

THE COFINALITY OF THE SYMMETRIC GROUP AND THE COFINALITY OF ULTRAPOWERS

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ABSTRACT. We prove that $\mathfrak{mcf} < \text{cf}(\text{Sym}(\omega))$ and $\mathfrak{mcf} > \text{cf}(\text{Sym}(\omega)) = \mathfrak{b}$ are both consistent relative to ZFC. This answers a question by Banach, Repovš and Zdomsky and a question from [MS11].

1. INTRODUCTION

We compare the cardinal \mathfrak{mcf} , the minimal cofinality of the ultrapower $(\omega, <)$ by a non-principal ultrafilter on ω , and the cofinality of the symmetric group on ω , $\text{cf}(\text{Sym}(\omega))$. These two cardinal invariants are closely related: Both are cofinalities and hence regular. In ZFC, both cardinals have value in the interval $[\mathfrak{g}, \mathfrak{d}]$, namely Blass and Mildenerger [BM99] showed $\mathfrak{mcf} \geq \mathfrak{g}$, Brendle and Losada [BL03] showed $\text{cf}(\text{Sym}(\omega)) \geq \mathfrak{g}$, and Simon Thomas [Tho95] showed $\text{cf}(\text{Sym}(\omega)) \leq \mathfrak{d}$. In their relations to \mathfrak{b} the two cardinals behave differently: Obviously $\mathfrak{b} \leq \mathfrak{mcf}$, whereas Sharp and Thomas [ST95, Theorem 1.6] showed that $\text{cf}(\text{Sym}(\omega)) < \mathfrak{b}$ is consistent relative to ZFC. Before our research, in all investigated forcing extensions we have had $\text{cf}(\text{Sym}(\omega)) \leq \mathfrak{mcf}$ and in the forcing extensions in which both $\text{cf}(\text{Sym}(\omega)) \geq \mathfrak{b}$ and $\mathfrak{mcf} \geq \mathfrak{b}$, the two cardinal characteristics $\text{cf}(\text{Sym}(\omega))$ and \mathfrak{mcf} coincide. The inequality $\text{cf}(\text{Sym}(\omega)) \leq \mathfrak{mcf}$ is partially due to a mathematical reason: Banach, Repovš and Zdomsky showed [BRZ11, Theorem 1.3]: If D is not nearly coherent to a Q -point then $\text{cf}(\text{Sym}(\omega)) \leq \text{cf}((\omega, <)^\omega / D)$. In particular if there is no Q -point then $\text{cf}(\text{Sym}(\omega)) \leq \mathfrak{mcf}$.

Here we show that indeed an extra assumption is necessary. Our first forcing shows the relative consistency of $\aleph_1 = \mathfrak{b} = \mathfrak{mcf} < \aleph_2 = \text{cf}(\text{Sym}(\omega))$.

In our second forcing we show how to separate the two cardinals in the second direction above \mathfrak{b} : $\aleph_1 = \mathfrak{b} = \text{cf}(\text{Sym}(\omega)) < \mathfrak{mcf}$ is consistent. We use versions of the oracle-c.c. in the \aleph_1 - \aleph_2 -scenario.

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There are some known forcings establishing the relative consistency of $\mathfrak{b} < \mathfrak{mcf}$: Three interesting forcings for $\aleph_1 = \mathfrak{b} < \mathfrak{mcf}$ are given in [SS93, SS94]. Since $\mathfrak{b} \leq \mathfrak{u}$ [PS87] and since NCF is equivalent to $\mathfrak{u} < \mathfrak{mcf}$ [Mil01] the NCF-models show the relative consistency of $\mathfrak{b} < \mathfrak{mcf}$. In [MS11] we showed that also $\mathfrak{b}^+ < \mathfrak{mcf}$ is possible. In the second forcing extension of that work we arranged $\mathfrak{b}^+ < \mathfrak{mcf} = \text{cf}(\text{Sym}(\omega))$. In the other forcing extensions for $\mathfrak{b} < \mathfrak{mcf}$ the value of $\text{cf}(\text{Sym}(\omega))$ has not yet been computed or is possibly not determined by the forcing or by NCF.

We recall the definitions: We denote by ${}^\omega\omega$ the set of functions from ω to ω . For $f, g \in {}^\omega\omega$ we write $f \leq^* g$ and say g eventually dominates f if $(\exists n)(\forall k \geq n)(f(k) \leq g(k))$. A set $B \subseteq {}^\omega\omega$ is called *unbounded* if there is no g that dominates all members of B . The *bounding number* \mathfrak{b} is the minimal cardinality of an unbounded set.

Definition 1.1. *Let D be a non-principal ultrafilter over ω . By ultrapower we mean the usual modeltheoretic ultrapower: The structure $(\omega, <)^{\omega}/D$ is defined on the domain $\{[f]_D : f \in {}^\omega\omega\}$ where $[f]_D = \{g \in {}^\omega\omega : \{n : f(n) = g(n)\} \in D\}$. The order relation is $[f]_D \leq_D [g]_D$ iff $\{n : f(n) \leq g(n)\} \in D$. We write $\text{cf}((\omega, <)^{\omega}/D)$ for the minimal size of a set that is cofinal in \leq_D . The minimal cofinality of an ultrapower of ω , \mathfrak{mcf} , is defined as the*

$$\mathfrak{mcf} = \min\{\text{cf}((\omega, <)^{\omega}/D) : D \text{ non-principal ultrafilter over } \omega\}.$$

We define the relation \leq_D also on the space ${}^\omega\omega$ by letting $f \leq_D g$ iff $\{n : f(n) \leq g(n)\} \in D$.

Definition 1.2. *The group of permutations of ω is denoted by $\text{Sym}(\omega)$. If $\text{Sym}(\omega) = \bigcup_{i < \kappa} G_i$, $\kappa = \text{cf}(\kappa) > \aleph_0$, $\langle G_i : i < \kappa \rangle$ is strictly increasing, and each G_i is a proper subgroup of $\text{Sym}(\omega)$, we call $\langle G_i : i < \kappa \rangle$ an increasing decomposition. We call the minimal κ such that an increasing decomposition of length κ exists the cofinality of the symmetric group, and denote it $\text{cf}(\text{Sym}(\omega))$.*

Definition 1.3. *A subset \mathcal{G} of $[\omega]^\omega$ is called groupwise dense if*

- (1) $(\forall X \in \mathcal{G})(\forall Y \subseteq^* X)(Y \text{ infinite} \rightarrow Y \in \mathcal{G})$, and
- (2) for every partition of ω into finite intervals $\Pi = \{[\pi_i, \pi_{i+1}) : i \in \omega\}$ there is an infinite set A such that $\bigcup\{[\pi_i, \pi_{i+1}) : i \in A\} \in \mathcal{G}$.

The groupwise density number, \mathfrak{g} , is the smallest number of groupwise dense families with empty intersection.

An ultrafilter U over ω is called a *Q-point*, if given any strictly increasing function $f: \omega \rightarrow \omega$ there is an $X \in U$ such that $\forall n, X \cap [f(n), f(n+1))$ has just one element. The existence of a *Q-point* is independent of ZFC, see, e.g., [Can90] for existence and [Mil80] for non-existence. An ultrafilter D is *nearly coherent to an ultrafilter U* if there is a finite-to-one function $f: \omega \rightarrow \omega$ such that $f(D) = f(U)$. Here $f(D) = \{E : f^{-1}[E] \in D\}$. Throughout we write $g[X]$ for the set $\{g(x) : x \in X\}$ and $g^{-1}[Y] = \{x : g(x) \in Y\}$. The principle NCF says that any two non-principal ultrafilters over ω are nearly

coherent. Its consistency is established in [BS87, BS89, Bla89]. A *base* for an ultrafilter is a subset \mathcal{B} of \mathcal{U} such that $(\forall Y \in \mathcal{U})(\exists X \in \mathcal{B})(X \subseteq Y)$. The character of an ultrafilter is the smallest size of a base. The *ultrafilter characteristic* \mathfrak{u} is the smallest character of a non-principal ultrafilter.

In forcing the *stronger* condition is the *larger* one. For a forcing order \mathbb{P} and a formula φ , we say \mathbb{P} forces φ if the weakest condition in \mathbb{P} forces φ .

2. $\text{Con}(\mathfrak{b} = \text{cf}(\omega^\omega/D) < \text{cf}(\text{Sym}(\omega)))$

In this section we prove:

Theorem 2.1. *The constellation $\aleph_1 = \mathfrak{b} = \mathfrak{mcf} < \text{cf}(\text{Sym}(\omega))$ is consistent relative to ZFC.*

We essentially use oracle c.c. [She98, Ch. 4], but in addition to the oracle sequence we construct a sequence $\langle D_\alpha : \alpha < \omega_1 \rangle$ which approximates a name \underline{D} for an ultrafilter. We construct a notion of forcing \mathbb{P} such that for a \mathbb{P} -generic filter \mathbf{G} , $D[\mathbf{G}]$ will be an ultrafilter witnessing $\mathfrak{mcf} = \aleph_1$. The construction of \mathbb{P} is done via an approximation forcing AP , so that $\mathbb{P} = AP * \mathbb{Q}$, where \mathbb{Q} is an AP -name for the AP -generic object.

We recall some oracle technique of [She98, Chapter IV]. Let S be a stationary subset of ω_1 . We fix S throughout this section. A set $\mathcal{D} \subseteq \mathcal{P}(S)$ is called a *filter over S* if $\emptyset \notin \mathcal{D}$, $S \in \mathcal{D}$, \mathcal{D} is closed under finite intersections and closed under supersets. A filter \mathcal{D} over S is called *normal* if it contains all sets of the form $[\alpha, \omega_1) \cap S$, $\alpha < \omega_1$, and is closed under *diagonal intersections*. We recall, given a sequence $\langle D_\delta : \delta \in S \rangle$, its diagonal intersection is the following set

$$\Delta_{\delta \in S} D_\delta = \{ \gamma \in S : \gamma \in \bigcap_{\delta \in \gamma \cap S} D_\delta \}.$$

For a filter \mathcal{D} over ω_1 and $X, Y \subseteq \omega_1$ we let $X = Y \text{ mod } \mathcal{D}$ if $(X \cap Y) \cup ((\omega_1 \setminus X) \cap (\omega_1 \setminus Y)) \in \mathcal{D}$, and $X \subseteq Y \text{ mod } \mathcal{D}$ if $X \setminus Y = \emptyset \text{ mod } \mathcal{D}$.

We recall the notion of a \diamond_S^- -sequence. A sequence $\bar{P} = \langle P_\delta : \delta \in S \rangle$ is called a \diamond_S^- -sequence if $P_\delta \subseteq \mathcal{P}(\delta)$ is countable and for any $X \subseteq \aleph_1$

$$\{ \delta \in S : X \cap \delta \in P_\delta \} \text{ is a stationary subset of } S.$$

It is well known that \diamond_S^- and \diamond_S are equivalent (see [Kun80, Ch. III]).

We fix a sufficiently large regular cardinal χ , indeed $\chi \geq (2^{\aleph_2})^+$ suffices. We fix a well-order $<_\chi$ on $H(\chi)$.

Definition 2.2. *We assume that $S \subseteq \omega_1$ is stationary and \diamond_S .*

(1) (See [She98, IV, Def. 1.1]) *An S -oracle is a sequence $\bar{M} = \langle M_\delta : \delta \in S \rangle$ such that*

- (a) M_δ is countable and transitive and $\delta + 1 \subseteq M_\delta$.
- (b) $i_\delta : (M_\delta, \in, (<_\chi)^{M_\delta}) \hookrightarrow_{\text{elem}} (H(\chi), \in, <_\chi)$ is elementary.
- (c) $M_\delta \models \delta$ is countable.

- (d) for $\delta < \varepsilon \in S$, $M_\delta \subseteq M_\varepsilon$.
- (e) For any $A \subseteq \omega_1$ the set $\{\delta \in S : A \cap \delta \in M_\delta\}$ is stationary in ω_1 .
- (2) Let M be a countable elementary submodel of $H(\chi)$. A real $\eta \in \omega^\omega$ is called a Cohen real over M iff for any $D \in M$ that is dense in $\mathbb{C} = \{p : (\exists n)(p \restriction n \rightarrow \omega)\}$ (ordered by end-extension) there is an n such that $\eta \restriction n \in D$. Equivalently, for any meagre set $F \subseteq \omega^\omega$ that is coded in M , e.g., by a sequence of nowhere dense trees, we have $\eta \notin F$.
- (3) We say that $\langle \bar{M}, \bar{N}, \bar{\eta} \rangle$ is an S -oracle triple if
- (a) $\bar{M} = \langle M_\delta : \delta \in S \rangle$ is an S -oracle,
- (b) $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$,
- (c) for $\delta \in S$, η_δ is Cohen over M_δ ,
- (d) $\bar{N} = \langle N_\delta : \delta \in S \rangle$,
- (e) $N_\delta = M_\delta[\eta_\delta]$.
- (4) Let \bar{M} be an S -oracle sequence. For $A \subseteq H(\omega_1)$, we let

$$I_{\bar{M}}(A) = \{\alpha \in S : A \cap \alpha \in M_\alpha\}$$

and

$$\mathcal{D}_{\bar{M}} = \{X \subseteq \omega_1 : (\exists A \subseteq \omega_1)(X \supseteq I_{\bar{M}}(A))\}.$$

From now on until the end of the section let $S \subseteq \omega_1$ be stationary and assume \diamond_S . For L -structures \mathcal{A}, \mathcal{M} , we write $\mathcal{A} \prec \mathcal{M}$ if \mathcal{A} is an elementary substructure of \mathcal{M} . Since for L -structures $\mathcal{A}, \mathcal{B}, \mathcal{M}$ with $\mathcal{A}, \mathcal{B} \prec \mathcal{M}$ and $\mathcal{A} \subseteq \mathcal{B}$ also $\mathcal{A} \prec \mathcal{B}$ holds, we have that the structures on any oracle sequence are \prec -increasing.

If $f: A \rightarrow B$ is a function and $C \subseteq A$, then we write $f[C]$ for $\{f(c) : c \in C\}$. We recall the following important properties of $\mathcal{D}_{\bar{M}}$.

