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Specialising Aronszajn trees by countable approximations

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Abstract. We show that there are proper forcings based upon countable trees of creatures that specialise a given Aronszajn tree.

0. Introduction

The main point of this work is finding forcing notions specialising an Aronszajn tree, which are creature forcings, tree-like with halving, but being based on ω_1 (the tree) rather than ω. Techniques to specialise a given Aronszajn tree are often useful for building models of the Souslin hypothesis SH, i.e., models in which there is no Souslin tree. The present work grew from attempts at showing the consistency of SH together with ♣ (see [6, I.7.1]), a question by Juhász. This stays open.

Creature forcing tries to enlarge and systemise the family of very nice forcings. There is "the book on creature forcing" [4], and for uncountable forcings the work is extended in [3] and [5] and Sh:F514. At first glance it cannot be applied for specialising an Aronszajn tree, because we have to add a subset of ω_1 rather than a subset of ω. Here we adopt it to ω_1. We dispense with some of the main premises made in the previous work and show new technical details. The work may also be relevant to cardinal characteristics of ${}^{\omega_1}2$, but this is left for future work.

The norm of creatures (see Definition 1.7) we shall use is natural for specialising Aronszajn trees. It is convenient if there is some $\alpha < \omega_1$ such that the union of the domains of the partial specialisation functions that are attached to any branch of the tree-like forcing condition is the initial segment of the Aronszajn tree $\mathbf{T}_{<\alpha}$, i.e., the union of the levels less than α. However, allowing for every branch of a given condition finitely many possibilities $\mathbf{T}_{<\alpha_i}$ with finite sets u_i sticking out of

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$\mathbf{T}_{<\alpha_i}$ is used for density arguments that show that the generic filter leads to a total specialisation function.

1. Tree creatures

In this section we define the tree creatures which will be used later to describe the branching of the countable trees that will serve as forcing conditions. We prove three important technical properties about gluing together (Claim 1.9), about filling up (Claim 1.10) and about changing the base together with thinning out (Claim 1.11) of creatures. We shall define the forcing conditions only in the next section. They will be countable trees with finite branching, such that each node and its immediate successors in the tree are described by a creature in the sense of Definition 1.5. Roughly spoken, in our context, a creature will be an arrangement of partial specialisation functions with some side conditions.

We reserve the symbol $(T, \lhd_T)$ for the trees in the forcing conditions, which are trees of partial specialisation functions of some given Aronszajn tree $(\mathbf{T}, <_{\mathbf{T}})$. A specialisation function is a function $f : \mathbf{T} \to \omega$ such that for all $s, t \in \mathbf{T}$, if $s <_{\mathbf{T}} t$, then $f(s) \neq f(t)$, see [2, p. 244].

χ stands for some sufficiently high regular cardinal, and $\mathcal{H}(\chi)$ denotes the set of all sets of hereditary cardinality less than χ. For our purpose, $\chi = (2^\omega)^+$ is enough.

Throughout this work we make the following assumption:

Hypothesis 1.1. $\mathbf{T}$ *is an Aronszajn tree ordered by* $<_{\mathbf{T}}$*, and for* $\alpha < \omega_1$ *the level* α *of* $\mathbf{T}$ *satisfies:*

$$\mathbf{T}_\alpha \subseteq [\omega\alpha, \omega\alpha + \omega).$$

Throughout this work, $\mathbf{T}$ will be fixed. We define the following finite approximations of specialisation maps:

Definition 1.2. *For* $u \subseteq \mathbf{T}$ *and* $n < \omega$ *we let*

$$\mathrm{spec}_n(u) = \{\eta \mid \eta : u \to [0, n) \ \wedge \ (\eta(x) = \eta(y) \to \neg(x <_{\mathbf{T}} y))\}.$$

We let $\mathrm{spec}(u) = \bigcup_{n<\omega} \mathrm{spec}_n(u)$ *and* $\mathrm{spec} = \mathrm{spec}^{\mathbf{T}} = \bigcup\{\mathrm{spec}(u) : u \subset \mathbf{T}, u \text{ finite}\}$.

Choice 1.3 *We choose three sequences of natural numbers* $\langle n_{k,i} : i < \omega\rangle$*,* $k = 1, 2, 3$*, such that the following growth conditions are fulfilled:*

$$(i+1) \cdot n_{1,i} < n_{3,i}, \tag{1.1}$$

$$n_{2,i} < n_{1,i+1}, \tag{1.2}$$

$$n_{1,i} \cdot n_{1,i} \leq n_{1,i+1}, \tag{1.3}$$

$$n_{1,i} \leq n_{2,i}. \tag{1.4}$$

We fix them for the rest of this work.

We compare with the book [4] in order to justify the use of the name creature. However, we cannot just cite that work, because the framework developed there is not suitable for the approximation of uncountable domains $\mathbf{T}$.

Definition 1.4. *(1.) [4, 1.1.1] A triple* $t = (\mathrm{nor}[t], \mathrm{val}[t], \mathrm{dis}[t])$ *is a weak creature for* $\mathbf{H}$ *if*

(a) $\mathrm{nor}[t] \in \mathbb{R}^{\geq 0}$,

(b) Let $\mathbf{H} = \bigcup_{i\in\omega} \mathbf{H}(i)$ *and let* $\mathbf{H}(i)$ *be sets. Let* $\lhd$ *be the strict initial segment relation.*
$\mathrm{val}[t]$ *is a non-empty subset of*

$$\left\{ \langle x, y\rangle \in \bigcup_{m_0<m_1<\omega} [\prod_{i<m_0} \mathbf{H}(i) \times \prod_{i<m_1} \mathbf{H}(i)] \,:\, x \lhd y \right\}.$$

(c) $\mathrm{dis}[t] \in \mathcal{H}(\chi)$.

(2.) nor *stands for norm,* val *stands for value, and* dis *stands for distinguish.*

In our case, we drop the component dis (in the case of simple creatures in the sense of Definition 1.5) or it will be called k (in the case of creatures), an additional coordinate, which is a natural number. In order to stress some parts of the weak creatures t more than others, we shall write $\mathrm{val}[t]$ in a slightly different form and call it a simple creature, $\mathbf{c}$.

As we will see in the next definition, in this work (b) of 1.4 is not fulfilled: For us val is a non-empty subset of $\{\langle x, y\rangle \in \mathrm{spec} \times \mathrm{spec} \,:\, x \lhd_T y\}$ for some strict partial order $\lhd_T$ as in Definition 2.1. Though the members of spec are finite partial functions, they cannot be written with some $n \in \omega$ as a domain, since spec is uncountable and we want to allow arbitrary finite parts. Often properness of a tree creature forcing follows from the countability of $\mathbf{H}$. Note that our analogue to $\mathbf{H}$ is not countable. In Section 3 we shall prove that the notions of forcing we introduce are proper for other reasons.

Nevertheless the simple creature in the next definition is a specific case for the value of a weak creature in the sense of 1.4 without item (1.)(b), and the creature from the next definition can be seen as a case of a value and a distinction part of a weak creature.

Definition 1.5. *(1) A simple creature is a tuple* $\mathbf{c} = (i(\mathbf{c}), \eta(\mathbf{c}), \mathrm{rge}(\mathrm{val}(\mathbf{c})))$ *with the following properties:*

(a) The first component, $i(\mathbf{c})$*, is called the kind of* $\mathbf{c}$ *and is just a natural number.*

(b) The second component, $\eta(\mathbf{c})$*, is called the base of* $\mathbf{c}$*. We require (*$\eta(\mathbf{c}) = \emptyset$ *and* $i(\mathbf{c}) = 0$*) or (*$i(\mathbf{c})$ *is the smallest* i *such that* $|\mathrm{dom}(\eta(\mathbf{c}))| < n_{2,i-1}$*), and* $\eta(\mathbf{c}) \in \mathrm{spec}_{n_{3,i-1}}$.

(c) The range of the value of $\mathbf{c}$*,* $\mathrm{rge}(\mathrm{val}(\mathbf{c}))$*, is a non-empty subset of* $\{\eta \in \mathrm{spec}_{n_{3,i}} \,:\, \eta(\mathbf{c}) \subsetneq \eta \;\wedge\; |\mathrm{dom}(\eta)| < n_{2,i}\}$*, such that* $|\mathrm{rge}(\mathrm{val}(\mathbf{c}))| < n_{1,i}$.
So we have $\mathrm{val}(\mathbf{c}) = \{\eta(\mathbf{c})\} \times \mathrm{rge}(\mathrm{val}(\mathbf{c}))$*. That the domain is a singleton, is typical for tree-creating creatures.*

(d) If $\eta_1 \in \mathrm{rge}(\mathrm{val}(\mathbf{c}))$ *and* $x \in \mathrm{dom}(\eta_1) \setminus \mathrm{dom}(\eta(\mathbf{c}))$ *then there is some* $\eta_2 \in \mathrm{rge}(\mathrm{val}(\mathbf{c}))$ *such that* $x \in \mathrm{dom}(\eta_2) \to \eta_1(x) \neq \eta_2(x)$.

(2) A creature $\mathbf{c}^+$ *is a tuple* $(i(\mathbf{c}^+), \eta(\mathbf{c}^+), \mathrm{rge}(\mathrm{val}(\mathbf{c}^+)), k(\mathbf{c}^+))$ *where* $(i(\mathbf{c}^+), \eta(\mathbf{c}^+), \mathrm{rge}(\mathrm{val}(\mathbf{c}^+)))$ *is a simple creature, and* $k(\mathbf{c}^+) \in \omega$ *is an additional coordinate.*

(3) An (simple) i-creature is a (simple) creature with $i(\mathbf{c}^+) = i$ *(*$i(\mathbf{c}) = i$*).*

(4) If $\mathbf{c}^+$ *is a creature we mean by* $\mathbf{c}$ *the simple creature such that* $\mathbf{c}^+ = (\mathbf{c}, k(\mathbf{c}^+))$.

(5) The set of creatures is denoted by K^+*, and the set of simple creatures is denoted by* K.

Remark 1.6. Property 1.5(d) is equivalent to $\eta(\mathbf{c}) = \bigcap\{\eta : \eta \in \mathrm{rge}(\mathrm{val}(\mathbf{c}))\}$, and also $i(\mathbf{c})$ is determined by $\eta(\mathbf{c})$ and hence from $\mathrm{rge}(\mathrm{val}(\mathbf{c}))$. Thus, in our specific case, every simple creature is determined by the range of its value.

For a real number r we let $m = \lceil r \rceil$ be the smallest natural number such that $m \geq r$. So, for negative numbers r, $\lceil r \rceil = 0$. We let lg denote the logarithm function to the base 2. Let $\log_2(x) = \lceil \lg(x) \rceil$ for $x > 0$, and we set $\log_2 0 = 0$.

Definition 1.7. *(1) For a simple i-creature* $\mathbf{c}$ *we define* $\mathrm{nor}^0(\mathbf{c})$ *as the maximal natural number k such that if* $a \subseteq n_{3,i}$ *and* $|a| \leq k$ *and* $B_0, \ldots, B_{k-1}$ *are branches of* $\mathbf{T}$*, then there is* $\eta \in \mathrm{val}(\mathbf{c})$ *such that*

(α) $(\forall x \in (\bigcup_{\ell<k} B_\ell \cap \mathrm{dom}(\eta)) \setminus \mathrm{dom}(\eta(\mathbf{c})))(\eta(x) \notin a)$,

(β) $\dfrac{|\mathrm{dom}(\eta)|}{n_{2,i}} \leq \dfrac{1}{2^k}$.

(2) We let $\mathrm{nor}^*(\mathbf{c}) = \log_2(\frac{n_{1,i(\mathbf{c})}}{|\mathrm{val}(\mathbf{c})|})$*, and* $\mathrm{nor}^{\frac{1}{2}}(\mathbf{c}) = \min(\mathrm{nor}^0(\mathbf{c}), \mathrm{nor}^*(\mathbf{c}))$.

(3) We define $\mathrm{nor}^1(\mathbf{c}) = \log_2(\mathrm{nor}^0(\mathbf{c}))$*, and* $\mathrm{nor}^2(\mathbf{c}) = \log_2(\mathrm{nor}^{\frac{1}{2}}(\mathbf{c}))$.

