FORCING DIAMOND AND APPLICATIONS TO ITERABILITY

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Abstract. We show that higher Sacks forcing at a regular limit cardinal and club Miller forcing at an uncountable regular cardinal both add a diamond sequence. We answer the longstanding question, whether $\kappa = \kappa^{<\kappa} \geq \aleph_1$ implies that κ-supported iterations of κ-Sacks forcing do not collapse κ^+ and are κ-proper in the affirmative. The results pertain to other higher tree forcings.

1. Introduction

Tree forcings like Silver forcing, Sacks forcing, Miller forcing or Laver forcing are used to arrange combinatorial properties of the power set of $\mathbb{R}$. Baumgartner [1], Kanamori [8] and later many researchers found analogues for an uncountable regular cardinal κ instead of ω that share at least part of the properties of their relatives at ω. The extent of the analogy depends on properties of κ. Here we are mainly interested in conditions that ensure the preservation of κ^+ and a version of κ-properness (see Definition 2.5) for iterations with supports of size $\leq \kappa$.

Baumgartner [1, Theorem 6.7] showed that the κ-supported product of κ-Silver forcing does not collapse κ^+ under $\diamondsuit_\kappa$. Kanamori showed that iterating κ-Sacks forcing with supports of size $\leq \kappa$ does not collapse κ^+ if $\diamondsuit_\kappa$ [8, Theorem 3.2] holds or if κ is strongly inaccessible [8, Section 6]. The same proofs work also for numerous $(< \kappa)$-closed forcings in which forcing conditions are trees with club many splitting nodes. Iterations may be replaced by $(\leq \kappa)$-supported products [8, Section 5].

Shelah [21] showed that $\kappa^\kappa = \kappa = \lambda^+ \geq \aleph_2$ implies $\diamondsuit_\kappa(\kappa \cap \mathrm{cof}(\mu))$ for any regular $\mu \neq \mathrm{cf}(\lambda)$. Hence for successor cardinals $\kappa = \kappa^{<\kappa} \geq \aleph_2$, the conditions that Baumgartner and Kanamori used for their iterability proofs are fulfilled.

In [12] we showed that in Kanamori's iterability theorem (see Theorem 1.4 below) the condition ($\diamondsuit_\kappa$ or κ is inaccessible) can be replaced by the slightly

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weaker $(\mathsf{DI})_\kappa$ (see Definition 2.3). There are regular limit cardinals $\kappa = \kappa^{<\kappa}$ with $\neg(\mathsf{DI})_\kappa$, see [6].

Here we show that $\kappa = \kappa^{<\kappa} \geq \omega_1$ suffices as a premise for κ-properness and not collapsing κ^+ in $\leq \kappa$-supported iterations of higher Sacks forcing. We do this by showing that $\diamondsuit_\kappa$ is forced by the first iterand of the respective forcings. A particularly simple case is Theorem 1.1 for κ weakly Mahlo. A more complex relative of Theorem 1.1 is Theorem 1.2 for any regular limit κ. The latter combined with Theorem 1.3 for $\kappa = \aleph_1$ and for κ that are not strongly inaccessible proves Corollary 1.5. In the case of Theorem 1.3 we can work with a fixed stationary set of potential splitting levels, and we will explain this by introducing W-versions of the tree forcings in Definition 6.1.

Our second result is: For club Miller and for club Laver forcing, the premise $\aleph_1 \leq \kappa^{<\kappa} = \kappa$ suffices for forcing a diamond and ensures iterability, see Theorem 1.6.

For regular uncountable κ under $\kappa^{<\kappa} > \kappa$ the forcing $\mathbb{Q}^{\text{Sacks}}_\kappa$ collapses κ^+ by [12, Section 4]. The combinatorial background Lemma 6.6 of Theorem 1.3 yields in Proposition 6.9 another type of names for collapsing functions under $\kappa^{<\kappa} > \kappa$ for regular κ that works also for the W-variants $\mathbb{Q}^{\text{Sacks}}_{(\kappa,W)}$ (see Definition 6.1). We do not consider singular κ here. For singular κ, higher tree forcings share features of Namba forcing, see, e.g. the Namba trees of height ω_1 used in [11].

Our first two theorems pertain to limit cardinals κ.

Theorem 1.1. *If κ is weakly Mahlo and $S \subseteq \{\delta < \kappa : \delta \text{ regular limit}\}$ is stationary in κ, then $\mathbb{Q}^{\text{Sacks}}_\kappa \Vdash \diamondsuit_\kappa(S)$. The same holds for $\mathbb{Q}^{\text{Silver}}_\kappa$.*

Theorem 1.2. *If κ is a regular limit cardinal, $\mu < \kappa$ is a regular cardinal, and $S \subseteq \kappa \cap \operatorname{cof}(\mu)$ is stationary in κ, then $\mathbb{Q}^{\text{Sacks}}_\kappa \Vdash \diamondsuit_\kappa(S)$. The same holds for $\mathbb{Q}^{\text{Silver}}_\kappa$.*

We note that for any regular limit κ, $\{\alpha < \kappa : |\alpha| = \alpha > \operatorname{cf}(\alpha)\}$ is stationary in κ, and hence for some regular $\mu < \kappa$, $\{\alpha < \kappa : \alpha = |\alpha|\} \cap \operatorname{cof}(\mu)$ is stationary in κ. This shows that Theorem 1.1 is not relevant for the proof of Corollary 1.5, since for any regular limit κ with $\kappa^{<\kappa} = \kappa$ we can invoke Theorem 1.2.

Working with the approachability ideal, we present a different name of a diamond in Section 6, and with this we settle the case of $\kappa = \aleph_1$ in Kanamori's question. We apply following theorem with $\kappa = \aleph_1$ under CH with $\sigma = \aleph_0$. The theorem also shows that in the case of κ being a successor cardinal $\kappa = \lambda^+$, a diamond on a stationary subset of $\kappa \cap \operatorname{cof}(\operatorname{cf}(\lambda))$ is forced.

The subforcings $\mathbb{Q}^{\text{Sacks}}_{(\kappa,W)}$ from Definition 6.1, $W \subseteq \kappa$, W stationary, of $\mathbb{Q}^{\text{Sacks}}_\kappa$ respect weaker demands on splitting nodes than $\mathbb{Q}^{\text{Sacks}}_\kappa$ and are still $(< \kappa)$-complete. The forcing $\mathbb{Q}^{\text{Sacks}}_{(\kappa,W)}$ for $W = \kappa$ is $\mathbb{Q}^{\text{Sacks}}_\kappa$.

The names of the diamonds in the next theorem are based on approachability and Bernstein combinatorics.

Theorem 1.3. *Assume that $\kappa^{<\kappa} = \kappa \geq \aleph_1$. Let $W \subseteq \kappa$ be stationary. Suppose there are $\sigma < \kappa$ and $S \subseteq \kappa$ with the following properties:*

(∗) *(a)* $\kappa = \kappa^{<\kappa} = 2^\sigma$,
(b) $2^{<\sigma} < \kappa$, *and*
(c) $S \subseteq \kappa \cap \mathrm{cof}(\mathrm{cf}(\sigma))$, *and S is stationary in κ and $S \in \check{I}[\kappa]$.*

Then $\mathbb{Q}^{\mathrm{Sacks}}_{(\kappa,W)} \Vdash \diamondsuit_\kappa(S)$.

Analogues of Theorem 1.3 for the W-variants of κ-Silver forcing, club κ-Miller forcing and Laver forcing hold. The approachability ideal $\check{I}[\kappa]$ is reviewed in Subsection 2.3.

We recall:

Theorem 1.4 (Kanamori, [8])**.** *Assume $\kappa^{<\kappa} = \kappa \geq \aleph_1$. Let γ be an ordinal and let $\mathbb{P} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \gamma, \beta < \gamma \rangle$ be a $(\leq \kappa)$-support iteration such that for $\beta < \gamma$, $\mathbb{P}_\beta \Vdash \underset{\sim}{\mathbb{Q}}_\beta = \mathbb{Q}^{\mathrm{Sacks}}_\kappa$. Assume $\diamondsuit_\kappa$. Then $\mathbb{P}_\gamma$ has the following properties.*

(1) $\mathbb{P}_\gamma$ does not collapse κ^+.
(2) $\mathbb{P}_\gamma$ is κ-proper.

Combining Kanamori's theorem with Theorem 1.2, Theorem 1.3 and with Shelah's result on the diamond on successor cardinals λ^+ with $2^\lambda = \lambda^+$ [21], we derive the following.

Corollary 1.5. *Theorem 1.4 holds without the assumption of $\diamondsuit_\kappa$ in the ground model.*

This answers Kanamori's question from [8]. It applies to Silver, Miller and Laver forcing at κ as well. It applies to the W-variants if (∗) of Theorem 1.3 holds.

Our next theorem shows that for club κ-Miller/Laver forcing, for any uncountable κ with $\kappa^{<\kappa} = \kappa$ there is a name of a diamond that is much simpler than the names used in Theorem 1.2 and Theorem 1.3. Part (3) answers [12, Question 2.17]. In the case of κ being strongly inaccessible iterability was proved by Kanamori [8, Section 6] for the Sacks version, and by Friedman and Zdomskyy work [5] for the Miller version. Part (3) of the following theorem extends their result to uncountable κ with $\kappa^{<\kappa} = \kappa$.

Theorem 1.6. *Assume $\kappa^{<\kappa} = \kappa \geq \aleph_1$.*

(1) Both $\mathbb{Q}^{\mathrm{Miller}}_\kappa$ and $\mathbb{Q}^{\mathrm{Laver}}_\kappa$ force $\diamondsuit_\kappa$.
(2) If $S \subseteq \kappa$ is stationary, then $\mathbb{Q}^{\mathrm{Miller}}_\kappa$ forces $\diamondsuit_\kappa(S)$ and the same holds for $\mathbb{Q}^{\mathrm{Laver}}_\kappa$.
(3) The iterability theorem holds as in Corollary 1.5.

The article [9] by Khomskii et. el. focuses on higher Laver forcing.

Organisation of the paper. In Section 2 we review definitions. In Section 3 we proof Theorem 1.1 and we show that diamond in the one-step-extension leads to Corollary 1.5. In Section 4 we proof Theorem 1.2. In Section 5 we prove Theorem 1.6. In Section 6 we introduce $\mathbb{Q}^{\rm Sacks}_{(\kappa,W)}$ and prove Theorem 1.3 and related results about collapsing functions for $\mathbb{Q}^{\rm Sacks}_{(\kappa,W)}$.

2. Background

Now we review the mentioned notions.

2.1. **Combinatorics and Properness.**

Definition 2.1. Let κ be a cardinal. For a regular cardinal $\mu < \kappa$, we let $\kappa \cap \mathrm{cof}(\mu)$ stand for $\{\alpha \in \kappa : \mathrm{cf}(\alpha) = \mu\}$.

Definition 2.2. Let κ be a cardinal of uncountable cofinality and let S be a stationary subset of κ. We let $\diamondsuit_\kappa(S)$ be the following statement: There is a sequence $\langle d_\delta : \delta \in S\rangle$ such that $d_\delta \in {}^\delta 2$ and such that for any $x \in {}^\kappa 2$ the set $\{\delta \in S : d_\delta = x \restriction \delta\}$ is stationary. For $\diamondsuit_\kappa(\kappa)$ we write just $\diamondsuit_\kappa$.

We recall a weakening of the diamond, called DI.

Definition 2.3 (See [15, 19, 20])**.** For a regular uncountable κ we let $(\mathsf{DI})_\kappa(S)$ mean the following: There is a sequence $\bar{\mathcal{D}} = \langle \mathcal{D}_\delta : \delta \in S\rangle$ such that $\mathcal{D}_\delta \subseteq {}^\delta\delta$ is of cardinality $< \kappa$ and for every $x \in {}^\kappa\kappa$ there are stationarily many $\delta \in S$ such that $x \restriction \delta \in \mathcal{D}_\delta$. For $(\mathsf{DI})_\kappa(\kappa)$ we write $(\mathsf{DI})_\kappa$.

Inaccessibility implies $(\mathsf{DI})_\kappa$.

Definition 2.4. An uncountable limit cardinal κ is called *weakly Mahlo* if κ is a regular limit cardinal (i.e., κ is weakly inaccessible) and the set of regular limit cardinals below κ is stationary in κ.

Definition 2.5. Let $\mathcal{H}(\theta) = (H(\theta), \in, <_\theta)$, and $N \prec \mathcal{H}(\theta)$ and $\mathbb{Q} \in N$, $p \in \mathbb{Q} \cap N$. A condition q is called $(N,\mathbb{Q})$-generic above p if $q \geq p$ and for any dense subset D of $\mathbb{Q}$, if $D \in N$, then $q \Vdash \underset{\sim}{\mathbf{G}} \cap D \cap N \neq \emptyset$.

Let $\kappa^{<\kappa} = \kappa$. A notion of forcing $\mathbb{Q}$ is called κ-proper if for any sufficiently large θ there is a club (in $[H(\theta)]^\kappa$) of $N \prec H(\chi)$, $|N| = \kappa$ with ${}^{<\kappa}N \subseteq N$ such that: If $\kappa, p, \mathbb{Q} \in N$, and $p \in \mathbb{Q} \cap N$, then there is a stronger $(N,\mathbb{Q})$ generic condition q.

2.2. **Notation for Tree Forcing.**

Definition 2.6. Let κ be an infinite cardinal.

(1) We write ${}^{\kappa>}\kappa = \{t \colon \alpha \to \kappa : \alpha < \kappa\}$. If $s, t \in {}^{\kappa>}\kappa$ we call s an inital segment of t if $t \restriction \mathrm{dom}(s) = s$. A *tree (on κ)* is a non-empty subset of ${}^{\kappa>}\kappa$ that is closed under initial segments. We use the symbol $\trianglelefteq$ for the initial segment relation and the symbol $\triangleleft$ for the corresponding strict relation.

(2) Let $T \subseteq {}^{\kappa>}\kappa$ be a tree and $s \in T$. We let

$$T^{\langle s \rangle} = \{t \in T : t \trianglelefteq s \vee s \trianglelefteq t\}.$$

(3) The elements of a tree are called nodes. A node that has at least two immediate $\triangleleft$-successors in p is called a splitting node of p. The set of splitting nodes of p is denoted by $\mathrm{split}(p)$.

(4) Let $T \subseteq {}^{\kappa>}\kappa$ be a tree that contains a splitting node. We let the trunk of T, $\mathrm{tr}(T)$, be the $\trianglelefteq$-least splitting node of T.

