FORCING DIAMOND AND APPLICATIONS TO ITERABILITY

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ABSTRACT. We show that higher Sacks forcing at a regular not strong inaccessible cardinal and club Miller forcing at an uncountable regular cardinal both add a diamond sequence. We answer the longstanding question, whether $\kappa = \kappa^{<\kappa} \geq \aleph_1$ implies that κ-supported iterations of κ-Sacks forcing do not collapse κ^+ and are κ-proper in the affirmative. The results pertain to other higher tree forcings.

1. INTRODUCTION

Tree forcings like Silver forcing, Sacks forcing, Miller forcing or Laver forcing are used to arrange combinatorial properties of the power set of $\mathbb{R}$. Baumgartner [1], Kanamori [9] and later many researchers found analogues for an uncountable regular cardinal κ instead of ω that share at least part of the properties of their relatives at ω. The extent of the analogy depends on properties of κ. We focus on regular uncountable κ. Here we are mainly interested in conditions that ensure the preservation of κ^+ and a version of κ-properness (see Definition 2.5) for iterations with supports of size $\leq \kappa$.

Baumgartner [1, Theorem 6.7] showed that the κ-supported product of κ-Silver forcing does not collapse κ^+ under $\diamondsuit_\kappa$. Kanamori showed that iterating κ-Sacks forcing with supports of size $\leq \kappa$ does not collapse κ^+ if $\diamondsuit_\kappa$ holds [9, Theorem 3.2] or if κ is strongly inaccessible [9, Section 6]. The same proofs work also for numerous $(< \kappa)$-closed forcings in which forcing conditions are trees with club many splitting nodes which allow suitable sets of immediate successors. Iterations may be replaced by $(\leq \kappa)$-supported products [9, Section 5].

Shelah [22] showed that $\kappa^{<\kappa} = \kappa = \lambda^+ \geq \aleph_2$ implies $\diamondsuit_\kappa(\kappa \cap \mathrm{cof}(\mu))$ for any regular $\mu \neq \mathrm{cf}(\lambda)$. Hence for successor cardinals $\kappa = \kappa^{<\kappa} \geq \aleph_2$, the conditions that Baumgartner and Kanamori used for their iterability proofs are fulfilled.

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In [14] we showed that in Kanamori's iterability theorem (see Theorem 1.4 below) the condition ($\diamondsuit_\kappa$ or κ is inaccessible) can be replaced by the slightly weaker $(\mathsf{DI})_\kappa$ (see Definition 2.3). There are regular limit cardinals $\kappa = \kappa^{<\kappa}$ with $\neg(\mathsf{DI})_\kappa$, see [7].

Here we show that $\kappa = \kappa^{<\kappa} \geq \omega_1$ suffices as a premise for κ-properness and not collapsing κ^+ in $\leq \kappa$-supported iterations of higher Sacks forcing. We do this by showing that $\diamondsuit_\kappa$ holds in the ground model or is forced by the first iterand of the respective forcings. A particularly simple case of a forcing name of a witness of $\diamondsuit_\kappa$ is Theorem 1.1 for κ weakly Mahlo. The latter yields the proof of Corollary 1.5 for weakly Mahlo cardinals κ. In Theorem 1.2 we give a name for a diamond sequence for any not strongly inaccessible cardinal κ with $\kappa^{<\kappa} = \kappa$. In the presented version of Theorem 1.2, we extend the family of Kanamori-style Sacks forcings by working with a fixed stationary set of potential splitting levels, see Definition 5.1. Stationarity cannot be waived here.

Our second result is: For club Miller and for club Laver forcing, the premise $\aleph_1 \leq \kappa^{<\kappa} = \kappa$ suffices for forcing $\diamondsuit_\kappa$ and it ensures iterability, see Theorem 1.6.