(4) In order not to fall into specific computations, we use functions f that exhibit the following properties, in order to define norms on (non-simple) creatures:

$(*)_1$ $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$*, where* $\mathbb{R}^+$ *is the set of strictly positive reals.*

$(*)_2$ *f fulfils the following monotonicity properties: If* $n_1 \geq n_2 \geq k_2 \geq k_1$ *then* $f(n_1, k_1) \geq f(n_2, k_2)$.

$(*)_3$ *(For the 2-bigness, see Claim 1.12)* $f(\frac{n}{2}, k) \geq f(n, k) - 1$.

$(*)_4$ $n \leq k \to f(n, k) \leq 0$.

$(*)_5$ *(For the halving property, see Definition 3.3) For all n, k: If* $f(n, k) \geq 1$*, then there is some* $k'(n, k) = k'$ *such that* $k < k' < n$ *and for all* n'*, if* $k' < n' < n$ *and* $f(n', k') \geq 1$*, then*

$$f(n, k') \geq \frac{f(n, k)}{2}, \text{ and}$$

$$f(n', k) = f(n', k') + f(k', k) \geq 1 + \frac{f(n, k)}{2} - 1 = \frac{f(n, k)}{2}.$$

For example, $f(n, k) = \lg(\frac{n}{k})$ for $k \leq n$, and $f(n, k) = 0$ otherwise, and $k'(n, k) = \lceil \sqrt{nk} \rceil$, fulfil these conditions. For a creature $\mathbf{c}^+$ we define its norm

$$\mathrm{nor}_f(\mathbf{c}^+) = \mathrm{nor}(\mathbf{c}^+) = f(\mathrm{nor}^{\frac{1}{2}}(\mathbf{c}), k(\mathbf{c}^+)).$$

Remark 1.8. 1. Note that property (1)(d) of simple creatures (Definition 1.5) follows from $\mathrm{nor}^0(\mathbf{c}) > 0$. So we will not check this property any more, but restrict ourselves to creatures with strictly positive nor^0.

2. Definition 1.7(1) speaks about infinitely many requirements, by ranging over all k-tuples of branches of $\mathbf{T}$. However, at a crucial point in the proof of Claim 1.10 this boils down to counting the possibilities for $a \subseteq n_{3,i}$.

3. From conditions of nor^0 together with additional conditions we shall draw conclusions on nor^0. Often the formulation is smoother for $\mathrm{nor}^{\frac{1}{2}}$ because the additional premises in Claims 1.9 and 1.10 are of the type $\mathrm{nor}^*(\mathbf{c}) \geq k$. Claim 1.11 works for all norms. We hope that the variety of norms will be helpful for future applications.

The next claim shows that we can extend the functions in the value of a creature and at the same time decrease the norm of the creature only by a small amount.

Claim 1.9. Assume that

(a) $\eta^* \in \mathrm{spec}$,
(b) $\mathbf{c}$ is a simple i-creature with base η^*, $\mathrm{nor}^0(\mathbf{c}) > 0$,
(c) $k^* > 1$, $|\,\mathrm{rge}(\mathrm{val}(\mathbf{c}))| \cdot k^* < n_{1,i}$,
(d) for each $\eta \in \mathrm{rge}(\mathrm{val}(\mathbf{c}))$ and $k < k^*$ we are given $\eta \subseteq \rho_{\eta,k} \in \mathrm{spec}_{n_{3,i}}$ with $|\,\mathrm{dom}(\rho_{\eta,k})| < n_{2,i}$,
(e) for each $\eta \in \mathrm{rge}(\mathrm{val}(\mathbf{c}))$, if $k_1 < k_2 < k^*$ and $x_1 \in \mathrm{dom}(\rho_{\eta,k_1}) \setminus \mathrm{dom}(\eta)$ and $x_2 \in \mathrm{dom}(\rho_{\eta,k_2}) \setminus \mathrm{dom}(\eta)$, then x_1, x_2 are $<_{\mathbf{T}}$-incomparable,
(f) $\ell^* = \max\{|\,\mathrm{dom}(\rho_{\eta,k})| + 1 \,:\, \eta \in \mathrm{rge}(\mathrm{val}(\mathbf{c})) \wedge k < k^*\}$.

Then there is a simple i-creature $\mathbf{d}$ given by

$$\mathrm{rge}(\mathrm{val}(\mathbf{d})) = \{\rho_{\eta,k} \,:\, k < k^*, \eta \in \mathrm{rge}(\mathrm{val}(\mathbf{c}))\}.$$

We have $\eta(\mathbf{d}) = \eta^*$, and $\mathrm{nor}^0(\mathbf{d}) \geq m_0 \stackrel{\mathrm{def}}{=} \min\left\{\mathrm{nor}^0(\mathbf{c}), \log_2(\frac{n_{2,i}}{\ell^*}) - 1, k^* - 1\right\}$.

Proof. First of all we are to check Definition 1.5(1). Clauses (a),(b), and (c) follow immediately from the premises of the claim. From premise (e) and from the properties of $\mathbf{c}$ it follows that $\eta(\mathbf{d}) = \eta^*$. Therefore $\mathbf{d}$ satisfies clause (d).

Now for the norm: We check clause (α) of Definition 1.7. Let branches $B_0, \dots B_{m_0-1}$ of $\mathbf{T}$ and a set $a \subseteq n_{3,i}$ be given, $|a| \leq m_0$. Since $m_0 \leq \mathrm{nor}^0(\mathbf{c})$, there is some $\eta \in \mathrm{rge}(\mathrm{val}(\mathbf{c}))$ such that $(\forall x \in \left(\bigcup_{\ell<m_0} B_\ell\right) \cap \mathrm{dom}(\eta) \setminus \mathrm{dom}(\eta(\mathbf{c})))(\eta(x) \notin a)$. We fix such an η. Now for each $\ell < m_0$, we let

$$w_{\eta,\ell} = \{j < k^* \,:\, \exists x \in B_\ell \cap \mathrm{dom}(\rho_{\eta,j}) \setminus \mathrm{dom}(\eta)\}.$$

Now we have that $|w_{\eta,\ell}| \leq 1$ because otherwise we would have $k_1 < k_2 < k^*$ in $w_{\eta,\ell}$ and $x_i \in B_\ell \cap \mathrm{dom}(\rho_{\eta,k_i}) \setminus \mathrm{dom}(\eta)$. As x_1 and x_2 are $<_{\mathbf{T}}$-comparable, this is contradicting the requirement (e) of 1.9.

Since $m_0 < k^*$, there is some $j \in k^* \setminus \bigcup_{\ell<m_0} w_{\eta,\ell}$. For such a j, $\rho_{\eta,j}$ is as required.

We check clause (β) of Definition 1.7. We take the $\rho_{\eta,j}$ as chosen above. Then we have

$$\frac{|\operatorname{dom}(\rho_{\eta,j})|}{n_{2,i}} \leq \frac{\ell^*}{n_{2,i}} = \frac{1}{2^{\lg\left(\frac{n_{2,i}}{\ell^*}\right)}} \leq \frac{1}{2^{\log_2\left(\frac{n_{2,i}}{\ell^*}\right)-1}} \leq \frac{1}{2^{m_0}},$$

as $m_0 \leq \log_2\left(\frac{n_{2,i}}{\ell^*}\right) - 1$. □

Whereas the previous claim will be used only in Section 3 in the proof on properness (see Claim 3.9), the following two claims will be used in the next section for density arguments in the forcings built from creatures.

Claim 1.10. Assume

(a) $\mathbf{c}$ is a simple i-creature.
(b) $k = \operatorname{nor}^0(\mathbf{c}) \geq 1$ and $k \leq n_{1,i}$.
(c) $x_0, \dots x_{m-1} \in \mathbf{T}$, $1 \leq m \leq \min(k, \frac{n_{2,i}}{2^k})$.
(d) $|\operatorname{rge}(\operatorname{val}(\mathbf{c}))| \cdot \frac{k!}{(k-m)!} \leq n_{1,i}$.
(e) If $\eta \in \operatorname{rge}(\operatorname{val}(\mathbf{c}))$, then $|\{y \in \operatorname{dom}(\eta) : (\exists m' < m)(x_{m'} <_T y)\}| < i$ and $|\operatorname{dom}(\eta)| < n_{2,i} - m$.

Then there is $\mathbf{d}$ such that

(1) $\eta(\mathbf{d}) = \eta(\mathbf{c})$,
(2) $\operatorname{rge}(\operatorname{val}(\mathbf{d})) \subseteq \{\nu \in \operatorname{spec}_{n_{3,i}} : |\operatorname{dom}(\nu)| < n_{2,i}, (\exists \eta \in \operatorname{rge}(\operatorname{val}(\mathbf{c})))(\eta \subseteq \nu \wedge \operatorname{dom}(\nu) = \operatorname{dom}(\eta) \cup \{x_0, \dots, x_{m-1}\})\}$,
(3) $|\operatorname{rge}(\operatorname{val}(\mathbf{d}))| < n_{1,i}$,
(4) $\mathbf{d}$ is a simple i-creature,
(5) $\operatorname{nor}^0(\mathbf{d}) \geq k - m$.

Proof. Independently of $\eta \in \operatorname{rge}(\operatorname{val}(\mathbf{c}))$, we take for $m' < m$, $z_{m'} \in n_{3,i} \setminus (\operatorname{rge}((\eta(\mathbf{c})) \cup \bigcup_{\eta \in \operatorname{rge}(\operatorname{val}(\mathbf{c}))}\{\eta(y) : x <_{\mathbf{T}} y\} \cup \{z_{m''} : m'' < m'\})$. Since $|\operatorname{rge}(\operatorname{val}(\mathbf{c}))| < n_{1,i}$ and by (d) and since by (1.1) $n_{2,i-1} + (i-1) \cdot n_{1,i} + k < n_{3,i}$ there is such a $z_{m'}$, and indeed, which is important for getting a creature that fulfils (5), there are at least $k - m'$ such $z_{m'}$'s. Now for every $\eta \in \operatorname{rge}(\operatorname{val}(\mathbf{c}))$, we take all these choices $\nu_{\eta,\bar{z}} = \eta \cup \{(x_{m'}, z_{m'}) : m' < m\}$ into $\operatorname{rge}(\operatorname{val}(\mathbf{d}))$. Hence we can choose all $\nu_{\eta,\bar{z}}$ so that we avoid any given a of size $k - m$ with all the $z_{m'}$'s.

Now we check the norm: Let $B_0, \dots, B_{k-m-1}$ be branches of $\mathbf{T}$ and let $a \subseteq n_{3,i(\mathbf{c})}$, $|a| \leq k - m$. We have to find $\nu \in \operatorname{rge}(\operatorname{val}(\mathbf{d}))$ such that $(\forall \ell < k - m)(\forall y \in \operatorname{dom}(\nu) \cap B_\ell \setminus \operatorname{dom}(\eta(\mathbf{c}))(\nu(y) \notin a)$ and $|\operatorname{dom}(\nu)| \leq \frac{n_{2,i}}{2^{k-m}}$. For $m' < m$ we choose $B_{k-m+m'}$, a branch containing $x_{m'}$. We take for $m' < m$, $z_{m'} \in n_{3,i} \setminus (\operatorname{rge}(\eta(\mathbf{c})) \cup \bigcup_{\eta \in \operatorname{rge}(\operatorname{val}(\mathbf{c}))}\{\eta(y) : x_{m'} <_{\mathbf{T}} y\} \cup a \cup \{z_{m''} : m'' < m'\})$. We set $a' = a \cup \{z_0, \dots, z_{m-1}\}$.

By premise (b), we find $\eta \in \operatorname{rge}(\operatorname{val}(\mathbf{c}))$ for a' and $B_0, \dots, B_{\ell-1}$ such that

(1) $(\forall \ell < k - 1)(\forall x \in \operatorname{dom}(\eta) \cap B_k \setminus \operatorname{dom}(\eta(\mathbf{c})))(\eta(x) \notin a')$ and
(2) $|\operatorname{dom}(\eta)| \leq \dfrac{n_{2,i}}{2^k}$.

