \aleph_1$.*
3. *$\mathbb{P}$ preserves stationary subsets of $\aleph_1$.*
4. *If $n \in P_{\mathrm{even}} \cup P_{\mathrm{odd}}$ and $T \in \mathbb{P}$, then the set of nodes $t \in T$ such that $\mathrm{dom}\, t \leq n$ has cardinality strictly less than $\aleph_{d(n)}$.*

Sketch of Proofs. Point *(1)* comes from the fact that there are infinitely many k such that $d(k) = n$ and Proposition 3.2: A genericity argument defines a cofinal function whose domain consists of these k's. Point *(2)* comes from a fusion argument using Lemma 3.4, where we build $T' \leq T$ with at most $|T| = \aleph_\omega$-many possible decisions for $\dot{\alpha}$. Point *(3)* is a variation of the arguments presented by Cummings-Magidor [6] and Krueger [11] using an open game; it is enough that the splitting nodes all split into sets of size $> \aleph_1$. Point *(4)* comes from sub-item (b) of item 3 in Definition 3.1. $\square$

Our focus here is on the distinction between a boundedness lemma and an *un*boundedness lemma. The former deals with the good scale on $\prod_{m \text{ even}} \aleph_m$ and the latter deals with the bad scale on $\prod_{m \text{ odd}} \aleph_m$.

Lemma 3.6. [6] *If V is a ground model with a scale $\vec{f} = \langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ on the product $\prod_{m \text{ even}} \aleph_m$, then the generic function b added by $\mathbb{P}$ is an exact upper bound of $\vec{f}$.*

Proof. To fact that for all b dominates all $h \in \prod_{m \text{ even}} \aleph_m$ follows from a straightforward genericity argument. On the other hand, suppose that $T \in \mathbb{P}$ and $T \Vdash$ "$h <^* \dot{b}$" with $n(T) \in P_{\mathrm{even}}$. We define a fusion sequence $\langle T_n : n < \omega \rangle$ as follows: Let $T_0 = T$ and let $\langle k(n) : n < \omega \rangle$ enumerate $P_{\mathrm{even}} \setminus n(T)$. Suppose we are given T_n. For all $t \in T_n$ with $\mathrm{dom}\, t = k(n+1)$, for all η such that $t^\frown \eta \in T$, use Lemma 3.3 to choose $U_\eta \leq_0 T \restriction (t^\frown \eta)$ deciding $\dot{h}(k(n+1))$. Since

$T \Vdash$ "$\dot{h} <^* \dot{b}$", it follows without generality (for large n) that for a stationary set $X_t \subset \aleph_{k(n+1)}$ and all $\eta \in X_t$, $U_\eta \Vdash$ "$\dot{h}(k(n+1)) = \beta(t)$" for some $\beta(t)$. Let $T_{n+1} = \bigcup\{U_\eta : \eta \in X_t, t \in T_n, \mathrm{dom}(t) = k(n+1)\}$. Let T' be the fusion of the T_n's. For each n, let $g(n) = \sup\{\beta(t) : t \in T_n, \mathrm{dom}(t) = k(n)\}$, where $g(n) < \aleph_{d(n)}$ by item 4 of Facts 3.5. Let α be large enough that $g <^* f_\alpha$. Then $T' \Vdash$ "$\dot{h} <^* f_\alpha$". $\square$

Lemma 3.7. *If V is a ground model and $\dot{h}$ is a $\mathbb{P}$-name for a function in the product $\prod_{m \text{ odd}} \aleph_m^V$, then there is some $g \in V$ such that $\dot{h} <^* g$.*