We extend our results to $\diamondsuit_\kappa(S)$ under specific conditions on a stationary set S in the ground model. Here the approachability ideal on κ is used in the constructions of diamond names in Theorem 1.6, Theorem 1.2. We work with continuously increasing chains of elementary submodels N_α, $\alpha < \kappa$. Guessing names of subsets of κ as being equal to entries of a name of a diamond sequence takes place at $\delta = \kappa \cap N_\alpha \in S$ if S is an element of the approachability ideal $\check{I}[\kappa]$, see Definition 2.13. Since our forcings preserve stationarity in κ (see Remark 3.11), for preserving the stationarity of S it not be in the approachability ideal.

For regular uncountable κ under $\kappa^{<\kappa} > \kappa$ the forcing $\mathbb{Q}_\kappa^{\mathrm{Sacks}}$ collapses κ^+ by [14, Section 4]. The combinatorial background Lemma 5.7 of Theorem 1.2 yields in Proposition 5.9 another type of names for collapsing functions under $\kappa^{<\kappa} > \kappa$ and an additional hypothesis for regular κ that works also for the W-variants $\mathbb{Q}_{(\kappa,W)}^{\mathrm{Sacks}}$ (see Definition 5.1).

We do not consider singular κ here. For singular κ, higher tree forcings share features of Namba forcing, see, e.g. the Namba trees of height ω_1 used in [13].

Our first theorem pertains to weakly Mahlo cardinals κ (see Definition 2.4). In this theorem, approachability is automatically given, since the set of regular limit cardinal $\delta < \kappa$ is an element of the approachability ideal $\check{I}[\kappa]$.

Theorem 1.1. *If κ is weakly Mahlo and $S \subseteq \{\delta < \kappa : \delta$ regular limit$\}$ is stationary, then $\mathbb{Q}_\kappa^{\mathrm{Sacks}} \Vdash \diamondsuit_\kappa(S)$. The same holds for $\mathbb{Q}_\kappa^{\mathrm{Silver}}$.*

For the general case of an uncountable κ with $\kappa^{<\kappa} = \kappa$, we work with a Bernstein type of name of a diamond in Section 5. With this we settle the case of $\kappa = \aleph_1$ and the limit case in Kanamori's question. In the other cases of κ, the iterability question is already settled by Shelah's diamond on successor cardinals in [22].

The subforcings $\mathbb{Q}^{\text{Sacks}}_{(\kappa,W)}$ from Definition 5.1, $W \subseteq \kappa$, W stationary, of $\mathbb{Q}^{\text{Sacks}}_{\kappa}$ respect weaker demands on splitting nodes than $\mathbb{Q}^{\text{Sacks}}_{\kappa}$ and are still $(<\kappa)$-complete (see Lemma 5.2). The forcing $\mathbb{Q}^{\text{Sacks}}_{(\kappa,W)}$ for $W = \kappa$ is $\mathbb{Q}^{\text{Sacks}}_{\kappa}$. Our main result is:

Theorem 1.2. *Assume that $\kappa^{<\kappa} = \kappa \geq \aleph_1$ and that W is a stationary subset of κ. Suppose there is cardinal σ with the following properties:*

(a) $\kappa = 2^\sigma$,
(b) $2^{<\sigma} < \kappa$.

Then for any stationary $S \subseteq \kappa \cap \operatorname{cof}(\operatorname{cf}(\sigma))$ we have $\mathbb{Q}^{\text{Sacks}}_{(\kappa,W)} \Vdash \diamondsuit_\kappa(S)$.

In the proof, approachability of sufficiently many $\delta \in S$ will be essential. By Theorem 2.17, for σ as in the theorem we have $\kappa \cap \operatorname{cof}(\operatorname{cf}(\sigma)) \in \check{I}[\kappa]$, where the latter is the approachability ideal on κ. The approachability ideal is reviewed in Subsection 2.3.

If κ is not strongly inaccessible, then there is a minimal cardinal $\sigma < \kappa$ such that $2^\sigma = \kappa$. Since κ is regular, $2^{<\sigma} < \kappa$. Hence the items in the premises of Theorem 1.2 are extant. Summing up, we get the following.

Corollary 1.3. *If $\kappa^{<\kappa} = \kappa > \aleph_0$ and κ is not strongly inaccessible, then $\mathbb{Q}^{\text{Sacks}}_{(\kappa,W)} \Vdash \diamondsuit$.*

Analogues of Theorem 1.2 for κ-Silver forcing, club κ-Miller forcing and Laver forcing hold.