Now $\nu_{\eta,\bar{z}} = \nu$ is a witness for the norm. We have $\frac{n_{2,i}}{2^k} + m \leq \frac{n_{2,i}}{2^{k-m}}$, which follows from the premises on m. The only thing to show is that ν is really a specialisation function. So let $y \in \mathrm{dom}(\eta)$ and $y <_{\mathbf{T}} x_{m'}$. If $y \in \mathrm{dom}(\eta) \setminus \mathrm{dom}(\eta(\mathbf{c}))$, then $\nu(y) = \eta(y) \neq \nu(x_{m'}) = z_{m'}$, because y is on the branch leading to $x_{m'}$ and because of (1). If $y >_{\mathbf{T}} x_{m'}$, then we have $\nu(y) \neq \eta(x_{m'})$ simultaneously for all η's by our choice of the $z_{m'}$'s. □

Suppose we have filled up the range of the value of a creature according to one of the previous claims. Then we want that these extended functions can serve as bases for suitable creatures as well. This is provided by the next claim.

Claim 1.11. Assume that

(a) $\mathbf{c}$ is a simple i-creature.
(b) $\eta^* \supseteq \eta(\mathbf{c})$, $\eta^* \in \mathrm{spec}_{n_{3,i-1}}$ (note that we do not suppose that $\eta^* \in \mathrm{rge}(\mathrm{val}(\mathbf{c}))$). Furthermore we assume $|\,\mathrm{dom}(\eta^*)| \leq n_{2,i(\mathbf{c})-1}$.
(c) We set

$$\ell_2^* = |\,\mathrm{dom}(\eta^*) \setminus \mathrm{dom}(\eta(\mathbf{c}))|,$$

and

$$\begin{aligned}\ell_1^* = |\{y\ :(\exists \nu \in \mathrm{rge}(\mathrm{val}(\mathbf{c})))(y \in \mathrm{dom}(\nu) \setminus \mathrm{dom}(\eta(\mathbf{c})))\\ \wedge\ (\exists x \in \mathrm{dom}(\eta^*) \setminus \mathrm{dom}(\eta(\mathbf{c})))(x <_{\mathbf{T}} y)\}|,\end{aligned}$$

and we assume that $\ell_1^* + \ell_2^* < \mathrm{nor}^0(\mathbf{c})$.

We define $\mathbf{d}$ by $\eta(\mathbf{d}) = \eta^*$ and

$$\mathrm{rge}(\mathrm{val}(\mathbf{d})) = \{\nu \cup \eta^*\ :\ \nu \in \mathrm{rge}(\mathrm{val}(\mathbf{c})) \wedge \nu \cup \eta^* \in \mathrm{spec}_{n_{3,i}}, |\,\mathrm{dom}(\nu \cup \eta^*)| < n_{2,i}\}.$$

Then

(α) $\mathbf{d}$ is a simple i-creature.
(β) $\mathrm{nor}^0(\mathbf{d}) \geq \mathrm{nor}^0(\mathbf{c}) - \ell_2^* - \ell_1^*$.

Proof. Item (α) follows from the requirements on η^* and from the estimates on the norm, see below. For item (β), we set $k = \mathrm{nor}^0(\mathbf{c}) - \ell_1^* - \ell_2^*$. We let $B_0, \dots B_{k-1}$ be branches of $\mathbf{T}$ and $a \subseteq n_{3,i(\mathbf{c})}$, $|a| \leq k$. We set $\ell^* = \ell_1^* + \ell_2^*$. We let $\langle y_\ell\ :\ \ell < \ell_1^* \rangle$ list $Y = \{y\ :\ \exists \nu(\nu \in \mathrm{rge}(\mathrm{val}(\mathbf{c})) \wedge y \in \mathrm{dom}(\nu)) \wedge \exists x(x \in \mathrm{dom}(\eta^*) \setminus \mathrm{dom}(\eta(\mathbf{c})) \wedge x \leq_{\mathbf{T}} y)\}$ without repetition. Let $B_k, \dots, B_{k+\ell_1^*-1}$ be branches of $\mathbf{T}$ such that $y_\ell \in B_{k+\ell}$ for $\ell < \ell_1^*$. Let $\langle x_\ell\ :\ \ell < \ell_2^* \rangle$ list $\mathrm{dom}(\eta^*) \setminus \mathrm{dom}(\eta(\mathbf{c}))$. Take for $\ell < \ell_2^*$, $B_{k+\ell_1^*+\ell}$ such that $x_\ell \in B_{k+\ell_1^*+\ell}$. We set $a' = a \cup \{\eta^*(x_\ell)\ :\ \ell < \ell_2^*\}$. Since $\mathrm{nor}^0(\mathbf{c}) \geq k + \ell^*$ there is some $\nu \in \mathrm{rge}(\mathrm{val}(\mathbf{c}))$ such that $\forall x \in ((\mathrm{dom}(\nu) \setminus \mathrm{dom}(\eta(\mathbf{c}))) \cap \bigcup_{\ell < k+\ell^*} B_\ell)(\nu(x) \notin a')$. Then, if $x \notin \mathrm{dom}(\eta^*)$, $(\nu \cup \eta^*)(x) \notin a$. Moreover $|\,\mathrm{dom}(\nu \cup \eta^*)| \leq \frac{n_{2,i}}{2^{k+\ell_1*}} + \ell_2^* \leq \frac{n_{2,i}}{2^k}$, if $\frac{n_{2,i}}{2^k}$ is large enough. (This premise will always be fulfilled in our applications, because $n_{1,i} \leq n_{2,i}$. We just perform all our operations on forcing conditions only at high levels i, compared to the size of the given η^*. This will be done in the next section.)

We have to show that $\nu \cup \eta^*$ is a partial specialisation: Since η^* and ν are specialisation maps, we have to consider only the case $x \in \mathrm{dom}(\eta^*) \setminus \mathrm{dom}(\eta(\mathbf{c}))$

and ($y \in Y$ or ($y \in \mathrm{dom}(\nu) \setminus \mathrm{dom}(\eta^*)$ and $y <_{\mathbf{T}} x$)). If $y \in Y$, then we have $\nu(y) \neq \eta^*(x_\ell)$ for all $\ell < \ell_2^*$. If $y \in \mathrm{dom}(\nu) \setminus \mathrm{dom}(\eta^*)$ and $y <_{\mathbf{T}} x$, then y is in a branch leading to some x_ℓ, $\ell < \ell_2^*$, and hence again $\nu(y) \neq \eta^*(x_\ell)$, $\ell < \ell_2^*$. □

In the applications, the proofs of the density properties, ℓ_2^* will be small compared to the norm (we add ℓ_2^* points to the domain of the functions in the range of the value of a creature with sufficiently high norm) and $\ell_1^* \leq |u|$, were u is the set that sticks out of $\mathbf{T}_{<\alpha(p)}$ (see Definition 2.2 and Remark 2.5). We will suppose that these two are small in comparison to $\mathrm{nor}^0(\mathbf{c})$, so that the premises for Claim 1.11 are fulfilled.

An analogous version of Claim 1.10 with $\mathrm{nor}^{\frac{1}{2}}$ instead of nor^0 holds as well. The analogous requirements to premises (c) and (d) in 1.10 are even easier: If we work with $\mathrm{nor}^{\frac{1}{2}}$ and use $n_{1,i} \leq n_{2,i}$ from equation (1.4) in the Choice 1.3, then $1 \leq m \leq k$ is enough in premise (c). Premise (d) is included in $\mathrm{nor}^{\frac{1}{2}}(\mathbf{c}) = k$ for sufficiently large k.

Claims 1.9, 1.10, and 1.11 for $\mathrm{nor}^{\frac{1}{2}}$ instead of nor^0 are proved by easy but a bit tedious accounting of $\mathrm{nor}^*(\mathbf{c}) = \log_2(\frac{n_{1,i(\mathbf{c})}}{|\mathrm{val}(\mathbf{c})|})$. Just see that $|\mathrm{val}(\mathbf{c})|$ increases only in a controllable way in Claim 1.9 and in Claim 1.10 and does not increase at all in Claim 1.11. Hence also if nor^* is the part determining the minimum in $\mathrm{nor}^{\frac{1}{2}}$, the latter falls at most by $\lg(k^*)$ in 1.9 from $\mathbf{c}$ to $\mathbf{d}$, and at most by $\log_2(\frac{k!}{(k-m)!})$ in 1.10 and does not decrease in 1.11. Since premises 1.9(c) and 1.10(d) are conditions on the largeness of nor^*, they can be combined with the premises on nor^0 to one conditions speaking about $\mathrm{nor}^{\frac{1}{2}}$. This combined norm falls only by a small amount as well.

The next claim will help to find large homogeneous subtrees of the trees built from creatures that will later be used as forcing conditions.

Claim 1.12. (1) The 2-bigness property [4, Definition 2.3.2]. If $\mathbf{c}$ is a simple i-creature with $\mathrm{nor}^1(\mathbf{c}) \geq k+1$, and $\mathbf{c}_1$, $\mathbf{c}_2$ are simple i-creatures such that $\mathrm{val}(\mathbf{c}) = \mathrm{val}(\mathbf{c}_1) \cup \mathrm{val}(\mathbf{c}_2)$, then $\mathrm{nor}^1(\mathbf{c}_1) \geq k$ or $\mathrm{nor}^1(\mathbf{c}_2) \geq k$. The same holds for nor^2.

(2) If $\mathbf{c}^+$ is a i-creature with $\mathrm{nor}(\mathbf{c}) \geq k+1$, and $\mathbf{c}_1^+$, $\mathbf{c}_2^+$ are i-creatures such that $\mathrm{val}(\mathbf{c}) = \mathrm{val}(\mathbf{c}_1) \cup \mathrm{val}(\mathbf{c}_2)$, and $k(\mathbf{c}_1^+) = k(\mathbf{c}_2^+) = k(\mathbf{c}^+)$, then $\mathrm{nor}(\mathbf{c}_1^+) \geq k$ or $\mathrm{nor}(\mathbf{c}_2^+) \geq k$.

Proof. (1) We first consider nor^0. Let $j = 2^k$. We suppose that $\mathrm{nor}^0(\mathbf{c}_1) < j$ and $\mathrm{nor}^0(\mathbf{c}_2) < j$ and derive a contradiction: For $\ell = 1, 2$ let branches $B_0^\ell, \dots, B_{j-1}^\ell$ and sets $a^\ell \subseteq n_{3,i}$ exemplify this.

Let $a = a^1 \cup a^2$ and let, by $\mathrm{nor}^0(\mathbf{c}) \geq 2j$, $\eta \in \mathrm{rge}(\mathrm{val}(\mathbf{c}))$ be such that for all $x \in (\mathrm{dom}(\eta) \cap \bigcup_{\ell=1,2} \bigcup_{i=0}^{j-1} B_i^\ell) \setminus \mathrm{dom}(\eta(\mathbf{c}))$ we have $\eta(x) \notin a$. But then for that $\ell \in \{1, 2\}$ for which $\eta \in \mathrm{rge}(\mathrm{val}(\mathbf{c}_\ell))$ we get a contradiction to $\mathrm{nor}^0(\mathbf{c}_i) < j$. Hence (1) follows for nor^1. nor^* increases or stays when taking subsets of $\mathrm{val}(\mathbf{c})$, and hence we have the analogous result for nor^2.

Since the k-components of the creatures coincide, part (2) follows from the behaviour of $\mathrm{nor}^{\frac{1}{2}}$ that was shown in part (1) and from the requirements on f in Definition 1.7(4): $f(\frac{n}{2}, k) \geq f(n, k) - 1$. □

2. Forcing with tree-creatures

Now we define a notion of forcing with ω-trees $\langle \mathbf{c}_t^+ : t \in (T, \lhd_T)\rangle$ as conditions. The nodes t of these trees $(T, \lhd_T) = (\mathrm{dom}(p), \lhd_p)$ and their immediate successors are described by certain creatures $\mathbf{c}_t^+$ from Definition 1.5.

