ON COMPACTNESS OF WEAK SQUARE AT SINGULARS OF UNCOUNTABLE COFINALITY

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Abstract. Cummings, Foreman, and Magidor proved that Jensen's square principle is non-compact at $\aleph_\omega$, meaning that it is consistent that $\square_{\aleph_n}$ holds for all $n < \omega$ while $\square_{\aleph_\omega}$ fails. We investigate the natural question of whether this phenomenon generalizes to singulars of uncountable cofinality. Surprisingly, we show that under some mild hypotheses, the weak square principle $\square^*_\kappa$ is in fact compact at singulars of uncountable cofinality.

1. Introduction and Background

The properties of singulars of uncountable cofinality are notoriously different from those of countable cofinality. A prime example is Silver's theorem that GCH cannot fail for the first time at a singular of uncountable cofinality. In contrast, Magidor showed that GCH can fail for the first time at $\aleph_\omega$. There is therefore a natural question of whether this phenomenon generalizes to more complex structures.[1]

Here we focus on the combinatorial properties of inner models, notably square principles. Jensen originally distilled the principle $\square_\kappa$ (where κ is some given cardinal) to study the properties of Gödel's Constructible Universe L [6]. Many variations of $\square_\kappa$ have been studied since then. (Precise definitions will be given below, but Cummings-Foreman-Magidor [2] is the canonical reference for this area.) There is in general a tension between square principles and large cardinals, one instance of which is that $\square_\kappa$ fails if κ is larger than a supercompact cardinal. Moreover, the failure of $\square_\kappa$ at a singular cardinal κ requires considerable consistency strength from large cardinals [12]. The models of interest in this area realize some compatibility of both square principles and the compactness properties exhibited by large cardinals.

In this paper we will address the compactness of square principles themselves: whether or not $\square_\kappa$ necessarily holds for some cardinal κ if $\square_\delta$ holds for sufficiently many cardinals $\delta < \kappa$. Cummings, Foreman, and Magidor proved that it is consistent that $\square_{\aleph_n}$ holds for $1 \leq n < \omega$ but that $\square_{\aleph_\omega}$ fails [3]. Later, Krueger improved the result by obtaining a bad scale on $\aleph_\omega$ in a similar model [7]. But such results can also go in the other direction: Cummings, Foreman, and Magidor also proved that if $\square_{\aleph_n}$ holds for all $n < \omega$, there is an object that to some extent resembles a $\square_{\aleph_\omega}$-sequence but with a weaker coherence property [4]. The main result of this paper is along these lines:

[1] See [8] and [9] for recent examples.

Theorem 1.1. *Suppose that κ is a singular strong limit of cofinality $\lambda > \omega$ such that for some stationary set $S \subseteq \kappa$, $\square^*_\delta$ holds for all $\delta \in S$ and $\prod_{\delta \in S} \delta^+$ carries a good scale. Then $\square^*_\kappa$ holds.*

This represents some progress on a question raised by Golshani online regarding a supposed Silver's Theorem for special Aronszajn trees [5]: at any cardinal δ, $\square^*_\delta$ is equivalent to the existence of a special δ^+-Aronszajn tree [1].

The difference between the results of Cummings-Foreman-Magidor and Theorem 1.1 is that the resulting sequence is fully coherent—not just coherent at points of uncountable cofinality. In other words, we are able to obtain some compactness for a canonical object by obtaining exactly that canonical object in the end. We nonetheless depend on the goodness of scales, as do Cummings-Foreman-Magidor.

Note that the use of stationarity in Theorem 1.1 is necessary. Starting from $V \models$ "κ supercompact", we could work in $V[\mathrm{Col}(\aleph_1, <\kappa)]$ and force with product of square-adding posets $\prod_{\alpha<\omega_1} \mathbb{S}_{\aleph_{\alpha+1}}$ to get to a model W. This model would have a bad scale carried by $\aleph_{\omega_1}$ for the following reason: the added squares could be threaded by a product $\prod_{\alpha<\omega_1} \mathbb{T}_{\aleph_{\alpha+2},\aleph_{\alpha+1}}$ (where the threads added to the squares originally of length $\aleph_{\alpha+2}$ have length $\aleph_{\alpha+1}$). This will preserve regularity of $\aleph^W_{\omega_1}$ using the fact that if τ is a regular cardinal such that $\mathbb{P}$ has size $\leq \tau$ and $\mathbb{Q}$ is τ^+-distributive, then $\Vdash_{\mathbb{P}}$ "$\mathbb{Q}$ is τ^+-distributive". Standard lifting arguments then show that there is a bad scale on $\aleph^W_{\omega_1}$ in the extension by the product of threads, but this implies that there is already a bad scale in W, and hence that $\square^*_{\aleph_{\omega_1}}$ fails.

We also note that there is a contrast to Theorem 1.1 in the case of $\aleph_\omega$: if $\kappa_0 = \aleph_0$ and $\langle \kappa_n : 1 < n < \omega \rangle$ is a sequence of supercompact cardinals in some ground model, then in an extension by $\prod_{n<\omega} \mathrm{Col}(\kappa_0, <\kappa_n)$, we have that all scales on $\aleph_\omega$ are good, $\square^*_{\aleph_\omega}$ fails, and $\square^*_{\aleph_n}$ holds for all $n < \omega$.[2]

For the remainder of the introduction, we will focus on definitions. In Section 2 we will prove Theorem 1.1.

1.1. Definitions.

We define square sequences in terms of a hierarchy introduced by Schimmerling [13].