We recall:

Theorem 1.4 (Kanamori, [9])**.** *Assume $\kappa^{<\kappa} = \kappa \geq \aleph_1$. Let γ be an ordinal and let $\mathbb{P} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \gamma, \beta < \gamma \rangle$ be a $(\leq \kappa)$-support iteration such that for $\beta < \gamma$, $\mathbb{P}_\beta \Vdash \underset{\sim}{\mathbb{Q}}_\beta = \mathbb{Q}^{\text{Sacks}}_\kappa$. Assume $\diamondsuit_\kappa$ or that κ is strongly inaccessible. Then $\mathbb{P}_\gamma$ has the following properties.*

(1) $\mathbb{P}_\gamma$ is κ-proper.
(2) $\mathbb{P}_\gamma$ does not collapse κ^+.

Actually, for our version of κ-properness in Definition 2.5, the first conclusion implies the second. Combining Kanamori's theorem with Corollary 1.3 we derive the following.

Corollary 1.5. *For not strongly inaccessible κ, Theorem 1.4 holds without the assumption of $\diamondsuit_\kappa$ in the ground model.*

This answers Kanamori's question from [9]. It applies to Silver, Miller and Laver forcing at κ as well. It applies to the W-variants.

Our next theorem shows that for club κ-Miller/Laver forcing, for any uncountable κ with $\kappa^{<\kappa} = \kappa$ there is a name of a diamond that is much simpler than the names used in Theorem 1.1 and Theorem 1.2. Theorem 1.6(3) answers [14, Question 2.17], whether $\kappa^{<\kappa} = \kappa$ implies the preservation of κ^+

and κ-properness. In the case of κ being strongly inaccessible iterability was proved by Kanamori [9, Section 6] for the Sacks version, and by Friedman and Zdomskyy work [6] for the Miller version. The article [10] by Khomskii et. el. focuses on interesting versions of higher Laver forcing.

Theorem 1.6. *Assume $\kappa^{<\kappa} = \kappa \geq \aleph_1$.*

(1) Both $\mathbb{Q}_\kappa^{\rm Miller}$ and $\mathbb{Q}_\kappa^{\rm Laver}$ force $\diamondsuit_\kappa$.
(2) If $S \in \check{I}[\kappa]$ is stationary, then $\mathbb{Q}_\kappa^{\rm Miller}$ forces $\diamondsuit_\kappa(S)$ and the same holds for $\mathbb{Q}_\kappa^{\rm Laver}$.
(3) The iterability theorem holds as in Corollary 1.5.

Organisation of the paper. In Section 2 we review definitions. In Section 3 we prove Theorem 1.1, and we show that diamond in the one-step-extension leads to Corollary 1.5. In Section 4 we prove Theorem 1.6. In Section 5 we introduce $\mathbb{Q}_{(\kappa,W)}^{\rm Sacks}$ and prove Theorem 1.2 and a related result connecting cardinal arithmetic with collapsing functions for $\mathbb{Q}_{(\kappa,W)}^{\rm Sacks}$.

2. Background

Now we review the mentioned notions.

2.1. Combinatorics and Properness.

Definition 2.1. Let κ be a cardinal. For a regular cardinal $\mu < \kappa$, we let $\kappa \cap {\rm cof}(\mu) = \{\alpha \in \kappa : {\rm cf}(\alpha) = \mu\}$.

Definition 2.2. Let κ be a cardinal of uncountable cofinality and let S be a stationary subset of κ. The symbol $\diamondsuit_\kappa(S)$ abbreviates the following statement: There is a sequence $\langle d_\delta : \delta \in S\rangle$ such that $d_\delta \in {}^\delta 2$ and such that for any $x \in {}^\kappa 2$ the set $\{\delta \in S : d_\delta = x \restriction \delta\}$ is stationary. For $\diamondsuit_\kappa(\kappa)$ we write just $\diamondsuit_\kappa$.