First we collect some general notation about trees. The trees here are not the Aronszajn trees of the first section, but trees T of finite partial specialisation functions, ordered by $\lhd_T$ which is a subrelation of $\subset$. Some of these trees will serve as forcing conditions.

Definition 2.1. *(1) A tree* $(T, \lhd_T)$ *is a set* $T \subseteq \mathrm{spec}$*, such that for any* $\eta \in T$, $(\{\nu : \nu \lhd_T \eta\}, \lhd_T)$ *is a finite linear order and such that in* T *there is one least element, called the root,* $\mathrm{rt}(T)$*. If* $\eta \lhd_T \nu$ *then* $\eta \subset \nu$*. Every* $\eta \in T \setminus \mathrm{rt}(T)$ *has just one immediate* $\lhd_T$*-predecessor in* T*. We shall only work with finitely branching trees.*

(2) We define the successors of η *in* T*, the restriction of* T *to* η*, the splitting points of* T *and the maximal points of* T *by*

$$\begin{aligned}
\mathrm{suc}_T(\eta) &= \{\nu \in T : \eta \lhd_T \nu \wedge \neg(\exists \rho \in T)(\eta \lhd_T \rho \lhd_T \nu)\},\\
T^{\langle \eta \rangle} &= \{\nu \in T : \eta \trianglelefteq_T \nu\},\\
\mathrm{split}(T) &= \{\eta \in T : |\mathrm{suc}_T(\eta)| \geq 2\},\\
\max(T) &= \{\nu \in T : \neg(\exists \rho \in T)(\nu \lhd_T \rho\}.
\end{aligned}$$

(3) The n*-th level of* T *is*

$$T^{[n]} = \{\eta \in T : \eta \textit{ has } n \lhd_T\textit{-predecessors}\}.$$

The set of all branches through T *is*

$$\begin{aligned}
\lim(T) = \{\langle \eta_k : k < \ell\rangle : \ell \leq \omega \wedge (\forall k < \ell)(\eta_k \in T^{[k]})\\
\wedge (\forall k < \ell - 1)(\eta_k \lhd_T \eta_{k+1})\\
\wedge \neg(\exists \eta_\ell \in T)(\forall k < \ell)(\eta_k \lhd_T \eta_\ell)\}.
\end{aligned}$$

A tree is well-founded if there are no infinite branches through it.

(4) A subset F *of* T *is called a front of* T *if every branch of* T *passes through this set, and the set consists of* $\lhd_T$*-incomparable elements.*

Definition 2.2. *We define a notion of forcing* $Q = Q_{\mathbf{T}}$*.* $p \in Q$ *iff*

(i) p *is a function from a subset of* $\mathrm{spec} = \mathrm{spec}^{\mathbf{T}}$ *(see Definition 1.2) to* ω*.*

(ii) $p^{[]} = (\mathrm{dom}(p), \lhd_p)$ *is a tree with* ω *levels, the* ℓ*-th level of which is denoted by* $p^{[\ell]}$*.*

(iii) $p^{[]}$ *has a root, the unique element of level 0, called* $\mathrm{rt}(p)$*.*

(iv) We let

$$i(p) \stackrel{def}{=} \min\{i : |\mathrm{dom}(\mathrm{rt}(p))| < n_{2,i-1} \wedge \mathrm{rt}(p) \in \mathrm{spec}_{n_{3,i-1}}\}.$$

Then for any $\ell < \omega$ and $\eta \in p^{[\ell]}$ the set

$$\mathrm{suc}_p(\eta) = \{\nu \in p^{[\ell+1]} \,:\, \eta \lhd_p \nu\}$$

is $\mathrm{rge}(\mathrm{val}(\mathbf{c}))$ *for a simple $(i(p)+\ell)$-creature* $\mathbf{c}$ *with base η. We denote this simple creature by $\mathbf{c}_{p,\eta}$ and let $\mathbf{c}^+_{p,\eta} = (\mathbf{c}_{p,\eta}, p(\eta))$. Furthermore, we require $p(\eta) \leq \mathrm{nor}^{\frac{1}{2}}(\mathbf{c}_{p,\eta})$.*

(v) If $\eta \in \mathrm{dom}(p)$ and $\nu \in \mathrm{dom}(p)$ and if $\eta \cup \nu \in \mathrm{spec}$, then $\eta \cup \nu \in \mathrm{dom}(p)$. It is a superset of both η and of ν, but in $\lhd_p$ it has only one immediate predecessor. If neither $\eta \subseteq \nu$ nor $\nu \subseteq \eta$, then at most one of η and ν can be $\lhd_p$-less than $\eta \cup \nu$, while both are $\subseteq$-less than $\eta \cup \nu$. Every $\eta \in \mathrm{spec}$ appears at most once in $\mathrm{dom}(p)$.

(vi) For some $k < \omega$ for every $\eta \in p^{[k]}$ there is $\alpha < \omega_1$ and a finite $u \subseteq \mathbf{T} \setminus \mathbf{T}_{<\alpha}$ such that for every ω-branch $\langle \eta_\ell \,:\, \ell < \omega\rangle$ of $p^{[]}$ satisfying $\eta_k = \eta$ we have $\bigcup_{\ell\in\omega} \mathrm{dom}(\eta_\ell) \setminus u = \mathbf{T}_{<\alpha}$.

(vii) For every ω-branch $\langle \eta_\ell \,:\, \ell \in \omega\rangle$ of $p^{[]}$ we have $\lim_{\ell\to\omega} \mathrm{nor}(\mathbf{c}^+_{p,\eta_\ell}) = \omega$.

The order $\leq = \leq_Q$ is given by letting $p \leq q$ (q is stronger than p, we follow the Jerusalem convention) iff $i(p) \leq i(q)$ and there is a projection $\mathrm{pr}_{q,p}$ which satisfies

(a) $\mathrm{pr}_{q,p}$ is a function from $\mathrm{dom}(q)$ to $\mathrm{dom}(p)$.

(b) $\eta \in q^{[\ell]} \Rightarrow \mathrm{pr}_{q,p}(\eta) \in p^{[\ell+i(q)-i(p)]}$.

(c) If η_1, η_2 are both in $q^{[]}$ and if $\eta_1 \unlhd_q \eta_2$, then $\mathrm{pr}_{q,p}(\eta_1) \unlhd_p \mathrm{pr}_{p,q}(\eta_2)$.

(d) $q(\eta) \geq p(\mathrm{pr}_{q,p}(\eta))$.

(e) If $\eta \in q^{[]}$ then $\eta \supseteq \mathrm{pr}_{q,p}(\eta)$.

(f) If $\nu \in q^{[\ell]}$ and $\rho \in q^{[\ell+1]}$ and $\nu \lhd_q \rho$, $\mathrm{pr}_{q,p}(\nu) = \eta$, $\mathrm{pr}_{q,p}(\rho) = \tau$, then $\mathrm{dom}(\tau) \cap \mathrm{dom}(\nu) = \mathrm{dom}(\eta)$.

Definition 2.3. *For $p \in Q$ and $\eta \in \mathrm{dom}(p)$ we let*

$$p^{\langle\eta\rangle} = p \restriction \{\rho \in \mathrm{dom}(p) \,:\, \eta \unlhd \rho\}.$$

Let us give some informal description of the $\leq$-relation in Q: The stronger condition's domain is via $\mathrm{pr}_{q,p}$ mapped homomorphically w.r.t. the tree orders into $\mathrm{dom}(p^{\langle \mathrm{pr}_{q,p}(\mathrm{rt}(q))\rangle})$. The root can grow as well. According to (b), the projection preserves the levels in the trees but for one jump in heights (the ℓ's in $p^{[\ell]}$), due to a possible lengthening of the root. The partial specialisation functions sitting on the nodes of the tree are extended (possibly by more than one extension per function) in q as to compared with the ones attached to the image under pr, but by (b) the extensions are so small and so few that it preserves the kind i of the creature given by the node and its successors, and according to (f) the new part of the domain of the extension is disjoint from the domains of the old partial specification functions living higher up in the projection of the new tree to the old tree.

Let us compare our setting with the forcings given in the book [4]: There the $\leq$-relation of the forcing is based on a sub-composition function (whose definition is not used here, because we just deal with one particular forcing notion) whose inputs are well-founded subtrees of the weaker condition. This well-foundedness condition [4, 1.1.3] is not fulfilled: if we look at (e) and (f) in the definition of $\leq$ we see that we have to look at all the branches of p that are in the range of $\mathrm{pr}_{q,p}$ in order to see whether some $\nu \in q^{[\ell]}$ fulfils (f) of the definition of $p \leq q$. On the other hand, the projections shift all the levels by the same amount $i(q) - i(p)$, and are not arbitrary finite contractions as in most of the forcings in the book [4].

Definition 2.4. *(1) $p \in Q$ is called normal iff for every ω-branch $\langle \eta_\ell : \ell \in \omega \rangle$ of $p^{[]}$ the sequence $\langle \mathrm{nor}(\mathbf{c}^+_{p,\eta_\ell}) : \ell \in \omega \rangle$ is non-decreasing.*
(2) $p \in Q$ is called smooth iff in clause (vi) of Definition 2.2 the number k is 0 and u is empty.
(3) $p \in Q$ is called weakly smooth iff in clause (vi) of Definition 2.2 the number k is 0.

Remark 2.5. If $p \in Q$ is smooth then there is some $\alpha < \omega_1$ such that for every ω-branch $\langle \eta_\ell : \ell \in \omega \rangle$ of $p^{[]}$ we have $\bigcup_{\ell<\omega} \mathrm{dom}(\eta_\ell) = \mathbf{T}_{<\alpha}$. This α is denoted by $\alpha(p)$.

Fact 2.6. (1) If p is weakly smooth Then: If $p \leq q$ and $\eta \in \mathrm{dom}(p)$, $\nu \in \mathrm{dom}(q)$, $\eta = \mathrm{pr}_{q,p}(\nu)$ and $\eta \triangleleft \tau \in \mathrm{dom}(p)$, then $\mathrm{dom}(\nu) \cap \mathrm{dom}(\tau) = \mathrm{dom}(\eta)$.
(2) If $p \leq q$ and p is weakly smooth then
$\nu \in \mathrm{dom}(q) \to \mathrm{dom}(\nu) \cap (\mathbf{T}_{<\alpha(p)} \cup u) = \mathrm{dom}(\mathrm{pr}_{q,p}(\nu))$.

Proof. (1): If p is weakly smooth, then all branches of $p^{[]}$ have the same union of domains, and hence it is immaterial whether ρ and ν from 2.2(f) are in the range of $\mathrm{pr}_{q,p}$ or not. (2) follows from (1). □

Definition 2.7. *For $0 \leq n < \omega$ we define the partial order $\leq_n$ on Q by letting $p \leq_n q$ iff*

(i) $p \leq q$,
(ii) $i(p) = i(q)$,
(iii) $p^{[\ell]} = q^{[\ell]}$ for $\ell < n$, and $p \restriction \bigcup_{\ell<n} p^{[\ell]} = q \restriction \bigcup_{\ell<n} q^{[\ell]}$,
(iv) if $\mathrm{pr}_{q,p}(\eta) = \nu$, then
- *$\eta = \nu$ and $\mathbf{c}^+_{q,\eta} = \mathbf{c}^+_{p,\nu}$*
- *or $\mathrm{nor}(\mathbf{c}^+_{q,\eta}) \geq n$.*

We state and prove some basic properties of the notions defined above.

Claim 2.8. (1) If $p \leq q$ and p is weakly smooth, then $\mathrm{pr}_{q,p}$ is unique.
(2) If $p \in Q$ and $\ell \in \omega$ then $|p^{[\ell]}| < n_{1,i(p)+\ell}$.
(3) $(Q, \leq_Q)$ is a partial order.
(4) If $p \leq q$ and $\mathrm{pr}_{q,p}(\eta) = \nu$, then $i(\mathbf{c}_{q,\eta}) = i(\mathbf{c}_{p,\nu})$.
(5) If $p \leq q$ and $\mathrm{pr}_{q,p}(\eta) = \nu$, then $\mathrm{nor}^0(\mathbf{c}_{q,\eta}) \leq \mathrm{nor}^0(\mathbf{c}_{p,\nu})$.
(6) $(Q, \leq_n)$ is a partial order.
(7) $p \leq_{n+1} q \to p \leq_n q \to p \leq q$.
(8) If $\mathbf{c}$ is a simple i-creature with $k \leq \mathrm{nor}^0(\mathbf{c})$, then there is a simple i-creature $\mathbf{c}'$ with $k = \mathrm{nor}^0(\mathbf{c}')$ and $\mathrm{val}(\mathbf{c}') \subseteq \mathrm{val}(\mathbf{c})$.