Lemma 2.3. ([She98, IV, Claim 1.4]) *The set $\{I_{\bar{M}}(A) : A \subseteq \omega_1\}$ is closed under finite intersections. The filter $\mathcal{D}_{\bar{M}}$ contains every end segment of ω_1 , is normal, and contains any club subset of S , and for every $A \subseteq H(\aleph_1)$, $I_{\bar{M}}(A) \in \mathcal{D}_{\bar{M}}$.*

Proof. We prove only the very last statement; the others are proved in [She98, IV, Claim 1.4]. By \diamond_S , $|H(\omega_1)| = \omega_1$. Let $f: H(\omega_1) \rightarrow \omega_1$ be the $<_{\chi}$ -least bijection. Let $C = \{\delta \in \omega_1 : \delta \text{ limit and } (\forall \alpha < \delta)(f[M_\alpha] \subseteq \delta)\}$. The set $\text{acc}(C)$ of accumulation points of C is club in ω_1 . Now we consider $A \subseteq H(\omega_1)$. By definition, $I_{\bar{M}}(f[A]) \in \mathcal{D}_{\bar{M}}$. For any $\delta \in S \cap \text{acc}(C)$ such that $f[A] \cap \delta \in M_\delta$ we have

$$M_\delta \ni (i_\delta^{-1}(f^{-1}))[(f[A] \cap \delta)] = \bigcup_{\alpha < \delta} (f^{-1} \restriction f[M_\alpha])[(f[A] \cap \alpha)] = \bigcup_{\alpha < \delta} A \cap \alpha = A \cap \delta.$$

Thus we have $I_{\bar{M}}(A) \supseteq I_{\bar{M}}(f[A]) \cap \text{acc}(C)$. By [Jec03, Lemma 14.4], for any club C' in ω_1 , any normal filter over S contains the set $S \cap C'$. Since $\text{acc}(C)$ is a club and $\mathcal{D}_{\bar{M}}$ is a normal filter, $\text{acc}(C) \in \mathcal{D}_{\bar{M}}$ and thus $I_{\bar{M}}(A) \in \mathcal{D}_{\bar{M}}$. \square

We recall when a notion of forcing \mathbb{P} has the \bar{M} -c.c.

Definition 2.4. ([She98, Ch. IV, Def. 1.5]) *Let \bar{M} be an S -oracle sequence and let \mathbb{P} be a notion of forcing. We define when \mathbb{P} satisfies the \bar{M} -c.c. by cases:*

- (a) *If $|\mathbb{P}| \leq \aleph_0$, always.*
- (b) *If $|\mathbb{P}| = \aleph_1$ and if for every injective $\pi: \mathbb{P} \rightarrow \omega_1$ the set*
- $$\left\{ \delta \in S : (\forall A \in M_\delta \cap \mathcal{P}(\delta)) \left(((\pi^{-1})[A] \text{ is predense in } (\pi^{-1})[\delta]) \right. \right. \\ \left. \left. \rightarrow ((\pi^{-1})[A] \text{ is predense in } \mathbb{P}) \right) \right\}$$

is an element of $\mathcal{D}_{\bar{M}}$.

- (c) *$\mathbb{P}'' \subseteq_{\text{ic}} \mathbb{P}$ means that \mathbb{P}'' is an incompatibility preserving suborder of \mathbb{P} , i.e., for any $p, q \in \mathbb{P}''$, $p \leq_{\mathbb{P}''} q$ iff $p \leq_{\mathbb{P}} q$ and $p \perp_{\mathbb{P}''} q$ iff $p \perp_{\mathbb{P}} q$.*
- (d) *If $|\mathbb{P}| > \aleph_1$ and for every $\mathbb{P}^\dagger \subseteq \mathbb{P}$ if $|\mathbb{P}^\dagger| \leq \aleph_1$ then here are \mathbb{P}'' such that $|\mathbb{P}''| = \aleph_1$ and $\mathbb{P}^\dagger \subseteq \mathbb{P}'' \subseteq_{\text{ic}} \mathbb{P}$ and $\pi: \mathbb{P}'' \rightarrow \omega_1$ as in (b).*

Oracle sequences are not continuous. The requirement $\delta \in M_\delta$ precludes continuity.

Lemma 2.5. *Assume S is stationary and \diamond_S .*

- (1) *There is an oracle triple.*
- (2) *Let $\langle \bar{M}, \bar{N}, \bar{\eta} \rangle$ be an oracle triple. Then*

$$I := \{ \delta \in S : \{ (\varepsilon, \eta_\varepsilon) : \varepsilon < \delta \} \in M_\delta \} \in \mathcal{D}_{\bar{M}}.$$

- (3) *If $\langle \bar{M}, \bar{N}, \bar{\eta} \rangle$ is an S -oracle triple then $\langle N_\varepsilon : \varepsilon \in I \rangle$ is an I -oracle, with the exception that $(N_\varepsilon, \varepsilon)$ is not necessarily an elementary substructure of $H(\chi)$.¹*

Proof. (1) Let $\langle P_\delta : \delta \in S \rangle$ be a \diamond_S^- -sequence. Again we fix the $<_\chi$ -least bijection $f: H(\omega_1) \rightarrow \omega_1$. We choose M_δ, i_δ by induction on δ . Suppose that $M_\gamma, i_\gamma, \gamma < \delta$, have been chosen. Let $M'_\delta \prec (H(\chi), \varepsilon, <_\chi)$ be a countable elementary substructure with $\langle M_\gamma, i_\gamma : \gamma < \delta \rangle, \delta, P_\delta \in M'_\delta$. Then $\delta+1 \subseteq M'_\delta$. We let M_δ be the Mostowski collapse of M'_δ . The Mostowski collapse maps P_δ to itself. Moreover, since P_δ is countable, $P_\delta \subseteq M_\delta$, and hence $X \cap \delta \in P_\delta$ implies $X \cap \delta \in M_\delta$. By now, we have taken care of Def. 2.2.(2) (a). For being definite, we let the Cohen forcing \mathbb{C} be the set of finite partial functions from ω to 2, ordered by extension. By the Rasiowa–Sikorski theorem (e.g., [Jec03, Lemma 14.4]) there is a Cohen-generic filter G_δ over M_δ . Then the function $\eta_\delta = \bigcup \{ p : p \in G_\delta \} \in {}^\omega 2$ is a Cohen real over M_δ . We let $M_\delta[G_\delta] = N_\delta$.

(2) The set $A = \{ (\varepsilon, \eta_\varepsilon) : \varepsilon \in S \} \subseteq H(\omega_1)$. We fix a club C such for $\delta \in C$, $f[\{ (\varepsilon, \eta_\varepsilon) : \varepsilon < \delta \}] \subseteq \delta$. By Lemma 2.3 we have $I_{\bar{M}}(A) \in \mathcal{D}_{\bar{M}}$. By

¹In Theorem 2.8 below we will rework the proof of the omitting types theorem for the particular types that shall be omitted and see that the requirement that $(N_\varepsilon, \varepsilon)$ fulfil sufficiently much of ZFC and be transitive suffices for our application.

normality $C \cap I_{\bar{M}}(A) \in \mathcal{D}_{\bar{M}}$. By the choice of C , $C \cap I_{\bar{M}}(A) \subseteq \{\delta : \{(\varepsilon, \eta_\varepsilon) : \varepsilon < \delta\} \in M_\delta\}$ and thus the latter is in $\mathcal{D}_{\bar{M}}$.

(3) Since $\mathcal{D}_{\bar{M}}$ is a normal filter, by [Jec03, Lemma 811], its elements are stationary sets. Hence I is stationary. For $\delta < \varepsilon$, $\delta \in S$, $\varepsilon \in I$, we have $N_\delta \subseteq M_\varepsilon \subseteq N_\varepsilon$. Hence $\langle N_\varepsilon : \varepsilon \in I \rangle$ is increasing. \square

From now until the end of the section we fix an S -oracle triple $(\bar{M}, \bar{N}, \bar{\eta})$. Note that for $\delta \in I$, $(\forall \alpha < \delta)(M_\alpha[\eta_\alpha] \in M_\delta)$.

Oracle triples allow for the application of the ‘‘Omitting Types Theorem’’:

Lemma 2.6. *(The Omitting Types Theorem, see [She98, Ch. IV, Lemma 2.1]) Assume \diamond_S . Suppose the $\psi_i(x)$, $i < \omega_1$, are Π_2^1 formulas on reals with a real parameter possibly. Suppose further that there is no solution to $\bigwedge_{i < \omega_1} \psi_i(x)$ in \mathbf{V} and even if we add a Cohen real to \mathbf{V} there will be none. Then there is an S -oracle \bar{M}' such that for any forcing \mathbb{P} ,*

$$\text{if } \mathbb{P} \text{ has the } \bar{M}'\text{-c.c then in } \mathbf{V}^{\mathbb{P}} \text{ there is no solution to } \bigwedge_i \psi_i(x).$$

We let $\psi(x, \eta_i)$ say the following

$$(2.1) \quad \begin{aligned} &x = (y, h) \wedge y \in {}^\omega 2 \text{ and } h \in {}^\omega \omega \text{ is increasing and} \\ &(\forall^\infty n)(\eta_i \upharpoonright [h(n), h(n+1)) \neq y \upharpoonright [h(n), h(n+1))). \end{aligned}$$

By [BJ95, Theorem Ch. 2], any meagre subset of 2^ω has a superset of the form

$$M_{(h,y)} = \{z \in {}^\omega 2 : (\forall^\infty n)z \upharpoonright [h(n), h(n+1)) \neq y \upharpoonright [h(n), h(n+1))\}$$

for some strictly increasing function h and some $y \in {}^\omega 2$. The formula $\psi(x, \eta_i)$ says that η_i is in the meagre set $M_{(h,y)}$. So the type Ψ to be omitted is

$$(2.2) \quad \bigwedge_{i \in I} \psi(x, \eta_i).$$

Actually, we will have a strong form of omission: There is a set Y in a normal filter such that for each $i \in Y$, $x = (y, h) \in M_i[\mathbb{P}]$,

$$(\exists^\infty n)\eta_i \upharpoonright [h(n), h(n+1)) = \eta_i \upharpoonright [h(n), h(n+1)).$$

Since $\mathbb{P} \in M_0$ and $\mathbb{P} \subseteq \bigcup\{M_i : i < \omega\}$, thus $\{\eta_i : i \in Y\}$ is not meagre in $\mathbf{V}^{\mathbb{P}}$.

We check that premise of the omitting types theorem is fulfilled in a very local form.

Lemma 2.7. *Let M be a countable transitive model that can be elementarily embedded into $H(\chi)$, and let $\eta \in \mathbf{V}$ be a Cohen real over M . Then there is no $p \in \mathbb{C}$ such that p forces in Cohen forcing over \mathbf{V} that η is not Cohen over $M[\mathbb{C}]$.*

Proof. If $\eta \in \mathbf{V}$ is Cohen over M and c is Cohen over \mathbf{V} then c is also Cohen over $M[\eta]$. So $M[\eta][c]$ is an iterated Cohen extension and (η, c) is M -generic

for $\mathbb{C} * \mathbb{C}$. Since $\mathbb{C} \times \mathbb{C}$ densely embeds into $\mathbb{C} * \mathbb{C}$, the order of the two Cohen reals does not matter. So c is forced to be Cohen over $M[\eta]$. \square

By Lemma 2.7, the omitting types theorem shows that there is an oracle \bar{N} for the preservation of η_i 's Coheness over M_i . We review the proof of the omitting types theorem for the preservation of Coheness in order to show that $N_i = M[\eta_i]$ is a strong enough oracle. ²

Theorem 2.8. *Let \bar{M}, \bar{N}, S, I be as in Definition 2.2 and Lemma 2.5(2). For each \mathbb{P}^\dagger with the \bar{N} -c.c. there is a set $Y \in \mathcal{D}_{\bar{N}}$ such that for any $i \in Y$, η_i is Cohen over $M_i[\mathbb{P}^\dagger]$.*

Proof. We work with the type given in (2.2). We assume $\mathbb{P}^\dagger = \omega_1$. Then by the oracle-c.c.

$$Y' = \left\{ \delta \in S : (\forall A \in N_\delta \cap \mathcal{P}(\delta)) \left(((A \text{ is predense in } (\delta)) \rightarrow ((A \text{ is predense in } \mathbb{P})) \right) \right\}$$

is an element of $\mathcal{D}_{\bar{N}}$.

Let τ be a \mathbb{P}^\dagger -name for a real. Since $\mathbb{P}^\dagger = \omega_1$ has the c.c.c. we can assume that $\tau \in H(\omega_1)$. Let $p \in \mathbb{P}^\dagger$. Let Y be the set of $\delta \in Y'$ such that

- (a) $\tau \in M_\delta$,
- (b) $\tau = \tau^{(N_\delta, \delta)}$,
- (c) $\mathbb{P}^\dagger \cap \delta \subseteq_{ic} \mathbb{P}^\dagger$.

Then $Y \in \mathcal{D}_{\bar{N}}$. Let G be \mathbb{P}^\dagger -generic over \mathbf{V} and $\delta \in Y$. Then $G \cap \delta$ is $\mathbb{P}^\dagger \cap \delta$ -generic over N_δ . Since $\mathbb{P}^\dagger \cap \delta$ is equivalent to Cohen forcing, by Lemma 2.7, $N_\delta[G \cap \delta] \models \neg\psi(\mathcal{I}[G \cap \delta], \eta_\delta)$. Since $\mathbb{P}^\dagger \cap \delta \subseteq_{ic} \mathbb{P}^\dagger$, we have $\mathcal{I}[G \cap \delta] = \mathcal{I}[G]$. By absoluteness, $N_\delta[G] \models \neg\psi(\mathcal{I}[G], \eta_\delta)$. \square

For building up a name for an ultrafilter witnessing $\mathfrak{mcf} = \aleph_1$ we introduce some notions for handling names.

Definition 2.9. *Let \mathbb{P} be a ccc forcing of size at most \aleph_1 .*

- (1) *A canonical \mathbb{P} -name for a subset of ω is a name of the form $\tau = \{ \langle \check{n}, p \rangle : p \in A_n \}$, where the $A_n \subseteq \mathbb{P}$ are countable maximal antichains.*
- (2) *A canonical \mathbb{P} -name for a subset of $\mathcal{P}(\omega)$ is a name of the form $K = \{ \langle \tau, q \rangle : q \in A_\tau, \tau \in X \}$, where X is a set of canonical \mathbb{P} -names τ for subsets of ω , for maps π as in (3), and for each $\tau \in X$, the set A_τ is a countable antichain in \mathbb{P} .*
- (3) *Let $\pi: \mathbb{P} \rightarrow \omega_1$ be injective. We let $\pi[\mathbb{P}] = \mathbb{P}'$ and define a partial order (or a quasi order) on \mathbb{P}' such that π is an isomorphism from $(\mathbb{P}, <_{\mathbb{P}})$ to $(\mathbb{P}', <_{\mathbb{P}'})$. Then we lift π to a map $\bar{\pi}: \mathbf{V}^{\mathbb{P}} \rightarrow \mathbf{V}^{\mathbb{P}'}$ -names by letting $\bar{\pi}(\tau) = \{ \langle \bar{\pi}(\sigma), \pi(p) \rangle : \langle \sigma, p \rangle \in \tau \}$.*

²The sequence of the N_i is not an oracle literally, since its entries are not necessarily elementary subsets of $H(\theta)$. However, they are transitive models of a sufficiently large fragment of ZFC. Theorem 2.8 shows that this is sufficient for our specific types. Henceforth we will also call \bar{N} an oracle sequence.