We recall a weakening of $\diamondsuit^-(S)$ (see [12, Chapter III]), called DI.

Definition 2.3 (Shelah, see [17, 20, 21])**.** For a regular uncountable κ we let $(\mathsf{DI})_\kappa(S)$ mean the following: There is a sequence $\bar{\mathcal{D}} = \langle \mathcal{D}_\delta : \delta \in S\rangle$ such that $\mathcal{D}_\delta \subseteq {}^\delta\delta$ is of cardinality $< \kappa$ and for every $x \in {}^\kappa\kappa$ there are stationarily many $\delta \in S$ such that $x \restriction \delta \in \mathcal{D}_\delta$. For $(\mathsf{DI})_\kappa(\kappa)$ we write $(\mathsf{DI})_\kappa$.

Inaccessibility implies $(\mathsf{DI})_\kappa$.
For any stationary S and any stationary $S' \subseteq S$, $\diamondsuit(S')$ implies $\diamondsuit(S)$.

Definition 2.4. An uncountable limit cardinal κ is called *weakly Mahlo* if κ is a regular limit cardinal (i.e., κ is weakly inaccessible) and the set of uncountable regular limit cardinals below κ is stationary in κ.

Definition 2.5. Let $\mathcal{H}(\theta) = (H(\theta), \in, <_\theta)$, and $N \prec \mathcal{H}(\theta)$ and $\mathbb{Q} \in N$, $p \in \mathbb{Q} \cap N$. A condition q is called $(N, \mathbb{Q})$-*generic above* p if $q \geq p$ and for any dense subset D of $\mathbb{Q}$, if $D \in N$, then $q \Vdash \underset{\sim}{\mathbf{G}} \cap D \cap N \neq \emptyset$.

Let $\kappa^{<\kappa} = \kappa$. A notion of forcing $\mathbb{Q}$ is called κ-*proper* if for any sufficiently large θ there is a club (in $[H(\theta)]^\kappa$) of $N \prec H(\chi)$ with ${}^{<\kappa}N \subseteq N$ such that: If

$\kappa, p, \mathbb{Q} \in N$, and $p \in \mathbb{Q} \cap N$, then there is a stronger $(N, \mathbb{Q})$ generic condition q.

2.2. **Notation for Tree Forcing.** Our notions of forcing are written in Israeli style: $p \leq q$ means that q is stronger than p. We write $\mathbb{P} \Vdash \varphi$ if any condition in $\mathbb{P}$ forcing φ. Equivalently on can say the weakest condition of $\mathbb{P}$ forces φ.

Definition 2.6. Let κ be an infinite cardinal.

(1) We write ${}^{\kappa>}\kappa = \{t \colon \alpha \to \kappa : \alpha < \kappa\}$. If $s, t \in {}^{\kappa>}\kappa$ we call s *an inital segment of* t and write $s \trianglelefteq t$ if $t \restriction \operatorname{dom}(s) = s$. We use the symbol $\triangleleft$ for the corresponding strict relation. A *tree (on* ${}^{\kappa>}\kappa$*)* is a non-empty subset of ${}^{\kappa>}\kappa$ that is closed under initial segments, equipped with the initial segment order. For $t \in {}^{\kappa>}\kappa$, we write $\operatorname{dom}(t)$ or $\lg(t)$ for the domain of t.

(2) A tree p on ${}^{\kappa>}\kappa$ is called *unbounded* if

$$(\forall t \in p)(\forall \alpha < \kappa)(\exists t' \in p)(\operatorname{dom}(t') \geq \alpha \wedge t' \trianglerighteq t).$$

(3) Let $p \subseteq {}^{\kappa>}\kappa$ be a tree and $s \in p$. We let

$$p^{\langle s \rangle} = \{t \in p : t \trianglelefteq s \vee s \trianglelefteq t\}.$$

(4) The elements of a tree are called nodes. A node that has at least two immediate $\triangleleft$-successors in p is called a *splitting node of* p. The set of splitting nodes of p is denoted by $\operatorname{split}(p)$.