(9) For every $p \in Q$ there is a $q \geq p$ such that for all η and ν

$$\mathrm{pr}_{q,p}(\eta) = \nu \rightarrow \mathrm{nor}^0(\mathbf{c}_{q,\eta}) = \min\{\mathrm{nor}^0(\mathbf{c}_{p,\rho}) : \nu \trianglelefteq_p \rho \in \mathrm{dom}(p)\}.$$

(10) For every (not necessarily normal) p we have that $\lim_{n\to\omega} \min\{\mathrm{nor}(\mathbf{c}^+_{p,\eta}) : \eta \in p^{[n]}\} = \infty$.

(11) If $p \in Q$ and $\eta \in p^{[\ell]}$ then $|\mathrm{dom}(\eta)| < n_{2,i(p)+\ell-1}$ or $\ell = 0$ and $i(p) = 0$ and $\eta = \emptyset$.

Proof. (1) By induction on ℓ we show that $\mathrm{pr}_{q,p} \restriction \bigcup_{\ell' \leq \ell} p^{[\ell']}$ is unique: It is easy to see that for weakly smooth p, $\mathrm{pr}_{q,p}(\mathrm{rt}(q))$ is the $\subseteq$-maximal element of p that is a subfunction of $\mathrm{rt}(q)$. By Definition 2.2(v) such a maximum exists. Then we proceed level by level in $q^{[]}$, and again Definition 2.2(v) yields uniqueness of $\mathrm{pr}_{q,p}$.

(2) This is also proved by induction on ℓ. Note that for $\eta \in p^{[\ell]}$ we have that $|\mathrm{rge}(\mathrm{val}((\eta))| \leq n_{1,i(p)+\ell-1}$. We have $|p^{[0]}| = 1$ and by Definition 1.5(c), $|p^{[\ell+1]}| \leq |p^{[\ell]}| \cdot n_{1,i(p)+\ell} \leq n_{1,i(p)+\ell} \cdot n_{1,i(p)+\ell} \leq n_{1,i(p)+\ell+1}$, by equation (1.3).

(3) Given $p \leq q$ and $q \leq r$ we define $\mathrm{pr}_{r,p} = \mathrm{pr}_{q,p} \circ \mathrm{pr}_{r,q}$. It is easily seen that this function is as required.

(4) Let ℓ be such that $\eta \in q^{[\ell]}$. Then $i(\mathbf{c}_{q,\eta}) = i(q) + \ell$ and $\nu \in p^{[\ell+i(q)-i(p)]}$. Hence $i(\mathbf{c}_{p,\nu}) = i(p) + \ell + i(q) - i(p) = i(q) + \ell$.

(5) Suppose $\mathrm{nor}^0(\mathbf{c}_{q,\eta}) > \mathrm{nor}^0(\mathbf{c}_{p,\nu})$. Let $k = \mathrm{nor}^0(\mathbf{c}_{q,\eta})$ and let $i = i(\mathbf{c}_{q,\eta}) = i(\mathbf{c}_{p,\nu})$. Suppose that $a \subseteq n_{3,i}$ and the branches $B_0, \dots, B_{k-1}$ of T exemplify that $\mathrm{nor}^0(\mathbf{c}_{p,\nu}) < k$. Hence for all $\tau \in \mathrm{suc}_p(\nu)$

(α) there is $x \in (\mathrm{dom}(\tau) \cap \bigcup_{\ell=0}^{k-1} B_\ell) \setminus \mathrm{dom}(\nu)$ such that $\tau(x) \in a$, or

(β) $|\mathrm{dom}(\tau)| > \frac{n_{2,i}}{2^k}$.

Suppose that a and $B_0, \dots, B_{k-1}$ exemplify $\mathrm{nor}^0(\mathbf{c}_{q,\eta}) < k$. Let $\tau \in \mathrm{suc}_q(\eta)$, and $\mathrm{pr}_{q,p}(\tau) = \tau'$. Suppose 1.7(1) ($\alpha$) is the case for τ'. Then the same a and $B_0, \dots, B_{k-1}$ exemplify (α) for τ and $\mathbf{c}_{q,\eta}$, because we have $\eta \supseteq \nu = \mathrm{pr}_{q,p}(\eta)$ and $\mathrm{pr}_{q,p}[\mathrm{suc}_q(\eta)] \subseteq \mathrm{suc}_p(\nu)$. The same x will show that $\exists x \in (\mathrm{dom}(\tau) \cap \bigcup_{\ell=0}^{k-1} B_\ell) \setminus \mathrm{dom}(\eta)$ such that $\tau(x) \in a$, if we verify that $x \not\in \mathrm{dom}(\eta)$. But we have for all $\tau \in \mathrm{suc}_q(\eta)$ that $\mathrm{dom}(\eta) \cap \mathrm{dom}(\tau) = \mathrm{dom}(\nu)$ by 2.2(f), and hence $x \not\in \mathrm{dom}(\eta)$.

Suppose 1.7(1)(β) is the case for τ'. Then $\tau' \in \mathrm{suc}_p(\nu)$ and $\tau \supseteq \tau'$, and hence $|\mathrm{dom}(\tau)| \geq \frac{n_{2,i}}{2^k}$.

(6) Suppose that $p \leq_n q \leq_n r$ and $\mathrm{pr}_{r,q}(\sigma) = \eta$ and $\mathrm{pr}_{q,p}(\eta) = \nu$. By (1) and (3) we have that $\mathrm{pr}_{r,p}(\sigma) = \nu$, and now it is easy to check the requirements for $p \leq_n r$.

(7) Obvious.

(8) We may assume that $\mathrm{nor}^0(\mathbf{c}) > k$, because otherwise $\mathbf{c}$ itself is as required. Look at

$$Y = \{\mathbf{d} : \mathbf{d} \text{ is a simple } i\text{-creature and } \mathrm{val}(\mathbf{d}) \neq \emptyset \text{ and}$$
$$\mathrm{nor}^0(\mathbf{d}) \geq k \text{ and } \mathrm{val}(\mathbf{d}) \subseteq \mathrm{val}(\mathbf{c})\}.$$

Since $\mathbf{c} \in Y$, it is non-empty, and it has a member $\mathbf{d}$ with a minimal number of elements. We assume towards a contradiction that $\mathrm{nor}^0(\mathbf{d}) > k$. We choose $\eta^* \in \mathrm{rge}(\mathrm{val}(\mathbf{d}))$. We let $\mathrm{rge}(\mathrm{val}(\mathbf{d}^*)) = \mathrm{rge}(\mathrm{val}(\mathbf{d})) \setminus \{\eta^*\}$.

Claim: $\mathbf{d}^* \neq \emptyset$. Otherwise we choose $x \in \mathrm{dom}(\eta^*) \setminus \mathrm{dom}(\eta(\mathbf{c}))$. If $\eta^* \neq \eta(\mathbf{c})$ such an x exists. $\eta^* 0 \eta(\mathbf{c})$ is excluded because $\eta(\mathbf{c}) \notin \mathrm{rge}(\mathrm{val}(\mathbf{c}))$. Now we let B_0 be a branch of T to which x belongs and set $a = \{\eta^*(x)\}$. They witness that $\mathrm{nor}^0(\mathbf{d}) \not\geq 1$, so $\mathrm{nor}^0(\mathbf{d}) = 0$, which contradicts the assumption that $\mathrm{nor}^0(\mathbf{d}) > k > 0$.

Claim: $\mathrm{nor}^0(\mathbf{d}^*) \geq k$. Otherwise there are branches $B_0, \dots, B_{k-1}$ and a set $a \subseteq n_{3,i}$ witnessing $\mathrm{nor}^0(\mathbf{d}^*) \not\geq k$. Let $x \in \mathrm{dom}(\eta^*) \setminus \mathrm{dom}(\eta(\mathbf{d}))$ and let B_k be a branch such that $x \in B_k$ and set $a' = a \cup \{\eta^*(x)\}$. The $B_0, \dots, B_k$ and a' witness that $\mathrm{nor}^0(\mathbf{c}) \not\geq k + 1$. Hence $\mathbf{d}^*$ is a member of Y with fewer elements than $\mathbf{d}$, contradiction.

(9) Follows from (8). We can even take $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$. First see: For no m the set $\{\eta \in p$ such that for densely (in $p^{[]}$) many $\eta' \trianglerighteq_p \eta$ we have that $\mathrm{nor}^0(\mathbf{c}_{p,\eta'}) < m\}$. is anywhere dense. Otherwise we can choose a branch $\langle \eta_\ell : \ell \in \omega \rangle$ such that there is some $m \in \omega$ such that for all $\ell < \omega$, $\mathrm{nor}^0(\mathbf{c}_{p,\eta_\ell}) < m$.

Now we choose by induction of ℓ, $\mathrm{dom}(q_\ell) \subseteq \mathrm{dom}(p)$, such that $\mathrm{dom}(q_\ell)$ has no infinite branch and hence is finite, though we do not have a bound on its height.

First step: Say $\min\{\mathrm{nor}^0(\mathbf{c}_{p,\eta}) : \eta \in \mathrm{dom}(p)\} = k$ and it is reached in $\eta \in \mathrm{dom}(p)$. We take $q^{[0]} = \{\eta\}$.

([1]) Then we take for any $\eta' \in \mathrm{rge}(\mathrm{val}(\mathbf{c}_{p,\eta}))$ some $\eta'' \supseteq \eta'$ such that $\eta'' \in \mathrm{dom}(p)$ and such that for all $\tilde{\eta} \supseteq \eta''$, if $\tilde{\eta} \in \mathrm{dom}(p)$ then $\mathrm{nor}^0(\mathbf{c}_{p,\tilde{\eta}}) \geq k + 1$. By the mentioned nowhere-density result, this is possible. We put such an η'' in $q^{[\ell]}$, if it is in $p^{[\ell + i(q) - i(p)]}$.

([2]) Then we look at the ν in the branch between η and η'' in $\mathrm{dom}(p)$. If $\mathrm{nor}^0(\mathbf{c}_{p,\nu}) > k$ we take according to (8) a subset of $\mathrm{rge}(\mathrm{val}(\mathbf{c}_{p,\nu}))$ with norm k and put this into $\mathrm{dom}(q)$. We have to put successors to all $\nu' \in \mathrm{rge}(\mathrm{val}(\mathbf{c}_{p,\nu}))$ for all ν in question into $\mathrm{dom}(q_1)$. This is done as in ([1]), applied to ν instead of η. With all the ν in this subset we do the procedure in ([1]), and repeat and repeat it. In finitely many (intermediate) steps we reach a subtree $\mathrm{dom}(q_\ell)$ of $\mathrm{dom}(p)$ without any ω-branches such that all its leaves fulfil $\eta'' \in \mathrm{dom}(p)$ and such that for all $\tilde{\eta} \supseteq \eta''$, if $\tilde{\eta} \in \mathrm{dom}(p)$ then $\mathrm{nor}^0(\mathbf{c}_{p,\tilde{\eta}}) \geq k + 1$, and all its nodes η fulfil $\mathrm{nor}^0(\mathbf{c}_{q_1,\eta}) \geq k$. By König's lemma, this tree $\mathrm{dom}(q_1)$ is finite.

([3]) With the leaves of $\mathrm{dom}(q_\ell)$ and $k+2$ instead of $k+1$, we repeat the choice procedure in ([1]) and ([2]). We do it successively for all $k \in \omega$. The union of the $\mathrm{dom}(q_\ell)$, $\ell \in \omega$, is a q as desired in (9).

(10) This follows from König's lemma: Since $p^{[]}$ is finitely branching, there is a branch though every infinite subset.