Definition 1.2. We say that $\langle \mathcal{C}_\alpha \mid \alpha \in \lim(\kappa^+) \rangle$ is a $\square_{\kappa,\lambda}$-*sequence* if for all limit $\alpha < \kappa^+$:

(1) each $C \in \mathcal{C}_\alpha$ is a club subset of α with $\mathrm{ot}(C) \leq \kappa$;
(2) for every $C \in \mathcal{C}_\alpha$, if $\beta \in \lim(C)$, then $C \cap \beta \in \mathcal{C}_\beta$;
(3) $1 \leq |\mathcal{C}_\alpha| \leq \lambda$.

The principle $\square_{\kappa,1}$ is the original $\square_\kappa$, and $\square_{\kappa,\kappa}$ is the weak square, denoted $\square^*_\kappa$.

Definition 1.3. If μ is a cardinal and $S \subset \lim(\mu^+)$ is stationary, then we say that $\langle C_\alpha : \alpha \in S \rangle$ is a *partial square sequence* if for all $\alpha \in S$:

(1) C_α is closed and unbounded in α;
(2) $\mathrm{ot}(C_\alpha) \leq \mu$;
(3) if $\beta \in S$ and $\gamma \in \lim C_\alpha \cap \lim C_\beta$, then $C_\alpha \cap \gamma = C_\beta \cap \gamma$.

Definition 1.4.

[2] The reasons these properties hold follow: Magidor and Shelah showed that all scales on $\aleph_\omega$ are good in this model [10], $\square^*_{\aleph_\omega}$ fails because the strong reflection property holds (see Section 4 of [2]), and $\square^*_{\aleph_n}$ holds for all $n < \omega$ because of GCH by a theorem of Specker.

(1) If τ is a cardinal and $f, g : \tau \to \mathrm{ON}$, then $f <^* g$ if there is some $j < \tau$ such that $f(i) < g(i)$ for all $i \geq j$. The analogous definitions hold for $>^*$ and $=^*$.
(2) Given a singular cardinal κ, we say that a strictly increasing sequence $\vec{\kappa} = \langle \mu_i : i < \operatorname{cf} \kappa \rangle$ of regular cardinals converging to κ is a *product* when we regard $\prod_{i<\operatorname{cf}\kappa} \mu_i$ as a space.
(3) Given a product $\vec{\kappa} = \prod_{i<\operatorname{cf}\kappa} \mu_i$, a sequence $\langle f_\alpha : \alpha < \kappa^+ \rangle$ is a *scale* on $\vec{\kappa}$ if:
   (a) for all $\alpha < \kappa^+$, $f_\alpha \in \vec{\kappa}$, i.e. $f_\alpha(i) < \mu_i$ for all $i < \operatorname{cf} \kappa$;
   (b) for all $\beta < \alpha < \kappa^+$, $f_\alpha <^* f_\beta$;
   (c) for all $g \in \vec{\kappa}$, there is some $\alpha < \kappa^+$ such that $g <^* f_\alpha$ (i.e. $\langle f_\alpha : \alpha < \kappa^+ \rangle$ is cofinal in the product $\vec{\kappa}$).

   We also say that the product $\vec{\kappa}$ *carries* $\vec{f}$.
(4) We will use the term *pseudo-scale* for an object resembling a scale that is not necessarily cofinal in its product $\vec{\kappa}$, i.e. it satisfies (a) and (b) of the previous item.
(5) Given a scale (or pseudo-scale) $\vec{f} = \langle f_\alpha : \alpha < \kappa^+ \rangle$, $\alpha < \kappa^+$ is *good* if there is some unbounded $A \subset \alpha$ with $\operatorname{ot} A = \operatorname{cf} \alpha$ and some $j < \operatorname{cf} \kappa$ such that for all $i \geq j$, $\langle f_\beta(i) : \beta \in A \rangle$ is strictly increasing.
(6) If there is a club $D \subset \kappa^+$ such that every $\alpha \in D$ with $\operatorname{cf} \alpha > \operatorname{cf} \kappa$ is a good point of $\vec{f}$, then $\vec{f}$ is a *good scale*. An analogous definition applies for good pseudo-scales.

The reason for defining pseudo-scales is that the cofinality clause of the definition of a scale will be largely irrelevant for our purposes. The next fact is what we use to obtain failure of $\square^*_\kappa$ in Conjecture 1.

Fact 1.5. *If κ is singular, then $\square^*_\kappa$ implies that all pseudo-scales on κ are good.*[3]

2. ZFC Results

In this section we will prove the main results of the paper. We clarify notions of continuity in Subsection 2.1, then we prove Theorem 1.1 in Subsection 2.2, and then we sketch an analogous theorem for partial squares in Subsection 2.3.

2.1. **Continuity.** Our goal in this section is to obtain a strong concept of the continuity used by Cummings, Foreman, and Magidor for scales on a singular cardinal κ of cofinality λ. The material concerning points α such that $\operatorname{cf}(\alpha) > \lambda$ is the same as theirs, but we want to consider some issues that arise when $\operatorname{cf}(\alpha) \leq \lambda$. Specifically, continuity is trivial if $\operatorname{cf}(\alpha) < \lambda$, and we would like to modify the concept of continuity for the situation where $\operatorname{cf}(\alpha) = \lambda$ so that the square sequences we define are coherent.

Fix a singular κ of cofinality $\lambda > \omega$. We will consider some fixed stationary $S \subseteq \lambda$ and a product $\vec{\kappa} = \prod_{i \in S} \mu_i$. This formulation will be important when we are considering $\alpha \in \kappa^+ \cap \operatorname{cof}(\lambda)$. Fix a pseudo-scale $\vec{f}$ on $\vec{\kappa}$.