Definition 2.7 (Kanamori's Higher Sacks Forcing, [8])**.** Let κ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$. Conditions in the forcing order $\mathbb{Q}_\kappa^{\mathrm{Sacks}}$ are trees $p \subseteq {}^{\kappa>}2$ with the following additional properties:

(1) (Perfectness) For any $s \in p$ there is an extension $t \trianglerighteq s$ in p such that t has two immediate successors.

(2) (Closure of splitting) For each increasing sequence of length $< \kappa$ of splitting nodes, the union of the nodes on the sequence is a splitting node of p as well.

A condition q is stronger than p if $q \subseteq p$.

The forcing $\mathbb{Q}_\kappa^{\mathrm{Sacks}}$ has a dense subset with the following closure property: For every increasing sequence $\langle t_i : i < \lambda \rangle$ of length $\lambda < \kappa$ of nodes $t_i \in p \in \mathbb{Q}_\kappa^{\mathrm{Sacks}}$ we have that the limit of the sequence $\bigcup\{t_i : i < \lambda\}$ is also a node in p.

Club Silver forcing is called $R(1, \kappa)$ in [1, Theorem 6.7].

Definition 2.8 (Club Silver forcing)**.** Let κ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$. Conditions in the forcing order $\mathbb{Q}_\kappa^{\mathrm{Silver}}$ are partial functions $f\colon \mathrm{dom}(f) \to 2$ where $\mathrm{dom}(f)$ is a non-stationary subset of κ.

Stronger conditions are extensions of the function f.

Equivalently one can see a Silver condition f as a set of nodes of a higher Silver tree $T_f = \{t \in {}^{\kappa>}\kappa : t \restriction \mathrm{dom}(f) = f \restriction \mathrm{dom}(t)\}$. We can restrict $\mathbb{Q}_\kappa^{\mathrm{Silver}}$ to the dense set of conditions f for which $\kappa \setminus \mathrm{dom}(f)$ is a club. For these T_f, the limit of any increasing sequence of splitting nodes is a splitting node. This shows that Theorem 1.1 and Theorem 1.2 work also for Silver forcing.

Definition 2.9 (Club Miller Forcing/Club Laver Forcing)**.** Let κ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$. Conditions in the forcing order $\mathbb{Q}_\kappa^{\mathrm{Miller}}$ are trees $p \subseteq {}^{\kappa>}\kappa$ with the following additional properties:

(1) (Club filter superperfectness) For any $s \in p$ there is an extension $t \trianglerighteq s$ in p such that

$$\mathrm{osucc}_p(t) := \{\alpha \in \kappa : t^\frown\langle\alpha\rangle \in p\} \text{ contains a club in } \kappa.$$

We require that each node has either only one direct successor or splits into a club.

(2) (Closure of splitting) For each increasing sequence of length $< \kappa$ of splitting nodes, the union of the nodes on the sequence is a splitting node of p as well.

A condition q is stronger than p if $q \subseteq p$.

In the case of $\mathbb{Q}_\kappa^{\rm Laver}$ we strengthen (1) to the following: there is a node $s \in p$, called the trunk of p and denoted as $\mathrm{tr}(p)$, and for any $t \in p$ with $s \subseteq t$ the set $\mathrm{osucc}_p(t)$ contains a club in κ.

We remark that club Miller forcing $\mathbb{Q}_\kappa^{\rm Miller}$ is well-known: Clauses (1) and (2) imply that any $p \in \mathbb{Q}_\kappa^{\rm Miller}$ has for any $s \in p$ and any height α some $t \in p$ with domain α. Again, the forcing $\mathbb{Q}_\kappa^{\rm Miller}$ has a dense subset with the following closure property: For every increasing sequence $\langle t_i : i < \lambda \rangle$ of length $\lambda < \kappa$ of nodes $t_i \in p \in \mathbb{Q}_\kappa^{\rm Miller}$ we have that the limit of the sequence $\bigcup \{t_i : i < \lambda\}$ is also a node in p. These clauses are sometimes added to the definition, see e.g., Brendle, Brooke-Taylor, Friedman, Montoya [2, Def. 74], where the forcing is called $\mathbb{MI}_\kappa^{\rm Clubfilter}$. Friedman and Zdomskyy [5] add the requirement that the successor set of a limit splitting node is a subset of the intersection of the $\lhd$-preceding splitting nodes. The set of these conditions is dense in $\mathbb{MI}_\kappa^{\rm Clubfilter}$. The recent article [9] is concerned with higher Laver forcing.

Definition 2.10. Let κ, μ be cardinals. Let $p \subseteq {}^{\kappa>}\mu$ be a tree, i.e., closed downwards. We let $[p] = \{b \in {}^\kappa\mu : \forall \alpha \in \kappa, b \restriction \alpha \in p\}$. The set$[p]$ is called the rump, body or set of κ-branches of p. Note that $p \mapsto [p]$ is not an absolute function.

Note that $p \mapsto [p]$ is not an absolute function. Since forcing conditions are perfect trees, in the generic extension there are new branches.

2.3. **Review of $I[\kappa]$.** We review the approachability ideal $I[\kappa]$ and its variant $\check{I}[\kappa]$ (from [14, Definition 6, page 360, page 377]) that is suitable also for the description of regular limit cardinals κ. Our review focuses on results that we use in Section 6.

Definition 2.11 (The Approachability Ideal on Successors [16])**.** Let $\bar{a} = \langle a_\alpha : \alpha < \kappa \rangle$ enumerate a subset of $\kappa^{<\kappa}$. The ideal $I[\kappa](\bar{a})$ is be the set of $S \subseteq \kappa$ such that for a club $C \subseteq \kappa$ for any $\delta \in S \cap C$, there is a set $A_\delta \subseteq \delta$ that is cofinal in δ with $\mathrm{ot}(A_\delta) = \mathrm{cf}(\delta) < \delta$ and satisfies $\{A_\delta \cap \beta : \beta < \delta\} \subseteq \{a_\alpha : \alpha < \delta\}$. The *approachability ideal* $I[\kappa]$ is the union of all the $I[\lambda](\bar{a})$, $\bar{a}$ as above. If $\kappa^{<\kappa} = \kappa$, we let $\langle a_\alpha : \alpha < \kappa \rangle$ be an enumeration of $\kappa^{<\kappa}$ and set $I[\kappa] = I[\kappa](\bar{a})$.

Remark 2.12. Equivalently we can require in addition that the A_δ be closed. The reason is, that we can choose $\bar{a}$ so that if there is a sequence of unbounded witnesses $\langle A_\delta : \delta \in S \rangle$ for $S \in I[\kappa](\bar{a})$ then there is also a sequence of club witnesses $\langle C_\delta : \delta \in S \rangle$ for $S \in I[\kappa](\bar{b})$ for a slightly richer sequence $\bar{b} \in {}^\kappa([\kappa]^{<\kappa})$. For a detailed proof we refer to [17, Lemma 4.4].

Most of the literature on $I[\kappa]$ in [16], [14], [4] focusses on the case of κ being a successor cardinal. For a successor cardinal κ, the regular cardinals below κ form a non-stationary set, and dropping the clause $\mathrm{ot}(C_\delta) = \mathrm{cf}(\delta) < \delta$ Definition 2.11 yields an equivalent notion. We work with a version that dispenses with $\mathrm{cf}(\delta) < \delta$.

Definition 2.13 (From [16], [14, Definition 6 and page 377], [17]). Let $\bar{a} = \langle a_\alpha : \alpha < \kappa \rangle$ enumerate a subset of $\kappa^{<\kappa}$. The ideal $\check{I}[\kappa](\bar{a})$ is be the set of $S \subseteq \kappa$ such that for a club $C \subseteq \kappa$ for any $\delta \in S \cap C$, there is a club $C_\delta \subseteq \delta$ that is cofinal in δ with $\mathrm{ot}(C_\delta) = \mathrm{cf}(\delta)$ and satisfies $\{C_\delta \cap \beta : \beta < \delta\} \subseteq \{a_\alpha : \alpha < \delta\}$. The *approachability ideal* $\check{I}[\kappa]$ is the union of all the $\check{I}[\lambda](\bar{a})$, $\bar{a}$ as above. If $\kappa^{<\kappa} = \kappa$, we let $\langle a_\alpha : \alpha < \kappa \rangle$ be an enumeration of $\kappa^{<\kappa}$ and have $\check{I}[\kappa] = \check{I}[\kappa](\bar{a})$.

Note, that $\{\delta < \kappa : \delta \text{ is regular}\} \in \check{I}[\kappa]$. We just take $\bar{a} = \{\alpha : \alpha \in \kappa\}$ and for regular $\delta < \kappa$, $C_\delta = \delta$.

Many authors call $\check{I}[\kappa]$ now $I[\kappa]$, see e.g. [13], [3], [10]. Many equivalent definitions of the ideal are given in [17] and [4].

Definition 2.14. Let κ be a regular cardinal. A κ-approximating sequence $\mathfrak{M} = \langle M_i : i < \kappa \rangle$ is a continuously increasing sequence of $M_i \prec (H(\chi), \in, <^*_\chi, \kappa, \dots)$ for some regular cardinal $\chi > 2^{2^\kappa}$ with $\langle M_i : i < j \rangle \in M_{j+1}$ and $|M_i| < \kappa$ and a countable signature. We define $S[\mathfrak{M}]$ to be the set of $\delta < \kappa$ such that δ, $M_\delta \cap \kappa = \delta$ and there is a cofinal subset A of δ of order-type $\mathrm{cf}(\delta)$ with the property that every proper initial segment of A is in M_δ.

The set $S(\mathfrak{M})$ is called $S_2(\mathfrak{M})$ in [14, Definition 6].

Theorem 2.15 (Shelah, Eisworth [4, Theorem 3.6]). *$S \in \check{I}[\kappa]$ iff there is a club E in κ, such that $S \cap E \subseteq S(\mathfrak{M})$ for a κ-approximating sequence $\mathfrak{M}$.*

Definition 2.16. For a subset E of κ, let $\mathrm{acc}(E) = E \cap \{\alpha < \kappa : \alpha = \sup(E \cap \alpha)\}$, and $\mathrm{nacc}(E) = E \setminus \mathrm{acc}(E)$.

Theorem 2.17 (Shelah [16], Eisworth [4, Theorem 3.7]). *Let κ be a regular uncountable cardinal. For any $S \subseteq \kappa$ the following are equivalent:*

(1) $S \in I[\kappa]$,
(2) There is a sequence $\langle C_\alpha : \alpha < \kappa \rangle$ and a closed unbounded $E \subseteq \kappa$ such that for any $\alpha < \kappa$,
  (a) C_α is a closed but not necessarily unbounded subset of α,
  (b) for any $\beta \in \mathrm{nacc}(C_\alpha)$ we have $C_\beta = C_\alpha \cap \beta$,
  (c) if $\alpha \in E \cap S$ then α is singular and C_α is a closed unbounded subset of α of order-type $\mathrm{cf}(\alpha)$.

The following theorem shows that at any uncountable $\kappa = \kappa^{<\kappa}$ the premises of Theorem 1.3 are fulfilled for suitable λ.

Theorem 2.18 (Shelah [16], [17]). *let $\lambda < \kappa$ be cardinals such that κ is regular and $\kappa^{<\lambda} \leq \kappa$. Then there is a stationary set $S \subseteq \kappa \cap \mathrm{cof}(\mathrm{cf}(\lambda))$ with $S \in \check{I}[\kappa]$.*

Proof. We let $\langle a_\alpha : \alpha < \kappa \rangle$ enumerate ${}^{\lambda >}\kappa$, such that each element appears κ often. We let

$$\begin{aligned} S = &\{\delta \in \kappa \cap \operatorname{cof}(\operatorname{cf}(\lambda)) : (\exists \eta \in {}^{\operatorname{cf}(\lambda)}\delta) \\ &\quad \big(\sup(\operatorname{range}(\eta)) = \delta \wedge (\forall i < \operatorname{cf}(\lambda))(\exists j < \delta)\eta \restriction i = a_j\big)\}. \end{aligned}$$

By definition, $S \in \check{I}[\kappa](\bar{a})$. We show that S is stationary. Let $C \subseteq \kappa$ be a club. By induction on $i < \operatorname{cf}(\lambda)$ we choose $\eta_i \in {}^{i+1}\kappa$ and $\delta_i \in \kappa$ with the following properties for any $i < \operatorname{cf}(\lambda)$,

(a) $\delta_i \in C$
(b) for $i < j < \operatorname{cf}(\lambda)$, $\delta_i < \delta_j$,
(c) $\eta_i = \langle \delta_j : j \leq i \rangle$,
(d) there is $k \leq \delta_{i+1}$ with $\eta_i = a_k$.

$i = 0$: We let $\delta_0 \in C$. We let $\eta_0 = \{(0, \delta_0)\}$.

Successor step: $i = j + 1$. We choose $\delta_i \in C \setminus (\delta_j + 1)$ such that there is some $k \leq \delta_{j+1}$ with $\eta_j = \langle \delta_\ell : \ell \leq j \rangle = a_k$. Then we let $\eta_i = \eta_j \cup \{(i, \delta_i)\}$.

Limit step $i < \operatorname{cf}(\lambda)$: We let $\delta_i = \sup\{\delta_j : j < i\}$ and $\eta_i = \bigcup\{\eta_j : j < i\} \cup \{(i, \delta_i)\}$. Then we pick δ_{i+1} such that for some $k \leq \delta_{i+1}$, $\eta_i = a_k$.

Then $\eta = \bigcup\{\eta_i : i < \operatorname{cf}(\lambda)\}$ witnesses $\delta = \sup\{\delta_i : i < \operatorname{cf}(\lambda)\} \in S \cap C$. □

2.4. **Forcing.** Our notions of forcing are written in Israeli style: $p \leq q$ means that q is stronger than p. We write $\mathbb{P} \Vdash \varphi$ if any condition in $\mathbb{P}$ forcing φ. Equivalently on can say the weakest condition of $\mathbb{P}$ forces φ.

3. The Case of κ being Weakly Mahlo

We consider regular limit cardinals κ that are not necessarily strong limits. For κ being weakly Mahlo there is an two step derivation of a name of a diamond, which we present in this section. We show that diamond in the one-step-extension leads to Corollary 1.5.

Definition 3.1. Let δ be an ordinal of unbountable cofinality. Let $S \subseteq \delta$ be stationary in δ.

(1) The quantifier $\forall^{\text{club}}\alpha \in S, \varphi(\alpha)$ says that there is a club C in δ such that $S_\varphi = \{\alpha \in S : \varphi(\alpha)\} \supseteq S \cap C$.
(2) We define the quantifier $\exists^{\text{stat}}\alpha \in S, \varphi(\alpha)$ as $S_\varphi = \{\alpha \in S : \varphi(\alpha)\}$ is a stationary subset of δ.