(5) Let $p \subseteq {}^{\kappa>}\kappa$ be a tree that contains a splitting node. We let the *trunk of* p, $\operatorname{tr}(p)$, be the $\trianglelefteq$-least splitting node of p.

(6) Analogously we define trees $p \subseteq {}^{\kappa>}2$.

Definition 2.7 (Kanamori's Higher Sacks Forcing, [9])**.** Let κ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$. Conditions in the forcing order $\mathbb{Q}_\kappa^{\mathrm{Sacks}}$ are trees $p \subseteq {}^{\kappa>}2$ with the following additional properties:

(1) (Perfectness) For any $s \in p$ there is an extension $t \trianglerighteq s$ in p such that t has two immediate successors.

(2) (Closure of splitting) For each increasing sequence of length $< \kappa$ of splitting nodes, the union of the nodes on the sequence is a splitting node of p as well.

A condition q is stronger than p if $q \subseteq p$.

The forcing $\mathbb{Q}_\kappa^{\mathrm{Sacks}}$ has a dense subset with the following closure property: For every increasing sequence $\langle t_i : i < \lambda \rangle$ of length $\lambda < \kappa$ of nodes $t_i \in p \in \mathbb{Q}_\kappa^{\mathrm{Sacks}}$ we have that the limit of the sequence $\bigcup\{t_i : i < \lambda\}$ is also a node in p. These p are called $(<\kappa)$*-closed trees.* For every condition p one can take the closure of $\operatorname{spl}(p)$ under initial segments and thus get a $(<\kappa)$-closed subtree. The latter is a stronger condition. Henceforth we work with the dense subforcing of $(<\kappa)$-closed conditions.

Definition 2.7 (1) and (2) imply that any $p \in \mathbb{Q}_\kappa^{\mathrm{Sacks}}$ is unbounded.

Definition 2.8 (Club Silver forcing)**.** Let κ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$. Conditions in the forcing order $\mathbb{Q}_\kappa^{\rm Silver}$ are partial functions $f\colon \operatorname{dom}(f) \to 2$ where $\operatorname{dom}(f)$ is a non-stationary subset of κ.

Stronger conditions are extensions of the function f.

Club Silver forcing is called $R(1, \kappa)$ in [1, Theorem 6.7].

Equivalently one can see a Silver condition f as a set of nodes of a higher Silver tree $T_f = \{t \in {}^{\kappa>}\kappa : t \restriction \operatorname{dom}(f) = f \restriction \operatorname{dom}(t)\}$. We can restrict $\mathbb{Q}_\kappa^{\rm Silver}$ to the dense set of conditions f for which $\kappa \setminus \operatorname{dom}(f)$ is a club. For these T_f, the limit of any increasing sequence of splitting nodes is a splitting node. This shows that an analogue to Theorem 1.1 holds also for Silver forcing.

Definition 2.9 (Club Miller Forcing/Club Laver Forcing)**.** Let κ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$.

(A) Conditions in the forcing order $\mathbb{Q}_\kappa^{\rm Miller}$ are trees $p \subseteq {}^{\kappa>}\kappa$ with the following additional properties:
   (1) (Club filter superperfectness) For any $s \in p$ there is an extension $t \trianglerighteq s$ in p such that $\operatorname{osucc}_p(t) := \{\alpha \in \kappa : t^\frown\langle\alpha\rangle \in p\}$ contains a club in κ. We require that each node has either only one direct successor or splits into a club.
   (2) (Closure of splitting) For each increasing sequence of length $< \kappa$ of splitting nodes, the union of the nodes on the sequence is a splitting node of p as well.

(B) A condition q is stronger than p, we write $q \geq p$, if $q \subseteq p$.

(C) Conditions in $\mathbb{Q}_\kappa^{\rm Laver}$ fulfil (2) and the following strengthening of (1): there is a $\trianglelefteq$-least node $s \in p$ such that for any $t \in p$ with $s \subseteq t$ the set $\operatorname{osucc}_p(t)$ contains a club in κ. The node s is called *the trunk of* p and denoted as $\operatorname{tr}(p)$.