(11) Follows from Definitions 1.5 and 2.2. □

The next lemma states that Q fulfils some fusion property:

Lemma 2.9. *Let $\langle n_i : i \in \omega \rangle$ be a strictly increasing sequence of natural numbers. We assume that for every i, $q_i \leq_{n_i} q_{i+1}$, and we set $n_{-1} = 0$. Then $q = \bigcup_{i<\omega} \bigcup_{n_{i-1} \leq n < n_i} (q_i) \restriction q_i^{[n]} \in Q$ and for all i, $q \geq_{n_i} q_i$.*

Proof. Clear by the definitions. □

The fusion lemma is usually applied in the following setting:

Conclusion 2.10. Suppose $p \in Q$ is given and we are to find $q \geq p$ such that q fulfils countably many tasks. For this it is enough to find for any single task and any p_0 and $k^* \in \omega$ some $q \geq_{k^*} p_0$ that fulfils the task.

Now we want to fill up the domains of the partial specialisation functions and to show that smooth conditions are dense:

Lemma 2.11. *If $p \in Q$ and $m < \omega$ then for some smooth $q \in Q$ we have $p \leq_m q$. Moreover, if* $\bigcup\{\mathrm{dom}(\eta) : \eta \in p^{[]}\} \subseteq \mathbf{T}_{<\alpha}$ *then we can demand that* $\bigcup\{\mathrm{dom}(\eta) : \eta \in q^{[]}\} = \mathbf{T}_{<\alpha}$. *Moreover, $\eta \in q^{[]}$ implies* $\mathrm{nor}^1(\mathbf{c}_{q,\eta}) \geq \mathrm{nor}^1(\mathbf{c}_{p,\mathrm{pr}_{q,p}(\eta)}) - 1$*, and* $q(\eta) = p(\mathrm{pr}_{q,p}(\eta))$ *implies that* $\mathrm{nor}(\mathbf{c}^+_{q,\eta}) \geq \mathrm{nor}(\mathbf{c}^+_{p,\mathrm{pr}_{q,p}(\eta)}) - 1$.

Proof. We first use the definition of $p \in Q$: By item (v) there is some $k < \omega$ for every $\eta \in p^{[k]}$ there is $\alpha(\eta) < \omega_1$ and a finite $u_\eta \in \mathbf{T} \setminus \mathbf{T}_{<\alpha(\eta)}$ such that for every ω-branch $\langle \eta_\ell : \ell < \omega\rangle$ of $p^{[]}$ satisfying $\eta_k = \eta$ we have $\bigcup_{\ell \in \omega} \mathrm{dom}(\eta_\ell) \setminus u_\eta = \mathbf{T}_{<\alpha(\eta)}$. We fix such a k and such u_η's and set$u = \bigcup_{\eta \in p^{[k]}} u_\eta$. Let $\{x^\eta_k : k < \omega)\}$ enumerate $\mathbf{T}_{<\alpha} \setminus (\mathbf{T}_{<\alpha(\eta)} \cup u_\nu)$ without repetition. Since m is arbitrary, it is enough to find $q \geq_m p$ such that the first ℓ of the x^η_k's are in its domain. So we aim for such a condition.

We can find n such that

$(*)_1$ $m \leq n < \omega, k \leq n$,
$(*)_2$ $|u| < n$,
$(*)_3$ for every $\nu \in p^{[n]}$, we have $\mathrm{nor}^0(\mathbf{c}_{p,\nu}) > m$,
$(*)_4$ if $\eta \in p^{[n]}$, $\eta \subseteq \nu \in \mathrm{dom}(p)$ then $\mathrm{dom}(\nu) \setminus \mathrm{dom}(\eta)$ is disjoint from u.

For each $\eta \in p^{[n]}$ let $w^+_\eta = \{\nu : \eta \lhd_T \nu \in \mathrm{dom}(p) \wedge \mathrm{nor}^1(\mathbf{c}_{p,\nu}) + \ell + n > \mathrm{nor}^1(\mathbf{c}_{p,\eta})\}$, $w_\eta = \{\nu \in w^+_\eta : (\not\exists \rho)(\eta \trianglelefteq_T \rho \lhd_T \nu \wedge \rho \in w^+_\eta\}$, and let $\tilde{\eta}$ be the predecessor of η in $p^{[k]}$. So $w := \bigcup\{w_\eta : \eta \in p^{[n]}\}$ is a front of $p^{[]}$. For each $\nu \in w$ let $\nu \in p^{[\ell(\nu)]}$ (so $\ell(\nu) \geq n$) and let $\alpha(\nu) = \alpha(\tilde{\eta})$ and $u_\nu = u_{\tilde{\eta}}$ when $\nu \in w_\eta$.

For each $\eta \in p^{[n]}$, $\nu \in w_\eta$,

$$|\{\tilde{\rho} \in u_\nu : (\exists k < \ell) x^{\tilde{\eta}}_k \lhd_{\mathbf{T}} \tilde{\rho}\}| + |\{x^{\tilde{\eta}}_k : k < \ell\}| < n + \ell. \qquad \boxtimes$$

We let $\rho \in \mathrm{dom}(p)$ be a candidate if

(a) $(\rho \subseteq \nu \vee \nu \subseteq \rho \in p^{[]})$ and $\alpha = \alpha(\nu)$ or
(b) $\nu \subseteq \rho \wedge \alpha(\nu) < \alpha \wedge \exists \ell > \ell(\nu) \exists \tau \in p^{[\ell]}(\nu \subseteq \tau \wedge \mathrm{dom}(\rho) = \mathrm{dom}(\tau) \cup \{x^{\tilde{\eta}}_k : k < \ell$ and $\rho(x^{\tilde{\eta}}_k) < n_{3,i(p)+k})$.

We let $q^{[]}_*$ be the set of all candidates and choose $q^{[]} \subseteq q^{[]}_*$ by successively climbing upwards in the levels of $p^{[]}$, using first Claim 1.10 for the immediate successors of an already chosen node, and then using Claim 1.11 to make these new successors the bases of the creatures attached to them. We choose q rich enough as in Claim 1.10 but also small enough as to have sufficiently high $\mathrm{nor}^*(\mathbf{c}_{q,\rho})$. We set $q(\rho) = p(\tau)$ if ρ, τ are as above.

For checking the conditions for $p \leq q$ and on the norms note that $\boxtimes$ above gives clause of the premises of Claim 1.10 on a given level and of Claim 1.11 on its successor level. By the choice of q, it is smooth. $\square$

Conclusion 2.12. Forcing with Q specialises $\mathbf{T}$.

3. Decisions taken by the tree creature forcing

In this section we prove that Q is proper and ${}^{\omega}\omega$-bounding. Indeed we prove that Q has "continuous reading of names" (this is the property stated in 3.9), which implies Axiom A (see [1]) and properness.

Claim 3.1. (1) If $p \in Q$ and $\{\eta_1, \dots, \eta_n\}$ is a front of p, then $\{p^{\langle \eta_1 \rangle}, \dots, p^{\langle \eta_n \rangle}\}$ is predense above p.

(2) If $\{\eta_1, \dots, \eta_n\}$ is a front of p and $p^{\langle \eta_\ell \rangle} \le q_\ell \in Q$ for each ℓ, then there is $q \ge p$ with $\{\eta_1, \dots, \eta_n\} \subseteq q^{[]}$ such that for all ℓ we have that $q^{\langle \eta_\ell \rangle} = q_\ell$. Hence $\{q^{\langle \eta_\ell \rangle} : 1 \le \ell \le n\}$ is predense above q.

Claim 3.2. If $p \in Q$ and $X \subseteq \mathrm{dom}(p)$ is upwards closed in $\lhd_p$, and $\forall \eta \in \mathrm{dom}(p)\, \mathrm{nor}^0(\mathbf{c}_{p,\eta}) > 0$, then there is some q such that

(a) $p \le_0 q$, and either $(\exists \ell) q^{[\ge \ell]} \subseteq X$ or $\mathrm{dom}(q) \cap X = \emptyset$,
(b) $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$ and $q = p \restriction \mathrm{dom}(q)$,
(c) for every $\nu \in \mathrm{dom}(q)$, if $\mathbf{c}_{q,\nu} \ne \mathbf{c}_{p,\nu}$, then $\mathrm{nor}^2(\mathbf{c}_{q,\nu}) \ge \mathrm{nor}^2(\mathbf{c}_{p,\nu}) - 1$ and $\mathrm{nor}(\mathbf{c}^+_{q,\nu}) \ge \mathrm{nor}(\mathbf{c}^+_{p,\nu}) - 1$.

Proof. We will choose $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$ and then let $q = p \restriction \mathrm{dom}(q)$. For each ℓ we first choose by downward induction on $j \le \ell$ subsets $X_{\ell,j} \subseteq p^{[\le \ell]}$ and a colouring $f_{\ell,j}$ of $X_{\ell,j} \cap p^{[j]}$ with two colours, 0 and 1. The choice is performed in such a way that $X_{\ell,j-1} \subseteq X_{\ell,j}$ and such that $p^{[i]} \subseteq X_{\ell,j}$ for $i \le j$.

We choose $X_{\ell,\ell} = p^{[\le \ell]}$ and for $\nu \in p^{[\ell]}$ we set $f_{\ell,\ell}(\nu) = 0$ iff $(\exists \ell')(p^{\langle \nu \rangle})^{[\ge \ell']} \subseteq X$ and $f_{\ell,\ell}(\nu) = 1$ otherwise.

Suppose that $X_{\ell,j}$ and $f_{\ell,j}$ are chosen. For $\eta \in p^{[j-1]} \cap X_{\ell,j}$ we have

$$\begin{aligned}\mathrm{rge}(\mathrm{val}(\mathbf{c}_{\eta,p})) = &\{\nu \in \mathrm{rge}(\mathrm{val}(\mathbf{c}_{\eta,p})) : f_{\ell,j}(\nu) = 0\} \cup \\ &\{\nu \in \mathrm{rge}(\mathrm{val}(\mathbf{c}_{\eta,p})) : f_{\ell,j}(\nu) = 1\}\end{aligned}$$

Note that the sets would be all the same if we intersect with $X_{\ell,j}$, because $p^{[j]} \subseteq X_{\ell,j}$. By Claim 1.12 at least one of the two sets gives a creature $\mathbf{c}$ with $\mathrm{nor}^2(\mathbf{c}) \ge \mathrm{nor}^2(\mathbf{c}_{\eta,p}) - 1$. The same holds for nor.

So we keep in $X_{\ell,j-1} \cap p^{[j]}$ only those of the majority colour and close this set downwards in $p^{[]}$. This is $X_{\ell,j-1}$. We colour the points on $p^{[j-1]} \cap X_{\ell,j-1}$ with $f_{\ell,j-1}$ according to these majority colors, i.e., $f_{\ell,j-1}(\eta) = i$ iff $\{\nu \in \mathrm{rge}(\mathrm{val}(\mathbf{c}_{\eta,p})) : f_{\ell,j}(\nu) = i\} \subseteq X_{\ell,j-1}$. We work downwards until we come to the root of p and keep $f_{\ell,0}(\mathrm{rt}(p))$ in our memory.

We repeat the procedure of the downwards induction on j for larger and larger ℓ.

If there is one ℓ where the root got colour 0, we are, because X is upwards closed, in the first case of the alternative in the conclusion (a). If for all ℓ the root got colour 1, we have for all ℓ finite subtrees t such that for all $\nu \in t$, $p^{\langle \nu \rangle} \cap t$ has at least original norm -1 at its root. By König's Lemma (initial segments of trees are taking from finitely many possibilities) we build a condition q such that all of its nodes are not in X, and thus (a) is proved. The item (b) is clear. Item (c) follows from our choice of q and from 1.12. □

The next claim is very similar to 3.2. We want to find $q \geq_m p$, and therefore we have to weaken the homogeneity property in item (a) of 3.2.