Definition 3.2. Let G be a $\mathbb{P}$-generic filter over V and assume that $\mathbb{P}$ is one of our named forcings. The following function $\eta \colon \kappa \to \kappa$ is called *the generic branch*: $\eta = \bigcup\{\operatorname{stem}(p) : p \in G\}$. We let $\underset{\sim}{\eta}$ be name for η.

We name a combinatorial principle $\boxplus_{\kappa,S}$. This asserts that there are stationarily many $\delta \in S$ for which δ can be partitioned into δ-many parts such that each of them is stationary in δ, via a partition that does not depend on δ.

Definition 3.3. Let κ be a weakly Mahlo cardinal and let $S \subseteq \{\delta \in \kappa : \delta \text{ is a regular limit cardinal}\}$.

$\boxplus_{\kappa,S}$ is the following statement: There is a function $f\colon \kappa \to \kappa$ such that for any $\alpha < \kappa$, $f(\alpha) < \min(\alpha, 1)$ and

$$(\exists^{\rm stat}\delta \in S)(\forall \beta < \delta) \\ (S_{\delta,\beta} := \{\gamma \in \delta : f(\gamma) = \beta\} \text{ is stationary in } \delta). \tag{3.1}$$

Now the proof of Theorem 1.1 consists of Lemma 3.4 and Lemma 3.6.

Lemma 3.4. *If κ is weakly Mahlo and $S \subseteq \{\delta < \kappa : \delta$ is a regular limit cardinal$\}$ then $\boxplus_{\kappa,S}$.*

Proof. We let

$$f(\gamma) = \begin{cases} \beta, & \text{if } \operatorname{cf}(\gamma) = \aleph_{\beta+1}; \\ 0, & \text{else.} \end{cases} \tag{3.2}$$

and Statement (3.1) holds in the slightly stronger form

$$(\forall \delta \in S)(\forall \beta < \delta)(\{\gamma < \delta : f(\gamma) = \beta\} \text{ is stationary in } \delta).$$

□

Definition 3.5. For $E \subseteq \kappa$ we write $\operatorname{acc}^+(E) = \{\alpha \in \kappa : \alpha = \sup(E \cap \alpha)\}$ and $\operatorname{acc}(E) = E \cap \operatorname{acc}^+(E)$.

We state the following lemma for Sacks forcing $\mathbb{Q}^{\rm Sacks}_\kappa$. It holds for any of the four types of tree forcings. For Miller forcing and for Laver forcing, we work with one fixed partition of κ into two stationary sets T_0, T_1. This partition is used to define the trunk lengthenings: For $j = 0, 1$, $\underset{\sim}{\eta}(\varepsilon) = j$ in Equation (3.3), in Clause $(*)_3$(e), and in Equations (3.5), (3.7) is replaced by $\underset{\sim}{\eta}(\varepsilon) \in T_j$.

Lemma 3.6. *Let $\kappa > \aleph_0$ and $S \subseteq \kappa$ be stationary. If $\boxplus_{\kappa,S}$ holds, then $\mathbb{Q}^{\rm Sacks}_\kappa \Vdash \diamondsuit_\kappa(S)$.*

Proof. We let $\boxplus_{\kappa,S}$ witnessed by f and let for $\delta \in S$, $S_{\delta,\beta} = \{\varepsilon < \delta : f(\varepsilon) = \beta\}$. For stationarily many $\delta \in S$, for any $\beta < \delta$, $S_{\delta,\beta}$ a stationary subset of δ. Let S' be a stationary set of these good δ. We define the name $\langle \underset{\sim}{\nu}_\delta : \delta \in S\rangle$ for a sequence by letting for $\delta \in S'$, $\beta < \delta$, $j = 0, 1$,

$$\mathbb{Q}^{\rm Sacks}_\kappa \Vdash \text{“}\underset{\sim}{\nu}_\delta(\beta) = j \;\leftrightarrow\; (\forall^{\rm club}\varepsilon \in S_{\delta,\beta})(\underset{\sim}{\eta}(\varepsilon) = j)\text{”}. \tag{3.3}$$

For $\delta \in S \setminus S'$, we can let $\underset{\sim}{\nu}_\delta$ be a name for the zero sequence of length δ.

$$\mathbb{Q}^{\rm Sacks}_\kappa \Vdash \text{“}\langle \underset{\sim}{\nu}_\delta : \delta \in S\rangle \text{ is a } \diamondsuit_\kappa(S)\text{-sequence.”}$$

Towards this suppose that

$$p \Vdash \text{“}\underset{\sim}{x} \in {}^\kappa 2, \text{ and } \underset{\sim}{D} \text{ is a club subset of } \kappa.\text{”}$$

We show that there is some $q \geq p$ that forces $\delta \in \underset{\sim}{D} \cap S'$ and $\underset{\sim}{x} \restriction \delta = \underset{\sim}{\nu}_\delta$.

We let $\chi = \beth_\omega(\kappa)$ and let $<^*_\chi$ be a well-ordering of $H(\chi)$. We choose a κ-approximating (see Definition 2.14) sequence $\langle N_\varepsilon : \varepsilon < \kappa\rangle$ with

$$\mathbf{c} = (\kappa, p, \bar{\underset{\sim}{\nu}}, \underset{\sim}{x}, \underset{\sim}{D}, S) \in N_0.$$

We let $E = \{\alpha < \kappa : N_\alpha \cap \kappa = \alpha\}$. We pick any δ with

$$\delta \in S' \cap \mathrm{acc}(E)$$

For $\varepsilon < \delta$, we let $\kappa_\varepsilon = N_\varepsilon \cap \kappa$.

By induction on $\varepsilon \leq \delta$ we chose a condition p_ε such that for any ε the following holds

($\circledast$) p_ε is the $<^*_\chi$-least element such that
- (a) $p_\varepsilon \geq p$. (We shall prove that $p_\varepsilon \in N_{\varepsilon+1}$ in our construction but it is better to not state it in our demands.)
- (b) $p_\varepsilon \geq p_\zeta$ for $\zeta < \varepsilon$.
- (c) $\mathrm{lg}(\mathrm{tr}(p_\varepsilon)) \geq \kappa_\varepsilon$.
- (d) p_ε forces values to $\underset{\sim}{x} \restriction \kappa_\varepsilon$, $\underset{\sim}{D} \cap \kappa_\varepsilon$ and $\min(\underset{\sim}{D} \setminus \kappa_\varepsilon)$ call them x_ε, e_ε, γ_ε respectively.
- (e) For limit ordinals $\varepsilon < \delta$, $\mathrm{tr}(p_\varepsilon)(\kappa_\varepsilon) = x_\varepsilon(f(\kappa_\varepsilon))$.

We can carry the induction since $\mathbb{Q}$ is $(< \kappa)$-complete and for clause (e) we recall $f(\kappa_\varepsilon) < \kappa_\varepsilon$. More fully, let us prove ($\circledast_\varepsilon$) by induction on ε,

$\circledast_\varepsilon$
- $\mathbf{m}_\varepsilon = \langle p_\zeta, x_\zeta, e_\zeta, \gamma_\zeta : \zeta < \varepsilon\rangle$ exists and is unique and $\zeta < \varepsilon$ implies $\mathbf{m}_{\zeta+1} \in N_{\zeta+1}$.
- $\mathbf{m}_\varepsilon$ is defined in $(H(\chi), \in, <^*_\chi)$ by a formula $\varphi = \varphi(x, \bar{y})$ with x for $\mathbf{m}_\varepsilon$ and $\bar{y} = (y_0, y_1)$ with $y_0 = \bar{N} \restriction \varepsilon$ and $y_1 = \mathbf{c}$ from $(\circledast)_1$(e).

Case 1 $\varepsilon = 0$.

$\mathbf{m}_0 = \langle\rangle$ and $\mathrm{lg}(\bar{N} \restriction 0) = 0$.

Case 2 $\varepsilon = \zeta + 1$.

Now p_ε is the $<^*_\chi$ first element of $\mathbb{Q}$ satisfying clauses $(\circledast)_3$(a) - (d). There is no requirement (e), since ε is a successor. Clearly such a p exists and hence one of them must be $<^*_\chi$-least. As each element of $H(\chi)$ mentioned above is computable from $\bar{N} \restriction \varepsilon$, it belongs to $N_\varepsilon = N_{\zeta+1}$ since $\bar{N} \restriction (\zeta + 1) \in N_{\zeta+1}$ by $(\circledast)_1$(d)

Case 3 ε limit.

This is the only place at which we use the specific choice of $\mathbb{Q} = \mathbb{Q}^{\mathrm{Sacks}}_\kappa$ and not just $(< \kappa)$-complete forcings that force $\underset{\sim}{\eta} \notin \mathbf{V}$ but that the limit of splitting nodes is a splitting node. By $(< \kappa)$-completeness $q = \bigcap_{\zeta<\varepsilon} p_\zeta$ in $\mathbb{Q}^{\mathrm{Sacks}}_\kappa$. Also for $\zeta < \varepsilon$, $\mathrm{lg}(\mathrm{tr}(p_\zeta)) \geq \kappa_\zeta = N_\zeta \cap \kappa \in N_{\zeta+1}$. Hence $\mathrm{lg}(\mathrm{tr}(p_\zeta)) \in [\kappa_\zeta, \kappa_{\zeta+1})$ for $\zeta < \varepsilon$. Also for

$$\forall \zeta \leq \xi < \varepsilon (\mathrm{tr}(p_\xi) \in \mathrm{split}(p_\zeta))$$

by induction hypothesis. Hence by the definition of the Sacks forcing Definition 2.7 clause number (2) we have $\forall \zeta < \varepsilon (\bigcup\{\mathrm{tr}(p_\xi) : \xi < \varepsilon\} \in \mathrm{split}(p_\zeta)$ and $\bigcup\{\mathrm{tr}(p_\xi) : \xi < \varepsilon\} \in \mathrm{split}(q)$. We have $\mathrm{lg}(\mathrm{tr}(q)) = \kappa_\varepsilon$, since $\bar{N}$ is continuous and $\kappa_\varepsilon = N_\varepsilon \cap \kappa$. Moreover $q \Vdash x_{\kappa_\varepsilon} = \underset{\sim}{x} \restriction \kappa_\varepsilon$. We can compute $\mathrm{tr}(q)$, κ_ε and $f(\kappa_\varepsilon)$ from $(\bar{N} \restriction \varepsilon, \mathbf{c})$. Moreover

$$q \Vdash x_{\kappa_\varepsilon} = \underset{\sim}{x} \restriction \kappa_\varepsilon \wedge \kappa_\varepsilon \in \underset{\sim}{D}. \tag{3.4}$$

by the induction hypothesis (⊛)(a) – (d) and since ε is a limit.

Hence $\mathrm{tr}(q)$ has two immediate successors of length $\kappa_\varepsilon + 1$ and we can let $p_\varepsilon \geq q$ and

$$\mathrm{tr}(p_\varepsilon)(\kappa_\varepsilon) = x_{\kappa_\varepsilon}(f(\kappa_\varepsilon)). \tag{3.5}$$

This is clause (⊛)(e) that we have to fulfil.

Now we carried the induction. We let $q = \bigcap_{\varepsilon<\delta} p_\varepsilon$. We show that

$$q \Vdash \underset{\sim}{x} \restriction \delta = \underset{\sim}{\nu}_\delta. \tag{3.6}$$

Equation (3.4) implies at the limit δ: $q \Vdash \underset{\sim}{x} \restriction \delta = \bigcup\{x_\varepsilon : \varepsilon < \delta\}$.

We fix $\beta < \delta$. We verify that for club many ε in the stationary set $S_{\delta,\beta}$ we have

$$q \Vdash (\kappa_\varepsilon = \varepsilon \wedge f(\varepsilon) = \beta) \to \eta(\varepsilon) = \underset{\sim}{x}(\beta) = x_\varepsilon(\beta). \tag{3.7}$$

This follows from Equations (3.4) and (3.5) that we made true at club many κ_ε, and thus as club many ε, since $\varepsilon \mapsto \kappa_\varepsilon$ is a continuously increasing function on $\varepsilon < \delta$ and $\kappa_\delta = \delta$.

□

Now we turn to Corollary 1.5.

We notice that Lemma 3.7 and Lemma 3.9 hold also for any of our forcings. They could be mixed along an iteration. We call the first iterand $\mathbb{P}_1$.

Lemma 3.7. *Let $\mathbb{P}$ be a $\leq \kappa$-supported iteration of iterands of $\mathbb{Q}_\kappa^{\mathrm{Sacks}}$. For proving Corollary 1.5, it suffices to prove $\mathbb{P}_1 \Vdash \diamondsuit_\kappa$ and that the forcing $\mathbb{P}_1$ does not collapse κ^+.*

Proof. If $\kappa > \aleph_1$ is a successor cardinal, [21] gives the diamond in $\mathbf{V}$. Now let κ be a regular limit cardinal. Let $\mathbf{G}$ be $\mathbb{P}_1$ generic over $\mathbf{V}$. In $\mathbf{V}[\mathbf{G}]$ we apply Theorem 1.4 to the $(\leq \kappa)$-support iteration $\langle \mathbb{P}_\alpha/\mathbf{G}, \underset{\sim}{\mathbb{Q}}_\beta/\mathbf{G} : \alpha \in [1,\delta], \beta \in [1,\delta)\rangle$. □

For defining fusion sequences, we use a notion that is suitable for $p \subseteq {}^{\kappa>}\kappa$ and which could be simplified for $p \subseteq {}^{\kappa>}2$.

Definition 3.8. We assume $\kappa = \kappa^{<\kappa}$. We conceive a forcing notion as a tree $p \subseteq \kappa^{<\kappa}$ or $\subseteq 2^{<\kappa}$. Recall, splitting means splitting into a club. For $\alpha < \kappa$ we let

$$\mathrm{spl}_\alpha(p) = \big\{t \in \mathrm{split}(p) : \mathrm{ot}(\{s \subsetneq t : s \in \mathrm{split}(p)\}) = \alpha\big\}$$

and with a fixed enumeration $\{\eta_\alpha : \alpha < \kappa\}$ of ${}^{\kappa>}\kappa$ we define

$$\mathrm{cl}_\alpha(p) := \{s \in p : (\exists \gamma \leq \alpha)(\exists t \in \mathrm{spl}_\gamma(p))(s \subseteq t) \wedge (\exists \beta \leq \alpha)(s = \eta_\beta)\}.$$

We let $p \leq_\alpha q$ if $p \leq q$ and $\mathrm{cl}_\alpha(p) = \mathrm{cl}_\alpha(q)$.