Proposition 2.1. *If $\operatorname{cf} \alpha > \operatorname{cf} \kappa$ and α is a good point, then for any cofinal $B \subset \alpha$ with $\operatorname{ot} B = \operatorname{cf} \alpha$, there is some $B^* \subseteq B$ such that B^* witnesses goodness of α.*

This follows from what is known as "The Sandwich Argument."

[3] This is in Cummings' survey [1], but without the distinction involving pseudo-scales.

Proof. Suppose $A \subset \alpha$ witnesses goodness. Let $\tau = \operatorname{cf} \alpha$ and enumerate $A' := \langle \alpha_\xi : \xi < \tau \rangle \subset A$ and $B' := \langle \beta_\xi : \xi < \tau \rangle \subset B$ in such a way that for all $\xi < \tau$, $f_{\alpha_\xi} \leq^* f_{\beta_\xi} <^* f_{\alpha_{\xi+1}}$. Observe that A' also witnesses goodness of α with respect to some j'. For each $\xi < \tau$, let $j_\xi \geq j'$ be such that $i \geq j_\xi$ implies $f_{\alpha_\xi}(i) \leq f_{\beta_\xi}(i) < f_{\alpha_{\xi+1}}(i)$. Then there is some unbounded $X \subset \tau$ and some $j < \lambda$ such that for all $\xi < \tau$, $j_\xi = j$. Since j also witnesses goodness with respect to A', this means that if $\xi, \eta \in X$ and $\xi < \eta$, then for all $i \geq j$, we have $f_{\beta_\xi}(i) < f_{\alpha_{\xi+1}}(i) \leq f_{\alpha_\eta}(i) \leq f_{\beta_\eta}(i)$. We have proved the proposition with $B^* = \langle \beta_\xi : \xi \in X \rangle$. □

Modulo a short argument, this implies:

Proposition 2.2. *If a product $\vec{\kappa}$ carries a good scale $\vec{f}$, then there is a scale $\vec{g}$ such that* every α *with* $\operatorname{cf} \alpha > \operatorname{cf} \kappa$ *is a good point of $\vec{g}$.*

Definition 2.3. Suppose $\vec{f} = \langle f_\alpha : \beta < \alpha \rangle$ is a $<^*$-increasing sequence on the product $\vec{\kappa} = \prod_{i \in S} \mu_i$, and that $A \subset \alpha$ is unbounded for some $\alpha < \kappa^+$ with $\operatorname{ot} A = \operatorname{cf} \alpha$.

- $\vec{f}_A$ denotes the function $i \mapsto \sup_{\beta \in A} f_\beta(i)$;
- if $\operatorname{cf} \alpha = \operatorname{cf} \kappa$ and $A = \langle \beta_i : i < \operatorname{cf} \kappa \rangle$, $\vec{f}^{\Delta}_A$ denotes the function $i \mapsto \sup_{j<i} f_{\beta_j}(i)$.

Definition 2.4. If f and g are functions on a product $\vec{\kappa}$, we write $f =^*_\Delta g$ if there is a club $C \subseteq \lambda$ such that for all $i \in C \cap S$, $f(i) = g(i)$. The definition for $f <^*_\Delta g$ is analogous.

Definition 2.5. A scale $\vec{f} = \langle f_\alpha : \alpha < \kappa^+ \rangle$ is *totally continuous* if the following hold:

- if $\operatorname{cf} \alpha < \operatorname{cf} \kappa$, then for all cofinal $A \subset \alpha$ with $\operatorname{ot} A = \operatorname{cf} \alpha$, $(\vec{f} \restriction \alpha)_A =^* f_\alpha$;
- if $\operatorname{cf} \alpha = \operatorname{cf} \kappa$, then for all *clubs* $A \subset \alpha$ such that $\operatorname{ot} A = \operatorname{cf} \alpha$, we have $f_\alpha =^*_\Delta (\vec{f} \restriction \alpha)^{\Delta}_A$;
- if $\operatorname{cf} \alpha > \operatorname{cf} \kappa$, then α is a good point, f_α is an exact upper bound of $\langle f_\beta : \beta < \alpha \rangle$, and for all cofinal $A \subset \alpha$ witnessing goodness of α, we have $(\vec{f} \restriction \alpha)_A =^* f_\alpha$.

Even though these cases are different, we will say *by continuity* if we invoke any of them.

Now we work towards:

Lemma 2.6. *If* $\operatorname{cf} \kappa = \lambda > \omega$*, $S \subset \lambda$ is stationary, and $\vec{\kappa} = \prod_{i \in S} \mu_i$ is a product of regular cardinals on κ that carries a good scale, then it carries a totally continuous good scale.*

Fix a $<^*$-increasing sequence $\vec{f} = \langle f_\alpha : \beta < \alpha \rangle$ on a product $\prod_{i < \operatorname{cf} \kappa} \mu_i$. The following is straightforward:

Proposition 2.7. *Suppose $\alpha < \kappa^+$,* $\operatorname{cf} \alpha < \operatorname{cf} \kappa$*, $A, B \subset \alpha$ are unbounded and* $\operatorname{ot} A = \operatorname{ot} B = \operatorname{cf} \alpha$*. Then $\vec{f}_A =^* \vec{f}_B$.*

Proposition 2.8. *If* $\operatorname{cf} \alpha > \operatorname{cf} \kappa$ *and $A \subset \alpha$ witnesses goodness, then $\vec{f}_A$ is an exact upper bound of $\langle f_\beta : \beta \in A \rangle$.*