(D) For a condition $p \in \mathbb{Q}_\kappa^{\rm Miller}$ or $q \in \mathbb{Q}_\kappa^{\rm Laver}$, $\operatorname{tr}(p)$ is just the $\trianglelefteq$-least splitting node.

Again, Definition 2.9 (1) and (2) imply that conditions are unbounded trees, and the $(<\kappa)$-closed trees form a dense subset. Unboundedness and $(<\kappa)$-closedness are sometimes added to the definition, see e.g., Brendle, Brooke-Taylor, Friedman, Montoya [2, Def. 74], where the forcing is called $\mathbb{MI}_\kappa^{\rm Clubfilter}$. Friedman and Zdomskyy [6] add the requirement that the successor set of a limit splitting node is a subset of the intersection of the $\triangleleft$-preceding splitting nodes. The set of these conditions is dense in $\mathbb{MI}_\kappa^{\rm Clubfilter}$. The recent article [10] is concerned with several versions of higher Laver forcing.

Definition 2.10. Let κ, μ be cardinals, $\mu > 0$, $\kappa \geq \omega$. Let $p \subseteq {}^{\kappa>}\mu$ be a tree. We let $[p] = \{b \in {}^\kappa\mu : \forall \alpha \in \kappa, b \restriction \alpha \in p\}$. The set $[p]$ is called *the rump, body or set of κ-branches of p.*

Note that for $\mu \geq 2$, $p \mapsto [p]$ is not an absolute function. For the forcing notions considered to far, by a density argument, in the generic extension there are new elements of $[p]$.

Lemma 2.11 ([9, Lemma 2.9]). *Any sequence $\langle p_\alpha : \alpha < \gamma\rangle$ for $\gamma < \kappa$ with $p_\alpha \le p_\beta$ for $\alpha < \beta < \gamma$ has an upper bound in $\le_{\mathbb{P}}$. The intersection $\bigcap\{p_\alpha : \alpha < \gamma\}$ is indeed a weakest upper bound.*

Proof. We carry out the proof for Sacks forcing. The other proofs are based on similar ideas. We go by induction over γ. Let $\gamma < \kappa$ and let $\langle p_\alpha : \alpha < \gamma\rangle$ be an ascending sequence of $(< \kappa)$-closed conditions. By induction hypothesis, we can assume that the sequence is continuous, that means that for limit $\delta < \gamma$, $p_\delta = \bigcap\{p_\alpha : \alpha < \delta\}$. We show that $p = \bigcap\{p_\alpha : \alpha < \gamma\}$ is a condition. We have $\emptyset \in p$ and it is easy to see that p fulfils Definition 2.7(2). We have to show that for each $s \in p$ there is a splitting node above s. Let $\langle \gamma_i : i < \mathrm{cf}(\gamma)\rangle$ be an ascending sequence in γ. We go by induction on i. Let $t_{-1} = s$. Suppose that $\langle t_j : j < i\rangle$ is defined such that $s \trianglelefteq t_{-1} \in p$ and for $j \ge 0$, $t_j \in \mathrm{spl}(p_{\gamma_j}) \cap p$ and $t_k \trianglelefteq t_j$ for $k < j < i$. If i is a limit, then let $t_i = \bigcup\{t_j : j < i\}$. By Definition 2.7(2) and $(< \kappa)$-closure we have $t_i \in \mathrm{spl}(p_{\gamma_j}) \cap p$ for any $j < i$. By continuity of our ascending sequence $\langle p_\beta : \beta < \gamma\rangle$ and by $(< \kappa)$-closure we have $t_i \in \bigcap\{\mathrm{spl}(p_{\gamma_j}) : j < i\} \cap p = \mathrm{spl}(p_{\gamma_i}) \cap p$.

If $i = k + 1$ ($k = -1$ is possible) we let t_i be the $\trianglelefteq$-least splitting node of p_{γ_i} with $t_i \trianglerighteq t_k, s$. Note that $t_i \in \bigcap\{p_\beta : \beta < \gamma\}$ by the unboundedness of conditions and since no p_β for $\beta \ge \gamma_i$ has a splitting node strictly between t_k and t_i. So $t_i \in \mathrm{spl}(p_{\gamma_i}) \cap p$ with $t_k \trianglelefteq t_i$ is found, and the induction is carried out.