Lemma 3.9. *Under $\kappa^{<\kappa} = \kappa$, the forcing $\mathbb{P}_1$ is κ-proper and does not collapse κ^+.*

Proof. Let $\chi > 2^\kappa$ is a regular cardinal. Let $p \in \mathbb{Q}_\kappa^{\rm Miller}$ and let $\underset{\sim}{\tau}$ be a name for function from κ into κ^+. Here the cardinal successor is interpreted in the ground model. We pick an $N \prec H(\theta)$ of size κ with ${}^{<\kappa}N$ with κ, p, $\mathbb{P}_1 \in N$ and let $\langle I_\varepsilon : \varepsilon < \kappa\rangle$ list all dense subseteq of $\mathbb{P}_1$ that are elements of N. Now by induction on $\varepsilon < \kappa$ we choose conditions p_ε, and sets $\{a_{s^\frown\langle i\rangle} : s \in {\rm spl}_\varepsilon(p_\varepsilon), i \in {\rm osucc}_{p_\varepsilon}(s)\} \subseteq {}^{(\varepsilon+1)}\kappa$ with the following properties:

(a) $p_\varepsilon \in N$.
(b) $p_0 = p$.
(c) If $\varepsilon < \delta$, then $p_\varepsilon \leq_\varepsilon p_\delta$.
(d) At limits ε, $p_\varepsilon = \bigcap\{p_\delta : \delta < \varepsilon\}$.
(e) if $s \in {\rm cl}_\varepsilon(p_\varepsilon) \cap {\rm spl}_\varepsilon(p_\varepsilon)$, then for every $i \in {\rm osucc}_{p_\varepsilon}(s)$, the condition $p_{\varepsilon+1}^{\langle s^\frown\langle i\rangle\rangle}$ is in I_ε and forces $\underset{\sim}{\tau} \restriction (\varepsilon+1) = a_{s^\frown\langle i\rangle}$.

In the end the fusion $q = \bigcap\{p_\varepsilon : \varepsilon < \kappa\} = \bigcup\{{\rm cl}_\varepsilon(p_\varepsilon) : \varepsilon < \kappa\}$ is am N-generic condition, since it forces for any $\varepsilon < \kappa$ that one of the $q^{\langle s^\frown\langle i\rangle\rangle}$, $s \in {\rm cl}_\varepsilon(q) \cap {\rm spl}_\varepsilon(q) = {\rm cl}_\varepsilon(p_\varepsilon) \cap {\rm spl}_\varepsilon(p_\varepsilon)$, $i \in {\rm osucc}_{p_\varepsilon}(s)$, is in $\mathbf{G} \cap I_\varepsilon \cap N$. For each $\varepsilon < \kappa$, we have for any $s \in {\rm cl}_\varepsilon(q) \cap {\rm spl}_\varepsilon(q) = {\rm cl}_\varepsilon(p_\varepsilon) \cap {\rm spl}_\varepsilon(p_\varepsilon)$, $i \in {\rm osucc}_{p_\varepsilon}(s)$, $q^{\langle s^\frown\langle i\rangle\rangle} \geq p_{\varepsilon+1}^{\langle s^\frown\langle i\rangle\rangle}$. The condition $p_{\varepsilon+1}$ forces that $\underset{\sim}{\tau} \restriction (\varepsilon+1)$ is one of the values in

$$K_\varepsilon = \{a_{s^\frown\langle i\rangle} : s \in {\rm spl}_\varepsilon(q) = {\rm spl}_\varepsilon(p_\varepsilon), i \in {\rm osucc}_{p_\varepsilon}(s)\}$$

that are given by $p_{\varepsilon+1}^{\langle s^\frown\langle i\rangle\rangle}$, $s \in {\rm spl}_\varepsilon(q) = {\rm spl}_\varepsilon(p_\varepsilon)$, $i \in {\rm osucc}_{p_\varepsilon}(s)$. For any ε, the stronger condition q forces this. By $\kappa^{<\kappa} = \kappa$, we have $|K_\varepsilon| \leq \kappa$. Since $|\bigcup\{K_\varepsilon : \varepsilon < \kappa\}| \leq \kappa$, the forcing preserves κ^+ as a cardinal. □

This concludes the proof of Corollary 1.5 in the weakly Mahlo case. In the complementary case, we finish with Theorem 1.2.

Remark 3.10. In the tree forcings considered here, any stationary subset S of κ stays stationary in any $\mathbb{P}_1$-extension. This is so since $\mathbb{P}_1$ is strongly $(< \kappa)$-distributive, i.e., for any sequence $\langle D_\beta : \beta < \kappa\rangle$ of dense subsets of $\mathbb{Q}_\kappa^{\rm Sacks}$ and any $p \in \mathbb{P}_1$, there is a sequence $\langle p_\beta : \beta < \kappa\rangle$ such that for $\beta < \kappa$, $p_\beta \in D_\beta$ and $p_0 \leq p$, see [7, Lemma 3.8].

4. The Case of a Regular Limit Cardinal κ

In this section we prove Theorem 1.2. Let κ be a regular limit cardinal. If the set of regular cardinals below κ is stationary, then Theorem 1.1 applies. If not, the set $S_{{\rm sing},\kappa}$ of singular cardinals in κ contains a club in κ. In any case, $S_{{\rm sing},\kappa}$ is stationary in κ. We apply the regressive function ${\rm cf}\colon S_{{\rm sing},\kappa} \to \kappa$ and find a regular cardinal μ and such that $S_{{\rm sing},\kappa} \cap {\rm cof}(\mu)$ is stationary in κ. Also for a weakly Mahlo cardinal, the set $S_{{\rm sing},\kappa}$ of singular cardinals is stationary, and hence for some μ, also $S_{{\rm sing},\kappa} \cap {\rm cof}(\mu)$ is stationary in κ. So for the main iterability theorem, we can do without Theorem 1.1.

First we recall club guessing.

Theorem 4.1 ([18, Def. III.1.3, Claim III.2.7, page 128])**.** *Let $\mu < \kappa$, κ be a regular limit cardinal, and $\mathrm{cf}(\mu) = \mu$. Let $S \subseteq \kappa \cap \mathrm{cof}(\mu)$ be stationary in κ. There is a sequence $\langle C_\delta : \delta \in S\rangle$ with the following properties: For any club E there is some $\delta \in E \cap S$ such that $C_\delta \subseteq E \cap \delta$. Moreover, for club many $\delta \in S$, $\sup\{\mathrm{cf}(\alpha) : \alpha \in C_\delta\} = |\delta|$.*

So we can thin out S to such a club C as in the last sentence and for each $\delta \in S \cap C$, we choose a cofinal club subsequence $\langle \alpha_{\delta,i} : i < \mu\rangle$ in C_δ such that $\lim_{i<\mu}\langle \mathrm{cf}(\alpha_{\delta,i+1}) : i < \mu\rangle = |\delta|$.

Fact 4.2. *For any weakly inaccessible κ, $\aleph_\kappa = \kappa$ and $C_{\mathrm{fix},\kappa} = \{\alpha, \kappa : \aleph_\alpha = \alpha\}$ is club in κ.*

Proof. First κ is a limit cardinal, so there is a limit ordinal λ, such that $\kappa = \aleph_\lambda$. In addition κ is regular, hence $\kappa = \mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$. So we have $\lambda = \kappa$. Since the $\aleph$-operation is continuous, the set $C_{\mathrm{fix},\kappa}$ is closed. We show that it is unbounded. To this end, let $\alpha < \kappa$. Then $\aleph_\alpha < \aleph_\kappa = \kappa$. Hence we can define $\alpha_0 = \alpha$, $\alpha_{n+1} = \aleph_{\alpha_n}$ for $n < \omega$. Then $\bigcup\{\alpha_n : n < \omega\} \in C_{\mathrm{fix},\kappa}$. □

If $S \subseteq C_{\mathrm{fix},\kappa}$, then the chosen sequences from above fulfil $\lim_{i<\mu}\langle \mathrm{cf}(\alpha_{\delta,i+1}) : i < \mu\rangle = \delta$. Such sequences will be important below.

We start with a combinatorial principle (in ZFC) and a name for a possible diamond sequence.

Definition 4.3. We let $\forall^{\mathrm{club}} i < \mu, \varphi(i)$ mean

- for any large enough $i < \omega$, $\varphi(i)$, if $\mu = \omega$,
- $\{i < \mu : \varphi(i)\}$ contains a club, if $\mu > \omega$.

We let $\exists^{\mathrm{stat}} \alpha < \delta, \varphi(\alpha)$ mean that the set of α with $\varphi(\alpha)$ is stationary in δ.

Definition 4.4. Let $\kappa > \mu$, κ be a regular limit cardinal, μ regular cardinals, and let $S \subseteq \kappa \cap \mathrm{cof}(\mu)$ be stationary in κ. Also $\mu = \omega$ is possible.

$\boxplus_{\kappa,\mu,S}$ is the following statement: There are $\bar{C}$, $\bar{\alpha}$ and f with the following properties:

(a) $\bar{C} = \langle C_\delta : \delta \in S\rangle$. C_δ is a club in δ of order type μ.

(b) For each $\delta \in S$, the sequence $\langle \alpha_{\delta,i} : i < \mu\rangle$ is an increasing continuous enumeration of C_δ. We write

$$\bar{\alpha} = \langle\langle \alpha_{\delta,i} : i < \mu\rangle : \delta \in S\rangle.$$

(c) For each $\delta \in S$, the sequence $\langle \mathrm{cf}(\alpha_{\delta,i+1}) : i < \mu\rangle)$ is strictly increasing with limit δ. By the choice of S, δ is a cardinal.

(d) Suppose that E is a club of κ. Then for stationarily many $\delta \in S$ we have:
   - for any large enough $i < \omega$, $\alpha_{\delta,i+1} \in E$, if $\mu = \omega$;
   - $\forall^{\mathrm{club}} i < \mu, \alpha_{\delta,i+1} \in E$, if $\mu > \omega$.

(e) The function $f\colon \kappa \to \kappa$ satisfies $f(\beta) < \min(\beta, 1)$ for any $\beta < \kappa$.

(f) For $\delta \in S' := S \cap C_{\mathrm{fix},\kappa}$ and $\beta < \delta$, we let

$$S_{\delta,i,\beta} := \{\gamma < \alpha_{\delta,i+1} : f(\gamma) = \beta\}.$$

We require: For any $\delta \in S'$ and any $\beta < \delta$ the statement

$$S_{\delta,i,\beta} \text{ is stationary in } \alpha_{\delta,i+1}.$$

holds

- for any large enough $i < \omega$, if $\mu = \omega$;
- $\forall^{\text{club}} i < \mu$, if $\mu > \omega$.

Lemma 4.5. *Let κ be weakly inaccessible and $\mu < \kappa$ be regular and let $S \subseteq \kappa \cap \text{cof}(\mu)$ be stationary. Then $\boxplus_{\kappa,\mu,S}$ holds. Indeed, for the function f as below for any club E in κ, for stationarily many $\delta \in S \cap E$ clauses (d), (e) and (f) hold simultanously, i.e, for any $\beta < \delta$,*

$$S_{\delta,i,\beta} \text{ is stationary in } \alpha_{\delta,i+1} \wedge \alpha_{\delta,i+1} \in E$$

holds

- *for any large enough $i < \omega$, if $\mu = \omega$;*
- *$\forall^{\text{club}} i < \mu$, if $\mu > \omega$.*

Proof. Given S, we first form $S' = S \cap C_{\text{fix},\kappa}$. Then we apply Theorem 4.1 to S' and get $\bar{C} = \langle\langle \alpha_{\delta,i} : i < \mu\rangle : \delta \in S'\rangle$ that fulfils $\boxplus_{\kappa,\mu,S}$(a), (b), (c)[1] and (d) for some $\delta \in S'$. Here (d) is true for all $i < \mu$. We let $f\colon \kappa \to \kappa$ be defined via

$$f(\gamma) = \begin{cases} \beta, & \text{if } \text{cf}(\gamma) = \aleph_{\beta+1};\\ 0, & \text{else.}\end{cases}$$

We show that $\boxplus_{\kappa,\mu,S}$(f) holds: Now $\delta \in S'$ is a fixed point of the $\aleph$-operation. For any $\beta < \delta$, $\aleph_{\beta+1} < \delta$. Since $\sup_{i<\mu} \text{cf}(\alpha_{\delta,i+1}) = \delta$, there is an end segment of $i < \mu$ such that for each i in this end segment, $\text{cf}(\alpha_{\delta,i+1}) > \aleph_{\beta+1}$. Thus the two bullet points in $\boxplus_{\kappa,\mu,S}$(f) are true: For i in this end segment, $S_{\delta,i,\beta} = \{\gamma < \alpha_{\delta,i+1} : f(\gamma) = \beta\}$ is stationary in $\alpha_{\delta,i+1}$. $\square$

Lemma 4.6. *Let κ be weakly inaccessible and $\mu < \kappa$ be regular. Let $S \subseteq \kappa \cap \text{cof}(\mu)$ be stationary. If $\boxplus_{\kappa,\mu,S}$ holds, then $\mathbb{Q}_\kappa^{\text{Sacks}} \Vdash \diamondsuit_\kappa(S)$.*

Proof. We let $S' = S \cap \{\alpha < \kappa : \aleph_\alpha = \alpha\}$ and fix $\bar{C} = \langle C_\delta : \delta \in S'\rangle$, $\bar{\alpha}$, f as in $\boxplus_{\kappa,\mu,S}$. Recall that $\underset{\sim}{\eta}$ is defined in Definition 3.2 and $S_{\delta,i,\beta}$.

We define the name $\langle \underset{\sim}{\nu}_\delta : \delta \in S\rangle$ by letting for $\delta \in S'$ and for any $\beta < \delta$, $j = 0, 1$,

$$\begin{aligned} &\mathbb{Q}_\kappa^{\text{Sacks}} \Vdash \text{“}\underset{\sim}{\nu}_\delta(\beta) = j \text{ iff}\\ &\qquad \forall^{\text{club}} i \in \mu\big(\exists^{\text{stat}} \alpha < \alpha_{\delta,i+1}(f(\alpha) = \beta)\\ &\qquad \to \forall^{\text{club}} \alpha < \alpha_{\delta,i+1}(f(\alpha) = \beta \to \underset{\sim}{\eta}(\alpha) = j)\big)\text{”}. \end{aligned} \tag{4.1}$$

Again for $\delta \in S \setminus S'$ we can take any name $\underset{\sim}{\nu}_\delta$ for a function in ${}^\delta 2$.

We show

$$\mathbb{Q}_\kappa^{\text{Sacks}} \Vdash \text{“}\langle \underset{\sim}{\nu}_\delta : \delta \in S\rangle \text{ is a } \diamondsuit_\kappa(S)\text{-sequence.”}$$

[1] For arranging (c), we possibly thin out C_δ to a sub-club.