For canonical names τ, \underline{K} as above, $\bar{\pi}(\tau) \in H(\omega_1)$, $\bar{\pi}(\underline{K}) \subseteq H(\omega_1)$. Thus according to Lemma 2.3, $I_{\bar{M}}(\bar{\pi}(\underline{K})) \in \mathcal{D}_{\bar{M}}$. The names $\bar{\pi}(\underline{K})$ and $\bar{\pi}(\tau)$ are canonical.

Definition 2.10. *Let \bar{M} be an S -oracle sequence and $\mathbb{P}' \subseteq \omega_1$.*

(1) *We let τ be a canonical \mathbb{P}' -name of a subset of ω . We let for $\delta \in \omega_1$,*

$$\tau^{(M_\delta, \delta)} = \begin{cases} \tau; & \text{if } \tau \text{ is a } \mathbb{P}' \cap \delta\text{-name, and } \tau \in M_\delta \\ \text{undefined;} & \text{otherwise.} \end{cases}$$

(2) *For a canonical \mathbb{P}' -name $\underline{K} = \{(\tau, q) : q \in A_\tau, \tau \in X\}$ for a subset of $\mathcal{P}(\omega)$ and $\delta < \omega_1$ we define the M_δ -part as follows:*

$$\underline{K}^{(M_\delta, \delta)} = \{(\tau, q) : (\tau, q) \in \underline{K}, q \in \mathbb{P}' \cap \delta, \tau \text{ is a } \mathbb{P}' \cap \delta\text{-name, } \tau \in M_\delta, A_\tau \subseteq \mathbb{P}' \cap \delta, A_\tau \in M_\delta\}.$$

Note that for a canonical \mathbb{P}' -name we have $\underline{K}^{(M_\delta, \delta)} \subseteq M_\delta$, however, in general $\underline{K}^{(M_\delta, \delta)}$ is not an element of M_δ . By Lemma 2.3 we have though

$$\{\delta \in S : \langle (\varepsilon, \underline{K}^{(M_\varepsilon, \varepsilon)}) : \varepsilon < \delta \rangle \in M_\delta\} \in \mathcal{D}_{\bar{M}}.$$

Now we are ready to define the set K^1 of pairs that serve as conditions in the first iterand of our final two-step forcing. The order on K^1 will be defined in Definition 2.18.

Definition 2.11.

(1) *For an S -oracle triple $(\bar{M}, \bar{N}, \bar{\eta})$ as above we let K^1 be the set of all $(\mathbb{P}, \underline{D})$ with the following properties:*

- (a) \mathbb{P} is a c.c.c. forcing with a nonstationary domain $\mathbb{P} \subseteq \omega_1$.
- (b) \underline{D} is a canonical \mathbb{P} -name of a non-principal ultrafilter over ω .
- (c) $Y(\mathbb{P}, \underline{D}) \in \mathcal{D}_{\bar{N}}$, where $Y(\mathbb{P}, \underline{D})$ is the set of $\delta \in S$ such that items
 - (α) to (ε) hold:
 - (α) $\mathbb{P} \cap \delta \in M_\delta$.
 - (β) If $E \subseteq \mathbb{P} \cap \delta$ and $E \in N_\delta$ and E is predense in $\mathbb{P} \cap \delta$ then E is predense in \mathbb{P} (so we have that \mathbb{P} has the \bar{N} -oracle-c.c.).
 - (γ) $\underline{D}^{(M_\delta, \delta)} \in M_\delta$ and $M_\delta \models \text{“}\underline{D}^{(M_\delta, \delta)} \text{ is a canonical } \mathbb{P} \cap \delta\text{-name of an ultrafilter over } \omega\text{”}$.
 - (δ) $N_\delta \models (\mathbb{P} \cap \delta \Vdash \text{“}\eta_\delta \text{ is Cohen-generic over } M_\delta[\mathbf{G}_{\mathbb{P} \cap \delta}] \text{”})$.
 - (ε) $\underline{D}^{(N_\delta, \delta)} \in N_\delta$ is a canonical $\mathbb{P} \cap \delta$ -name of an ultrafilter over ω such that

$$\mathbb{P} \cap \delta \Vdash (\forall f \in M_\delta[\mathbf{G}_{\mathbb{P} \cap \delta}] \cap {}^\omega \omega)(f \leq_{\underline{D}^{(N_\delta, \delta)}} \eta_\delta).$$

(2) *For an oracle triple $(\bar{M}, \bar{N}, \bar{\eta})$ we let K^2 be the set of $(\mathbb{P}, \underline{D}) \in H(\aleph_2)$ such that there are a non-stationary $\mathbb{P}' \subseteq \omega_1$ and a bijective $\pi: \mathbb{P}' \rightarrow \mathbb{P}$ and $(\mathbb{P}', \underline{D}') \in K^1$, π is an isomorphism from \mathbb{P}' onto \mathbb{P} with $\bar{\pi}(\underline{D}') = \underline{D}$.*

Remark 2.12. Since we do not add new types that have to be omitted in the course of the iteration, one fixed oracle $\bar{N} \in \mathbf{V}$ is sufficient.

We recall the the successor step and the direct limit step for oracle-c.c.

Lemma 2.13. (*Lemma [She98, IV 3.2]*) *If \mathbb{P} has the \bar{M} -c.c. and \mathbb{P} forces that \mathbb{Q} has the $\langle M_\delta[\mathbb{P}] : \delta \in S \rangle$ -c.c. then $\mathbb{P} * \mathbb{Q}$ has the \bar{M} -c.c.*

Lemma 2.14. (*Lemma [She98, IV 3.10]*): *If $\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \beta \rangle$ is a finite support iteration such that has the \bar{M} -c.c. and for $\alpha < \beta$ the forcing \mathbb{P}_α forces that \mathbb{Q}_α has the $\langle M_\delta[\mathbb{P}_\alpha] : \delta \in S \rangle$ -c.c. then \mathbb{P}_β has the \bar{M} -c.c.*

If $\pi: \mathbb{P}' \rightarrow \mathbb{P}$ is an isomorphism between forcing orders, we use it also for its natural extension that maps \mathbb{P} -names to \mathbb{P}' -names.

Lemma 2.15. *Let $(\bar{M}, \bar{N}, \bar{\eta})$ be an S -oracle triple and let K^1 be as in Definition 2.11(1). Assume*

- (a) $(\mathbb{P}, \underline{D}) \in H(\aleph_2)$, \mathbb{P} is a forcing notion, $\mathbb{P} \in H(\omega_2)$ and $\underline{D} \in H(\omega_2)$ is a canonical \mathbb{P} -name of an ultrafilter over ω .
- (b) \mathbb{P}'_ℓ is a notion of forcing whose domain is a non-stationary subset of ω_1 for $\ell = 1, 2$.
- (c) π_ℓ is an isomorphism from \mathbb{P}'_ℓ onto \mathbb{P} for $\ell = 1, 2$.
- (d) \underline{D}'_ℓ is a \mathbb{P}'_ℓ -name of a subset of $\mathcal{P}(\omega)$ such that π_ℓ maps \underline{D}'_ℓ onto \underline{D} .

Then $(\mathbb{P}'_1, \underline{D}'_1) \in K^1$ iff $(\mathbb{P}'_2, \underline{D}'_2) \in K^1$.

Proof. The map $\pi = \pi_2^{-1} \circ \pi_1$ is an isomorphism from \mathbb{P}'_1 onto \mathbb{P}'_2 , and its lifting $\bar{\pi}$ maps \underline{D}'_1 to \underline{D}'_2 . According to Lemma 2.3,

$$Z = \{\delta \in S : \pi \upharpoonright \delta \text{ is a one-to-one mapping from } \mathbb{P}'_1 \cap \delta \text{ to } \mathbb{P}'_2 \cap \delta \\ \text{and } \pi \upharpoonright \delta \in M_\delta\}$$

belongs to $\mathcal{D}_{\bar{M}}$. If $\delta \in Z$ then $\delta \in Y(\mathbb{P}'_1, \underline{D}'_1)$ iff $\delta \in Y(\mathbb{P}'_2, \underline{D}'_2)$, since the defining properties of the sets $Y(\mathbb{P}'_\ell, \underline{D}'_\ell)$ are preserved by isomorphisms of forcing orders. \square

This shows that in Definition 2.11(2) the following is true: If the demand holds for some pair (\mathbb{P}', π) then it holds for every such pair. The primed partial orders in Lemma 2.15 shall ensure that the domain is a non-stationary subset of ω_1 . Canonical \mathbb{P}' -names for reals and for filters over ω are actual subsets of $H(\omega_1)$. According to Lemma 2.15, their properties are invariant under bijections of ω_1 . Since any property of the forcing is named modulo $\mathcal{D}_{\bar{N}}$ the particular choice of the injections does not matter. For the actual construction of forcing posets it is convenient to use non-stationary domains for the $\mathbb{P}' \in K^1$, since non-stationarity is preserved by countable unions and by diagonal unions.

The property in Def. 2.11(1)(c)(ε) ensures that \underline{D} will be forced to be an ultrafilter such that the weakest condition in the two-step forcing forces

$\text{cf}(\omega^\omega/\underline{D}) = \aleph_1$, as witnessed by $\langle \eta_\delta : \delta \in S \rangle$. Technically it is more convenient to prove property (δ) by induction and then derive property (ε) from property (δ) , though property (ε) is more directly related to $\text{cf}(\omega^\omega/\underline{D}) = \aleph_1$. In the case of an \leq^* -increasing sequence $\langle \eta_\delta : \delta < S \rangle$ unboundedness is preserved in limits of finite support iterations if each initial segments preserves it [BJ95, Ch. 6, §4]. So it might be possible to prove by induction property (ε) and the negation of (δ) . We have not investigated this issue.

Concerning the preservation of (δ) , we will frequently use [BJ95, Chapter 6 Section 4]:

Lemma 2.16. *Let $\mathbb{P}_n \leq \mathbb{P}_{n+1}$ for $n \in \omega$ and let \mathbb{P} be the direct limit of $\langle \mathbb{P}_n : n \in \omega \rangle$. If $\mathbb{P}_n \Vdash$ “ η_δ is Cohen generic over $M_\delta[G_{\mathbb{P}_n}]$ ” for all n , then $\mathbb{P} \Vdash$ “ η_δ is Cohen generic over $M_\delta[G_{\mathbb{P}}]$.”*

Let $\text{unif}(\mathcal{M})$ denote the smallest cardinality of a non-meagre set. The following proposition gives the additional information that $\text{unif}(\mathcal{M}) = \aleph_1$ in our forcing extensions, as witnessed by $\{\eta_\delta : \delta \in S\}$.

Proposition 2.17. *If $(\mathbb{P}, \underline{D}) \in K^2$ then \mathbb{P} forces that $\{\eta_\delta : \delta \in S\}$ is a non-meagre subset of ω^2 .*

Proof. Let $p \in \mathbb{P}$ force that $\{\eta_\delta : \delta \in S\}$ is meagre. Let τ be a name for a meagre F_σ -set. By the c.c.c., there is a $\delta \in Y(\mathbb{P}, \underline{D})$ such that $\tau, p \in M_\delta$, $p \in \mathbb{P} \cap \delta$, τ is a $\mathbb{P} \cap \delta$ -name, and $p \Vdash \{\eta_\varepsilon : \varepsilon \in S\} \subseteq \tau$. Then $p \Vdash_{\mathbb{P}} \eta_\delta \in \tau$. Since $\delta \in Y(\mathbb{P}, \underline{D})$, clause (β) in the definition of $Y(\mathbb{P}, \underline{D})$ yields also $p \Vdash_{\mathbb{P} \cap \delta} \eta_\delta \in \tau$. This is a contradiction to Definition 2.11(1)(c)(δ) of the definition of $Y(\mathbb{P}, \underline{D})$. \square

Proposition 2.17 has a sort of an inverse direction for the class of Suslin forcings. A forcing $\mathbb{Q} \subseteq \omega^\omega$ is called Suslin if \mathbb{Q} is an analytic subset of ω^ω and the relations $\leq_{\mathbb{Q}}$ and $\perp_{\mathbb{Q}}$ are analytic sets in $\omega^\omega \times \omega^\omega$. For Suslin proper forcings, not making the ground model meagre is equivalent to preserving the genericity of a Cohen real over any countable model [Gol93, 6.21, 6.22], and then all non-meagre sets in the ground model stay non-meagre.

Now we introduce the approximation forcing $(AP, <_{AP})$:

Definition 2.18. *We let K^2 be as above.*

- (A) *Let $\mathbf{p} = (\mathbb{P}_{\mathbf{p}}, \underline{D}_{\mathbf{p}})$, $\mathbf{q} = (\mathbb{P}_{\mathbf{q}}, \underline{D}_{\mathbf{q}}) \in K^2$. We define $\mathbf{p} \leq_{AP} \mathbf{q}$, that is \mathbf{q} is stronger than \mathbf{p} , if*
- (a) $\mathbb{P}_{\mathbf{p}} \leq \mathbb{P}_{\mathbf{q}}$,
 - (b) $\Vdash_{\mathbb{P}_{\mathbf{q}}} \underline{D}_{\mathbf{p}} \subseteq \underline{D}_{\mathbf{q}}$.
- (B) *For $i = 1, 2$, we let forcing order of approximations be $AP^i = (K^i, \leq_{AP})$. We let $AP = AP^2$.*

The following is the parallel of the basic claim on oracle c.c. forcing, [She98, Ch. IV, Claim 3.2]. The forcing \mathbb{P}_i does not mean iteration up to stage i . The variable i , ranging over $\omega + 1$ or $\omega_1 + 1$ or ω_2 , is just an index

for \mathbb{P}_i being a component of $(\mathbb{P}_i, \underline{D}_i) \in K^2$. \mathbb{P}_i is an \bar{N} -oracle c.c. forcing and $|\mathbb{P}_i| \leq \aleph_1$.

Lemma 2.19.