Proof. Suppose $T \in \mathbb{P}$ and $T \Vdash$ "$\dot{h} \in \prod_{m \text{ odd}} \aleph_m^V$". Let $\langle k_e(n) : n < \omega\rangle$ enumerate $P_{\text{even}} \setminus n(T)$ and let $\langle k_o(n) : n < \omega\rangle$ enumerate $P_{\text{odd}} \setminus n(T)$. Assume without loss of generality that for all $n < \omega$, $k_e(n) < k_o(n)$. Define a fusion sequence $\langle T_n : n < \omega\rangle$ as follows: Let $T_0 = T$. Suppose we are given T_n. Then for all $t \in T_n$ with $\mathrm{dom}\, t = k_e(n)$, choose a collection Y_t of nodes $t' \supseteq t$ of length $< k_e(n+1)$ such that for all $t' \in Y_t$, t' has $\min\{\aleph_{d(n)}, \aleph_{\mathrm{dom}(t')}\}$-many immediate successors in Y_t. Then for all such t and $t' \in Y_t$ such that $\mathrm{dom}(t') = k_e(n+1)$, use Lemma 3.3 to choose $U_t \leq_0 T_n \restriction t$ forcing $\dot{h}(k_o(n)) = \beta(t')$. Then let $T_{n+1} = \bigcup\{U_{t'} : t' \in Y_t, t \in T_n, \mathrm{dom}(t) = k_e(n)\}$. Let T' be the fusion limit of the T_n's. For each n let $g(n) = \sup\{\beta(t') : t' \in Y_t, t \in T_n, \mathrm{dom}(t) = k_e(n)\}$. Then $T' \Vdash$ "$\dot{h} <^* g$". $\square$

3.2. **A Poset for Forcing a Good Scale.** Fix a singular κ of cofinality λ. We define a poset for forcing a good scale.

Definition 3.8. Given some $\vec{\kappa} = \prod_{i<\lambda} \mu_i$, let $\mathbb{G}(\vec{\kappa})$ be a partial order whose conditions have the form $\langle f_\beta : \beta \leq \alpha\rangle$ for some $\alpha < \kappa^+$ such that for all $\beta \leq \alpha$:

(1) $f_\beta \in \prod_{i<\lambda} \mu_i$;
(2) for all $\gamma < \beta$, $f_\gamma <^* f_\beta$;
(3) if $\mathrm{cf}(\beta) > \lambda$, then β is a good point with respect to $\langle f_\gamma : \gamma < \beta\rangle$.

Ordering is by end-extension: if $p, q \in \mathbb{G}(\vec{\kappa})$, then $p \leq q$ if and only if $p \restriction \mathrm{dom}\, q = q$. We drop the notation for $\vec{\kappa}$ when the context is clear.

Proposition 3.9. *$\mathbb{G}(\vec{\kappa})$ is λ^+-directed closed.*

Proof. $\mathbb{G}(\vec{\kappa})$ is *tree-like*, meaning that $p, q \in \mathbb{G}(\vec{\kappa})$ are compatible if and only if $p \leq q$ or $q \leq p$. Therefore it is enough to show that $\mathbb{G}(\vec{\kappa})$ is λ^+-closed. This follows from the facts that points β with $\mathrm{cf}(\beta) < \kappa$ are automatically good and that we do not require points β with $\mathrm{cf}(\beta) = \kappa$ to be good. $\square$

Proposition 3.10. *$\mathbb{G}(\vec{\kappa})$ is $(\kappa+1)$-strongly strategically closed.*

Proof. The play will take the form of a decreasing sequence $\langle p_\xi : \xi \leq \kappa\rangle$ in $\mathbb{G}(\vec{\kappa})$. We will let γ_ξ denote $\max \mathrm{dom}\, p_\xi$. Let j be large enough that $\mu < \kappa_j$.