Towards this suppose that

$$p \Vdash \text{“}\underset{\sim}{x} \in {}^{\kappa}2, \text{ and } \underset{\sim}{D} \text{ is a club subset of } \kappa.\text{”}$$

We show that there is some $q \geq p$ that forces $\delta \in \underset{\sim}{D} \cap S'$ and $\underset{\sim}{x} \restriction \delta = \underset{\sim}{\nu}_\delta$.

We let $\chi = \beth_\omega(\kappa)$ and let $<^*_\chi$ be a well-ordering of $H(\chi)$. We choose a κ-approximating sequence (Definition 2.14) $\langle N_\varepsilon : \varepsilon < \kappa \rangle$ with the following properties:

$$\mathbf{c} = (\kappa, p, S, \bar{C}, \bar{\alpha}, \bar{\underset{\sim}{\nu}}, \underset{\sim}{x}, \underset{\sim}{D}) \in N_0.$$

Again we let $E = \{\alpha < \kappa : N_\alpha \cap \kappa = \alpha\}$. We pick any δ with

$$\delta \in S' \cap \mathrm{acc}(E) \wedge \forall^{\mathrm{club}} i < \mu(\alpha_{\delta,i+1} \in \mathrm{acc}(E)). \tag{4.2}$$

For $\varepsilon < \delta$, we let $\kappa_\varepsilon = N_\varepsilon \cap \kappa$.

By induction on $\varepsilon \leq \delta$ we chose a condition p_ε such that for any ε for any $\zeta < \varepsilon$ the following holds

- ($\otimes$) p_ε is the $<^*_\chi$-least element such that
  - (a) $p_\varepsilon \geq p$
  - (b) $p_\varepsilon \geq p_\eta$ for $\zeta < \varepsilon$.
  - (c) $\lg(\mathrm{tr}(p_\varepsilon)) \geq \kappa_\varepsilon$.
  - (d) p_ε forces values to $\underset{\sim}{x} \restriction \kappa_\varepsilon$, $\underset{\sim}{D} \cap \kappa_\varepsilon$ and $\min(\underset{\sim}{D} \setminus \kappa_\varepsilon)$ call them x_ε, e_ε, γ_ε respectively.
  - (e) for limit ε, $\mathrm{tr}(p_\varepsilon)(\kappa_\varepsilon) = x_\varepsilon(f(\kappa_\varepsilon))$.

We can carry the induction since $\mathbb{Q}$ is $(< \kappa)$-complete and for clause ($\otimes$)(e) we recall $f(\kappa_\varepsilon) < \kappa_\varepsilon$.

This is literally like the proof of $\circledast_\varepsilon$ in the proof of Theorem 1.1.

After carrying the induction, we let $q = \bigcap_{\varepsilon<\delta} p_\varepsilon$. Since $\delta \in E$, we have $\kappa_\delta = \delta$.

We show that

$$q \Vdash \underset{\sim}{x} \restriction \delta = \underset{\sim}{\nu}_\delta. \tag{4.3}$$

According to (4.1), this means for any $\beta < \delta$, for any $j \in 2$,

$$\begin{aligned} &q \Vdash \text{“}\underset{\sim}{x}(\beta) = j \text{ iff} \\ &\quad \forall^{\mathrm{club}} i \in \mu\big(\exists^{\mathrm{stat}} \alpha < \alpha_{\delta,i+1}(f(\alpha) = \beta) \\ &\quad \to \forall^{\mathrm{club}} \alpha < \alpha_{\delta,i+1}(f(\alpha) = \beta \to \underset{\sim}{\eta}(\alpha) = j)\big)\text{”}. \end{aligned} \tag{4.4}$$

Since $\bar{N}$ is a continuous sequence, the function $\varepsilon \mapsto \kappa_\varepsilon$ is continuously increasing. Hence $\kappa_\delta = \delta$ and

$$q \Vdash \underset{\sim}{x} \restriction \delta = \bigcup \{x_{\kappa_\varepsilon} : \varepsilon < \delta\}. \tag{4.5}$$

Since q forces $\underset{\sim}{x}(\beta) = 0$ or q forces $\underset{\sim}{x}(\beta) = 1$, it suffices to check the forward direction in the “iff” in Statement (4.4). So suppose that $q \Vdash \underset{\sim}{x}(\beta) = j$. We verify that at club many $i < \mu$ there are stationarily many $\alpha < \alpha_{\delta,i+1}$ with $f(\alpha) = \beta$, and for club many $\varepsilon < \alpha_{\delta,i+1}$,

$$q \Vdash (\kappa_\varepsilon = \varepsilon \wedge f(\varepsilon) = \beta) \to \eta(\varepsilon) = \underset{\sim}{x}(\beta) = x_\varepsilon(\beta). \tag{4.6}$$

Since $\delta \in S'$ and since $\varepsilon \mapsto \kappa_\varepsilon$ is continuous from κ to κ, the premise in (4.1) (i.e., the definition of $\underaccent{\tilde}{\nu}_\delta$), $(\forall^{\rm club} i < \mu)(\exists^{\rm stat} \alpha < \alpha_{\delta,i+1})(f(\kappa_\alpha) = f(\alpha) = \beta)$ holds. This is not a forcing statement. Namely this statement holds for the i such as in $\boxplus_{\kappa,\mu,S}(e)$.

In each limit ε, we respected $(\otimes)$(e). By the choice of δ according to Statement (4.2), for club many $i < \mu$, $\alpha_{\delta,i+1} \in {\rm acc}(E)$ and hence for these i, the set of $\varepsilon < \alpha_{\delta,i+1}$ with $\kappa_\varepsilon = \varepsilon$ is a club in $\alpha_{\delta,i+1}$. So Statement (4.6) holds. Together with Statement (4.5) and the definition of the name in Statement (4.1) this shows that Statement (4.3) is true. $\square$

4.1. **Weakening $(< \kappa)$-Closure to a Strong Form of Strategic Closure.** We recall, a forcing $\mathbb{Q}$ is κ-strategically complete if the following holds: For any $p \in \mathbb{Q}$ there is a there is winning strategy in the game $G(p, \kappa)$ for player COM. The game $G(p, \kappa)$ is as follows. Player COM starts with $p_0 = 1_{\mathbb{P}}$ and player INC plays in any round $q_\alpha \geq p_\alpha$. In successor rounds COM plays $p_{\alpha+1} \geq q_\alpha$. In limit rounds $\delta < \kappa$, Player COM plays $p_\delta \geq q_\alpha$ for $\alpha < \delta$. Player COM wins if p_δ exists for any $\delta \in \kappa$, otherwise player INC wins.

If σ is a winning strategy for COM and and COM modifies this strategy by first picking a move according to σ and thereafter strengthening it, then this is a winning strategy as well, since INC could have played this stengthening.

Under $\boxplus_{\kappa,\mu,S}$, we may consider the following property.

${\rm Pr}(\kappa, \mu, S, \mathbb{Q})$: $\mathbb{Q}$ is a κ-strategically complete forcing and there is a name $\underaccent{\tilde}{\tau} = \langle \underaccent{\tilde}{\tau}_\varepsilon : \varepsilon < \kappa \rangle$, such that for any $p \in \mathbb{Q}$ there is a winning strategy for COM in $G(p, \kappa)$ with the following property: In any play played according to this strategy: For a club C in κ for $\varepsilon \in C$ for $j = 0, 1$, there are upper bounds $r_{\varepsilon,0}$, $r_{\varepsilon,1}$ of $\langle p_\zeta, q_\zeta : \zeta < \varepsilon \rangle$ with $r_{\varepsilon,j} \Vdash \underaccent{\tilde}{\tau}_\varepsilon = j$.

Theorem 4.7. *Suppose $\boxplus_{\kappa,\mu,S}$ and that $\mathbb{Q}$ is a κ-strategically closed forcing with ${\rm Pr}(\kappa, \mu, S, \mathbb{Q})$. Then $\mathbb{Q} \Vdash \diamondsuit_\kappa(S)$.*

Proof. (Sketch) We modify the original proof by adding that the strategy is an element of N_0, the first element of an κ-approximating sequence. We work with the following diamond $\langle \underaccent{\tilde}{\nu}_\delta : \delta \in S \rangle$ such that for $\delta \in S$ for any $q \in \mathbb{Q}$:

$$\begin{aligned} & q \Vdash \text{``}\underaccent{\tilde}{\nu}_\delta(\beta) = j \text{ iff} \\ & \quad \forall^{\rm club} i \in \mu \big(\exists^{\rm stat} \alpha < \alpha_{\delta,i+1}(f(\alpha) = \beta) \\ & \quad \rightarrow \forall^{\rm club} \alpha < \alpha_{\delta,i+1}(f(\alpha) = \beta \rightarrow \underaccent{\tilde}{\tau}_\alpha = j)\big)\text{''}. \end{aligned}$$

We proceed as in the proof of Theorem 1.2. There, clause $(\otimes)$(e) says $p_\varepsilon \Vdash {\rm tr}(p_\varepsilon)(\kappa_\varepsilon) = x_\varepsilon(f_\varepsilon(\kappa_\varepsilon))$. On the club set of ε with $\kappa_\varepsilon = \varepsilon$, now player COM chooses $j \in 2$ such that that $p_\varepsilon = r_{\varepsilon,j} \Vdash \tau_\varepsilon = x_\varepsilon(f_\varepsilon(\varepsilon))$.

$\square$

Remark 4.8. We wrote a "a strong form of strategic closure", since "$(< \kappa)$-strategically closed" means often that for each $\alpha < \kappa$, player COM has a winning strategy in $G(p,\alpha)$. A typical example is the forcing adding a $\square_{\aleph_1}$-sequence with $(<\aleph_2)$-sized closed initial segments for $\kappa=\aleph_2$. $(<\kappa)$-strategical completeness does not allow to carry out the above proof, since separate strategies for each $\delta<\kappa$ cannot be all contained as elements in an elementary submodel of size $<\kappa$.

5. Higher Miller Forcing with Splitting into a Club

Higher Miller forcing and higher Laver forcing are special among our forcings, since there is a name for a $\diamondsuit$-sequence in the respective forcing extensions that is much simpler than the other names for diamonds. Now we prove Theorem 1.6. The proof works for either of these two forcings.

Proof. (1) Let κ be a regular uncountable cardinal. We give a $\mathbb{Q}_\kappa^{\rm Miller}$-name that witnesses $\mathbb{Q}_\kappa^{\rm Miller}\Vdash\diamondsuit_\kappa$.

Let $\langle S_\alpha:\alpha\in\kappa\rangle$ be a partition of κ into stationary sets. For each $\alpha<\kappa$, we let $\langle t_{\alpha,\varepsilon}:\varepsilon<\kappa\rangle$ be an enumeration of ${}^\alpha\kappa$. For $i<\kappa$ we let $u_{\alpha,i}=t_{\alpha,\varepsilon}$ if $i\in S_\varepsilon$. Recall, $\underset{\sim}{\eta}$ is a name of the generic branch. Now we give a name for a $\diamondsuit_\kappa(S)$-sequence:

$$\mathbb{Q}_\kappa^{\rm Miller}\Vdash\bar{\underset{\sim}{d}}=\langle\underset{\sim}{d}_\alpha=u_{\alpha,\underset{\sim}{\eta}(\alpha)}:\alpha\in\kappa\rangle.$$

We show

$$\mathbb{Q}_\kappa^{\rm Miller}\Vdash\text{“}\bar{\underset{\sim}{d}}\text{ is a }\diamondsuit_\kappa\text{-sequence.”}$$

We assume $p\Vdash\underset{\sim}{x}\in{}^\kappa\kappa,\underset{\sim}{C}$ is a club in κ. We show that there are some $\alpha<\kappa$ and a stronger condition q that forces $\alpha\in\underset{\sim}{C}$ and $\underset{\sim}{x}\restriction\alpha=\underset{\sim}{d}_\alpha$. By induction on $n<\omega$ we choose p_n and $\alpha_n\in\kappa$ such that

(a) $p_0=p$,
(b) $p_n\le p_{n+1}$,
(c) $\alpha_n<\mathrm{dom}(\mathrm{tr}(p_n))\le\alpha_{n+1}$,
(d) $p_n\Vdash\alpha_n\in\underset{\sim}{C}$,
(e) p_{n+1} forces a value in $\mathbf{V}$ to $\underset{\sim}{x}\restriction\alpha_n$, we call it x_n.

The induction can be carried since the forcing is $(<\kappa)$-closed and hence does not add new elements to ${}^{\kappa>}\kappa$. Also by closure, the set $p_\omega=\bigcap\{p_n:n<\omega\}$ is a condition. We let $\alpha=\sup_n\alpha_n$. Then $\mathrm{dom}(\mathrm{tr}(p_\omega))=\alpha$. We let $\bigcup\{x_n:n<\omega\}=x_\omega$ and notice $x_\omega\in{}^\alpha\kappa$. By construction,

$$p_\omega\Vdash\underset{\sim}{x}\restriction\alpha=v_\omega\wedge\alpha\in\underset{\sim}{C}.$$

Now we strengthen p_ω by a trunk lengthening: The set $\mathrm{osucc}_{p_\omega}(\mathrm{tr}(p_\omega))$ is a club subset of κ and thus has non-empty intersection with each S_ε, $\varepsilon<\kappa$. We choose ε to be an ε with $t_{\alpha,\varepsilon}=x_\omega$. We pick some $i\in S_\varepsilon\cap\mathrm{osucc}_{p_\omega}(\mathrm{tr}(p_\omega))$. Then $u_{\alpha,i}=t_{\alpha,\varepsilon}$. Now

$$p_\omega^{\langle\mathrm{tr}(p_\omega)^\frown\langle i\rangle\rangle}\Vdash\underset{\sim}{\eta}(\alpha)=i\wedge\underset{\sim}{d}_\alpha=u_{\alpha,i}=t_{\alpha,\varepsilon}=x_\omega=\underset{\sim}{x}\restriction\alpha.$$

(2) Let a stationary set $S \subseteq \kappa$ be given. We work with the same $\langle S_\varepsilon : \varepsilon < \kappa \rangle$, $\langle t_{\alpha,\varepsilon} : \varepsilon < \kappa \rangle$ and $u_{\alpha,i}$ for $i \in S_\varepsilon$ and $\alpha < \kappa$ as above. We give a name for a $\diamondsuit_\kappa(S)$-sequence:

$$\bar{\underset{\sim}{d}} = \langle \underset{\sim}{d}_\alpha = u_{\alpha,\underset{\sim}{\eta}(\alpha)} : \alpha \in S \rangle.$$

We show

$$\mathbb{Q}^{\rm Miller}_\kappa \Vdash \text{“}\bar{\underset{\sim}{d}} \text{ is a } \diamondsuit_\kappa(S)\text{-sequence.”}$$

Let $p \Vdash$ “$\underset{\sim}{C}$ is club in κ and $\underset{\sim}{x} \subseteq \kappa$”. We show that there is $q \geq p$ and $\delta \in S$ with

$$q \Vdash \delta \in \underset{\sim}{C} \wedge \underset{\sim}{x} \restriction \delta = \underset{\sim}{d}_\delta. \tag{5.1}$$

We let $\chi = \beth_\omega(\kappa)$ and let $<^*_\chi$ be a well-ordering of $H(\chi)$. We choose a κ-approximating sequence (Definition 2.14) with

$$\mathbf{c} = (\kappa, p, \bar{\underset{\sim}{d}}, \underset{\sim}{x}, \underset{\sim}{D}, S) \in N_0.$$

Again we let $E = \{\alpha < \kappa : N_\alpha \cap \kappa = \alpha\}$. We pick any δ with

$$\delta \in S \cap \mathrm{acc}(E)$$

For $\varepsilon < \delta$, we let $\kappa_\varepsilon = N_\varepsilon \cap \kappa$.