Claim 3.3. If $p \in Q$, $k^* \in \omega$, and $X \subseteq \mathrm{dom}(p)$ is upwards closed, and $\forall \eta \in \mathrm{dom}(p)\ \mathrm{nor}^0(\mathbf{c}_{p,\eta}) > 0$, then there is some q such that

(a) $p \leq_{k^*} q$, and there is a front $\{\nu_0, \dots \nu_s\}$ of p which is contained in q and whose being contained in q ensures $p \leq_{k^*} q$, and such that for all ν_i we have: either $(\exists \ell)(q^{\langle \nu_i \rangle})^{[\geq \ell]} \subseteq X$ or $\mathrm{dom}(q^{\langle \nu_i \rangle}) \cap X = \emptyset$,
(b) $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$ and $q = p \restriction \mathrm{dom}(q)$,
(c) for every $\nu \in \mathrm{dom}(q)$, if $\mathbf{c}^+_{q,\nu} \neq \mathbf{c}^+_{p,\nu}$, then $\mathrm{nor}^2(\mathbf{c}_{q,\nu}) \geq \mathrm{nor}^2(\mathbf{c}_{p,\nu}) - 1$ and $\mathrm{nor}(\mathbf{c}^+_{q,\nu}) \geq \mathrm{nor}(\mathbf{c}^+_{p,\nu}) - 1$.

Proof. We repeat the proof of 3.2 for each $p^{\langle \nu_i \rangle}$. □

Now for the first time we make use of the coordinate $k(\mathbf{c}^+)$ of our creatures. The next lemma states that the creatures have the halving property (compare to [4, 2.2.7]).

Definition 3.4. *K^+ has the halving property, iff there is a function* half : $K^+ \to K^+$ *with the following properties:*

(1) $\mathrm{half}(\mathbf{c}^+) = (\mathbf{c}, k(\mathrm{half}(\mathbf{c}^+)))$,
(2) $\mathrm{nor}(\mathrm{half}(\mathbf{c}^+)) \geq \frac{\mathrm{nor}(\mathbf{c}^+)}{2} - 1$,
(3) if $\mathbf{c}'$ *is a simple creature and* $k \geq k(\mathrm{half}(\mathbf{c}^+))$ *and* $\mathrm{nor}(\mathbf{c}', k) > 0$*, then* $\mathrm{nor}(\mathbf{c}', k(\mathbf{c}^+)) \geq \frac{\mathrm{nor}(\mathbf{c}^+)}{2}$.

Lemma 3.5. *K^+ has the halving property.*

Proof. We set $k(\mathrm{half}(\mathbf{c}^+)) = \lceil k'(\mathrm{nor}^{\frac{1}{2}}(\mathbf{c}), k(\mathbf{c}^+)) \rceil \geq k(\mathbf{c}^+)$ as in 1.7(4). Then we have that $\mathrm{nor}(\mathrm{half}(\mathbf{c}^+)) = f(\mathrm{nor}^{\frac{1}{2}}(\mathbf{c}), k(\mathrm{half}(\mathbf{c}^+))) \geq \frac{\mathrm{nor}(\mathbf{c}^+)}{2} - 1$, by Definition 1.7(4).

If $\mathbf{c}'$ is a simple creature and $\mathrm{nor}(\mathbf{c}', k(\mathrm{half}(\mathbf{c}^+)) > 0$ and $\mathrm{nor}^{\frac{1}{2}}(\mathbf{c}') \leq \mathrm{nor}^{\frac{1}{2}}(\mathbf{c})$, then

$$\begin{aligned}\mathrm{nor}(\mathbf{c}', k(\mathbf{c}^+)) &= f(\mathrm{nor}^{\frac{1}{2}}(\mathbf{c}'), k(\mathbf{c}^+)) \\ &\geq f(\mathrm{nor}^{\frac{1}{2}}(\mathbf{c}'), k(\mathrm{half}(\mathbf{c}^+)) + f(\mathrm{nor}^{\frac{1}{2}}(\mathbf{c}), k(\mathrm{half}(\mathbf{c}^+))) \\ &\geq 1 + \frac{\mathrm{nor}(\mathbf{c}^+)}{2} - 1 \geq \frac{\mathrm{nor}(\mathbf{c}^+)}{2}.\end{aligned}$$

If $\mathrm{nor}^{\frac{1}{2}}(\mathbf{c}') > \mathrm{nor}^{\frac{1}{2}}(\mathbf{c})$, then the inequality follows from the monotonicity properties in Definition 1.7 (4). □

Claim 3.6. Assume that $\underset{\sim}{\tau}$ is a Q-name for an ordinal, and let a be a set of ordinals. Let $k, m \in \omega$. Let p, q be conditions such that

(a) $(\forall \nu \in p^{[\geq k]})(\mathrm{nor}(\mathbf{c}_{p,\nu}, p(\nu)) \geq 2m + 1)$,
(b) $\mathrm{dom}(q) = \mathrm{dom}(p)$, and for $\eta \in \mathrm{dom}(q)$, $k(\mathbf{c}^+_{q,\eta}) = k(\mathrm{half}(\mathbf{c}^+_{p,\eta}))$, $\mathbf{c}_{q,\eta} = \mathbf{c}_{q,\eta}$.

Then for any $r \in Q$: $q \leq r$ and $r \Vdash \underset{\sim}{\tau} \in a$ and $\nu = \mathrm{rt}(r)$ and $\eta = \mathrm{pr}_{r,p}(\nu) \in p^{[k]}$ imply that there is some r' such that

(α) $p^{\langle \eta \rangle} \leq r'$, $\nu = \mathrm{rt}(r')$
(β) $r' \Vdash \underset{\sim}{\tau} \in a$,
(γ) for every $\rho \in \mathrm{dom}(r')$, $\mathrm{nor}(\mathbf{c}_{r',\rho}, q'(\rho)) \geq m$.

Proof. So let $r \geq q$ and $r \Vdash \underset{\sim}{\tau} \in a$. We take some $n(*) \in \omega$ such that

$$(\forall \rho \in \mathrm{dom}(r))(\rho \in \bigcup_{n' \geq n(*)} r^{[n']} \rightarrow \mathrm{nor}(\mathbf{c}^+_{r,\rho}) > 2m).$$

We define r' by $\mathrm{dom}(r') = \mathrm{dom}(r)$ and $\rho \in \bigcup_{n' \geq n(*)} r^{[n']} \rightarrow \mathbf{c}^+_{r',\rho} = \mathbf{c}^+_{r,\rho}$, $\rho \in \bigcup_{n' < n(*)} r^{[n']} \rightarrow \mathbf{c}^+_{r',\rho} = (\mathbf{c}_{r,\rho}, k(\mathbf{c}^+_{p,\rho}))$.

q and q' force the same things, because we weakened q to q' only in an atomic part, because there are only finitely many k such that $(\mathbf{c}_{q,\rho}, k)$ is a creature with $0 \leq \mathrm{nor}^0(\mathbf{c}_{q,\rho})$.

From Lemma 3.5 we get $\rho \in \mathrm{dom}(q) \rightarrow \mathrm{nor}(\mathbf{c}^+_{q',\rho}) \geq m$. □

As a preparation for the following proof, we define isomorphism types of partial specialisation functions over conditions p:

Definition 3.7. *Let $\eta_0, \eta_1 \in$ spec and let $p \in Q$. We say η_0 is isomorphic to η_1 over p if there is some injective partial function $f: \mathbf{T} \rightarrow \mathbf{T}$ such that $x <_{\mathbf{T}} y$ iff $f(x) <_{\mathbf{T}} f(y)$ and $\mathrm{dom}(\eta_0) \cup \bigcup\{\mathrm{dom}(\eta) : \eta \in \mathrm{dom}(p)\} \subseteq \mathrm{dom}(f)$ and $f \restriction \bigcup\{\mathrm{dom}(\eta) : \eta \in \mathrm{dom}(p)\} = id$ and $f[\mathrm{dom}(\eta_0)] = \mathrm{dom}(\eta_1)$ and $\eta_0(x) = \eta_1(f(x))$ for all $x \in \mathrm{dom}(\eta_0)$.*

Fact 3.8. For each fixed p, there are only countably many isomorphism types for η over p. If the elements of $\mathrm{dom}(\eta_0)$ and of $\mathrm{dom}(\eta_1)$ are pairwise incomparable in $\mathbf{T}$ and if they are isomorphic over p with $\bigcup\{\mathrm{dom}(\eta) : \eta \in \mathrm{dom}(p)\} \subseteq \mathbf{T}_{<\alpha}$ for some countable α, and if there is some $r \geq p$ such that $\eta_0 \in r^{[]}$, then there is some $r' \geq p$ such that $\eta_1 \in (r')^{[]}$.

Claim 3.9. Suppose that $p_0 \in Q$ and that $m < \omega$ and that $\underset{\sim}{\tau}$ is a Q-name of an ordinal. Then there is some $q \in Q$ such that

(a) $p_0 \leq_m q$,
(b) for some $\ell \in \omega$ we have that for every $\eta \in q^{[\ell]}$ the condition $q^{\langle \eta \rangle}$ forces a value to $\underset{\sim}{\tau}$.

Proof. Choose $n(*)$ such that $\rho \in \bigcup_{n \geq n(*)} p_0^{[n]} \rightarrow \mathrm{nor}(\mathbf{c}^+_{p_0,\rho}) \geq m + 1$.

Then we define

$$X = \Bigg\{\rho \, : \rho \in \bigcup_{n \geq n(*)} p_0^{[n]} \wedge (\exists q)\Bigg(p_0^{\langle \rho \rangle} \leq_0 q \ \wedge \ q \text{ forces a value to } \underset{\sim}{\tau} \\ \wedge (\forall \nu \in q^{[]})(\mathrm{nor}(\mathbf{c}^+_{q,\nu}) \geq 1)\Bigg)\Bigg\}.$$

Let p_1 be chosen as in 3.3 for $(p_0, X, n(*))$. By a density argument, there is a front $\{\nu_0, \dots \nu_r\}$ of p_1 such that for all ν_i the first clause of the alternative in 3.3(a) holds with $k^* = 4m + 4$. W.l.o.g. let $n(*)$ be bigger than all the ℓ's from 3.3(a) plus the maximum of the height of ν_i in p_0 for all ν_i, $0 \leq i \leq r$.

For $\tilde{m} < \omega, r, s \in Q, \eta \in \mathrm{dom}(r)$, we denote the following property by $(*)^{\tilde{m},\eta}_{r,s}$:

$$(*)^{\tilde{m},\eta}_{r,s} \qquad \begin{aligned} & r^{\langle\eta\rangle} \leq_0 s \ \wedge \\ & \forall\nu(\eta \subseteq \nu \in \mathrm{dom}(s) \rightarrow \mathrm{nor}(\mathbf{c}^+_{s,\nu}) \geq \tilde{m} + 1) \ \wedge \\ & (\exists\ell \in \omega)(\forall\rho \in s^{[\ell]})(s^{\langle\rho\rangle} \text{ forces a value to } \underset{\sim}{\tau}). \end{aligned}$$

We choose by induction on $t < \omega$ a countable $N_t \prec (\mathcal{H}(\chi), \in)$ and an ordinal α_t and pairs (k_t, q_t) such that

(0) $q_0 = p_1$,
(1) $p_1, \mathbf{T}, \underset{\sim}{\tau} \in N_0$,
(2) $N_t \in N_{t+1}$,
(3) $N_t \cap \omega_1 = \alpha_t$,
(4) $\delta = \lim_{t\to\omega} \alpha_t$,
(5) k_t is increasing with t, $k_t \geq n(*)$,
(6) $q_t \in Q$ is smooth,
(7) $\alpha(q_t) = \alpha_t$,
(8) k_t is the first k strictly larger than all the k_{t_1} for $t_1 < t$ and such that $\rho \in q_t^{[\geq k]} \rightarrow \mathrm{nor}(\mathbf{c}^+_{q_t,\rho}) > 4m + 4t + 3$,
(9) $q_t \leq_{m+t+1} q_{t+1}$,
(10) if $\eta \in q_t^{[k_t]}$ and there is q ($\in V$) satisfying $(*)^{m+t,\eta}_{q_t,q}$, then $q = q_{t+1}^{\langle\eta\rangle}$ satisfies it,
(11) $q_t \in N_{t+1}$,
(12) if $\eta \in q_t^{[k_t]}$ and no q satisfies $(*)^{m+t,\eta}_{q_t,q}$, then $(q_{t+1}^{\langle\eta\rangle})^{[]} = (q_t^{\langle\eta\rangle})^{[]}$ and $\eta \subseteq \rho \in \mathrm{dom}(q_t)$ implies that $\mathbf{c}^+_{q_{t+1},\rho} = \mathrm{half}(\mathbf{c}^+_{q_t,\rho})$.