- (A) The structure (K^2, \leq_{AP}) is a partial order of cardinality $|H(\aleph_2)|$.
- (B) $K^2 \neq \emptyset$.
- (C) If $\mathbf{p}_n = (\mathbb{P}_n, \underline{D}_n) \in K^2$ for $n \in \omega$ and $\mathbf{p}_n \leq_{AP} \mathbf{p}_{n+1}$ then the set has an upper bound $\mathbf{p}_\omega = (\mathbb{P}_\omega, \underline{D}_\omega)$ with $\mathbb{P}_\omega = \bigcup \{\mathbb{P}_n : n \in \omega\}$.
- (D) (K^2, \leq_{AP}) is $(\omega_1 + 1)$ -strategically closed, that is, for every $\mathbf{p} \in AP$ the protagonist has a winning strategy in the following game $\mathfrak{D}(\mathbf{p})$: A play lasts $\omega_1 + 1$ moves. During the play the player COM, the protagonist, chooses for each $i \leq \omega_1$, $\mathbf{p}_i = (\mathbb{P}_i, \underline{D}_i) \in K^2$, and INC, the antagonist, chooses $\mathbf{q}_i \in K^2$ such that
- (a) $\mathbf{p}_i \leq_{AP} \mathbf{q}_i$,
 - (b) $(\forall j < i)(\mathbf{q}_j \leq_{AP} \mathbf{p}_i)$,
 - (c) $\mathbf{p}_0 = \mathbf{p}$.

The protagonist COM wins the game if they can always move. The hard case is the choice of \mathbf{p}_{ω_1} .

Proof. (A) and (B) are obvious.

(C) Let $\mathbf{p}_n = (\mathbb{P}_n, \underline{D}_n)$ and let $\langle \mathbf{p}_n : n \in \omega \rangle$ be \leq_{AP} -increasing. We choose $(\mathbb{P}_n, \pi_n, \mathbb{P}'_n, \underline{D}'_n)$ by induction on n with the following properties:

- (1) $\mathbb{P}'_n \subseteq \omega_1$ is not stationary
- (2) $\pi_n : \mathbb{P}'_n \rightarrow \mathbb{P}_n$ is an isomorphism of partial orders,
- (3) $(\bar{\pi})^{-1}(\underline{D}_n) = \underline{D}'_n$,
- (4) $\pi_n \subseteq \pi_{n+1}$,
- (5) $(\mathbb{P}'_n, \underline{D}'_n) \in K^1$.

Then we let $\mathbb{P}'_\omega = \bigcup_{n \in \omega} \mathbb{P}'_n$, and the latter is not stationary. Moreover we let $\pi_\omega = \bigcup_{n \in \omega} \pi_n$.

We fix for $n \in \omega$ a reduction $r_{\mathbb{P}'_\omega, \mathbb{P}'_n} : \mathbb{P}'_\omega \rightarrow \mathbb{P}'_n$ and we set $C = \{\delta \in S : \delta \text{ limit of } S \text{ and } (\forall n) r_{\mathbb{P}'_\omega, \mathbb{P}'_n}[\mathbb{P}'_\omega \cap \delta] \subseteq \delta\}$. Of course C is club in ω_1 . We let

$$(2.3) \quad Y = \bigcap_{k \in \omega} Y(\mathbb{P}'_k, \underline{D}'_k) \cap C.$$

By [She98, Ch. IV, Claim 3.2], the poset \mathbb{P}'_ω has the \bar{N} -oracle c.c, i.e., \mathbb{P}'_ω satisfies clause (c)(β) of Def. 2.11. By Lemma 2.16 the set Y is also a witness to clause (c)(δ) for $\mathbb{P}'_\omega \in K^1$.

We show that there is \underline{D}'_ω such that $(\mathbb{P}'_\omega, \underline{D}'_\omega)$ is an upper bound of $\langle \mathbf{p}'_n : n < \omega \rangle$ in \leq_{AP} . To this end we define an \mathbb{P}'_ω -name \underline{D}'_ω for an ultrafilter such that $\mathbf{p}_\omega = (\mathbb{P}'_\omega, \underline{D}'_\omega) \in K^1$ and $Y \subseteq Y(\mathbb{P}'_\omega, \underline{D}'_\omega)$. We let

$$\mathbb{P}'_\omega \Vdash \underline{E}' = \bigcup_{k \in \omega} \underline{D}'_k.$$

Since \mathbb{P}'_k is a complete suborder of \mathbb{P}'_ω the D'_k are names for filters and $0_{\mathbb{P}'_{k+1}} \Vdash D'_k \subseteq D'_{k+1}$ the weakest element of \mathbb{P}'_ω forces that E' is a \mathbb{P}'_ω -name for a filter.

We write $\text{next}(Y, \varepsilon)$ for the next element in Y after ε , i.e., $\text{next}(Y, \varepsilon) = \min\{\delta > \varepsilon : \delta \in Y\}$. By induction on $\delta \in Y$, we will define a canonical $\mathbb{P}'_\omega \cap \delta$ -name $D'_\omega(\delta) \in M_\delta$ such that

$$\mathbb{P}'_\omega \cap \delta \Vdash \text{“} D'_\omega(\delta) \supseteq \bigcup \{D'_\omega(\gamma) : \gamma \in Y \cap \delta\} \\ \text{and } D'_\omega(\delta) \text{ is an ultrafilter in } M_\delta\text{”}$$

and

$$\mathbb{P}'_\omega \cap \text{next}(Y, \delta) \Vdash \text{“} (\forall f \in M_\delta[\mathbb{P}'_\omega]) (\eta_\delta \geq_{D'_\omega(\text{next}(Y, \delta))} f) \\ \text{and } D'_\omega(\text{next}(Y, \delta)) \cap \mathcal{P}(\omega)^{N_\varepsilon} \text{ is an ultrafilter in } N_\varepsilon\text{”}$$

The restriction of names, mapping each name X to a name $X^{(M_\delta, \delta)}$, was defined in Definition 2.10(2). We will often write X^{M_δ} instead of $X^{(M_\delta, \delta)}$. For $k \in \omega$ we let

$$Y_k = \{\delta \in Y : D'_k(\delta) = D'_k^{M_\delta}\},$$

Then $Y_k \in \mathcal{D}_{\bar{N}}$ and thus also their intersection $Y' = \bigcap_{k \in \omega} Y_k$ is in $\mathcal{D}_{\bar{N}}$. For simplicity, we write just Y for Y' .

Assume that $\langle D'_\omega(\gamma) : \gamma \in Y \cap \delta \rangle$ has been defined. By the induction hypothesis on (\mathbf{p}'_k, π_k) , the \mathbb{P}'_k -names for ultrafilters D'_k are defined and increasing in k .

We first consider the limit steps in the induction. Let $\delta \in Y$ be a limit of Y . First case: $\langle D'_\omega(\gamma) : \gamma < Y \cap \delta \rangle \notin M_\delta$. Then we let

$$1_{\mathbb{P}'_\omega \cap \delta} \Vdash D'_\omega(\delta) = \bigcup \{D'_\omega(\gamma) : \gamma \in Y \cap \delta\}.$$

Second case: $\langle D'_\omega(\gamma) : \gamma \in Y \cap \delta \rangle \in M_\delta$. We first show

$$1 \Vdash_{\mathbb{P}'_\omega \cap \delta} \underline{F}'(\delta) := \underline{E}'^{M_\delta} \cup \bigcup \{D'_\omega(\gamma) : \gamma \in Y \cap \delta\} \text{ is a filter base.}''$$

We assume, for a contradiction, that there are a condition $p \in \mathbb{P}'_\omega$, $k \in \omega$, and a $\gamma \in Y \cap \delta$ and there are names X, X' , such that p forces that $X \in D'_k^{M_\delta}$ and $X' \in \underline{E}'^{M_\delta}$, $\gamma \in Y \cap \delta$ such that $X \cap X'$ is empty. Then $p \upharpoonright \mathbb{P}'_k \Vdash X \in D'_k \upharpoonright \delta$. Let \mathbf{G}_k be \mathbb{P}'_k -generic over N_δ with $p \upharpoonright \mathbb{P}'_k \in \mathbf{G}_k$. We let $Z[\mathbf{G}_k] = \{n : (\exists \tilde{q} \in \mathbb{P}'_\omega \cap \delta / \mathbf{G}_k)(\tilde{q} \geq p[\mathbf{G}_k] \wedge \tilde{q} \Vdash n \in X'[\mathbf{G}_k] \cap X)\}$. Since \mathbf{p}_k is a condition the name $D'_\omega(\gamma) \upharpoonright \delta$ is an ultrafilter compatible with $D'_k(\gamma)$. Therefore we have that $p \upharpoonright \mathbb{P}'_k \Vdash_{\mathbb{P}'_k}$ “ $Z[\mathbf{G}_k]$ is infinite.” Now we take $n \in \omega$, \tilde{q} as in the definition of $Z[\mathbf{G}_k]$, so that $\tilde{q} \Vdash n \in X \cap X'$. So we have a contradiction. Hence for any $\gamma \in Y \cap \delta$, the weakest condition forces that $\underline{E}' \upharpoonright \delta \cup D'_\omega(\gamma)$ is a filter basis. Since the names $D'_\omega(\gamma)$ are forced to be increasing with $\gamma \in Y \cap \delta$, also their union, $\underline{F}'(\delta)$, is forced to be a filter basis. Now we choose a name $D'_\omega(\delta) \in M_\delta$ for an ultrafilter that extends $\underline{F}'(\delta)$.

Now we consider the beginning and the successor steps of the induction. For the beginning, let $\gamma = -1$, $\underline{D}'_\omega(-1) = \underline{E}'$ and let $\delta = \min(Y)$, and for the successor let δ be the successor of $\gamma \in Y$, i.e., $\delta = \text{next}(Y, \gamma)$. Then $N_\gamma \in M_\delta$. We extend $\underline{D}'_\omega(\gamma)$ to $\underline{D}'_\omega(\delta) \in M_\delta$ so that $\underline{D}'_\omega(\delta)$ is a $\mathbb{P}' \cap \delta$ -name for an ultrafilter such that

$$\begin{aligned} 1_{\mathbb{P}' \cap \delta} \Vdash \underline{D}'_\omega(\delta) \supseteq \underline{F}(\delta) &:= (\underline{E}' \upharpoonright \delta) \cup \underline{D}'_\omega(\gamma) \\ \cup \{ \{ n \in \omega : \eta_\gamma(n) \geq \underline{f}(n) \} : \underline{f} \in M_\gamma \text{ a } \mathbb{P}'_\omega \cap \delta\text{-name for a function} \}. \end{aligned}$$

Since $\gamma \in Y$, we can restrict the considerations to $\mathbb{P}'_\omega \cap \gamma$ names \underline{f} . Again we show that the weakest condition forces that $\underline{F}(\delta)$ has the finite intersection property. Let $q_0 \in \mathbb{P}'_\omega \cap \delta$ be given. Let q_0 force that \underline{A}_1 be a name of a member of $\underline{D}'_k \upharpoonright \delta$ and $q_0 \Vdash \underline{A}_2 \in \underline{D}'_\omega(\delta)$ and $\underline{A}_3 = \{ n : \eta_\gamma(n) > \underline{f}(n) \}$. Now in M_δ we define a $(\mathbb{P}'_k \cap \delta)$ -name \underline{A}_{23} as follows: if $\mathbf{G}_k \subseteq \mathbb{P}'_{\mathbf{p}_k}$, $q_0 \upharpoonright \mathbb{P}'_k \in G_k$ is \mathbb{P}'_k -generic over M_δ we let

$$\begin{aligned} \underline{A}_{23}[\mathbf{G}_k] &= \{ n : (\exists \hat{q} \in (\mathbb{P}'_\omega \cap \delta) / \mathbf{G}_k) \\ &\quad (\hat{q} \geq q_0[\mathbf{G}_k] \wedge \hat{q} \Vdash (n \in \underline{A}_2[\mathbf{G}_k] \wedge \eta_\gamma(n) \geq \underline{f}[\mathbf{G}_{\mathbf{p}_k}](n))) \}. \end{aligned}$$

Then $q_0 \upharpoonright \mathbb{P}'_k \Vdash_{\mathbb{P}'_k} \underline{A}_1 \cap \underline{A}_{23}[\mathbf{G}_k]$ is infinite, since \mathbb{P}'_k is already an approximation and η_γ is Cohen generic also over $M_\gamma[\mathbb{P}'_k]$, and hence $M_\gamma[\mathbb{P}'_k] \models \eta_\gamma \not\leq_{\underline{D}'_k} \underline{f}$. We take $\hat{q} \in (\mathbb{P}'_\omega \cap \delta) / \mathbf{G}_k$ and n as in the definition of $\underline{A}_{23}[\mathbf{G}_k]$. Since $q_0 \upharpoonright \mathbb{P}'_k$ is \mathbb{P}'_k -generic over M_δ , we may assume that $\hat{q} \upharpoonright \mathbb{P}'_k \geq q_0$ and $\hat{q} \Vdash "n \in \underline{A}_1 \cap \underline{A}_{23}."$ Hence in M_δ there is a name for an ultrafilter $\underline{D}'_\omega(\delta)$ containing $\underline{F}(\delta)$, and we choose and fix the $<_\chi$ -least one and call it $\underline{D}'_\omega(\delta)$. Since $N_\gamma \subseteq M_\delta$ and $N_\gamma \in M_\delta$, $\underline{D}'_\omega(\delta) \cap \mathcal{P}(\omega)^{N_\gamma}$ is an ultrafilter in N_γ .

Now the induction on $\delta \in Y$ is carried out. We choose a name \underline{D}'_ω such that

$$\mathbb{P}'_\omega \Vdash \underline{D}'_\omega = \bigcup \{ \underline{D}'_\omega(\delta) : \delta \in Y \}.$$

We mirror the construction back to the class K^2 : by letting $\underline{D}_\omega = \bar{\pi}(\underline{D}'_\omega)$.