By induction on $\varepsilon \leq \delta$ we chose a condition p_ε such that for any ε for any $\zeta < \varepsilon$ the following holds

($\oplus$) p_ε is the $<^*_\chi$-least element such that
- (a) $p_\varepsilon \geq p$
- (b) $p_\varepsilon \geq p_\eta$ for $\zeta < \varepsilon$.
- (c) $\mathrm{lg}(\mathrm{tr}(p_\varepsilon)) \geq \kappa_\varepsilon$.
- (d) p_ε forces values to $\underset{\sim}{x} \restriction \kappa_\varepsilon$, $\underset{\sim}{D} \cap \kappa_\varepsilon$ and $\min(\underset{\sim}{D} \setminus \kappa_\varepsilon)$ call them x_ε, e_ε, γ_ε respectively.
- (e) for limit ε, $\mathrm{tr}(p_\varepsilon)(\kappa_\varepsilon) = x_\varepsilon(f(\kappa_\varepsilon))$.

As in the proof of Lemma 3.6 or of Theorem 1.2, it is shown that we can carry the induction.

Now we carried the induction. We let $q = \bigcap_{\varepsilon<\delta} p_\varepsilon$.

Since $\delta \in E$, we have $\kappa_\delta = \delta$.

Now we strengthen p_δ by a trunk lengthening: The set $\mathrm{osucc}_{p_\delta}(\mathrm{tr}(p_\delta))$ is a club subset of κ and thus has non-empty intersection with each S_ε, $\varepsilon < \kappa$. There is some $\varepsilon < \kappa$ with $t_{\alpha,\varepsilon} = x_\delta$. We pick some $i \in S_\varepsilon \cap \mathrm{osucc}_{p_\delta}(\mathrm{tr}(p_\delta))$. Then $u_{\alpha,i} = t_{\alpha,\varepsilon}$.

Putting all together, we get

$$p_\delta^{\langle \mathrm{tr}(p_\delta)^\frown \langle i \rangle \rangle} \Vdash \underset{\sim}{\eta}(\alpha) = i \wedge \underset{\sim}{d}_\alpha = u_{\alpha,i} = t_{\alpha,\varepsilon} = x_\delta = \underset{\sim}{x} \restriction \delta,$$

and $p_\delta^{\langle \mathrm{tr}(p_\delta)^\frown \langle i \rangle \rangle} = q$ witnesses Statement (5.1).

Part (3) is proved in the proof of Corollary 1.5. □

This concludes the proof of Theorem 1.6.

Remark 5.1. The forcing adds a κ-Cohen real $\mathbb{C}_\kappa$. This is shown in [2]. So there is a $\mathbb{C}_\kappa$-name of a diamond sequence in $V[\mathbb{C}_\kappa]$.

6. On $\mathbb{Q}^{\text{Sacks}}_{(\kappa,W)}$ and Successor Cardinals κ

Now we work with $\kappa^{<\kappa} = \kappa \geq \aleph_1$ and allow κ to be a successor cardinal. We present another type of name of a diamond. The technique works for a weakening of the demands on splitting nodes in forcing conditions. The diamonds guess at approachable ordinals. Therefore the approachability ideal becomes important. We show that under $\kappa^{<\kappa} > \kappa$ the forcing adds a collapse from $\kappa^{<\kappa}$ to κ. Our collapsing technique is different from [12, Section 4]. The results in this section pertain to any of the mentioned tree forcings and their W-variants as given by the pattern Definition 6.1. For simplicity, we focus on $\mathbb{Q}^{\text{Sacks}}_{(\kappa,W)}$.

Definition 6.1. Let $\kappa = \text{cf}(\kappa) > \omega$ and let $W \subseteq \kappa$ be stationary in κ. We let $\mathbb{Q} = \mathbb{Q}^{\text{Sacks}}_{(\kappa,W)}$ be the forcing notion that is defined as follows

(A) $p \in \mathbb{Q}^{\text{Sacks}}_{(\kappa,W)}$ if
  - (a) p is a non-empty subtree of ${}^{\kappa>}2$.
  - (b) p is closed under unions of $\triangleleft$ increasing sequences of lengths $(<\kappa)$. We say p is $(<\kappa)$-closed.
  - (c) The set of spitting nodes is dense. That is for any $\eta \in p$ there is $\nu \in p$ with $\eta \trianglelefteq \nu$ and $\nu^\frown\langle 0\rangle, \nu^\frown\langle 1\rangle \in p$.
  - (d) If δ is a limit ordinal and $\langle \eta_\varepsilon : \varepsilon < \delta\rangle$ is a $\triangleleft$-increasing sequence with $\bigwedge_{\varepsilon<\delta} \eta_\varepsilon \in \text{split}(p)$ and $\bigcup_{\varepsilon<\delta} \text{lg}(\eta_\varepsilon) \in W$, then $\bigcup_{\varepsilon<\delta} \eta_\varepsilon \in \text{split}(p)$.

(B) $p \leq q$ if $p \supseteq q$.

(C) The generic is $\underset{\sim}{\eta} = \bigcup\{\text{tr}(p) : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}}\}$.

Fact 6.2. *Let $W \subseteq \kappa$ be stationary.*

1) $\mathbb{Q}^{\text{Sacks}}_{(\kappa,W)}$ is closed under intersections of increasing chains of length $<\kappa$.

2) $\mathbb{Q} \Vdash \underset{\sim}{\eta} \in {}^\kappa 2$ and $\mathbf{V}[\underset{\sim}{\eta}] = \mathbf{V}[\underset{\sim}{\mathbf{G}}_{\mathbb{Q}}]$.

Proof. 1) Let $\langle p_\alpha : \alpha < \gamma\rangle$ be an increasing sequence of conditions and $\gamma < \kappa$. The intersection $q = \bigcap_{\alpha<\gamma} p_\alpha$ has properties Definition 6.1(a), (b) and (d). Since $\emptyset \in q$, it is non-empty. We have to show that q is a perfect tree. We first show

($*$) For any $s \in q$, q has a branch b containing s and containing cofinally many splitting nodes in p_α for each $\alpha < \gamma$.

Since q is $(<\kappa)$-closed, it suffice to show:

($*'$) For any $s \in q$ there is a strict extension $t \triangleright s$, $t \in q$ and $t \in \text{split}(q)$ or for any $\alpha < \gamma$, t is a limit of splitting nodes in p_α.

Let $s \in q$ be given. We choose $i \in 2$ such that $s' = s^\frown\langle i\rangle \in q$. First case: $s' \in \text{split}(p_\alpha)$ for any $\alpha < \gamma$. Then $s' \in \text{split}(q)$ and we are done. Second case: There is $j \in 2$, α_0 such that $s'^\frown\langle j\rangle \trianglelefteq \text{tr}(p_{\alpha_0})$. We let $s_0 = \text{tr}(p_{\alpha_0})$.

Given $\langle s_j = \text{tr}(p_{\alpha_i}) : j < i \leq \gamma\rangle$ that is $\trianglelefteq$-increasing and $\langle \alpha_j : j < i\rangle$ that is increasing, in the limit case, we let s_i be the union of the s_j and α_i be the supremum of the α_j. In the successor case $i = j + 1 < \gamma$, if $s_j \in \mathrm{split}(p_\alpha)$, $\alpha < \gamma$, then $(*')$ is proved with $t = s_j$. If not, we let $s_i = \mathrm{tr}(p_{\alpha_i}) \rhd s_j$ for the minimal $\alpha_i > \alpha_j$ such $\mathrm{tr}(p_{\alpha_i}) \rhd s_j$. If for any $i < \gamma$, we are always in the second case, then $t = s_\gamma$ is as in $(*')$.

So $(*')$ and $(*)$ are proved and hence for any $t \in q$ there is some $b \in [q]$ with $t \in b$ such that for any $\alpha < \gamma$ there are cofinally many splitting nodes of p_α on this branch b.

We show that q fulfils Definition 6.1(c). Let $t \in q$. We have to show that there is a splitting node above t. We take a branch b containing t as above. Now by Definition 6.1(c), for each $\alpha < \gamma$, the set $W_{b,\alpha} = \{\beta \in \kappa : (\exists t' \in b \cap \mathrm{split}(p_\alpha))(\mathrm{dom}(t') = \beta)\}$ is a superset of a club C_α in κ intersected with W. We can let $C_\alpha = \mathrm{acc}^+(W_{b,\alpha})$. Now the intersection of the $W_{b,\alpha}$, $\alpha < \gamma$, is a superset of $\bigcap\{C_\alpha : \alpha < \gamma\} \cap W$. Hence there is a splitting node $t' \in \mathrm{split}(q)$ on the branch b with $t' \trianglerighteq t$, namely any $t' \in b$ with $\mathrm{dom}(t') \in \bigcap\{C_\alpha : \alpha < \gamma\} \cap W \cap [\mathrm{dom}(t), \kappa)$ has these properties.

2) Let $\mathbf{G}$ be $\mathbb{Q}$-generic over $\mathbf{V}$. Then $\eta = \bigcup\{\mathrm{tr}(p) : p \in \mathbf{G}\}$ is a function from κ to 2, since for any $p \in \mathbb{Q}$ and any $t \in p$ also the subtree $p^{\langle t \rangle}$ is a condition, and if t and t' are incompatible nodes in p, the conditions $p^{\langle t \rangle}$ and $p^{\langle t' \rangle}$ are incompatible. The generic branch η contains the full information about $\mathbf{G}$ since for any generic filter $\mathbf{G}$ we have for any p: $p \in \mathbf{G}$ iff $\eta \in [p]$. For a detailed proof see [12, Proposition 1.2]. □

Fact 6.3. *Assume $\kappa > \aleph_0$ is regular and $W \subseteq \kappa$ is stationary and $\mathbb{Q} = \mathbb{Q}^{\mathrm{Sacks}}_{(\kappa,W)}$. If $\kappa = \kappa^{<\kappa}$, then $\mathbb{Q}$ is proper.*

Proof. This is proved by a routine fusion construction. The proof that $\mathbb{Q}$ is $(<\kappa)$-closed and has κ-long fusion sequences with limits and is similar to the proof of Lemma 3.9. □

We introduce some combinatorics that will be useful for defining names.

Definition 6.4. Suppose that $\delta \in \kappa$ and $\mathrm{cf}(\delta) = \mathrm{cf}(\sigma)$.

(A) A function $f \colon {}^{\sigma>}2 \to {}^{\delta>}2$ is called a σ-tree embedding of height δ, if the following holds:
  (a) for any $s, t \in {}^{\sigma>}2$, if $s \trianglelefteq t$, then $f(s) \trianglelefteq f(t)$.
  (b) For any $b \in {}^{\sigma}2$, $\bigcup\{f(b \restriction i) : i < \sigma\} \in {}^{\delta}2$.
  We let $\sigma = \lim\langle \sigma_i : i < \mathrm{cf}(\sigma)\rangle$ for an increasing continuous sequence. It suffices to know $f \restriction \bigcup\{{}^{\sigma_i}2 : i < \mathrm{cf}(\sigma)\}$ and define the other f values by a natural modification of (b).

(B) A σ-tree embedding of height δ is called one-to-one if in addition for any $s, t \in {}^{\sigma>}2$, if $s \perp t$ then $f(s) \perp f(t)$. We write $s \perp t$, if s and t are incomparable (which is the same as incompatible) in $\trianglelefteq$.

(C) Let f be a σ-tree embedding of height δ. Let $\langle \sigma_i : i < \mathrm{cf}(\sigma)\rangle$ be an increasing cofinal sequence in σ. A sequence $\langle (\sigma_i, \ell_i) : i < \mathrm{cf}(\sigma)\rangle$ is a height sequence for f, if for any $i < \mathrm{cf}(\sigma)$, for any $t \in {}^{\sigma_i}2$, $\mathrm{dom}(f(t)) \in [\ell_i, \ell_{i+1})$. Necessarily $\lim_{i<\mathrm{cf}(\sigma)} \ell_i = \delta$.

(D) Given a σ-tree embedding f of height δ, there is a lift to branches $\bar{f}\colon {}^\sigma 2 \to {}^\delta 2$ given by $\bar{f}(b) = \bigcup\{f(b \restriction \sigma_i) : i < \mathrm{cf}(\sigma)\}$.

Remark 6.5.

(a) Height sequences do not need to exist. If $2^{<\sigma} < \kappa$ and κ is regular, then for $i < \mathrm{cf}(\sigma)$, the heights in $\{f(t) : t \in 2^{\sigma_i}\}$ are bounded. In our inductive construction of a σ-tree of conditions in the proof of Theorem 1.3, we will naturally define two tree embeddings with the same height sequence. These are the (f_1, f_2) in the end of the proof of Theorem 1.3. In the proof of Proposition 6.9 there are just fronts of heights.

(b) We do not require that the tree emdeddings fulfil $f(s \cap t) = f(s) \cap f(t)$. The righthand side might be longer. Definition 6.1(A) entails that for limits of splitting nodes in a condition p with domain not in W we cannot expect prompt splitting.

The following lemma is used for names of diamonds and for names of collapsing functions.