Player II will play in such a way that for all even successors $\xi_1 < \xi_2 < \mu$, $p_{\xi_1}(\gamma_{\xi_1})(i) < p_{\xi_2}(\gamma_{\xi_2})(i)$ for all $i \geq j$. If ξ is an even successor and $\xi = \eta + 2$ then Player II will choose $p_\xi = p_{\eta+1}{}^\frown\langle \gamma_{\eta+1} + 1, h\rangle$ such that $p_{\eta+1}(\gamma_\xi) <^* h$ and $p_\eta(\gamma_\eta)(i) < h(i)$ for all $i \geq j$. If ξ is a limit and $\mathrm{cf}\,\xi \leq \mathrm{cf}\,\alpha$, then there is nothing to prove because $\mathbb{G}$ is κ^+-closed. If ξ is a limit and $\mathrm{cf}\,\xi > \mathrm{cf}\,\alpha$ and $\gamma_\xi = \sup_{\eta<\xi} \gamma_\eta$, then we need to show that γ_ξ is a good point of $\bigcup_{\eta<\xi} p_\eta$. This follows from the Sandwich Argument (Proposition 2.1). If $\xi = \kappa$, then we can find a lower bound by Proposition 3.9. $\square$

Proposition 3.11. *$\mathbb{G}(\vec{\kappa})$ preserves cardinals and cofinalities through κ^+.*

Proposition 3.12. *$\mathbb{G}(\vec{\kappa})$ adds a good scale to $\vec{\kappa}$.*

Proof. It is clear that the generic object added by $\mathbb{G}(\vec{\kappa})$ is a good pseudo-scale. A genericity argument shows that it is in fact a scale. □

3.3. **The Proof.** We will shortly describe a preparatory forcing. It resembles the one used by Cummings, Foreman, and Magidor for the non-compactness of square [4] insofar as it follows the proof of the consistency of Martin's Maximum [7].

First we recall Jensen's poset for adding $\square_\delta$.

Definition 3.13. $\mathbb{S}_\delta$ is the set of conditions p such that:

- $\operatorname{dom} p = \{\alpha \leq \delta : \alpha \text{ a limit}\}$ for some limit $\delta < \delta^+$;
- $p(\alpha)$ is a club of order type less than or equal to δ;
- $\forall \alpha \in \operatorname{dom} p, \forall \beta \in \lim p(\alpha)$, $p(\alpha) \cap \beta \in p(\beta)$.

For the ordering, $p \leq q$ if p end-extends q, meaning that $\max p \geq \max q$ and $p \restriction (\max \operatorname{dom} q + 1) = q$.

Fact 3.14. *$\mathbb{S}_\delta$ is $(\delta+1)$-strategically closed.*

Fix $d : \omega \to \omega \setminus \{0,1\}$ as in Definition 3.1. Let $\mathbb{S} := \prod_{n<\omega} \mathbb{S}_{\aleph_n}$ and let $\mathbb{G} = \mathbb{G}(\prod_{m \text{ even}} \aleph_m)$. We will consider an extension by $\mathbb{S} \times \mathbb{G}$ over the prepared model, and we will use this product for the preparation.

The preparation works as follows: Let κ be supercompact and fix a Laver function $\ell : \kappa \to \kappa$ such that for every x and $\nu \geq |\operatorname{tc}(x)|$, there is a ν-supercompact embedding $j : V \to M$ with critical point κ such that $j(\ell)(\kappa) = x$.

Now define an iteration $\mathbb{I} = \langle \mathbb{I}_\alpha, \dot{\mathbb{J}}_\alpha : \alpha < \kappa \rangle$ with revised countable support as follows:

1. $\mathbb{J}_\alpha$ is trivial if α is accessible or if $\not\Vdash_{\mathbb{I}_\alpha}$ "$\alpha = \aleph_2$".
2. If Case 1 does not holds and $\ell(\alpha)$ is an $\mathbb{I}_\alpha$-name for a semi-proper poset of the form $(\mathbb{S} \times \mathbb{G}) * \operatorname{Col}(\aleph_1, (\alpha^{+\omega+1})^V)$, or for a poset of the form $\operatorname{Col}(\aleph_1, \nu)$ where $\nu > \alpha$ is regular, then let $\dot{\mathbb{J}}_\alpha = \ell(\alpha)$.
3. If neither Case 1 nor Case 2 hold, then let $\dot{\mathbb{J}}_\alpha$ be a name for $\operatorname{Col}(\aleph_1, \aleph_2)$.