(D) Let $\mathbf{p} \in K^2$ be given. We write $\mathbf{p}_i = (\mathbb{P}_i, \underline{D}_i)$, $i < \omega_1$. The strategy of the protagonist is to choose in addition to $\mathbf{p}_i \geq_{AP} \mathbf{q}_j$ for $j < i$, on the side also $\mathbf{p}'_i = (\mathbb{P}'_i, \underline{D}'_i) \in K^1$ and $\pi_i: \mathbb{P}'_i \rightarrow \mathbb{P}_i$ and $\xi_i \in \omega_1$ with the following properties:

- (a) $\langle \xi_i : i < \omega_1 \rangle$ is continuously increasing.
- (b) $(\mathbb{P}'_i, \underline{D}'_i) \in K^1$, $\mathbb{P}'_i \setminus \bigcup \{ \mathbb{P}'_j : j < i \} \subseteq [\xi_i + 1, \omega_1)$.
- (c) π_i is a isomorphism from \mathbb{P}'_i onto \mathbb{P}_i mapping \underline{D}'_i onto \underline{D}_i .
- (d) for $j < i$, $\pi_j \subseteq \pi_i$ (so the \mathbb{P}'_i are \subseteq -increasing in ω_1).
- (e) for $j < i$, $(\mathbb{P}'_j, \underline{D}'_j) \leq_{AP^1} (\mathbb{P}'_i, \underline{D}'_i)$ and $(\mathbb{P}_j, \underline{D}_j) \leq_{AP} (\mathbb{P}_i, \underline{D}_i)$.
- (f) If $k < j \leq i$, $p \in \mathbb{P}'_k$ and $q \in \mathbb{P}'_j \cap \xi_i$ and p and q are compatible in \mathbb{P}'_i , then they are compatible with a witness in $\mathbb{P}'_i \cap \xi_i$. (Then the proof of [She98, Claim 3.2] for showing that also \mathbb{P}_i has the \bar{N} -c.c. works.)
- (g) If $i = j + 1 < \omega_1$ is a successor ordinal, then COM chooses $\mathbf{p}_i = \mathbf{q}_j$.

(h) If $i < \omega_1$ is a limit ordinal and if there is $j(*) < i$ such that

$$H = \bigcap \{Y(\mathbb{P}'_j, \mathcal{D}'_j) : j \in [j(*), i)\} \in \mathcal{D}_{\bar{N}},$$

then player COM takes for \mathbf{p}_i the limit of a countable cofinal sequence of \mathbf{q}_j 's in the manner described in (C). Thus

$$(2.4) \quad H \subseteq Y(\mathbb{P}'_i, \mathcal{D}'_i).$$

If there is no such $j(*)$, then COM can play just any lower bound of the countable sequence \mathbf{q}_j , $j < i$. For a set of $i \in \mathcal{D}_{\bar{N}}$ there is such a $j(*) < i$ with Equation (2.4).

Now if \mathbf{p}'_i , $i < \omega_1$, are defined, in the ω_1 -limit COM chooses \mathbb{P}'_{ω_1} as the direct limit. Then Equation (2.4) implies that

$$Y(\mathbb{P}'_{\omega_1}, \mathcal{D}'_{\omega_1}) \supseteq \bigtriangleup_{i < \omega_1} Y(\mathbb{P}'_i, \mathcal{D}'_i) \cap \{i : \xi_i = i\},$$

and hence $Y(\mathbb{P}'_{\omega_1}, \mathcal{D}'_{\omega_1}) \in \mathcal{D}_{\bar{N}}$. Hence

$$1_{\mathbb{P}'} \Vdash \mathcal{D}'_{\omega_1} = \bigcup_{i < \omega_1} \mathcal{D}'_i \text{ is an ultrafilter extending } \mathcal{D}'_i, i < \omega_1.$$

We mirror the primed objects via $\bigcup_{j < \omega_1} \pi_j$ back to K^2 and thus we get a forcing $\mathbb{P}_{\omega_1} = \bigcup \{\mathbb{P}_i : i < \omega\}$ and a \mathbb{P}_{ω_1} -name \mathcal{D}_{ω_1} for an ultrafilter over ω . The protagonist COM hence has won the play of the completeness game. \square

Definition 2.20. Let \mathbf{G}_{AP} be an AP-generic filter. In $\mathbf{V}[\mathbf{G}_{AP}]$ we let

$$\mathbb{Q} = \bigcup \{\mathbb{P} : (\exists \mathcal{D}) (\mathbb{P}, \mathcal{D}) \in \mathbf{G}_{AP}\}$$

and let \underline{E} be a \mathbb{Q} -name such that

$$\mathbb{Q} \Vdash \underline{E} = \bigcup \{\mathcal{D} : (\exists \mathbb{P}) (\mathbb{P}, \mathcal{D}) \in \mathbf{G}_{AP}\}.$$

We let \mathbb{Q} be an AP-name for \mathbb{Q} and we use the symbol \underline{E} also for an AP-name for \underline{E} .

Lemma 2.21.

- (a) $\Vdash_{AP} \mathbb{Q}$ is a ccc forcing of cardinality \aleph_2 ,
- (b) $\Vdash_{AP} \underline{E}$ is \mathbb{Q} -name of a non-principal ultrafilter and $\mathfrak{b} = \aleph_1$,
- (c) if $(\mathbb{P}, \mathcal{D}) \in AP$ then $(\mathbb{P}, \mathcal{D}) \Vdash_{AP} (\mathbb{Q} \Vdash \langle \eta_\delta : \delta \in S \rangle \text{ is a } \leq_{\underline{E}}\text{-increasing sequence and cofinal in } \omega^\omega / \underline{E})$.

Proof. For (a), see [She98, Ch. IV, Claim 1.6]. Now we prove (b). By the c.c.c. and the construction with direct limits, for every AP* \mathbb{Q} -name τ for a real there are a pair $\mathbf{p} = (\mathbb{P}, \mathcal{D}) \in AP$ and a condition $p \in \mathbb{P}$, and a \mathbb{P} -name real τ' for such that $(\mathbf{p}, p) \Vdash_{AP*\mathbb{Q}} \tau' = \tau$.

(c) We work with the approximation forcing AP^1 . Suppose for a contradiction that $((\mathbb{P}, \mathcal{D}), p) \Vdash_{AP^1*\mathbb{Q}} (\exists f \in {}^\omega\omega)(f \geq_{\underline{E}} \langle \eta_\delta : \delta \in S \rangle)$. Then there is $((\mathbb{P}', \mathcal{D}'), p') \geq_{AP^1} ((\mathbb{P}, \mathcal{D}), p)$ and there is a canonical \mathbb{P}' -name \underline{h} such that

$$(2.5) \quad ((\mathbb{P}', \mathcal{D}'), p') \Vdash_{AP^1*\mathbb{Q}} \underline{h} \geq_{\underline{E}} \langle \eta_\delta : \delta \in S \rangle.$$

Since \dot{h} is a name of a real in the c.c.c. forcing \mathbb{P}' , there are some $\delta_0 < \omega_1$, $\dot{h}' \in M_{\delta_0}$ such that \dot{h}' is a $\mathbb{P}' \cap \delta_0$ -name such that $((\mathbb{P}', \underline{D}'), p') \Vdash_{AP^* \ast \mathbb{Q}} \dot{h} = \dot{h}'$. We fix such a δ_0, \dot{h}' . Since $(\mathbb{P}', \underline{D}') \in K^1$, by Lemma 2.8 there is $\delta \geq \delta_0$ such that $N_\delta \models (\forall h \in M_\delta[G_{\mathbb{P}' \cap \delta}])(h \not\leq_{D'[G_{\mathbb{P}' \cap \delta}]} \eta_\delta)$. We take a condition $q \in \mathbb{P}' \cap \delta$, $q \geq_{\mathbb{P}'} p'$, forcing $\forall h \in M_\delta[G_{\mathbb{P}'}] h \not\leq_{D'} \eta_\delta$. Thus $((\mathbb{P}', \underline{D}'), q') \geq ((\mathbb{P}', \underline{D}'), p')$ and this is a contradiction to Equation (2.5). \square

Now we show that the union of the generic filter of the approximation forcing, i.e., the \mathbb{Q} as given in Lemma 2.21, fulfils $\Vdash_{AP^* \ast \mathbb{Q}} \text{cf}(\text{Sym}(\omega)) = \aleph_2$. The conditions of the form $((\mathbb{P}_*, \underline{D}_*), p)$ with $p \in \mathbb{P}_*$ are dense in $AP^* \ast \mathbb{Q}$.

A forcing destroying a given increasing cofinal chain of subgroups $\langle G_i : i < \omega_1 \rangle$ of $\text{Sym}(\omega)$ is written down in [MS11]. Such a forcing adds one particular real, a new permutation g that for cofinally many $i < \omega_1$ there is $f_i \in G_{i+1} \setminus G_i$ such that $g \circ f_i \circ g^{-1} \in G_i$. Thus in the extension we have $g \in \text{Sym}(\omega) \setminus \bigcup \{G_i : i < \omega_1\}$ and the sequence $\langle G_i : i < \omega_1 \rangle$ is not cofinal any more.

In the rest of this section we construct a variant of such a forcing that adds such a conjugator and at the same time has the \bar{N} -oracle c.c. We first show that we can work with convenient supports of permutations.

Lemma 2.22. *Suppose that chain of subgroups $\langle G_i : i < \omega_1 \rangle$ is an increasing chain of subgroups of $\text{Sym}(\omega)$ such that all permutations that move only finitely many elements are elements of G_0 . Suppose that $U \subseteq \omega_1$ is uncountable and there are*

$$\langle \zeta_i^1, \zeta_i^2, f_i^1, f_i^2 : i \in U \rangle \text{ and } g$$

with the following properties:

- (1) for $i < j \in U$, $i \leq \zeta_i^1 < \zeta_i^2 < j$,
- (2) for $i \in U$, $f_i^1 \in G_{\zeta_i^1}$ and $f_i^2 \in G_{\zeta_i^2} \setminus G_{\zeta_i^1}$, and
- (3) for $i \in U$, $(\forall^\infty n)((g \circ f_i^1)(n) = (f_i^2 \circ g)(n))$.

Then $g \in \text{Sym}(\omega) \setminus \bigcup \{G_i : i \in \omega_1\}$.

Proof. If $g \in G_{\zeta_i^1}$ for some $i \in U$, then by (3) also $f_i^2 \in G_{\zeta_i^1}$, contradiction. \square

For carrying this out we use some notions describing permutation groups.

Definition 2.23. *Let $f: \omega \rightarrow \omega$. $\text{supp}(f) = \{n : f(n) \neq n\}$.*

Observation 2.24. *If $f \in \text{Sym}(\omega)$, then $f[\text{supp}(f)] = \text{supp}(f)$.*

For $f \in \text{Sym}(\omega)$, we say f has order 2 if $f \circ f$ is the identity.

For arguing with given supports, we use:

Lemma 2.25. ([MS11, Lemma 3.3]) *If $\langle G_i : i < \omega_1 \rangle$ is an increasing sequence of proper subgroups of $\text{Sym}(\omega)$ with union $\text{Sym}(\omega)$, and G_0 contains all permutations with finite support, then for any $W \in [\omega]^{\aleph_0}$ the sequence*

$$\langle G_i \cap \{f \in \text{Sym}(\omega) : \text{supp}(f) \subseteq W \wedge f \text{ is of order } 2\} : i < \omega_1 \rangle$$

is not eventually constant.

Now we return to forcing.

Lemma 2.26. $\Vdash_{AP*\mathbb{Q}}$ “ $\text{cf}(\text{Sym}(\omega)) = \aleph_2$ ”.

Proof. Assume towards a contradiction:

- \oplus_1 $((\mathbb{P}_*, \underline{D}_*), p_*) \Vdash_{AP*\mathbb{Q}}$ “ $\langle \underline{G}_i : i < \omega_1 \rangle$ is an increasing sequence of proper subgroups of $\text{Sym}(\omega)$ with union $\text{Sym}(\omega)$, and \underline{G}_0 contains all permutations with finite support”.
- \oplus_2 By Lemma 2.25, \oplus_1 implies: $((\mathbb{P}_*, \underline{D}_*), p_*) \Vdash_{AP*\mathbb{Q}}$ “if $W \in [\omega]^{\aleph_0}$ then $\langle \underline{G}_i \cap \{f \in \text{Sym}(\omega) : \text{supp}(f) \subseteq W \wedge f \text{ is of order } 2\} : i < \omega_1 \rangle$ is not eventually constant”.
- \oplus_3 We let $\langle m_\eta : \eta \in {}^\omega \omega \rangle$ be a sequence of natural numbers without repetitions. For $\eta \in {}^\omega \omega$ we let $W(\eta) = \{m_{\eta \upharpoonright n} : n \in \omega\}$. Then for $\eta \neq \eta'$ and $k = \min\{n : \eta(n) \neq \eta'(n)\}$ we have $W(\eta) \cap W(\eta') = \{m_{\eta \upharpoonright n} : n < k\}$.

By induction on $i < \omega_1$ we choose $\mathbf{p}_i = (\mathbb{P}_i, \underline{D}_i) \in AP$, $\pi_i, \mathbf{p}'_i \in AP^1$, $\xi_i \in \omega_1$, and $(\mathbf{p}_i, \pi_i, \mathbf{p}'_i, \xi_i, \zeta_i^1, \zeta_i^2, \underline{f}_i^1, \underline{f}_i^2, \underline{\mathbb{R}}'_i, Y(\mathbb{P}'_i, \underline{D}'_i))$ such that

- (a) $\mathbf{p}_0 = \mathbf{p}_*$, $Y(\mathbb{P}'_0, \underline{D}'_0) = Y(\mathbb{P}_*, \underline{D}_*)$.
- (b) $\mathbf{p}_i = ((\mathbb{P}_i, \underline{D}_i), p_*) \in AP * \mathbb{Q}$ and $j < i \rightarrow \mathbf{p}_j \leq_{AP} \mathbf{p}_i$.
- (c) $\mathbf{p}'_i = ((\mathbb{P}'_i, \underline{D}'_i), p_*) \in AP^1 * \mathbb{Q}$ satisfies
 - (α) $\mathbb{P}'_0 \cap \{\xi_i : i < \omega_1\} = \emptyset$, the set of members of $\mathbb{P}'_i \setminus \bigcup\{\mathbb{P}'_j : j < i\} \subseteq [\xi_i + 1, \omega_1)$, hence $\mathbb{P}'_i \cap \xi_i = \mathbb{P}'_j \cap \xi_i$ for any $j \geq i$.
 - (β) $\pi_i: \mathbb{P}'_i \rightarrow \omega_1$ is a one-to-one function mapping \mathbb{P}'_i onto \mathbb{P}_i and mapping \underline{D}'_i onto \underline{D}_i .
 - (γ) if $j < i$, then $\pi_j \subseteq \pi_i$.
 - (δ) $\langle \xi_i : i < \omega_1 \rangle$ has the properties (a) to (d) of the proof of Lemma 2.19 (D) with respect to the sequence $\langle \mathbf{p}'_i, \pi_i : i < \omega_1 \rangle$.
 - (ε) The set $Y(\mathbb{P}'_i, \underline{D}'_i)$ witnesses that $(\mathbb{P}'_i, \underline{D}'_i) \in K^1$ as in Def. 2.11(1)(c).
- (d) At double successor steps of limit ordinals we add a new Cohen real: If $i = \omega j + 1$ then $\mathbb{P}'_{i+1} = \mathbb{P}'_i * ({}^\omega \omega, \triangleleft)$, we let ν_i be a name for $({}^\omega \omega, \triangleleft)$ -generic real. So ν_i is a Cohen real over $\mathbf{V}^{\mathbb{P}'_{\omega \cdot j}}$. Since $\mathbf{V}^{\mathbb{P}'_i}$ is unbounded in $\mathbf{V}^{\mathbb{P}'_{i+1}}$ by Lemma 2.7, there is a \mathbb{P}'_{i+1} -name for an ultrafilter \underline{D}_{i+1} . The set $Y(\mathbb{P}_{i+1}, \underline{D}_{i+1}) = Y(\mathbb{P}_i, \underline{D}_i) \cap [i+1, \omega) \in \mathcal{D}_{\bar{N}}$ witnesses that $(\mathbb{P}'_{i+1}, \underline{D}'_{i+1}) \in K^1$.
- (e) Also, if $i = \omega j + 1$ then we choose \underline{D}'_{i+1} such that $(\mathbb{P}'_{i+1}, \underline{D}'_{i+1}) \geq_{AP} (\mathbb{P}'_i, \underline{D}'_i)$ and $\langle \underline{G}_\ell \cap \mathcal{P}(\omega)^{\mathbb{P}'_j} : \ell < \omega_1 \rangle$ and even $\langle \underline{G}_\ell \cap \mathcal{P}(\omega)^{\mathbb{P}'_i} : \ell < \omega_1 \rangle$ is a \mathbb{P}'_i -name.
- (f) Also at double successors to limit ordinals we fix witnessing functions with the new Cohen ν_i as information in their support, i.e., if $i = \omega \cdot j + 1$ then