Lemma 6.6 (Bernstein Lemma)**.** *We assume that $\kappa = \kappa^{<\kappa}$ and $2^\sigma = \kappa$ and $\kappa^{2^{<\sigma}} = \kappa$. For each $\delta \in \kappa \cap \mathrm{cof}(\mathrm{cf}(\sigma))$ we let*

$$\begin{aligned}\mathcal{F}_{\sigma,\delta} =& \{(f_1, f_2) : f_1, f_2 \text{ are } \sigma\text{-tree embeddings}\\ &\text{of height } \delta \text{ and } f_1 \text{ is one-to-one}\}.\end{aligned}$$

Then there is some $h_\delta\colon {}^\delta 2 \to {}^\delta 2$ such that

$$\begin{aligned}&\forall (f_1, f_2) \in \mathcal{F}_{\sigma,\delta}\Big((\exists \eta \in {}^\sigma 2)\\ &(h_\delta(\bar{f}_1(\eta)) = \bar{f}_2(\eta) \wedge (\forall \alpha \in {}^\delta 2)(\exists \eta' \in {}^\sigma 2) h_\delta(\bar{f}_1(\eta')) = \alpha))\Big).\end{aligned}$$

Proof. For $\delta \in \kappa \cap \mathrm{cof}(\mathrm{cf}(\sigma))$ we have $|\mathcal{F}_{\sigma,\delta}| \le \kappa$. Note that here we use $2^{<\sigma} < \kappa$, $2^{<\delta} \le \kappa$ and $\kappa^{2^{<\sigma}} = \kappa^{<\kappa} = \kappa$.

We enumerate

$$\{(f_1, f_2, x) : (f_1, f_2) \in \mathcal{F}_{\sigma,\delta}, x \in {}^\delta 2\}$$

as $\langle (f_1^\alpha, f_2^\alpha, x_\alpha) : \alpha < \kappa\rangle$ such that each triple appears κ often. At step α we have to take care of (f_1^α, f_2^α) and we have to ensure that x_α gets into the range of $h_\delta \circ \bar{f}_1^\alpha$.

We define η_α, z_α and $h_\delta(\bar{f}_1^\alpha(\eta_\alpha)) := \bar{f}_2^\alpha(\eta_\alpha)$ and $h_\delta(\bar{f}_1^\alpha(z_\alpha)) := x_\alpha$ by induction on α. Suppose that $\langle (\eta_\beta, z_\beta, h_\delta(f_1^\beta(\eta_\beta)), h_\delta(\bar{f}_1^\alpha(z_\beta)) : \beta < \alpha\rangle$ is defined.

Since $\bar{f}_1^\alpha$ is one-to-one, there is some $\eta = \eta_\alpha \in {}^\sigma 2 \setminus \{\eta_\beta : \beta < \alpha\}$ such that $\bar{f}_1^\alpha(\eta) \ne \bar{f}_1^\beta(\eta_\beta)$ and for each $\beta < \alpha$. We let $h_\delta(\bar{f}_1^\alpha(\eta_\alpha)) = \bar{f}_2^\alpha(\eta_\alpha)$ and we can pick some $z_\alpha \in {}^\sigma 2 \setminus \{\eta_\beta : \beta \le \alpha\}$ and let $h_\delta(\bar{f}_1^\alpha(z_\alpha)) = x_\alpha$. Here we use that f_1^α is one-to-one. If after the induction the domain of h_δ is not yet the full set ${}^\delta 2$, we can define h_δ at the remaining arguments in an arbitrary manner. □

Remark 6.7. Theorem 1.3 and the results in previous sections are incomparable: E.g., for $\kappa = \lambda^+$ with $\mathrm{cf}(\lambda) = \aleph_0$ and $S \subseteq \kappa \cap \mathrm{cof}(\aleph_0)$, $S \in \check{I}[\kappa]$, we can apply Theorem 1.3 but not Theorem 1.2, whereas if $S \not\in \check{I}[\kappa]$ we can apply Theorem 1.2 but not Theorem 1.3. We work at $\delta \in S$ with an induction of length $\mathrm{cf}(\delta)$ in the case of approachability, or of length δ in general.

Proof of Theorem 1.3

Proof. Recall $S \in \check{I}[\kappa]$, $S \subseteq \kappa \cap \mathrm{cof}(\mathrm{cf}(\sigma))$, $2^{<\sigma} < \kappa$, $2^\sigma = \kappa = \kappa^{<\kappa} \geq \aleph_1$. For $\delta \in S$ we let h_δ be as in the Lemma 6.6. We define a $\mathbb{Q}$-name $\underset{\sim}{\bar{\nu}} = \langle \underset{\sim}{\nu}_\delta : \delta \in S \rangle$ by

$$\mathbb{Q} \Vdash \underset{\sim}{\nu}_\delta = h_\delta(\underset{\sim}{\eta} \restriction \delta).$$

We show that $\mathbb{Q}$ forces that $\underset{\sim}{\bar{\nu}}$ is a $\diamondsuit(S)$-sequence. Let

$$p \Vdash \underset{\sim}{x} \in {}^\kappa 2 \wedge \underset{\sim}{D} \text{ is a club in } \kappa.$$

We have to find a $\delta \in S$ and some $q \geq p$ such that

$$q \Vdash \delta \in \underset{\sim}{D} \wedge \underset{\sim}{\nu}_\delta = \underset{\sim}{x} \restriction \delta. \tag{6.1}$$

Suppose that $\kappa^{<\kappa} = \kappa > \aleph_0$, $2^\sigma = \kappa$ and $2^{<\sigma} < \kappa$.

Let for $\delta \in S$, σ be as in the assumptions. We let $\langle \sigma_i : i < \mathrm{cf}(\sigma) \rangle$ be a cofinal sequence in σ.

As $S \in \check{I}[\kappa]$ and $S \subseteq \kappa \cap \mathrm{cof}(\tau)$ there is $(E, \bar{A})$ such that

$(\odot)_0$
- (a) E is a club in κ
- (b) $\bar{A} = \langle A_\alpha : \alpha < \kappa \rangle$, $A_\alpha \subseteq \alpha$, A_α consists only of successor ordinals,
- (c) For any $\beta \in A_\alpha$ we have $A_\beta = A_\alpha \cap \beta$.
- (d) if $\alpha \in E \cap S$ then $\alpha = \sup(A_\alpha)$ and $\mathrm{ot}(a_\alpha) = \mathrm{cf}(\sigma)$.
- (e) if $\alpha \in \kappa \setminus (E \cap S)$ then $\mathrm{ot}(A_\alpha) < \mathrm{cf}(\sigma)$.

The existence of $\bar{A}$ is derived in the proof of Theorem 2.17 as given in [4, Theorem 3.7].

We fix a sequence $\langle \sigma_i : i < \mathrm{cf}(\sigma) \rangle$ that is continuously increasing and cofinal in σ.

We chose by induction on $\alpha < \kappa$ a sequence $\langle N_\alpha : \alpha < \kappa \rangle$ such that

$(\odot)_1$
- (a) $N_\alpha \prec (H(\chi), \in, <^*_\chi)$,
- (b) $|N_\alpha| < \kappa$,
- (c) $N_\alpha \cap \kappa \in \kappa$,
- (d) N_α is $\prec$-increasing and continuous, and $\langle N_\beta : \beta \leq \alpha \rangle \in N_{\alpha+1}$,
- (e) $\mathbf{c} = (\kappa, p, \bar{a}, \langle A_\delta : \alpha \in \kappa \rangle, \underset{\sim}{x}, \underset{\sim}{D}, E, S, \langle \sigma_i : i < \mathrm{cf}(\sigma) \rangle) \in N_0$.
- (f) For any $\beta < \kappa$, if $\beta \in N_\alpha$, then for any $j < \mathrm{cf}(\sigma)$, ${}^{\sigma_j}\beta \subseteq N_{\alpha+1}$. Since $2^{<\sigma} < \kappa$, we can add the clause, which will be used to derive $(\odot)_2$ below.

We can assume that $\tau \subseteq N_0$ and hence for any $i < \tau$, $\sigma_i \in N_0$. Let

$$C = \{\delta \in E : N_\delta \cap \kappa = \delta\}.$$

So C is a club of κ.

For $\delta \in C \cap S$ let $\langle \gamma_{\delta,i} : i < \mathrm{cf}(\sigma) \rangle$ list the closure of A_δ from $(\odot)_0$ in increasing order and let

$$N_{\delta,i} = N_{\gamma_{\delta,i}}.$$

Clearly

$$\langle N_{\delta,\varepsilon} : \varepsilon \leq i \rangle \in N_{\delta,i+1} \prec N_{\delta,j} \tag{6.2}$$

for any $i < j < \sigma$ thanks to the sequence $\bar{A}$ from $(\odot)_0$ and the requirements $(\odot)_1$ for $\alpha = \gamma_{\delta,i}$.

By induction on $i \leq \mathrm{cf}(\sigma)$ we chose a condition $(\bar{p}_i, \bar{x}_i, \bar{\gamma'}_i)$ where $\bar{p}_i = \langle p_{\sigma_i,\varrho} : \varrho \in {}^{\sigma_i}2 \rangle$, such that for any $i < \mathrm{cf}(\sigma)$ the following holds

$(\odot)_2$ p_i is the $<^*_\chi$-least element such that

- (a) $\bar{p}_i = \langle p_{\sigma_i,\varrho} : \varrho \in {}^{\sigma_i}2 \rangle$.
  (We shall prove that $\bar{p}_i \in N_{\delta,i+1}$ in our construction but it is better to not state it in our demands.)
- (b) For each $\varrho \in {}^{\sigma_i}2$, for any $j < i$, $p_{\sigma_j,\varrho\restriction\sigma_j} \leq p_{\sigma_i,\varrho}$.
- (c) $\bar{x}_i = \langle x_{\sigma_i,\varrho} : \varrho \in {}^{\sigma_i}2 \rangle$.
- (d) For $i < \mathrm{cf}(\sigma)$, $\varrho \in {}^{\sigma_i}2$, $p_{\sigma_i,\varrho} \Vdash \underset{\sim}{x} \restriction \gamma'_{\sigma_i,\varrho} = x_{\varepsilon,\varrho}$ for some $x_{\sigma_i,\varrho} \in {}^{\gamma_{\sigma_i,\varrho}}2 \cap \mathbf{V}$.
- (e) for any $\varrho \in {}^{\sigma_i}2$, $p_{\sigma_i,\varrho} \Vdash \ \gamma'_{\sigma_i,\varrho} \in \underset{\sim}{D}$.
- (f) for any $\varrho \in {}^{\sigma_i}2$, $p_{\sigma_i,\varrho} \geq p$.
- (g) for any $\varrho \in {}^{\sigma_i}2$, $\lg(\mathrm{tr}(p_{\sigma_i,\varrho})) \geq \gamma'_{\sigma_i,\varrho}$.
- (h) $\langle \mathrm{tr}(p_{\sigma_i,\varrho}) : \varrho \in {}^{\sigma_i}2 \rangle$ consists of pairwise $\trianglelefteq$-incomparable elements of $p \cap 2^{<\delta}$.
- (i) for any $\varrho \in {}^{\sigma_i}2 \cap N_{\delta,i+1}$,
  $\lg(\mathrm{tr}(p_{\sigma_i,\varrho}))$, $\gamma'_{\sigma_i,\varrho} \in [\sup(N_{\delta,i} \cap \kappa), \sup(N_{\delta,i+1} \cap \kappa))$.

We can carry the induction? We can carry the induction since $\mathbb{Q}$ is $(< \kappa)$-complete. More fully, let us prove $(\odot)_{3,i}$ by induction on i,

$(\odot)_{3,i}$

- $\mathbf{m}_i = \langle \bar{p}_j, \bar{x}_j, \bar{\gamma'}_j : j < i \rangle$ exists and is unique and $j < i$ implies $\mathbf{m}_{j+1} \in N_{\delta,j+1}$ and ${}^{\sigma_j}2$ and the ranges of each of $\bar{p}_j$, $\bar{x}_j$, $\bar{\gamma'}_j$, $j \leq i$ are subsets of $N_{\delta,j+1}$.
- $\mathbf{m}_i$ is defined in $(H(\chi), \in, <^*_\chi)$ by a formula $\varphi = \varphi(x, \bar{y})$ with x for $\mathbf{m}_i$ and $\bar{y} = (y_0, y_1)$ with $y_0 = \langle N_{\delta,j} : j < i \rangle$ and $y_1 = \mathbf{c}$ from $(\odot)_1$(e).

Case 1 $i = 0$.
$\mathbf{m}_0 = \langle \rangle$.
Case 2 $i = j + 1$.

Now $\bar{p}_i$ is the $<^*_\chi$ first ${}^{\sigma_i}2$-tuple of $\mathbb{Q}$ satisfying clauses $(\odot)_2$. As each element of $H(\chi)$ mentioned above is computable from $\langle N_{\delta,j} : j < i \rangle$, it belongs as an element to $N_{\delta,i} = N_{\delta,j+1}$ by Equation (6.2). We use the specific choice of $\mathbb{Q} = \mathbb{Q}^{\mathrm{Sacks}}_{(\kappa,W)}$ and not just $(< \kappa)$-complete forcings that force $\underset{\sim}{\eta} \notin \mathbf{V}$ but that above each node there is a higher splitting node. This entails that each entry of $\bar{p}_j$ gets ${}^{[\sigma_j,\sigma_i)}2$-many successors with pairwise incompatible trunks. By $(*)_\alpha(\mathrm{f})$ for $\alpha = \gamma_{\delta,i}$ we have $N_{\gamma_{\delta,i}+1} \subseteq N_{\gamma_{\delta,i+1}}$. Therefore, ${}^{\sigma_i}2$ and the ranges of $\bar{p}_i$, $\bar{x}_i$, $\bar{\gamma'}_i$ are subsets of $N_{\delta,i+1}$.

Case 3 i limit.

We again use the specific choice of $\mathbb{Q} = \mathbb{Q}^{\rm Sacks}_{(\kappa,W)}$ and not just $(<\kappa)$-complete forcings that force $\underset{\sim}{\eta} \notin \mathbf{V}$ but that above the limit of splitting nodes there is a higher splitting node. (This jump does not harm the continuity of f_1 from (6) but just makes it not necessarily $\wedge$-preserving. Since W is not necessarily closed, such jumps can appear.) By $(<\kappa)$-completeness of $\mathbb{Q}^{\rm Sacks}_{(\kappa,W)}$, we have $p_{i,\varrho} = \bigcap_{\zeta<i} p_{\sigma_\zeta,\varrho\restriction\zeta}$ in $\mathbb{Q}^{\rm Sacks}_{(\kappa,W)}$. Also for $\zeta < i$ $\varrho \in N_{\zeta+1}$, $\lg(\mathrm{tr}(p_{\sigma_\zeta,\varrho\restriction\zeta})) \in [\sup(N_{\delta,\zeta}\cap\kappa), \sup(N_{\delta,\zeta+1}\cap\kappa))$ so the sup over all $\zeta < i$, $\varrho \in N_{\delta,\zeta+1}$, is an element of $[\sup(N_{\delta,i}\cap\kappa), \sup(N_{\delta,i+1}\cap\kappa))$. $p_{i,\varrho} = \bigcup\{p_{\sigma_\zeta,\varrho\restriction\zeta} : \zeta < i\}$. Moreover $p_{i,\varrho} \Vdash x_{i,\varrho} = \underset{\sim}{x} \restriction \gamma_{i,\varrho}$. We can compute $\bar{p}_i$, $\bar{x}_i$, $\bar{\gamma'}_i$ from $(\langle N_{\delta,j} : j < i\rangle, \mathbf{c})$. At the same time the set ${}^{\sigma_i}2$ and ranges of the sequences $\bar{p}_i$, $\bar{x}_i$ and $\bar{\gamma'}_i$ are subsets of $N_{\delta,i+1}$ by $(*)_{\gamma_{\delta,i}}(\mathrm{f})$. So carried the induction.