This iteration is semiproper and has the κ-chain condition. Moreover, a poset in $V[\mathbb{I}]$ is semiproper if and only if it preserves stationary subsets of ω_1 (this is by Lemma 3 from the Martin's Maximum paper [7]). Now let $W = V[\mathbb{I}]$.

Proposition 3.15. *For all $n < \omega$, $\square_{\aleph_n}$ holds in $W[\mathbb{S} \times \mathbb{G}]$.*

Proof. Easton's Lemma (which works when closure is replaced by strategic closure) shows that $\mathbb{S}$ preserves cardinals and cofinalities, and it is clear that it adds $\square_{\aleph_n}$-sequences for all $n < \omega$. The fact that these remain $\square_{\aleph_n}$-sequences in the extension by $\mathbb{G}$ follows from the $\aleph_{\omega+1}$-distributivity of $\mathbb{G}$. □

Proposition 3.16. *In $W[\mathbb{S} \times \mathbb{G}]$, there is a good scale on $\prod_{m \text{ even}} \aleph_m$.*

Proof. This follows by Proposition 3.10 and Proposition 3.12. □

Most of the work here consists of the following:

Lemma 3.17. *In $W[\mathbb{S} \times \mathbb{G}]$, there is a bad (pseudo-)scale on $\prod_{m \text{ odd}} \aleph_m$.*

Proof. Fix a $\kappa^{+\omega+1}$-supercompact embedding $j : V \to M$. Let I be $\mathbb{I}$-generic over V, let G_1 be $\mathbb{S}$-generic over V, and let G_2 be $\mathbb{G}$-generic over $V[I][G_1]$. For the first part of the proof, we will show that there is an extension of $W[\mathbb{S}\times\mathbb{G}]$ in which $j : V \to M$ can be lifted to $j : V[\mathbb{I}][\mathbb{S}\times\mathbb{G}] \to M[j(\mathbb{I})][j(\mathbb{S})\times j(\mathbb{G})]$.

We perform the lift in stages, and the first is to get a lift with domain $V[I]$. We know that $V[I] \models$ "$(\mathbb{S}\times\mathbb{G}) * \mathbb{P}$ is semiproper" (in this model this is equivalent to preserving stationary subsets of ω_1). Therefore, by elementarity, we can see that $j(\mathbb{I})$ factors as $\mathbb{I} * (\mathbb{S}\times\mathbb{G}) * \mathbb{P} * \mathrm{Col}(\aleph_1, \aleph_{\omega+1}^W) * \mathbb{I}'$ where $\mathbb{I}'$ is semiproper. Let $(G_1 \times G_2) * H * K_1 * K_2$ be a generic for $j(\mathbb{I})/I$ (so K_1 is the generic with the indicated Lévy collapse component). Silver's classical lifting argument (see [2]) gives us an embedding $j : V[I] \to M[I * (G_1 \times G_2) * H * K_1 * K_2] = M[j(I)]$.

To lift j to have domain $V[I][G_1]$, we use a master condition argument. Since $M[j(I)] \models$ "$\mathrm{cf}(\aleph_n^W) = \omega$" for all $n < \omega$, there are cofinal sets $A_n \subset j``\aleph_n^W$ in $M[j(I)]$ of order-type $\omega$. We define $\bar{s} : n \mapsto (\bigcup_{s\in G_1} j``s(n))^\frown\langle j``\aleph_n^W, A_n\rangle$, and we can see that $\bar{s}$ is a condition because the coherence clause of Definition 3.13 holds vacuously. This gives us $j : V[I][G_1] \to M[j(I)][j(G_1)]$.