- (α) for $\ell = 1, 2$, \mathbf{p}'_{i+1} forces that $i < \zeta_i^1 < \zeta_i^2$,
 (β) and for $\ell = 1, 2$, \mathbf{p}'_{i+1} forces that $f_i^2 \in G_{\zeta_i^2} \setminus G_{\zeta_i^1}$, $f_i^1 \in G_{\zeta_i^1}$ is a \mathbb{P}'_{i+1} -name of a member of $\text{Sym}(\omega)$ of order 2 such that

$$\mathbb{P}'_{i+1} \Vdash \text{supp}(f_i^\ell) \subseteq w_i^\ell = W(\langle \ell \rangle \hat{\ } \nu_i).$$

Here $\langle \ell \rangle \hat{\ } \nu$ is the concatenation of the singleton $\langle \ell \rangle$ and ν i.e. $(\langle \ell \rangle \hat{\ } \nu)(k) = \ell$ if $k = 0$, and $= \nu(k - 1)$ else. Recall that for $\eta \in {}^\omega \omega$, $W(\eta)$ has been defined in \oplus_3 .

By Lemma 2.25, the desired names for countable ordinals ζ_i^1, ζ_i^2 and names f_i^1, f_i^2 exist. The triple $\mathbf{p}'_i \in AP * \mathbb{Q}$ stays unchanged.

- (g) At limit steps $i < \omega_1$, we let (\mathbb{P}'_i, D'_i) be a lower bound of (\mathbb{P}_j, D_j) , $j < i$, as in Lemma 2.19(C). We let $Y(\mathbb{P}'_i, D'_i) \in \mathcal{D}_{\bar{N}}$ be a witness to $(\mathbb{P}'_i, D'_i) \in K^1$.
 (i) Now finally we explain the order \mathbb{P}_{i+1} for countable limit ordinals i . We let

$$H = \bigcap \{Y(\mathbb{P}'_\varepsilon, D'_\varepsilon) : \varepsilon < i\}.$$

Then $H \in \mathcal{D}_{\bar{N}}$. We let Y_i, ξ_i be as follows

$$(2.6) \quad Y_i = \left\{ \delta \in H : (\forall j < i)(\xi_j < \delta) \wedge (\forall j_1 \in i) \right. \\ \left. \left((\zeta_{j_1}^1, \zeta_{j_1}^2, f_{j_1}^1, f_{j_1}^2) \in M_\delta \wedge N_{j_1} \in M_\delta \wedge \right. \right. \\ \left. \left. \zeta_{j_1}^1, \zeta_{j_1}^2, f_{j_1}^1, f_{j_1}^2 \text{ are } \mathbb{P}'_i \cap \delta\text{-names} \right) \right\}. \\ \xi_i = \min(Y_i).$$

Then $Y_i \in \mathcal{D}_{\bar{N}}$. Since any element of $\mathcal{D}_{\bar{N}}$ is unbounded in ω_1 , the ordinal ξ_i is well-defined. We define $\mathbb{R}'_i \in M_{\xi_i}$: \mathbb{R}'_i is a $\mathbb{P}'_i \cap \xi_i$ -name of a c.c.c. forcing notion. Recall that $w_\varepsilon^1, w_\varepsilon^2$, $\varepsilon < \xi_i$, ε successor ordinal, are defined in $\oplus_3(f)(\beta)$. The key fact to the \bar{N} -c.c. is that these names are so faintly related to the Cohen reals $\langle \eta_\delta : \delta \in S \rangle$.

The following is forced by $\mathbb{P}'_i \cap \xi_i$: A member of \mathbb{R}'_i has the form (u, g) such that

- (α) $u \subseteq \{\omega \cdot j + 1 : \omega \cdot j + 1 \in \xi_i\}$ is finite, g a finite partial permutation of order two, $\text{dom}(g) \subseteq \bigcup_{\varepsilon \in u} w_\varepsilon^2$, such that $\varepsilon \in u$ implies $\text{range}(g) \subseteq w_\varepsilon^1$.
 (β) Recall that for $\eta \in {}^{\omega > \omega} m_\eta$ has been defined in \oplus_3 . The sets $\text{dom}(g)$ and $\text{range}(g)$ are sufficiently large in the following sense:
- if $\delta \neq \varepsilon \in u$ then we fix n , such that $\nu_\delta \upharpoonright n \neq \nu_\varepsilon \upharpoonright n$ and then require that for $k = 1, 2$ the set $\{m_{\langle k \rangle \hat{\ } \nu_\delta \upharpoonright \ell} : \ell < n\} \subseteq \text{dom}(g) \cap \text{range}(g)$,
 - $\forall \varepsilon \in \text{dom}(g)$, if ε is Cohen coordinate (as in $\oplus_3(d)$) and $p(\varepsilon) \in 2^n$, $\ell \leq n$, $k = 1, 2$, then $m_{\langle k \rangle \hat{\ } p(\varepsilon) \upharpoonright \ell} \in \text{dom}(g) \cap \text{range}(g)$.

- (γ) If $\varepsilon \in u$ then $\text{dom}(g) \cap w_\varepsilon^2$ is closed under f_ε^1 and $\text{range}(g) \cap w_\varepsilon^1$ is closed under f_ε^2 .
- (δ) For $(u_1, g_1), (u_2, g_2) \in \mathbb{R}'_i$ we let $(u_1, g_1) \leq (u_2, g_2)$ iff
 - (i) $u_1 \subseteq u_2$,
 - (ii) $g_1 \subseteq g_2$,
 - (iii) $(\forall \varepsilon \in u_1)(\forall n \in w_\varepsilon^2 \cap (\text{dom}(g_2) \setminus \text{dom}(g_1)))(g_2(n) \in w_\varepsilon^1 \wedge f_\varepsilon^2(g_2(n)) = g_2(f_\varepsilon^1(n)))$.

We let $\mathbb{P}'_{i+1} = \mathbb{P}'_i * \mathbb{R}'_i$.

Since \mathbb{R}'_i is countable, \mathbb{P}'_{i+1} has the \bar{N} -c.c., and again by Lemma 2.7 we find \underline{D}'_{i+1} such that $(\mathbb{P}'_{i+1}, \underline{D}'_{i+1}) \in K^1$ with witness $Y(\mathbb{P}'_{i+1}, \underline{D}'_{i+1}) = Y_i \cap [\xi_i, \omega_1)$.

- \oplus_4 Once the induction is performed, we define $\mathbf{p}_{\omega_1} = (\mathbb{P}_{\omega_1}, \underline{D}_{\omega_1})$ and $\mathbf{p}'_{\omega_1} \in K^1$ and $\pi = \bigcup_{i < \omega_1} \pi_i$ which maps \mathbf{p}'_{ω_1} onto \mathbf{p}_{ω_1} as follows:
 - (a) $\mathbb{P}'_{\omega_1} = \bigcup \{(\mathbb{P}'_i \cap \xi_i) * \mathbb{R}'_i : i < \omega_1, i \text{ limit}\}$.
 - (b) $\mathbb{P}'_{\omega_1} \Vdash \underline{D}'_{\omega_1} = \bigcup \{\underline{D}'_i : i < \omega_1, i \text{ limit}\}$.
 - (c) $\pi = \bigcup_{i < \omega_1} \pi_i$ is an isomorphism from \mathbb{P}'_{ω_1} onto \mathbb{P}_{ω_1} mapping $\underline{D}'_{\omega_1}$ to \underline{D}_{ω_1} .
 - (d) $\bigwedge_{i < \omega_1} \mathbf{p}_i \leq \mathbf{p}_{\omega_1} \in K^2, \bigwedge_{i < \omega_1} \mathbf{p}'_i \leq \mathbf{p}'_{\omega_1} \in K^1$.

We show that $\mathbf{p}'_{\omega_1} \in K^1$. We let $Y(\mathbb{P}'_{\omega_1}, \underline{D}'_{\omega_1})$ be the diagonal intersection of the $Y(\mathbb{P}'_i, \underline{D}'_i)$ intersected with the set of i such that for any $j < i, \xi_j < i$. Since $\mathcal{D}_{\bar{N}}$ is a normal filter, $Y(\mathbb{P}'_{\omega_1}, \underline{D}'_{\omega_1}) \in \mathcal{D}_{\bar{N}}$. We show that this set witnesses Def. 2.11(1)(c). To this end, we prove the following claim.

Claim: Suppose that $i \in Y(\mathbb{P}'_{\omega_1}, \underline{D}'_{\omega_1})$. The forcing $\mathbb{P}'_i \cap \xi_i$ forces the following: If $i_1 < i, i_1 \in Y(\mathbb{P}'_i, \underline{D}'_i)$, then $\mathbb{R}'_{i_1} \subseteq_{ic} \mathbb{R}'_i$ and if $D_0 \in N_{i_1}$ is a predense subset of $\mathbb{P}'_{i_1} \cap \xi_{i_1} * \mathbb{R}'_{i_1}$ then D_0 is predense in $\mathbb{P}'_i \cap \xi_i * \mathbb{R}'_i$.

We prove this claim: $\mathbb{P}'_i \cap \xi_i \Vdash \mathbb{R}'_{i_1} \subseteq_{ic} \mathbb{R}'_i$ follows from the definition of the orders \mathbb{R}'_j .

Assume that $D_0 \in N_{i_1}$ is an open dense subset of $\mathbb{P}'_{i_1} \cap \xi_{i_1} * \mathbb{R}'_{i_1}$, and $p = (p \upharpoonright \xi_{i_1}, p(\xi_{i_1})) \in (\mathbb{P}'_i \cap i * \mathbb{R}'_i)$. We have to find a condition in $q \in D_0$ that is compatible with p . Assume that $p \cap \xi_{i_1} \Vdash_{\mathbb{P}'_{\xi_{i_1}}} p(i_1) = (u, g)$ and u, g are pinned down in \mathbf{V} , not names. After possibly strengthening p and g we can assume that g is so strong that it fulfils:

$$\text{dom}(g) \supseteq \{m_{p(\beta) \upharpoonright k} : \beta \in \text{supp}(p), \beta \text{ successor ordinal},$$

$$\beta \in u, k \leq |p(\beta)| \wedge \mathbb{P}'_\beta = \mathbb{P}'_{\beta-1} * (\omega^{>\omega}, \triangleleft)\};$$

$$\text{range}(g) \supseteq \{(f_\beta^1)(m_{p(\beta)}) : \beta \in \text{supp}(p), \beta \text{ successor ordinal}, \beta \in u,$$

$$k \leq |p(\beta)| \wedge \mathbb{P}'_\beta = \mathbb{P}'_{\beta-1} * (\omega^{>\omega}, \triangleleft)\}.$$

After possibly further strengthening p we can assume that $p \upharpoonright \xi_{i_1}$ determines ζ_β^j for $j = 1, 2$ and determines f_β^2 restricted to the set on the

right-hand side of the first equation, and determines f_β^1 on the right-hand side of the second equation for any $\beta \in u$. We assume the analogous strength of p' for all triples $(p', (u', g'))$ appearing later in the proof. We assume that $\text{dom}(g) \in \omega$ and that $\text{dom}(g)$ is larger than any $W_\varepsilon^2 \cap W_\zeta^2$ for $\varepsilon \neq \zeta \in u$ and that $\text{range}(g)$ is a superset of $W_\varepsilon^1 \cap W_\zeta^1$ for $\varepsilon \neq \zeta \in u$.

Now we choose $p_0 = (p \upharpoonright \xi_{i_1}, (u \cap \xi_{i_1}, g)) \in M_{\xi_{i_1}}$. We choose $q_0 = (q_0 \upharpoonright \xi_{i_1}, (u_{q_0}, g_{q_0})) \geq p_0$, $q_0 \in D \cap \xi_{i_1} \cap M_{\xi_{i_1}}$. Then q_0 does not determine more of the Cohen real ν_ε for $\varepsilon \in u_{q_0}$ than p_0 does. Then we take $q_1 \geq q_0$ such that

$$\begin{aligned} q_1 &= (q_0 \upharpoonright \xi_{i_1} \cup \{(\varepsilon, q_1(\varepsilon)) : \varepsilon \in u_{q_0} \setminus \xi_{i_1}\}, (u_{q_0}, g_{q_0})) \\ &\text{where for each } \varepsilon \in u \setminus \xi_{i_1}, \\ q_1(\varepsilon) &\Vdash W(0 \frown \nu_\varepsilon) \cap (\text{dom}(g_{q_0}) \setminus \text{dom}(g)) = \emptyset \wedge \\ &W(1 \frown \nu_\varepsilon) \cap (\text{range}(g_{q_0}) \setminus \text{range}(g)) = \emptyset. \end{aligned}$$

This special point (not in [She98, Ch. IV],[She06]) is that the ν_i , i successor of a countable limit ordinal, η_δ , $\delta \in S$, are just Cohen reals: Defining relevant generic objects that have a Cohen real as domain allows us to carry on the oracle-c.c. and thus to preserve the Cohenness of the η_δ . This main trick is also used in the next section. Now q_1 is compatible with p .