For $\varrho \in {}^\sigma 2$, we let

$$p_{\sigma,\varrho} = \bigcap\{p_{\sigma_i,\varrho\restriction\sigma_i} : i < \mathrm{cf}(\sigma)\}.$$

By $(\odot)_2(e)$, for each $\varrho \in {}^\sigma 2$,

$$p_{\sigma,\varrho} \Vdash \bigcup\{\gamma'_{\sigma_i,\varrho} : i < \mathrm{cf}(\sigma)\} \in \underset{\sim}{D}.$$

By $(\odot)_2(\mathrm{f})$, for any $\varrho \in {}^\sigma 2$ such that for any $i < \mathrm{cf}(\sigma)$, $\varrho \restriction \sigma_i \in N_{\delta,i+1}$.

$$\gamma'_{\sigma,\varrho} = \bigcup\{\gamma'_{\varepsilon,\varrho\restriction\varepsilon} : \varepsilon < \sigma\} = \sup(N \cap \kappa) = \delta.$$

By $(\odot)_{3,i}$ for $i < \mathrm{cf}(\sigma)$, any $\varrho \in {}^\sigma 2$ fulfils for any $i < \mathrm{cf}(\sigma)$, $\varrho \restriction \sigma_i \in N_{\delta,i+1}$.

We define $(f_1, f_2) \in \mathcal{F}_{\sigma,\delta}$ by letting for $i < \mathrm{cf}(\sigma)$, $\varrho \in {}^{\sigma_i}2$,

$$f_1(\varrho) = \mathrm{tr}(p_{\mathrm{dom}(\varrho),\varrho}).$$

$$f_2(\varrho) = x_{\mathrm{dom}(\varrho),\varrho}.$$

By the definition of the continuation of the tree embeddings we have: For any $\varrho \in {}^\sigma 2$: $\bar{f}_1(\varrho) = \mathrm{tr}(p_{\mathrm{dom}(\varrho),\varrho})$, $\bar{f}_2(\varrho) = x_{\mathrm{dom}(\varrho),\varrho}$, by $\odot_2$(i) we have $\mathrm{dom}(\varrho)) = \sigma$ and $\mathrm{dom}(\mathrm{tr}(p_{\mathrm{dom}(\varrho),\varrho})) = \delta$ if for any $i < \mathrm{cf}(\sigma)$, $\varrho \restriction \sigma_i \in N_{\delta,i+1}$.

Now by Lemma 6.6 there is some $\varrho \in {}^\sigma 2$ with for any $i < \mathrm{cf}(\sigma)$, $\varrho \restriction \sigma_i \in N_{\delta,i+1}$ such that

$$p_{\sigma,\varrho} \Vdash \bar{f}_1(\varrho) = \underset{\sim}{\eta} \restriction \delta \wedge h_\delta(\bar{f}_1(\varrho)) = \underset{\sim}{\nu}_\delta = \bar{f}_2(\varrho) = x_{\sigma,\varrho} = \underset{\sim}{x} \restriction \delta.$$

For the very last equality relation we use $(\odot)_2$(d). So $q = p_{\sigma,\varrho}$ and δ are as in (6.1). □

Now Kanamori's premise on iterability is true in the one-step extension:

Corollary 6.8. *We assume* $\aleph_1 \leq \kappa = \kappa^{<\kappa}$.

(a) *For* $\kappa = \aleph_1$ *for any stationary* S, W, *we have* $\mathbb{Q}^{\rm Sacks}_{(\kappa,W)} \Vdash \diamondsuit_{\aleph_1}(S)$.

(b) *For* κ *that is not a strong limit,* $\mathbb{Q}^{\rm Sacks}_{(\kappa,W)} \Vdash \diamondsuit_\kappa$.

Proof. (a) Any stationary $S \subseteq \aleph_1$ is in the approachability ideal. (b) By Theorem 2.18, there is a stationary set $S \subseteq \kappa \cap \mathrm{cof}(\mathrm{cf}(\sigma))$ with $S \in \check{I}[\kappa]$. $\square$

This concludes the proof of Corollary 1.5.

In the next proposition we show that the Bernstein technique Lemma 6.6 at $2^\sigma > \kappa$ may provide a name of a collapse of 2^σ to κ under additional hypotheses.

Proposition 6.9. *If W is stationary in κ and there is a cardinal $\sigma < \kappa$ such that $2^\sigma > \kappa$, $(2^\sigma)^{2^{<\sigma}} \leq 2^\sigma$, and $2^{<\sigma} < \kappa$, then $\mathbb{Q}^{\mathrm{Sacks}}_{(\kappa,W)}$ collapses this 2^σ to κ.*

Proof. We fix a regular $\chi > \beth_\omega(\kappa)$ and let $\mathcal{H}(\chi) = (H(\chi), \in, <^*_\chi)$. Now we use Lemma 6.6 for κ from their being now $\kappa' = 2^\sigma$. Using Lemma 6.6 we have for $\delta \in \kappa \cap \mathrm{cof}(\mathrm{cf}(\sigma))$, a function $h_\delta \colon {}^\sigma 2 \to {}^\delta 2$ such that if $f \colon {}^{\sigma>}2 \to {}^{\delta>}2$ is one-to-one σ-tree embedding of height δ, then $h''_\delta(\mathrm{range}(\bar{f})) = 2^{|\sigma|}$.

Now define a sequence of $\langle \underset{\sim}{\tau}_\delta : \delta \in \kappa \cap \mathrm{cof}(\mathrm{cf}(\sigma))\rangle$ of of $\mathbb{Q}$-names letting

$$\mathbb{Q}^{\mathrm{Sacks}}_{(\kappa,W)} \Vdash (\forall \delta \kappa \cap \mathrm{cof}(\mathrm{cf}(\sigma))(\underset{\sim}{\tau}_\delta = h_\delta(\underset{\sim}{\eta} \restriction \delta)$$

Given $p \in \mathbb{Q}^{\mathrm{Sacks}}_{(\kappa,W)}$ and $\alpha \in 2^\sigma$, we have to produce some $q \geq p$ and some $\delta \in W$ such that

$$q \Vdash \underset{\sim}{\tau}_\delta = \alpha.$$

To this end, we build a tree of conditions

$$\langle p_{\sigma_i,\varrho} : i < \mathrm{cf}(\sigma), \varrho \in {}^{\sigma_i}2\rangle.$$

such that, letting $p_{\sigma,\varrho} = \bigcap\{p_{\sigma_i,\varrho\restriction\sigma_i} : i < \mathrm{cf}(\sigma)\}$ we have for any $\alpha' \in 2^\sigma$ there is some $\varrho \in {}^\sigma 2$ with

$$p_{\sigma,\varrho} \Vdash \text{“}h_\delta(\underset{\sim}{\eta} \restriction \delta) = \alpha'\text{”},$$

so in particular for $\alpha' = \alpha$.

We construct a tree of conditions $p_{\sigma,\varrho}$, $\varrho \in {}^\sigma 2$ and a σ embedding f_1 of height δ sending ϱ to $\mathrm{tr}(p_{\mathrm{dom}(\varrho),\varrho})$.

By induction on $i \leq \mathrm{cf}(\sigma)$ we chose a triple $(\bar{p}_i, \bar{\gamma}_i, \delta_i)$ where $\bar{p}_i = \langle p_{\sigma_i,\varrho} : \varrho \in {}^{\sigma_i}2\rangle$,

such that for any $i < \mathrm{cf}(\sigma)$ for any $j < i$ the following holds:

($\odot$) p_i is the $<^*_\chi$-least element with (a) to (f), where

(a) $\bar{p}_i = \langle p_{\sigma_i,\varrho} : \varrho \in {}^{\sigma_i}2\rangle$;
(b) For each $i < \mathrm{cf}(\sigma)$ and each $\varrho \in {}^{\sigma_i}2$, for any $j < i$, $p_{\sigma_j,\varrho\restriction\sigma_j} \leq p_{\sigma_i,\varrho}$;
(c) for any $\varrho \in {}^{\sigma_i}2$, $p_{\sigma_i,\varrho} \geq p$;
(d) for any $\varrho \in {}^{\sigma_i}2$, $\lg(\mathrm{tr}(p_{\sigma_i,\varrho})) \geq \gamma_{\sigma_i,\varrho}$;
(e) $\langle \mathrm{tr}(p_{\sigma_i,\varrho}) : \varrho \in {}^{\sigma_i}2\rangle$ consists of pairwise $\trianglelefteq$-incomparable elements of $p \cap 2^{<\delta}$;
(f) We let $\delta_i = \sup\{\lg(\mathrm{tr}(p_{\sigma_i,\varrho})) : \varrho \in {}^{\sigma_i}2\}$. For any $\varrho \in {}^{\sigma_{i+1}}2$, $\lg(\mathrm{tr}(p_{\sigma_{i+1},\varrho})), \gamma_{\sigma_{i+1},\varrho} \in [\delta_i + 1, \kappa)$.

There are no problems in the inductive choice. Again $2^{\sigma_i} < \kappa$ is crucial for (f). Note that the $F_i := \{\mathrm{tr}(p_{\sigma_i,\varrho} : \varrho \in {}^{\sigma_i}2\}$ is a front of $p_i := \bigcup\{p_{\sigma_i,\varrho} : \varrho \in {}^{\sigma_i}2\}$.

Now suppose that the induction is performed. We let for $i < \mathrm{cf}(\sigma)$, $\varrho \in {}^{\sigma_i}2$, $\mathrm{tr}(p_{\sigma_i,\varrho}) := f_1(\varrho)$. Thus f_1 defines a σ-embedding of height δ. We let $f_2(\varrho) = \alpha'$ for any $\varrho \in {}^{\sigma_i}2$ for any $i < \mathrm{cf}(\sigma)$.

By the choice of h_δ, there is some $\varrho \in {}^{\sigma}2$ such

$$p_{\sigma,\varrho} \Vdash h_\delta(\bar{f}_1(\varrho)) = h_\delta(\underset{\sim}{\eta} \restriction \delta) = \bar{f}_2(\varrho) = \alpha'.$$

□

Remark 6.10. Proposition 6.9 is proved differently for ordinary κ-Sacks in [12], where Solovay partitions of stationary sets in pairwise disjoint stationary sets are used and Clause (2) Definition 2.7 is used.

References

[1] James Baumgartner. Almost-disjoint sets, the dense-set problem, and the partition calculus. *Ann. Math. Logic*, 9:401–439, 1976.

[2] Jörg Brendle, Andrew Brooke-Taylor, Sy-David Friedman, and Diana Carolina Montoya. Cichoń's diagram for uncountable cardinals. *Israel J. Math.*, 225(2):959–1010, 2018.

[3] James Cummings, Sy-David Friedman, Menachem Magidor, Assaf Rinot, and Dima Sinapova. The eightfold way. *J. Symb. Log.*, 83(1):349–371, 2018.

[4] Todd Eisworth. Successors of singular cardinals. In *Handbook of set theory. Vols. 1, 2, 3*, pages 1229–1350. Springer, Dordrecht, 2010.

[5] Sy-David Friedman and Lyubomyr Zdomskyy. Measurable cardinals and the cofinality of the symmetric group. *Fund. Math.*, 207(2):101–122, 2010.

[6] Mohammad Golshani. (Weak) diamond can fail at the least inaccessible cardinal. *Fund. Math.*, 256(2):113–129, 2022.

[7] Hannes Jakob. Disjoint Stationary Sequences on an Interval of Cardinals. arXiv:2309.01986, 2023.

[8] Akihiro Kanamori. Perfect-set forcing for uncountable cardinals. *Ann. Math. Logic*, 19(1-2):97–114, 1980.

[9] Yurii Khomskii, Marlene Koelbing, Giorgio Laguzzi, and Wolfgang Wohofsky. Laver trees in the generalized Baire space. *Israel J. Math.*, 255(2):599–620, 2023.

[10] John Krueger. Guessing models imply the singular cardinal hypothesis. *Proc. Amer. Math. Soc.*, 147(12):5427–5434, 2019.

[11] Maxwell Levine. On Namba Forcing and Minimal Collapses. arXiv:2408.03487, 2024.

[12] Heike Mildenberger and Saharon Shelah. Higher Miller forcing may collapse cardinals. *J. Symb. Log.*, 86(4):1721–1744, 2021.

[13] William J. Mitchell. $I[\omega_2]$ can be the nonstationary ideal on $\mathrm{Cof}(\omega_1)$. *Trans. Amer. Math. Soc.*, 361(2):561–601, 2009.

[14] Saharon Shelah. On successors of singular cardinals. In *Logic Colloquium '78 (Mons, 1978)*, volume 97 of *Stud. Logic Foundations Math*, pages 357–380. North-Holland, Amsterdam-New York, 1979.

[15] Saharon Shelah. Models with second order properties. IV. A general method and eliminating diamonds. *Annals of Pure and Applied Logic*, 25:183–212, 1983.

[16] Saharon Shelah. Appendix: on stationary sets (in "Classification of nonelementary classes. II. Abstract elementary classes"). In *Classification theory (Chicago, IL, 1985)*, volume 1292 of *Lecture Notes in Mathematics*, pages 483–495. Springer, Berlin, 1987.

Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December 1985; ed. Baldwin, J.T.

[17] Saharon Shelah. Reflecting stationary sets and successors of singular cardinals. *Archive for Mathematical Logic*, 31:25–53, 1991.

[18] Saharon Shelah. *Cardinal Arithmetic*, volume 29 of *Oxford Logic Guides*. Oxford University Press, 1994.

[19] Saharon Shelah. Applications of PCF theory. *Journal of Symbolic Logic*, 65:1624–1674, 2000. arxiv:math.LO/9804155.

[20] Saharon Shelah. More on the Revised GCH and the Black Box. *Annals of Pure and Applied Logic*, 140:133–160, 2006. arxiv:math.LO/0406482.

[21] Saharon Shelah. Diamonds. *Proceedings of the American Mathematical Society*, 138:2151–2161, 2010. arxiv:0711.3030.

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