Proof. It is straightforward that $\vec{f}_A$ is an upper bound. For exactness, suppose that $g <^* \vec{f}_A$. Let $j < \lambda$ witness goodness with respect to A as well as $g <^* \vec{f}_A$, and for all i with $j \leq i < \lambda$, let $\beta_i \in A$ be such that $g(i) < f_{\beta_i}(i)$. If $\beta = \sup_{j \leq i < \lambda} \beta_i$, then by goodness we have $g <^* f_\beta$. $\square$

Remark. If $\operatorname{cf} \alpha \leq \operatorname{cf} \kappa$, then $\langle f_\beta : \beta < \alpha \rangle$ has no exact upper bound: Let $\langle \beta_\xi : \xi < \operatorname{cf} \alpha \rangle$ be increasing and cofinal in α and let $\langle S_\xi : \xi < \operatorname{cf} \alpha \rangle$ be a partition of $\operatorname{cf} \kappa$ into disjoint unbounded sets. Define g such that $g(i) = f_{\beta_\xi}(i)$ if and only if $i \in S_\xi$. Then $g <^* f_\alpha$, but there is no $\beta < \alpha$ such that $g <^* f_\beta$.

Proposition 2.9. *If $\operatorname{cf} \alpha > \operatorname{cf} \kappa$, $A \subset \alpha$ witnesses goodness of α, and $A' \subset A$ is unbounded in α, then $\vec{f}_A =^* \vec{f}_{A'}$.*

Proof. It is immediate that $\vec{f}_{A'} \leq^* \vec{f}_A$. Suppose for contradiction that $\vec{f}_{A'} <^* \vec{f}_A$ as witnessed by $j < \lambda$. Assume that j is also large enough to witnesses goodness with respect to A, which implies that it witnesses goodness with respect to A' as well. Then for all i with $j \leq i < \lambda$, there is some $\beta_i \in A$ such that $\vec{f}_{A'}(i) < f_{\beta_i}(i) < \vec{f}_A(i)$. Let β be an element of A' greater or equal to $\sup_{j \leq i < \lambda} \beta_i < \alpha$. By goodness of A', $i \geq j$ implies that $f_{\beta_i}(i) \leq f_\beta(i)$, and so we have $f_\beta(i) \leq \vec{f}_{A'}(i) < f_\beta(i)$, a contradiction. $\square$

Proposition 2.10. *Suppose $\alpha < \kappa^+$, $\operatorname{cf} \alpha > \operatorname{cf} \kappa$ and $A, B \subset \alpha$ both witness goodness of α. Then $\vec{f}_A =^* \vec{f}_B$.*

Proof. Assume that j is large enough to witness goodness with respect to both A and B. Use the Sandwich Argument from Proposition 2.1 to find $A' \subset A$ and $B' \subset B$ such that $\vec{f}_{A'} =^* \vec{f}_{B'}$. Our result then follows from Proposition 2.9. $\square$

Proposition 2.11. *Suppose $\alpha < \kappa^+$, $\operatorname{cf} \alpha = \operatorname{cf} \kappa$, and C, D are both clubs in α such that $\operatorname{ot} C = \operatorname{ot} D = \operatorname{cf} \alpha$. Then $f^\Delta_C =^*_\Delta f^\Delta_D$.*

Proof. Suppose otherwise. Enumerate $C = \langle \beta_i : i < \operatorname{cf} \kappa \rangle$ and $D = \langle \gamma_i : i < \operatorname{cf} \kappa \rangle$. Then without loss of generality, $\{i < \operatorname{cf} \kappa : \vec{f}^\Delta_C(i) < \vec{f}^\Delta_D(i)\}$ is stationary in $\operatorname{cf} \kappa$. Let E be the club $\{i < \operatorname{cf} \kappa : \forall j_1, j_2 < i, \exists j^* < i \text{ witnessing } f_{\gamma_{j_1}} <^* f_{\gamma_{j_2}}\}$. Observe that if $i \in \lim E$, then $\langle f_{\gamma_j}(i) : j < i \rangle$ is strictly increasing, so for all $\delta < \sup_{j<i} f_{\gamma_j}(i)$, there is some $j' < i$ such that $\delta < f_{\gamma_{j'}}(i)$. Let $S := \lim E \cap \{i < \operatorname{cf} \kappa : \vec{f}^\Delta_C(i) < \vec{f}^\Delta_D(i)\}$.

Then for all $i \in S$, there is some $j < i$ such that $\vec{f}^\Delta_C(i) < f_{\gamma_j}(i)$. By Fodor's Lemma, there is a stationary $T \subset S$ and some $k < \operatorname{cf} \kappa$ such that for all $i \in T$, $\vec{f}^\Delta_C(i) < f_{\gamma_k}(i)$. If ℓ is large enough that $\gamma_k < \beta_\ell$, then there is some m such that for all $i \geq m$, $f_{\gamma_k}(i) < f_{\beta_\ell}(i)$. If $i > m, \ell$, then $f_{\gamma_k}(i) < f_{\beta_\ell}(i) \leq \vec{f}^\Delta_C(i)$. But T is of course unbounded, so this implies that we can find an i such that $f_{\gamma_k}(i) < \vec{f}^\Delta_C(i) < f_{\gamma_k}(i)$, a contradiction. $\square$

Proof of Lemma 2.6. We are working with a product $\vec{\kappa} := \prod_{i<\lambda} \mu_i$. Let $\vec{g} = \langle g_\alpha : \alpha < \kappa^+ \rangle$ be a good scale on this product. Then we define a totally continuous scale $\vec{f} = \langle f_\alpha : \alpha < \kappa^+ \rangle$ by induction as follows using the propositions from this section: If $\alpha = \beta + 1$, choose $\gamma < \kappa^+$ large enough that $f_\beta <^* g_\gamma$. Then let f_α be such that $g_\gamma <^* f_\alpha$. If α is a limit and $\operatorname{cf} \alpha < \lambda$, choose any A, a cofinal subset of α of order-type $\operatorname{cf} \alpha$. Then let $f_\alpha := \vec{f}_A$. (Proposition 2.7.) If α is a limit and $\operatorname{cf} \alpha = \lambda$, choose A to be any club subset of α of order-type $\operatorname{cf} \alpha$. Then let $f_\alpha := \vec{f}^\Delta_A$.