Now by Definition 2.7(2) we have $t = \bigcup\{t_j : j < \mathrm{cf}(\gamma)\} \in \mathrm{spl}(p)$. □

Definition 2.12. A notion of forcing $\mathbb{P}$ is called $(< \kappa)$-*closed* if for any $\gamma < \kappa$, any ascending sequence $\langle p_\alpha : \alpha < \gamma\rangle$ has an upper bound.

2.3. **Review of $I[\kappa]$.** We review the approachability ideal $I[\kappa]$ and its variant $\check{I}[\kappa]$ (from [16, Definition 6, page 360, page 377]), which is suitable also for the description of regular limit cardinals κ. Our review focuses on results that we use to evaluate names for diamond sequences.

Definition 2.13 (The Approachability Ideal on Successors [18]). Let $\bar{a} = \langle a_\alpha : \alpha < \kappa\rangle$ enumerate a subset of $\kappa^{<\kappa}$. The ideal $I[\kappa](\bar{a})$ is be the set of $S \subseteq \kappa$ such that for a club $C \subseteq \kappa$ for any $\delta \in S \cap C$, there is a set $A_\delta \subseteq \delta$ that is cofinal in δ with $\mathrm{ot}(A_\delta) = \mathrm{cf}(\delta) < \delta$ and satisfies $\{A_\delta \cap \beta : \beta < \delta\} \subseteq \{a_\alpha : \alpha < \delta\}$. The *approachability ideal* $I[\kappa]$ is the union of all the $I[\lambda](\bar{a})$, $\bar{a}$ as above.

Remark 2.14. Equivalently we can require in addition that the A_δ be closed. The reason is, that we can choose $\bar{a}$ so that if there is a sequence of unbounded witnesses $\langle A_\delta : \delta \in S\rangle$ for $S \in I[\kappa](\bar{a})$ then there is also a sequence of club witnesses $\langle C_\delta : \delta \in S\rangle$ for $S \in I[\kappa](\bar{b})$ for a slightly richer sequence $\bar{b} \in {}^\kappa([\kappa]^{<\kappa})$. For a detailed proof we refer to [19, Lemma 4.4].

If $\kappa^{<\kappa} = \kappa$, we let $\langle a_\alpha : \alpha < \kappa\rangle$ be an enumeration of $\kappa^{<\kappa}$ and get $I[\kappa] = I[\kappa](\bar{a})$.

Most of the literature on $I[\kappa]$ in [18], [16], [4] focusses on the case of κ being a successor cardinal. For a successor cardinal κ, the regular cardinals below κ form a non-stationary set, and weakening the clause $\mathrm{ot}(A_\delta) = \mathrm{cf}(\delta) < \delta$ to the simpler $\mathrm{ot}(A_\delta) = \mathrm{cf}(\delta)$ Definition 2.13 yields an equivalent notion of the approachability ideal in this case. We work with a version of $I[\kappa]$ that dispenses with $\mathrm{cf}(\delta) < \delta$ in any case. For distinction we write $\check{I}[\kappa]$ for this modified definition.

Definition 2.15 (See [18], [16, Definition 6 and page 377], [19]). Let $\bar{a} = \langle a_\alpha : \alpha < \kappa \rangle$ enumerate a subset of $\kappa^{<\kappa}$. The ideal $\check{I}[\kappa](\bar{a})$ is be the set of $S \subseteq \kappa$ such that for a club $C \subseteq \kappa$ for any $\delta \in S \cap C$, there is a club $C_\delta \subseteq \delta$ that is cofinal in δ with $\mathrm{ot}(C_\delta) = \mathrm{cf}(\delta)$ and satisfies $\{C_\delta \cap \beta : \beta < \delta\} \subseteq \{a_\alpha : \alpha < \delta\}$. The *approachability ideal* $\check{I}[\kappa]$ is the union of all the $\check{I}[\kappa](\bar{a})$, $\bar{a}$ as above. If $\kappa^{<\kappa} = \kappa$, we let $\langle a_\alpha : \alpha < \kappa \rangle$ be an enumeration of $\kappa^{<\kappa}$ and have $\check{I}[\kappa] = \check{I}[\kappa](\bar{a})$.