Finally, we need to lift j to have domain $V[I][G_1][G_2]$. Here we use a similar master condition argument. Let $\nu = (\kappa^{+\omega+1})^V$ and let $\rho = \sup j``\nu$. Observe that ρ is an ordinal of cardinality and cofinality $\aleph_1$ in the model $M[I][G_1][G_2][H][K_1][K_2][L_1][L_2]$ because $H * K_1 * K_2 * L_1 * L_2$ is a generic for a semi-proper forcing and we have $M[I][G_1][G_2][H][K_1] \models$ "$|\nu| = \aleph_1, \mathrm{cf}(\nu) = \aleph_1$", and this is preserved when adjoining K_2. Let $\bar{p} = \bigcup_{p\in G_2} j``p = j(\vec{f})$. By Lemma 3.6, the generic function $b$ added by $\mathbb{P}$ is an exact upper bound of $\vec{f}$, therefore $\{j(\xi) : \xi \in b\}$ is an exact upper bound of $j(\vec{f}) \restriction \rho$, so $\rho$ is a good point of $j(\vec{f})$, hence $\bar{p}$ is a master condition for $j``G_2$. This completes the lifting argument.

Now that we have lifted the embedding, we will use it to show that there is a bad pseudo-scale—in fact, that any pseudo-scale on $\prod_{m \text{ odd}} \aleph_m^V$ is bad in $V[I][G_1][G_2]$. In this model, let $\vec{f}$ be a pseudo-scale on $\prod_{m \text{ odd}} \aleph_m^V$ and let C be a club in $\aleph_{\omega+1}$. We have $\rho \in j(C)$ by elementarity. We will show that $M[j(I)][j(G_1)][j(G_2)] \models$ "ρ is a bad point of $j(\vec{f})$", from which the lemma will follow by elementarity.

We use the argument from the crux of Krueger's construction. In the model $M[I][G_1][G_2][H][K_1]$, we let $A \subseteq \rho$ be a cofinal set of order-type $\aleph_1$. Choose sets S_n cofinal in $\aleph_n^W$ of order-type ω that are added by the generic object of $\mathbb{P}$. By Lemma 3.7, every function $h \in M[I][G_1][G_2][H]$ in $\prod_{m \text{ odd}} S_m$ is bounded by a function in $\vec{f}$, and the converse also holds by a genericity argument. In other words, $\vec{f}$ is cofinally interleaved with functions from $\prod_{m \text{ odd}} S_m$. This remains true in $M[I][G_1][G_2][H][K_1]$ by countable closure of the Lévy collapse. Working in $M[I][G_1][G_2][H][K_1]$, let $A' \subset A$ be cofinal such that for all $\alpha, \beta \in A'$ with $\alpha < \beta$, there is some $h \in \prod_{m \text{ odd}} S_m$ with $f_\alpha^* <^* h <^* f_\beta^*$. If ρ were a good point of $j(\vec{f})$ in $M[j(I)][j(G_1)][j(G_2)]$, we could apply Proposition 2.1 to A' to find a cofinal $B \subseteq j``A'$ witnessing goodness, which means that if $A'' = \{\alpha : j(\alpha) \in B\}$, then $\langle f_\alpha(i) : \alpha \in A''\rangle$ is strictly increasing for large i. But this contradicts the fact that the S_n's are countable and cofinally interleaved with the f_α's. $\square$

This finishes the proof of Theorem 1.2. We conclude with the following:

Question 1. *Suppose that κ is a singular strong limit of uncountable cofinality λ such that $S := \{\delta < \kappa : \square_\delta^* \text{ holds}\}$ is stationary and of order-type λ. Does $\prod_{\delta\in S} \delta^+$ carry a good pseudo-scale?*

By Theorem 1.1, this question is almost equivalent (modulo a generalization and a strong limit assumption) to the question of Golshani mentioned above: a positive answer would mean that these hypotheses imply $\square^*_\kappa$, and a negative answer would mean that $\square^*_\kappa$ consistently fails in conjunction with these hypotheses.

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