(Proposition 2.11.) Lastly, suppose α is a limit and $\operatorname{cf}\alpha > \lambda$. Then α is a good point in terms of $\langle f_\beta : \beta < \alpha \rangle$ because it is cofinally interleaved with $\langle g_\beta : \beta < \alpha \rangle$. Hence we can choose any cofinal $A \subset \alpha$ and let $f_\alpha := \vec{f}_A$. (Proposition 2.8 and Proposition 2.10.) $\square$

2.2. **The Construction for Weak Square.** Commencing with the proof of Theorem 1.1, fix a singular κ with cofinality $\lambda > \omega$ such that $S^* := \{\delta < \kappa : \square^*_\delta \text{ holds}\}$ is stationary (and of order-type λ). It will be sufficient to assume that for all $\tau < \kappa$, $\tau^\lambda < \kappa$, and to assume that $\prod_{\delta \in S^*} \delta^+$ carries a good pseudo-scale.

Proposition 2.12. *There is a club $E \subset \kappa$ consisting of singular cardinals.*

Proof. If $E \subset \kappa$ is any club of order-type λ, then all ordinals in $E' := \lim(E) \setminus (\lambda+1)$ are greater than λ and have cofinality less than λ, so they are singular. Moreover, we can argue that there is a club $E'' \subset E'$ of cardinals. Otherwise, there is a stationary $T \subset E'$ and a regressive function $\delta \mapsto |\delta| < \delta$ on T. This function is constant with value υ on a stationary subset $T' \subset T$, but this contradicts that fact that T' is unbounded in κ. $\square$

Using Proposition 2.12, let $\langle \kappa_i : i < \lambda \rangle$ be a continuous, cofinal, and strictly increasing sequence of singular cardinals in κ. It follows that $S := \{i < \lambda : \kappa_i \in \lim(S^*)\}$ is stationary in λ. Note that $\prod_{i \in S} \kappa_i^+$ also carries a good pseudo-scale, so we can use Lemma 2.6 to find a totally continuous pseudo-scale $\vec{f} = \langle f_\alpha : \alpha < \kappa^+ \rangle$ on the same product.

Let $\vec{\mathcal{C}}_i = \langle \mathcal{C}^i_\xi : \xi < \kappa_i^+ \rangle$ witness $\square^*_{\kappa_i}$ for all $i \in S$. Since κ_i is a limit cardinal for all i, we can assume that for all such i these $\square^*_{\kappa_i}$-sequences have the property that $\operatorname{ot} C < \kappa_i$ for all $C \in \mathcal{C}^i_\xi$, $\xi < \kappa_i^+$ (see [1]). If $\alpha < \kappa^+$, we define $\mathcal{F}_\alpha$ as follows:

- If $\operatorname{cf}(\alpha) \neq \lambda$, we let $\mathcal{F}_\alpha$ be the set of functions F such that $\operatorname{dom} F = S$ and such that $\forall i \in S$, $F(i) \in \mathcal{C}^i_{f_\alpha(i)}$.
- If $\operatorname{cf}(\alpha) = \lambda$, we let $\mathcal{F}_\alpha$ be the set of functions F such that $\operatorname{dom} F = S$ and such that *for some* $h =^*_\Delta f_\alpha$, $\forall i \in S$, $F(i) \in \mathcal{C}^i_{h(i)}$.

Regardless of whether or not $\operatorname{cf}(\alpha) = \lambda$, we will say that some $h \in \prod_{i \in S} \kappa_i^+$ *witnesses* $F \in \mathcal{F}_\alpha$ if for all $i \in S$, $F(i) \in \mathcal{C}^i_{h(i)}$.

For each $\alpha < \kappa^+$ and $F \in \mathcal{F}_\alpha$, we define $C_F \subset \alpha$ as follows:

- If $\beta < \alpha$ and $\operatorname{cf}\beta \neq \lambda$, then $\beta \in C_F$ if and only if there is some $j < \lambda$ such that for all $i \in S \setminus j$, $f_\beta(i) \in \lim F(i)$.
- If $\beta < \alpha$ and $\operatorname{cf}\beta = \lambda$, then $\beta \in C_F$ if and only if there the set of limit ordinals $\gamma \in C_F$ with $\operatorname{cf}(\gamma) < \lambda$ is unbounded in β.

Now we define our $\square^*_\kappa$-sequence at α depending on the cofinality:

- If $\operatorname{cf}\alpha < \lambda$, then $\mathcal{C}_\alpha := \{C_F : F \in \mathcal{F}_\alpha \text{ and } C_F \text{ is unbounded in } \alpha\} \cup \{C \subset \alpha : C \text{ is a club in } \alpha \text{ and } \operatorname{ot} C < \lambda\}$.
- If $\operatorname{cf}\alpha = \lambda$, choose a club $C \subset \alpha$ such that $\operatorname{ot} C = \lambda$ and let $\mathcal{C}_\alpha := \{C_F : F \in \mathcal{F}_\alpha \text{ and } C_F \text{ is unbounded in } \alpha\} \cup \{C\}$.
- If $\operatorname{cf}\alpha > \lambda$, let $\mathcal{C}_\alpha := \{C_F : F \in \mathcal{F}_\alpha\}$.