Thus $Y(\mathbb{P}'_{\omega_1}, \mathcal{D}'_{\omega_1}) \in \mathcal{D}_{\bar{N}}$ is a witness for the oracle-c.c. of \mathbb{P}'_{ω_1} , as required in Def. 2.11(1)(c)(β). The other properties in Def. 2.11(1)(c) follow now for $i \in Y(\mathbb{P}'_{\omega_1}, \mathcal{D}'_{\omega_1})$ by the inductive definition of the \mathbb{P}'_i .

This finishes the construction of a stronger member in AP -forcing.

\oplus_5 Let

$$\begin{aligned} \underline{g} &= \bigcup \{g : \exists p \exists u (p, (u, g)) \in \mathbf{G}'_{\omega_1}\} \\ \underline{U} &= \bigcup \{u : \exists p \exists g (p, (u, g)) \in \mathbf{G}'_{\omega_1}\} \end{aligned}$$

We show:

$$((\mathbb{P}'_{\omega_1}, \mathcal{D}'_{\omega_1}), p_*) \Vdash_{AP^* \mathbb{Q}} |\underline{U}| = \aleph_1 \wedge \text{“} \underline{g} \notin \bigcup \{G_i : i < \omega_1\}\text{”}.$$

Proof: We fix a generic filter \mathbf{G}'_{ω_1} . By the construction of \mathbb{P}'_{ω_1} we have

$$(\forall i < j \in S \cap C) (f_i^\ell \in M_j \wedge f_i^\ell \text{ is a } \mathbb{P}'_{\omega_1} \text{ } j\text{-name}).$$

The forcing \mathbb{P}'_{ω_1} adds a $g: \bigcup_{\varepsilon \in U} w_\varepsilon^2 \rightarrow \bigcup_{\varepsilon \in U} w_\varepsilon^1$ that conjugates for $i \in U$, $f_i^1 \in G_{\zeta_i^1}$ and $f_i^2 \in G_{\zeta_i^2} \setminus G_{\zeta_i^1}$. If $i \in U$ then $\text{dom}(f_i^\ell) = w_i^\ell = W_{\langle \ell \rangle \frown \nu_i}$ and g conjugates f_i^1 and f_i^2 up to a finite mistake, by \oplus_3 item (i)(δ)(iii). So, for each $i \in U$, $g \circ f_i^1 \circ g^{-1} = f_i^2$ up to finitely many arguments. But g is in some subgroup G_j . So for $\zeta_i^1 > i > j$, $i \in X$, $f_i^2 \in G_{\zeta_i^1}$, contradiction. \square

End of proof of Theorem 2.1:

We assume that $S \subseteq \omega_1$ is stationary and $\mathbf{V} \models \diamond_S^-$. We extend \mathbf{V} with the forcing poset $AP * \mathbb{Q}$. By Lemma 2.21, $\mathfrak{mcf} = \aleph_1$ in the extension, and by Lemma 2.26, $\text{cf}(\text{Sym}(\omega)) = \aleph_2$.

3. ON $\text{Con}(\mathfrak{b} = \text{cf}(\text{Sym}(\omega)) < \mathfrak{mcf})$

Now we show that $\aleph_1 = \mathfrak{b} = \text{cf}(\text{Sym}(\omega)) < \aleph_2 = \mathfrak{mcf}$ is consistent relative to ZFC. In [MST06] we established that it is consistent relative to ZFC that $\aleph_1 = \mathfrak{b} = \mathfrak{g} < \aleph_2 = \mathfrak{mcf}$. Brendle and Losada showed that $\mathfrak{g} \leq \text{cf}(\text{Sym}(\omega))$ in ZFC, see [BL03]. So the following theorem gives another consistency proof for $\aleph_1 = \mathfrak{b} = \mathfrak{g} < \aleph_2 = \mathfrak{mcf}$.

Theorem 3.1. *It is consistent relative to ZFC that $\mathfrak{b} = \text{cf}(\text{Sym}(\omega)) < \aleph_2 = \mathfrak{mcf}$.*

For the proof we will again work with oracle c.c.-forcing. Let $D \subseteq [\omega]^\omega$ be a filter over ω . Then we write D^+ for the D -positive sets, i.e., $X \in D^+$ iff $X \cap Y$ is infinite for any $Y \in D$.

Lemma 3.2. *Let $\kappa \geq \aleph_2$ be a cardinal in \mathbf{V} . The $(A)_\kappa$ implies $(B)_\kappa$.*

$(A)_\kappa$ *For every filter $D \subseteq [\omega]^\omega$ over ω such that $\mathcal{P}(\omega)/D$ has the c.c.c. (that is: for every $A_i, i < \omega_1$, such that $A_i \in D^+$ there are $i \neq j$ such that $A_i \cap A_j \in D^+$) for every regular $\kappa_* < \kappa$, for every sequence $\langle f_i : i < \kappa_* \rangle$ of functions $f_i \in {}^\omega\omega$ there is $g \in {}^\omega\omega$ such that for unboundedly many $i < \kappa_*$, $\neg g \leq_D f_i$.*

$(B)_\kappa$ *After forcing with a ccc ${}^\omega\omega$ -bounding forcing \mathbb{Q} , in the extension $\mathbf{V}^\mathbb{Q}$ for every non-principal ultrafilter D on ω , $\text{cf}({}^\omega\omega/D) \geq \kappa$, and $\mathfrak{b}^\mathbf{V} = \mathfrak{b}^{\mathbf{V}^\mathbb{Q}}$.*

Proof. Assume $(A)_\kappa$ and that $q_0 \in \mathbb{Q}$ forces “ \underline{D} is an ultrafilter over ω and $\langle \underline{f}_\alpha : \alpha < \kappa_* \rangle$ is increasing modulo \underline{D} and $\kappa_* < \kappa$ ”. So κ_* is regular and uncountable in $\mathbf{V}^\mathbb{Q}$ and hence regular and uncountable in \mathbf{V} . We shall show that there is $q_* \geq q_0$,

$$(\square) \quad q_* \Vdash \exists f \in ({}^\omega\omega) \bigwedge_{\alpha < \kappa_*} \underline{f}_\alpha <_{\underline{D}} f,$$

and thus we will have established $(B)_\kappa$.

Since \mathbb{Q} is ${}^\omega\omega$ -bounding and ccc, we can take $g_\alpha \in \mathbf{V}$ for $\alpha \in \kappa_*$ such that $q_0 \Vdash_{\mathbb{Q}} “\underline{f}_\alpha \leq^* g_\alpha”$.

We let

$$E = \{A \in \mathcal{P}(\omega)^\mathbf{V} : (\exists q \in \mathbb{Q})(q \geq q_0 \wedge q \Vdash \check{A} \in \underline{D})\}$$

and we let

$$D' = \{A \in \mathcal{P}(\omega)^\mathbf{V} : q_0 \Vdash \check{A} \in \underline{D}\}.$$

Then we have $E, D' \in \mathbf{V}$ and the following holds:

(1) D' is a filter over ω .

- (2) $E \subseteq (D')^+$. Let $A \in E$, say $q \Vdash A \in \underline{D}$, $q \geq q_0$ and let $B \in D'$. Then $q \Vdash A \in \underline{D} \wedge B \in \underline{D}$, so $q \Vdash 'A \cap B \text{ is infinite}.'$ Since $A, B \in \mathbf{V}$, $A \cap B$ is infinite. Since this holds for every $B \in D'$, item (2) is proved.
- (3) $(D')^+ \subseteq E$. Suppose that $X \notin E$. Then $\forall q \in \mathbb{Q}$, $q \geq q_0$ implies that $q \not\Vdash X \in \underline{D}$, so $q_0 \Vdash X \notin \underline{D}$. Since \underline{D} is a name of an ultrafilter $q_0 \Vdash X^c \in \underline{D}$. So $X^c \in D'$ and $X \notin (D')^+$.
- (4) So together: $(D')^+ = E$.
- (5) q_0 forces that D' is a c.c.c. filter. Proof: Let $q_0 \Vdash_{\mathbb{Q}} A_\alpha \in (D')^+ = E$ for $\alpha \in \omega_1$, via $q_\alpha \geq q_0$. Since \mathbb{Q} is c.c.c there are $\alpha \neq \beta$ such that $q_\alpha \not\leq q_\beta$. Then there is $r \in \mathbb{Q}$, $r \Vdash A_\alpha \in \underline{D}, A_\beta \in \underline{D}$, and hence $r \Vdash A_\alpha \cap A_\beta \in \underline{D}$ since \underline{D} is forced to be a filter. So $A_\alpha \cap A_\beta \in D'^+$.

Let g be as in the condition $(A)_\kappa$, applied to D' and $\langle g_\alpha : \alpha < \kappa \rangle$, so for some cofinal set $u \subseteq \kappa_*$ we have for $\alpha \in u \subseteq \kappa_*$, $\neg g \leq_{D'} g_\alpha$. Hence for $\alpha \in u$, $q_0 \not\Vdash \{n : g(n) \leq g_\alpha(n)\} \in \underline{D}$ and there is $\tilde{q}_\alpha \geq q_0$, $\tilde{q}_\alpha \Vdash \{n : g(n) \leq g_\alpha(n)\} \notin \underline{D}$. Thus $\tilde{q}_\alpha \Vdash \{n : g(n) > g_\alpha(n)\} \in \underline{D}$ and the choice of g_α implies $\tilde{q}_\alpha \Vdash \{n : g(n) > \tilde{f}_\alpha(n)\} \in \underline{D}$. Since \mathbb{Q} has the c.c.c., we have $\text{cf}(\kappa_*) > \omega$. Therefore κ_* -many of the \tilde{q}_α are in the generic filter. So for any \mathbb{Q} -generic filter G with $q_0 \in G$ we have $\tilde{f}_\alpha[G] \leq_{D[G]} g$ for cofinally many $\alpha \in u$. Hence a condition $q_* \geq q_0$ forces this. Since the sequence $\langle \tilde{f}_\alpha : \alpha < \kappa_* \rangle$ is $\leq_{\underline{D}}$ -increasing, we get $q_* \Vdash '(\forall \alpha < \kappa_*)(\tilde{f}_\alpha \leq_{\underline{D}} g).'$ Thus Equation (\square) and the first statement of $(B)_\kappa$ are proved.

Since the forcing \mathbb{Q} is ${}^\omega\omega$ -bounding, we have $\mathfrak{b}^{\mathbf{V}} = \mathfrak{b}^{\mathbf{V}^{\mathbb{Q}}}$. \square

An example for such a \mathbb{Q} is the forcing adding \aleph_1 random reals, in a countable support iteration or with the measure algebra over 2^{ω_1} . From now on, we let \mathbb{Q} be one of these forcing for adding \aleph_1 random reals. In the extension $\mathbf{V}^{\mathbb{Q}}$ of Lemma 3.2 we have $\text{cf}(\text{Sym}(\omega)) = \aleph_1$ by [ST95, Theorem 1.6]. So if we succeed to establish the condition $(A)_\kappa$ of the lemma together with $\mathfrak{b} = \aleph_1$ for some $\kappa \geq \aleph_2$, Theorem 3.1 will be proved. We fix a stationary $S \subseteq \omega_1$ and take $\kappa = \aleph_2$ and we work again with oracle-c.c. forcings in order to establish the consistency of $(A)_{\aleph_2}$ and $\mathfrak{b} = \aleph_1$.

Lemma 3.3. *We assume that in \mathbf{V} , the set S is stationary in ω_1 and the two diamond principles \diamond_S and $\diamond_{\{\delta < \aleph_2 : \text{cf}(\delta) = \aleph_1\}}$ hold. Then there is an oracle c.c. forcing notion \mathbb{P} such that in $\mathbf{V}^{\mathbb{P}}$ we have $(A)_{\aleph_2}$ of the previous lemma, and $\mathfrak{b} = \omega_1$.*

Proof. We fix in \mathbf{V} a \leq^* -increasing sequence $\langle g_\delta : \delta < \omega_1 \rangle$ that is \leq^* -unbounded. We fix an oracle $\bar{M} = \langle M_\varepsilon : \varepsilon \in S \rangle$ such that the \bar{M} -c.c. ensures that the type $\bigwedge_{\delta < \omega_1} x \geq^* g_\delta$ is omitted. Indeed, $\langle g_\delta : \delta \in \omega_1 \rangle \in M'_0 \prec H(\chi)$ and M_0 being the Mostowski collapse of M'_0 suffices for this. In addition we fix a $\diamond_{\{\alpha < \aleph_2 : \text{cf}(\alpha) = \aleph_1\}}$ -sequence $\langle T_\alpha : \alpha \in \omega_2, \text{cf}(\alpha) = \aleph_1 \rangle \in M'_0$.

In the following α, α' will range over ω_2 , $i, j, \varepsilon, \zeta, \xi$ over ω_1 , and the letters β, γ, δ will denote particular functions with values in $\omega_2, \omega_1, \omega_1$. We fix

a bijection $b: 2^{<\omega} \rightarrow \omega$, a bijection $c: 2^\omega \cap \mathbf{V} \rightarrow \omega_1$ and another bijection $b_2: \aleph_2 \rightarrow (\mathcal{P}(H(\omega_1)))^2$. By \diamond_S and $\diamond_{\{\alpha < \aleph_2 : \text{cf}(\alpha) = \aleph_1\}}$ such bijections exist.

A finite support iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \beta < \omega_2, \alpha \leq \omega_2 \rangle$ is constructed by induction on $\alpha \leq \omega_2$ with the following properties:

- (1) $|\mathbb{P}_\alpha| \leq \aleph_1$ for $\alpha < \omega_2$
- (2) \mathbb{P}_α has the \bar{M} -c.c.

For an *odd stage* $\alpha \in \omega_2$ we force via $\mathbb{Q}_\alpha = \mathbb{C}$, and we conceive Cohen forcing \mathbb{C} in the form

$$\{p : p \text{ is a partial function from } 2^{<\omega} \text{ to } 2, |p| < \omega\}$$

and fix for $\eta \in 2^\omega \cap \mathbf{V}$ sets $A_{\alpha, \eta} = \{b((p(\eta \upharpoonright 0), \dots, p(\eta \upharpoonright n - 1))) : n \in \omega, p \in G\} \subseteq \omega$ in the extension by \mathbb{C} , where b is the bijection from above. Note that for $\eta \neq \eta'$, $A_{\alpha, \eta} \cap A_{\alpha, \eta'}$ is finite. We write $A'_{\alpha, \varepsilon} = A_{\alpha, c^{-1}(\varepsilon)}$. Then $|\mathbb{P}_{\alpha+1}| \leq \aleph_1$.