It is clear that the definition can be carried out as required. If we are given q_t we can easily find k_t. For each $\eta \in q_t^{[k_t]}$ we choose $q_{t,\eta} \in N_t$ such that $(*)^{m+t,\eta}_{q_t,q_{t,\eta}}$ if possible and in fact w.l.o.g. $q_{t,\eta} = q_t^{\langle\eta\rangle}$, otherwise we follow (12) and apply the halving function.

Having carried out the induction, we let $r = \bigcup_{t\in\omega}(q_t \restriction q_t^{(k_{t-1},k_t]})$. So, by (7), $r \in Q$ is smooth with $\alpha(r) = \delta$ and for every t we have $q_t \leq_{m+t+1} r$, and in particular $p_1 \leq_{m+1} r$.

Assume for a contradiction that we are in the bad case that r does not fulfil (b) of 3.9. So for all t there is $\eta \in q_t^{[k_t]}$ not fulfilling (10). We take $t = 0$.

For $\eta_0 \in r^{[k_0]}$ either clause (10) applies, then we have nothing to do, or clause (12) applies (and this happens at least once by our assumption), which means

$$\boxtimes \qquad \begin{aligned} & \text{there is no } s \text{ such that} \\ & r^{\langle\eta_0\rangle} \leq_{m+1} s \text{ and } (\exists\ell)(\forall\rho' \in s^{[\ell]})(s^{\langle\rho'\rangle} \text{ forces a value to } \underset{\sim}{\tau}). \end{aligned}$$

For each such η_0 now separately we carry out the following construction: Choose $q \geq_0 r^{\langle \eta_0 \rangle}$ in Q such that q forces a value to $\underset{\sim}{\tau}$ and that q has the property as in Claim 3.6. Note that also $\mathrm{dom}(r)$ has the property of X with p_0 replaced by r in the definition of X. So such a q exists.

As r is smooth, by the definition of $q \geq r$ (2.2(f)) we have that the additional information on partial specialisation functions that are in q but not in r does not have the domain in $\mathbf{T}_{<\delta}$.

Let $\nu_0 = \mathrm{rt}(q)$. Then by the choice of X, $i(\nu_0) = i(\eta_0)$. Moreover, $\eta_0 = \mathrm{dom}(\mathrm{rt}(q)) \cap \mathbf{T}_{<\delta} \subseteq \mathbf{T}_{<\alpha_0}$, and $i(p_0) = i(q_0) = i(r) = i(q) - k_0$. By the halving we have

$$(\forall \nu)(\nu_0 \subseteq \nu \in \mathrm{dom}(q) \rightarrow \mathrm{nor}(\mathbf{c}^+_{q,\nu}) \geq 2m + 2).$$

Now easily, if $0 \leq t$, $\nu_0 \leq \nu \in \mathrm{dom}(q)$, $\mathrm{pr}_{q,r}(\nu) \in r^{[k_t]}$, then $\mathrm{pr}_{q,r}(\nu) = \mathrm{pr}_{q,q_t}(\nu)$, $\mathrm{dom}(\nu) \cap \mathbf{T}_{<\alpha(r)} = \mathrm{dom}(\mathrm{pr}_{q,r}(\nu)) = \mathrm{dom}(\mathrm{pr}_{q,q_t}(\nu)) \subseteq \mathbf{T}_{<\alpha(q_t)}$.

From $\eta_0 \in r^{[k_0]}$ we get $\eta_0 \in q_0^{[k_0]}$. So by the choice of $\langle q_t : t \in \omega \rangle$ we know that there is no q with $(*)^{m,\eta_0}_{q_0,q}$, as otherwise $q_1^{\langle \eta_0 \rangle}$ would be like this and this property would be inherited by r.

Fix for some time $\nu \in \mathrm{suc}_q(\nu_0)$ and let $\eta = \mathrm{pr}_{q,q_1}(\nu) \in q_1^{[k_0+1]}$, so $\eta \in \mathrm{suc}_{q_1}(\eta_0)$.

So let $\mathrm{dom}(\nu) \setminus \mathrm{dom}(\eta) = \{x_0, \ldots, x_{\tilde{s}-1}\}$. We just saw that $x_0, \ldots, x_{\tilde{s}-1} \notin \mathbf{T}_{<\delta}$. Let us define $\bar{y} = \langle y_\ell : \ell < \tilde{s} \rangle$ is a candidate for an extended domain iff:

(a) y_ℓ are without repetitions,
(b) there is some $r_{\bar{y}}$ such that
  (0) $\mathrm{rt}(r_{\bar{y}}) = \eta \cup \{(y_\ell, \nu(x_\ell)) : \ell < \tilde{s}\} \in Q$, such that
  (1) $r_{\bar{y}} \geq q_t^{\langle \eta \rangle}$,
  (2) $(\forall \rho)(\eta \subseteq \rho \in \mathrm{dom}(r_{\bar{y}}) \rightarrow \mathrm{nor}(\mathbf{c}^+_{r_{\bar{y}},\rho}) > m + 1)$,
  (3) $r_{\bar{y}}$ forces a value to $\underset{\sim}{\tau}$
  (4) $\langle (y_\ell, \nu(x_\ell)) : \ell < \tilde{s} \rangle \in \mathrm{spec}^{\mathbf{T}}$ is isomorphic over $\mathbf{T}_{<\delta}$ to $\langle (x_\ell, \nu(x_\ell)) : \ell < \tilde{s} \rangle$.

We set

$$Y = Y_\eta = \{\bar{y} : \bar{y} \text{ is a candidate for an extension}\}.$$

Now we have that $\langle x_\ell : \ell < \tilde{s} \rangle \in Y$. This is exemplified by $q^{\langle \nu \rangle}$.

We have that $q_t \in \bigcup_{t<\omega} N_t = N$ and for all ℓ, $x_\ell \in \mathbf{T}_{\geq \delta}$, because the α_t are cofinal in δ and since $\alpha_t = \alpha(q_t)$.

Since $x_\ell \geq \delta$, counting isomorphism types over $\mathbf{T}_{<\delta}$ yields $|Y_\eta| = \aleph_1$.

By a fact on Aronszajn trees (Jech, or [6, III, 5.4]) we find $\langle y^\eta_{j,\ell} : \ell < s, j \in \omega_1 \rangle$ and a root Δ_η such that

(a) $\langle y^\eta_{j,\ell} : \ell \in \tilde{s} \rangle \in Y_\eta$ are without repetition,
(b) for $j \neq j'$, $\{y^\eta_{j,\ell} : \ell < \tilde{s}\} \cap \{y^\eta_{j',\ell} : \ell < \tilde{s}\} = \Delta_\eta$,
(c) if $j_1 \neq j_2$ and if $y^\eta_{j_1,\ell_1} \notin \Delta_\eta$ and $y^\eta_{j_2,\ell_2} \notin \Delta_\eta$ then they are incompatible in $<_{\mathbf{T}}$.

Let $\mathbf{c} = \{\mathrm{pr}_{q,q_t}(\nu) : \nu \in \mathrm{suc}_q(\nu_0)\}$. This is a simple $(i(q_0) + k_0 + 1)$-creature with $\mathrm{nor}(\mathbf{c}, q_0(\eta_0)) \geq m + 2$ by property (10) of (k_0, q_0). For each $\eta \in \mathrm{rge}(\mathrm{val}(\mathbf{c}))$ let $\langle y^\eta_{j,\ell} : \ell < \tilde{s}, j < \omega_1\rangle$ be as above and let r^η_j be a witness for $\langle y^\eta_{j,\ell} : \ell < \tilde{s}\rangle \in Y_\eta$.

Let $j^* = \mathrm{nor}^0(\mathbf{c}_{q_0,\eta_0})$.

For each $\eta \in \mathrm{rge}(\mathrm{val}(\mathbf{c}_{q_0,\eta_0}))$ choose a witness $\nu_\eta \in \mathrm{suc}_q(\nu_0)$ such that $\mathrm{pr}_{q,q_0}(\nu_\eta) = \eta$. Now we define a simple $i(\mathbf{c}_{q_0,\eta_0})$-creature $\mathbf{d}$ by

$$\eta(\mathbf{d}) = \eta(\mathbf{c}_{q_0,\eta_0})$$
$$\mathrm{rge}(\mathrm{val}(\mathbf{d})) = \{\eta \cup \{(y^\eta_{j,\ell}, \nu_\eta(y_\ell)) : \ell < \tilde{s}\} : \eta \in \mathrm{rge}(\mathrm{val}(\mathbf{c})), j < j^*\}.$$

Then we have by Claim 1.9 that $\mathbf{d}$ is a $i(\mathbf{c}_{q_0,\eta_0}))$-creature and $\mathrm{nor}^0(\mathbf{d}) \geq \min\{\mathrm{nor}^0(\mathbf{c}_{q_0,\eta_0}), \lg(\frac{n_{2,i(\mathbf{d})}}{\tilde{s}}) - 1, k^* - 1\} \geq m+1$. By the choice of ν_0, $\lg(\frac{n_{2,i(\mathbf{d})}}{\tilde{s}}) \geq m + 1$. Since nor^* drops at most by 1, we have $\mathrm{nor}(\mathbf{d}, q_0(\eta_0)) \geq m + 1$. Now we define $s \in Q$ as follows:

(α) $\mathrm{rt}(s) = \eta_0$, $s(\eta_0) = q_0(\eta_0)$,
(β) $\mathbf{c}_{s,\eta_0} = \mathbf{d}$,
(γ) if $\rho \in \mathrm{rge}(\mathrm{val}(\mathbf{d}))$ and if $\rho = \eta \cup \{(y^\eta_{j,\ell}, \nu_\eta(y_\ell)) : \ell < \tilde{s}\}\}$ then $s^{\langle\rho\rangle} = r^\eta_j$.

Clearly $s \in Q$ and $q_t^{\langle\eta_0\rangle} \leq_{m+1} s$ and for every $\eta \in s^{[\ell]}$ the condition $s^{\langle\eta\rangle}$ forces a value to $\underset{\sim}{\tau}$, in fact $\ell = 1$ is o.k., by the way the r^η_j were chosen. So we get a contradiction to $\boxtimes$ for η_0. □

Conclusion 3.10. Q is a proper ${}^\omega\omega$-bounding forcing adding reals that specialises a given Aronszajn tree.

Proof. We show that forcing with Q adds a new real: Let $x_n \in \mathbf{T}$, $n \in \omega$, be pairwise different arbitrary nodes of the Aronszajn tree $\mathbf{T}$. Let $\underset{\sim}{f}$ be a name for the generic specification function. By Conclusion 2.12, $\underset{\sim}{f}$ is defined on the whole $\mathbf{T}$. Every condition $\langle \mathbf{c}^+_t : t \in T\rangle$ determines $\underset{\sim}{f}(x_n)$ iff $x_n \in \mathrm{dom}(\mathrm{rt}(T))$, so only for finitely many x_n. Hence by a density argument, $\langle (n, \underset{\sim}{f}(x_n)) : n \in \omega\rangle$ is a new real. For a proof that continuous reading of names implies properness and ${}^\omega\omega$-bounding see Sections 2.3 and 3.1 in [4]. □

Now the preservation theorems for properness and ${}^\omega\omega$-bounding allow us to iterate forcings $Q = Q_\mathbf{T}$ with countable support, for various $\mathbf{T}$. Starting form a ground model with $2^{\aleph_1} = \aleph_2$ we can successively specialise all Aronszajn trees in the ground model and in all intermediate models of the iteration and thus get a model where all Aronszajn trees are special and $\mathfrak{b} = \aleph_1$ and $2^\omega = \aleph_2$. Since ♣ and CH together imply ◊ (see [6, Fact 7.3]), SH and ♣ together imply $2^\omega \geq \aleph_2$.

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