Lemma 2.13. *For all $\alpha \in \lim(\kappa^+)$ and $C \in \mathcal{C}_\alpha$, C is closed.*

Proof. It is enough to show that for all $\alpha \in \lim(\kappa^+)$ and $F \in \mathcal{F}_\alpha$, C_F is closed. The proof of this lemma does not depend on whether or not $\operatorname{cf}(\alpha) = \lambda$; that is, it does not depend on whether $F \in \mathcal{F}_\alpha$ is witnessed specifically by f_α or some $h =^*_\Delta f_\alpha$.

Let $\langle \beta_\xi : \xi < \tau \rangle \subseteq C_F$ be a strictly increasing sequence with supremum $\beta < \alpha$ where τ is regular. For each $\xi < \tau$, let j_ξ witness that $\beta_\xi \in C_F$, i.e. for all $i \geq j$, $f_{\beta_\xi}(i) \in \lim F(i)$.

Case 1: $\tau < \lambda$. If $j' = \sup_{\xi<\tau} j_\xi$, then for all $i \geq j'$, we have $\sup_{\xi<\tau} f_{\beta_\xi}(i) \in \lim F(i)$. By continuity, there is also some j'' such that for all $i \geq j''$, $f_\beta(i) = \sup_{\xi<\tau} f_{\beta_\xi}(i)$. Hence, if j is larger than j' and j'', then j witnesses that $\beta \in C_F$ by closure of $F(i)$ for $i \in S$.

Case 2: $\tau > \lambda$. By the Pigeonhole Principle there is some unbounded $Z \subset \tau$ and some $j' < \lambda$ such that $j_\xi = j'$ for all $\xi \in Z$. By Proposition 2.1, there is some j'' and some $Z' \subset Z$ such that $\{\beta_\xi : \xi \in Z'\}$ and j'' witness goodness. It then follows by continuity that for all $i \geq j''$, $f_\beta(i) = \sup_{\xi \in Z'} f_{\beta_\xi}(i)$. If $j \geq j', j''$, then j witnesses that $\beta \in C_F$ as in the previous case.

Case 3: $\tau = \lambda$. By Case 2, we can assume that $\mathrm{cf}(\beta_\xi) < \lambda$ for all $\xi < \lambda$. Then closure follows by definition. □

Lemma 2.14. *For all $\alpha \in \lim(\kappa^+)$, $C \in \mathcal{C}_\alpha$ is unbounded in α.*

Proof. It is sufficient to show that if $\mathrm{cf}\,\alpha > \mathrm{cf}\,\kappa$, then C_F is unbounded in α for an arbitrary $F \in \mathcal{F}_\alpha$. We will use the fact that $F \in \mathcal{F}_\alpha$ can only be witnessed by f_α. Consider some $\bar\alpha < \alpha$. We will find an element of C_F larger than $\bar\alpha$. By induction we define a sequence of ordinals $\langle \alpha_n : n < \omega \rangle$ in the interval $(\bar\alpha, \alpha)$, a $<^*$-increasing sequence of functions $\langle g_n : n < \omega \rangle$ in $\prod_{i \in S} \kappa_i^+$, and an undirected list of ordinals $\langle j_n : n < \omega \rangle$ in λ.

Suppose that α_n and g_n are defined. Let g_{n+1} be defined so that for all $i < \mathrm{cf}\,\kappa$, $g_{n+1}(i)$ an element of $F(i)$ larger than $f_{\alpha_n}(i)$. Using the facts that $g_{n+1} <^* f_\alpha$ and that f_α is an exact upper bound of $\langle f_\beta : \beta < \alpha \rangle$, find α_{n+1} so that $g_{n+1} <^* f_{\alpha_{n+1}}$, and let $j_{n+1} < \lambda$ witness this.

Let $\beta = \sup_{n<\omega} \alpha_n$, which in particular is larger than $\bar\alpha$. We claim that $\beta \in C_F$ as witnessed by $j := \sup_{n<\omega} j_n < \lambda$. For each $i < \lambda$ such that $i \geq j$, $\langle g_n(i) : i < \omega \rangle$ and $\langle f_{\alpha_n}(i) : n < \omega \rangle$ interleave each other, so $\sup_{n<\omega} f_{\alpha_n}(i) \in \lim F(i)$ for such i. For sufficiently large i, $f_\beta(i) = \sup_{n<\omega} f_{\alpha_n}(i)$ by continuity, so this completes the proof. □

Lemma 2.15. *For all $\alpha \in \lim(\kappa^+)$ and $C \in \mathcal{C}_\alpha$, if $\beta \in \lim C$, then $C \cap \beta \in \mathcal{C}_\beta$.*

Proof. The lemma is only substantial if $C = C_F$ for some $F \in \mathcal{F}_\alpha$, and it does not depend on whether $\mathrm{cf}(\alpha) = \lambda$. By assumption C_F is unbounded in β, so Lemma 2.13 implies that $\beta \in C_F$.