For *even* $\alpha < \omega_2$ we define \mathbb{Q}_α as follows: If $\text{cf}(\alpha) < \omega_1$, we let \mathbb{Q}_α be the trivial forcing, i.e. $\mathbb{Q}_\alpha = \{0\}$. Now let $\alpha > 0$. We assume that $\mathbb{P}_\alpha \subseteq \omega_1$. Then every canonical \mathbb{P}_α -name $(\underline{D}, \langle f_i : i < \omega_1 \rangle)$ for a subset of $\mathcal{P}(\omega)$ and an ω_1 -sequence of reals is a subset of $H(\omega_1)$. We say that $T \subseteq \alpha$ codes the canonical name $(\underline{D}, \langle f_i : i < \omega_1 \rangle)$ if $b_2[T] = (\underline{D}, \langle f_i : i < \omega_1 \rangle)$.

If $\text{cf}(\alpha) = \omega_1$ and \underline{T}_α is a canonical \mathbb{P}_α -name of a pair $(\underline{D}, \langle f_{\alpha, i} : i < \omega_1 \rangle)$ such

$$\mathbb{P}_\alpha \Vdash \text{“}\underline{D} \text{ contains the cofinite sets and } \mathcal{P}(\omega)/\underline{D} \text{ is c.c.c.} \text{”}$$

then we first fix in the ground model an increasing sequence $\langle \beta(\alpha, i) : i < \omega_1 \rangle$ that converges to α such that each $\beta(\alpha, i)$ is an odd member of ω_2 .

Next we define by induction on $i < \omega$ countable ordinals as follows:

$$(3.1) \quad \begin{aligned} \gamma(\alpha, 0) &= \min\{\varepsilon < \omega_1 : f_{\alpha, 0} \in \mathbf{V}^{\mathbb{P}^{\beta(\alpha, \varepsilon)}}\} \\ \gamma(\alpha, i) &= \min\{\varepsilon < \omega_1 : f_{\alpha, i} \in \mathbf{V}^{\mathbb{P}^{\beta(\alpha, \varepsilon)}} \wedge (\forall j < i)(\varepsilon > \gamma(\alpha, j))\} \end{aligned}$$

Later it will be important that the $\gamma(\alpha, i)$, $i < \omega_1$, are pairwise different.

Then for each $i < \omega_1$ we choose with the maximum principle a name $\delta(\alpha, i) \in \omega_1$ such that

$$(3.2) \quad \mathbb{P}_\alpha \Vdash (\omega \setminus A_{\beta(\alpha, \gamma(\alpha, i)), \delta(\alpha, i)}) \in \underline{D}.$$

We do not write the tildes under the names of the δ . For the existence of such $\delta(\alpha, i)$ we use the following claim.

Claim: For any $i < \omega_1$ there are coboundedly many ε such that

$$\mathbb{P}_\alpha \Vdash (\omega \setminus A_{\beta(\alpha, \gamma(\alpha, i)), \varepsilon}) \in \underline{D}.$$

Proof: Assume for a contradiction that $i < \omega_1$ is a counterexample to the claim. Then there are unboundedly many $\varepsilon \in \omega_1$ such that there is $p_\varepsilon \in \mathbb{P}_\alpha$ such that $p_\varepsilon \Vdash A_{\beta(\alpha, \gamma(\alpha, i)), \varepsilon} \in \underline{D}^+$. Since \mathbb{P}_α has the c.c.c. there is a \mathbb{P}_α -generic G that contains \aleph_1 many p_ε as above. Call this uncountable set of ε 's X . However for $\varepsilon \neq \varepsilon' \in X$, $\mathbb{P}_\alpha \Vdash A_{\beta(\alpha, \gamma(\alpha, i)), \varepsilon} \cap A_{\beta(\alpha, \gamma(\alpha, i)), \varepsilon'}$ is finite.

This contradicts the fact that $\mathbb{P}_\alpha \Vdash \mathcal{P}(\omega)/\bar{D}$ is c.c.c., and thus the claim is proved.

We use only one $\delta(\alpha, i)$ and its value in ω_1 is not important. However, for the $\gamma(\alpha, i)$, the pairwise inequality $\beta(\alpha, \gamma(\alpha, i)) \neq \beta(\alpha, \gamma(\alpha, j))$ for $i \neq j$ is important, so that there are no conflicts between the various instances of condition (6) below.

Once the sequence $\langle \gamma(\alpha, i), \delta(\alpha, i) : i < \omega_1 \rangle$ is chosen, we define in $\mathbf{V}^{\mathbb{P}_\alpha}$ the forcing \mathbb{Q}_α as follows: $p \in \mathbb{Q}_\alpha$ iff

- (1) $p = (u_p, h_p)$,
- (2) $u_p \subseteq \omega_1$ is finite,
- (3) $h_p \in {}^\omega \omega$.

$\mathbb{Q}_\alpha \Vdash p \leq q$ if

- (4) $u_p \subseteq u_q$ and
- (5) $h_p \trianglelefteq h_q$ and
- (6) if $\xi \in u_p$ and

$$m \in (\omega \setminus A_{\beta(\alpha, \gamma(\alpha, \xi)), \delta(\alpha, \xi)}) \cap (\text{dom}(h_q) \setminus \text{dom}(h_p))$$

then $f_{\alpha, \xi}(m) < h_q(m)$.

We show by induction on $\alpha \leq \omega_2$ that \mathbb{P}_α has the \bar{M} -c.c. and $|\mathbb{P}_\alpha| \leq \aleph_1$ for $\alpha < \omega_1$. Since we take direct limits, the limit steps are covered by [She98, Ch. IV, 3.2]. The start of the induction is trivial. Now we look at the successor steps $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{Q}_\alpha$.

Odd α : \mathbb{Q}_α is the Cohen forcing. Any countable forcing has the $\bar{M}[\mathbb{P}_\alpha]$ -c.c. Putting this together with the induction hypothesis, $\mathbb{P}_{\alpha+1}$ has the \bar{M} -c.c.

Even α : Since \mathbb{P}_α has the c.c.c., there is a set of representatives of \mathbb{P}_α -names of members of \mathbb{Q}_α of size at most \aleph_1 . Hence we can assume that $|\mathbb{P}_{\alpha+1}| \leq \aleph_1$. To simplify notation, we assume that $\mathbb{P}_\alpha \subseteq \omega_1$ and we assume $\mathbb{P}_\alpha \Vdash \mathbb{Q}_\alpha \cap \varepsilon = \{(u, p) \in \mathbb{Q}_\alpha : u \subseteq \varepsilon\}$. We fix a witness $Y(\mathbb{P}_\alpha) \in \mathcal{D}_{\bar{M}}$ for the \bar{M} -c.c. of \mathbb{P}_α , i.e., for every $\varepsilon \in Y(\mathbb{P}_\alpha)$ for every $I \in M_\varepsilon$ that is a dense subset of $\mathbb{P}_\alpha \cap \varepsilon$, I is dense in \mathbb{P}_α .

We intersect $Y(\mathbb{P}_\alpha)$ with the club $C \subseteq \omega_1$ of countable limit ordinals that are closed under the functions $\gamma(\alpha, \cdot)$ and $\delta(\alpha, \cdot)$ that are defined as in equations (3.1), (3.2). Since \mathbb{P}_α is c.c.c. such a club can be found in the ground model although $\delta(\alpha, \cdot)$ is a name.

Next we prove that $Y(\mathbb{P}_\alpha) \cap C$ witnesses that $\mathbb{P}_{\alpha+1}$ has the \bar{M} -c.c. Let $\varepsilon \in Y(\mathbb{P}_\alpha) \cap C$, $D \in M_\varepsilon$ be an open and dense subset of $(\mathbb{P}_\alpha \cap \varepsilon) * (\mathbb{Q} \cap \varepsilon)$. Let $p \in \mathbb{P}_{\alpha+1}$. We have to show that there is $q \in D$ that is compatible with p .

We write $p = (p \upharpoonright \alpha, (u_{p(\alpha)}, h_{p(\alpha)}))$ and we assume that $p \upharpoonright \alpha$ determines the finite sets $u_{p(\alpha)}$ and $h_{p(\alpha)}$ so that they to elements of $[\omega_1]^{<\omega}$ and ${}^\omega \omega$ and that it also determines $\gamma(\alpha, \xi)$ and $\delta(\alpha, \xi)$ for any $\xi \in u_{p(\alpha)}$.

The search for q proceeds in four steps:

First step: We apply the induction hypothesis. We let $D' = D \cap \mathbb{P}_\alpha$. $D' \in M_\varepsilon$ is dense and open in $\mathbb{P}_\alpha \cap \varepsilon$. Since \mathbb{P}_α has the \bar{M} -c.c. and $\varepsilon \in Y(\mathbb{P}_\alpha)$ there is $q' \in D' \cap M_\varepsilon$ that is compatible with $p \upharpoonright \alpha$. We fix a witness $r' \in \mathbb{P}_\alpha$ for compatibility.

Second step: We choose $(h', u_{p(\alpha)}) \geq p(\alpha)$ to take a record of r' on its finitely many Cohen coordinates by taking $n \in \omega$ so large such that

$$(3.3) \quad \begin{aligned} & (\forall m)(\forall \xi \in u_{p(\alpha)})(\forall \beta = \beta(\alpha, \gamma(\alpha, \xi)) \in \text{supp}(r')) \\ & ((r' \Vdash (m \notin A_{\beta(\alpha, \gamma(\alpha, \xi)), \delta(\alpha, \xi)})) \rightarrow m < n). \end{aligned}$$

Such an n exists since r' pins down only a finite part of the name $A_{\beta(\alpha, \gamma(\alpha, \xi)), \delta(\alpha, \xi)}$ for any $\xi \in u_{p(\alpha)}$ with $\beta(\alpha, \gamma(\alpha, \xi)) \in \text{dom}(r')$. Now we let $\text{dom}(h') = n$ and on $n \setminus \text{dom}(h_{p(\alpha)})$ we fix some $h'(k) \geq f_{\alpha, \xi}(k)$ for all $\xi \in u_{p(\alpha)}$. We let $q' = (h', u_{p(\alpha)})$.

Third step: We go again into $D \cap M_\varepsilon$. With the maximum principle we choose $q(\alpha) \in M_\varepsilon$ such that $q' \Vdash q(\alpha) \geq_{\mathbb{Q}_\alpha} (u_{p(\alpha)} \cap \varepsilon, h') \wedge q(\alpha) \in D_\alpha[\mathbb{P}_\alpha]$ and let $q = (q', q(\alpha))$. Then $q = (q', q(\alpha)) \in M_\varepsilon \cap D$.

Fourth step: We show that p and q are compatible. For any $\xi \in u_{p(\alpha)} \setminus \varepsilon$ we choose $q_1(\beta(\alpha, \gamma(\alpha, \xi))) \geq q'(\beta(\alpha, \gamma(\alpha, \xi)))$ such that

$$(3.4) \quad \begin{aligned} & q_1(\beta(\alpha, \gamma(\alpha, \xi))) \Vdash_{\mathbb{Q}_{\beta(\alpha, \gamma(\alpha, \xi))}} (\forall n \in \text{dom}(h_{q(\alpha)} \setminus \text{dom}(h')) \\ & (n \in A_{\beta(\alpha, \gamma(\alpha, \xi)), \delta(\alpha, \xi)})). \end{aligned}$$

We let

$$r = \left(q' \cup \{(\beta(\alpha, \gamma(\alpha, \xi)), q_1(\beta(\alpha, \gamma(\alpha, \xi)))) : \xi \in u_{p(\alpha)} \setminus \varepsilon\}, \right. \\ \left. (u_{p(\alpha)} \cup u_{q(\alpha)}, h_{q(\alpha)}) \right).$$

The condition r is well defined, since for any $\xi \in u_{p(\alpha)} \setminus \varepsilon$, the condition $q_1(\beta(\alpha, \gamma(\alpha, \xi))) \in \mathbb{P}_\alpha$ can be chosen to be compatible with $q'(\beta(\alpha, \gamma(\alpha, \xi)))$, by the choice of n as in Equation (3.3).

We show that $r \geq p, q$. First $r \upharpoonright \alpha \geq p \upharpoonright \alpha, q'$ and $q' = q \upharpoonright \alpha$. We show

$$r \upharpoonright \alpha \Vdash_{\mathbb{P}_\alpha} (u_{p(\alpha)} \cup u_{q(\alpha)}, h_{q(\alpha)}) \geq_{\mathbb{Q}_\alpha} (u_{q(\alpha)}, h_{q(\alpha)}), (u_{p(\alpha)}, h').$$

The first is trivial. For the latter, let $\xi \in u_{p(\alpha)}$. First case: $\xi \in M_\delta$. We chose (after Equation (3.3)) the function $h_{q(\alpha)}(k)$ such that it dominates $f_{\alpha, \xi}(k)$ on any coordinate k not in $\text{dom}(h_{p(\alpha)})$ such that $r' \Vdash k \notin A_{\beta(\alpha, \gamma(\alpha, \xi)), \delta(\alpha, \xi)}$. Thus $r \upharpoonright \alpha$ forces the relevant instances of clause (6) of $r(\alpha) \geq p(\alpha)$.

Second case: $\xi \in u_{p(\alpha)} \setminus \varepsilon$. Since clause (6) speaks only about $m \in \omega \setminus A_{\beta(\alpha, \gamma(\alpha, \xi)), \delta(\alpha, \xi)}$, Equation (3.4) implies $r \upharpoonright \alpha \Vdash_{\mathbb{P}_\alpha} r(\alpha) \geq q(\alpha)$. \square

Remark: We work with the assumption $\diamond_{\{\delta < \aleph_2 : \text{cf}(\delta) = \aleph_1\}}$. Alternatively, we could force as in the previous section by approximations of size \aleph_1 in a first step and thereafter force with the generic filter of the first forcing. The diamond $\diamond_{\{\delta < \aleph_2 : \text{cf}(\delta) = \aleph_1\}}$ hands down at stage α a possible \mathbb{P}_α -name for objects $D, \langle g_i : i < \aleph_1 \rangle$ as in property (A) $_{\aleph_2}$ of Lemma 3.2 and thus

allows to construct a finite support iteration up to stage ω_2 instead of using an approximation forcing in a first forcing step. So the partial order \mathbb{P} of the sketched alternative construction corresponds in the actually performed forcing $AP * \mathbb{Q}$ to the generic \mathbb{Q} of the approximation forcing AP .

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