Case 1: $\mathrm{cf}\,\beta \neq \lambda$: Let $j < \lambda$ witness $\beta \in C_F$, meaning that if $i \geq j$ then $f_\beta(i) \in \lim F(i)$. By the coherence of $\vec{\mathcal{C}}^i$ for $i \in S$, it follows that $F(i) \cap f_\beta(i) \in \mathcal{C}^i_{f_\beta(i)}$ for such i. Let F' be a function with domain S such that $F'(i) \in \mathcal{C}^i_{f_\beta(i)}$ for all $i \in S$ and such that $F'(i) = F(i) \cap f_\beta(i)$ for $i \geq j$ in particular. Then $F' \in \mathcal{F}_\beta$ and $C_{F'}$ is unbounded in β, so $C_{F'} \in \mathcal{C}_\beta$. If $\gamma < \beta$, let $j' < \lambda$ witness $f_\gamma <^* f_\beta$. Then if $i \geq j, j'$, it follows that $f_\gamma(i) \in F(i)$ if and only if $f_\gamma(i) \in F'(i)$. We conclude that $C_F \cap \beta = C_{F'}$.

Case 2: $\mathrm{cf}\,\beta = \lambda$: Choose a sequence $\langle \beta_i : i < \lambda \rangle \subset C_F \cap \beta$; by closure (Lemma 2.13, Case 1) we can assume that $\langle \beta_i : i < \lambda \rangle$ is closed and unbounded in λ, and that $\mathrm{cf}(\beta_i) < \lambda$ for all $i < \lambda$. By Proposition 2.11, we also know that $f_\beta =^*_\Delta (\vec{f} \upharpoonright \beta)^\Delta_{\langle \beta_i : i < \lambda \rangle}$, i.e. that there is a club $E \subset \lambda$ such that for all $i \in E$, $f_\beta(i) = \sup_{j<i} f_{\beta_j}(i)$. Let D be a club such that $D \subseteq E$ and such that for all

$i \in D, j < i$, there is some $j' < i$ witnessing that $\beta_j \in C_F$, and moreoever such that for all $i \in D, j_1, j_2 < i$, there is some $j < i$ witnessing that $f_{\beta_{j_1}} <^* f_{\beta_{j_2}}$. It follows that for all $i \in D$ and $j < i$, $f_{\beta_j}(i) \in \lim F(i)$, and therefore that for all $i \in D$, $f_\beta(i) \in \lim F(i)$. Then let F' be defined so that $F'(i) = F(i) \cap f_\beta(i)$ for $i \in D \cap S$ and $F'(i) = F(i)$ for $i \in S \setminus D$. Then it follows that $C_F \cap \beta = C_{F'}$: in particular, if $\gamma \in C_{F'}$, then f_γ is dominated by f_β on a club, so it must be the case that $\gamma < \beta$. Hence we find that $F' \in \mathcal{F}_\beta$ is witnessed by h such that $h(i) = f_\beta(i)$ for $i \in D$ and $h(i) = h'(i)$ for the h' witnessing $F \in \mathcal{F}_\alpha$ (hence $h =^*_\Delta f_\beta$). Therefore we have shown that $C_F \cap \beta \in \mathcal{C}_\beta$. □

Lemma 2.16. *For all $\alpha \in \lim(\kappa^+)$ and $C \in \mathcal{C}_\alpha$, $\operatorname{ot} C < \kappa$.*

Proof. It is sufficent to show that $\operatorname{ot} C_F < \kappa$ for all $F \in \mathcal{F}_\alpha$ and all $\alpha < \kappa^+$ (independently of whether $\operatorname{cf}(\lambda) = \alpha$). Recall that we assumed that the $\square^*_{\kappa_i}$-sequences $\langle \mathcal{C}^i_\xi : i < \kappa_i^+ \rangle$ were defined so that for all for all $i < \lambda, \xi < \kappa_i^+, C \in \mathcal{C}^i_\xi$, $\operatorname{ot} C < \kappa_i$.

Fix $\alpha < \kappa^+$. For every $i \in S$, there is some $j < i$ such that $\operatorname{ot} F(i) < \kappa_j$. This means that there is a stationary $T \subseteq S$ and some k such that for all $i \in T$, $\operatorname{ot} F(i) < \kappa_k$. If $\beta \in C_F$ and $i \in T$, let $g_\beta(i) = \operatorname{ot}(F(i) \cap f_\beta(i))$ for all i such that $f_\beta(i) \in F(i)$ and 0 otherwise. The set $\{g_\beta : \beta \in C_F\}$ has size $\kappa_k^\lambda < \kappa$ (we assumed this bit of cardinal arithmetic), so it is enough to observe that if $\beta, \beta' \in C_F$ and $\beta < \beta'$, then g_β and $g_{\beta'}$ are distinct. □

Lemma 2.17. *For all $\alpha \in \lim(\kappa^+)$, $|\mathcal{C}_\alpha| \leq \kappa$.*

Proof. Our assumption that $\tau^\lambda < \kappa$ for all $\tau < \kappa$ implies that $|\{C \subset \alpha : \operatorname{ot} C < \lambda\}| = \kappa$, so it is enough to show that $|\{C_F : F \in \mathcal{F}_\alpha\}| \leq \kappa$ for all $\alpha \in \lim(\kappa^+)$.

Fix $\alpha \in \lim(\kappa^+)$. We first argue for the case in which $\operatorname{cf}(\alpha) \neq \lambda$. For all $i \in S'$ enumerate $\mathcal{C}^i_{f_\alpha(i)} = \langle C^i_\zeta : \zeta < \kappa_i \rangle$. For stationary sets $T \subset S'$ and $\zeta < \kappa$, let

$$X^k_T = \{F \in \mathcal{F}_\alpha : \forall i \in T, \exists \zeta < \kappa_k \text{ such that } F(i) = C^i_\zeta \text{ and } \operatorname{ot}(C^i_\zeta) < \kappa_k\}.$$

We claim that for all $F \in \mathcal{F}_\alpha$, there are $T \subset S'$ and $k < \lambda$ such that $C_F \in X^k_T$. Let $F \in \mathcal{F}_\alpha$. For each F and $i \in S'$, there is some $j < i$ such that we have $F(i) = C^i_\zeta$ for some $\zeta < \kappa_j$ and $\operatorname{ot}(C^i_\zeta) < \kappa_j$ as well. It follows that there is a stationary $T \subset S'$ and $k < \lambda$ such that for all $i \in T$, $F(i) = C^i_\zeta$ and $\operatorname{ot}(C^i_\zeta) < \kappa_k$ for some $\zeta < \kappa_k$.

Because $2^\lambda = \lambda^\lambda < \kappa$, there are at most κ-many X^k_T's. Therefore it remains to show that for all such T, k, that $|\{C_F : F \in X^k_T\}| \leq \kappa$. Let G_F be the set of functions $g_\beta = f_\beta \restriction T$ for all $\beta \in C_F$. If $\beta \neq \beta'$, then $g_\beta \neq g_{\beta'}$, so if $F' \neq F$ then $G_F \neq G_{F'}$. Now, for $i \in T$, let $R^k_T(i) = \bigcup_{\zeta < \kappa_k} C^i_\zeta$. Then for all $F \in F^k_T$, $G_F \subseteq \prod_{i \in T} R^k_T(i)$. Moreover, $\prod_{i \in T} R^k_T(i)$ has cardinality $\kappa_k^\lambda < \kappa$. It follows that $|\{C_F : F \in X^k_T\}| \leq \kappa$.

Now we comment on the case in which $\operatorname{cf}(\alpha) = \lambda$. For all $F \in \mathcal{F}_\alpha$, there is some $S' \subset S$ be a stationary set such that for all $i \in S'$, $F(i) \in C^i_{f_\alpha(i)}$. The argument above can be done for all $F \in \mathcal{F}_\alpha$ such that there is an h witnessing $F \in \mathcal{F}_\alpha$ where $h \restriction S' = f_\alpha \restriction S'$. Since $2^\lambda < \kappa$, and we only need to consider 2^λ-many possible S', this is sufficient. □

This finishes the proof of Theorem 1.1.

2.3. **Sketching the Construction for Partial Square.** We observe that a result similar to Theorem 1.1 holds for partial squares:

Theorem 2.18. *Let κ be a singular strong limit cardinal of cofinality $\lambda > \omega$. Suppose there is a stationary set $S \subset \kappa$ such that $\square_\delta$ holds for all $\delta \in S$ and such that $\prod_{\delta \in S} \delta^+$ carries a good scale. Then there is a partial square sequence on $\kappa^+ \cap \operatorname{cof}(> \lambda)$.*

This can be proved with the same techniques as the previous theorem, and the setup is basically the same: We fix a singular strong limit κ with cofinality $\lambda > \omega$ such that $\{\delta < \kappa : \square_\delta \text{ holds}\}$ is stationary (and of order-type λ). Let $\langle \kappa_i : i < \lambda \rangle$ be continuous, cofinal, and strictly increasing in κ. We find that $S := \{i < \lambda : \square_{\kappa_i} \text{ holds}\}$ is stationary in λ, and we can construct a totally continuous scale $\vec{f} = \langle g_\alpha : \alpha < \kappa^+ \rangle$ on $\prod_{i \in S} \kappa_i^+$. Let $\mathcal{C}_i = \langle C^i_\xi : \xi < \kappa_i^+ \rangle$ witness $\square_{\kappa_i}$ for all $i \in S$. By Proposition 2.12, we can again assume that $\operatorname{ot} C^i_\xi < \kappa_i$ for all $\xi < \kappa_i^+, i < \lambda$. Now we can define the clubs of which our square sequence will consist. For each $\alpha \in \kappa^+ \cap \operatorname{cof}(> \lambda)$, let:

$$X_\alpha := \langle \beta < \alpha : \{i < \lambda : f_\beta(i) \in \lim C^i_{f_\alpha(i)}\} \text{ is co-bounded in } S \rangle.$$

Then we have an analog of Lemma 2.13:

Lemma 2.19. *For all $\alpha \in \lim(\kappa^+)$, if $\langle \beta_\xi : \xi < \tau \rangle \subset X_\alpha$'s and $\tau \neq \lambda$, then $\sup_{\xi < \tau} \beta_\xi \in X_\alpha$.*

Then let C_α be the closure of X_α inside α. The partial square sequence will be the sequence $\langle C_\alpha : \alpha \in \kappa^+ \cap \operatorname{cof}(> \lambda) \rangle$. Proofs of the various lemmas are analogous. Coherence for the case $\operatorname{cf}(\beta) = \lambda$ is easier since no witness needs to be constructed.

A Question. We expecet a stronger contrast with the countable case:

Conjecture 1. *Assuming large cardinals, it is consistent that $\aleph_\omega$ is a strong limit, there is a good scale on $\aleph_\omega$, $\square_{\aleph_n}$ holds for all $n < \omega$, and $\square^*_{\aleph_\omega}$ fails.*

We conclude with the following:

Question 2. *Suppose that κ is a singular strong limit of uncountable cofinality λ such that $S := \{\delta < \kappa : \square^*_\delta \text{ holds}\}$ is stationary and of order-type λ. Does $\prod_{\delta \in S} \delta^+$ carry a good pseudo-scale?*

By Theorem 1.1, this question is almost equivalent (modulo a generalization and a strong limit assumption) to the question of Golshani mentioned above: a positive answer would mean that these hypotheses imply $\square^*_\kappa$, and a negative answer would mean that $\square^*_\kappa$ consistently fails in conjunction with these hypotheses.

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