Note, that $\{\delta < \kappa : \delta \text{ is regular}\} \in \check{I}[\kappa]$. We just take $\bar{a} = \langle \alpha : \alpha \in \kappa \rangle$ and for regular $\delta < \kappa$, $C_\delta = \delta$.

Some authors call $\check{I}[\kappa]$ now $I[\kappa]$, see e.g. [15], [3], [11]. We continue to write $\check{I}[\kappa]$. Several equivalent definitions of the ideal are given in [19] and [4].

Definition 2.16. Let κ be a regular cardinal. A *κ-approximating sequence* $\mathfrak{M} = \langle M_i : i < \kappa \rangle$ is a continuously increasing sequence of $M_i \prec (H(\chi), \in, <^*_\chi, \kappa, R)_{R \in \tau}$ for some regular cardinal $\chi > 2^{2^\kappa}$ with $\langle M_i : i < j \rangle \in M_{j+1}$ and $|M_i| < \kappa$, $M_i \cap \kappa \in \kappa$. Here τ is a finite or countable signature.

The following theorem shows that at any uncountable $\kappa = \kappa^{<\kappa}$ the existence of a stationary set S as in Theorem 1.2 and Theorem 1.6 are fulfilled for suitable λ.

Theorem 2.17 (Shelah [18], [19]). *let $\lambda < \kappa$ be cardinals such that κ is regular and $\kappa^{<\lambda} \leq \kappa$. Then there is a stationary set $S \subseteq \kappa \cap \mathrm{cof}(\mathrm{cf}(\lambda))$ with $S \in \check{I}[\kappa]$. The approachability is witnessed by closed sets.*

Proof. We let $\langle a_\alpha : \alpha < \kappa \rangle$ enumerate ${}^{\lambda >}\kappa$, such that each element appears κ often. We let

$$\begin{aligned} S = &\{\delta \in \kappa \cap \mathrm{cof}(\mathrm{cf}(\lambda)) : (\exists \eta \in {}^{\mathrm{cf}(\lambda)}\delta) \\ &\quad \big(\sup(\mathrm{range}(\eta)) = \delta \wedge (\forall i < \mathrm{cf}(\lambda))(\exists j < \delta)(\eta \restriction i = a_j)\big)\}. \end{aligned}$$

By definition, $S \in \check{I}[\kappa](\bar{a})$. We show that S is stationary. Let $C \subseteq \kappa$ be a club. By induction on $i < \mathrm{cf}(\lambda)$ we choose $\eta_i \in {}^{i+1}\kappa$ and $\delta_i \in \kappa$ with the following properties for any $i < \mathrm{cf}(\lambda)$,

(a) $\delta_i \in C$
(b) for $i < j < \mathrm{cf}(\lambda)$, $\delta_i < \delta_j$,
(c) $\eta_i = \langle \delta_j : j \leq i \rangle$,
(d) there is $k \leq \delta_{i+1}$ with $\eta_i = a_k$.

$i = 0$: We let $\delta_0 \in C$. We let $\eta_0 = \{(0, \delta_0)\}$.

Successor step: $i = j + 1$. We choose $\delta_i \in C \setminus (\delta_j + 1)$ such that there is some $k \leq \delta_{j+1}$ with $\eta_j = \langle \delta_\ell : \ell \leq j \rangle = a_k$. Then we let $\eta_i = \eta_j \cup \{(i, \delta_i)\}$.

Limit step $i < \mathrm{cf}(\lambda)$: We let $\delta_i = \sup\{\delta_j : j < i\}$ and $\eta_i = \bigcup\{\eta_j : j < i\} \cup \{(i, \delta_i)\}$.

Then $\eta = \bigcup\{\eta_i : i < \mathrm{cf}(\lambda)\}$ and $C_\delta = \mathrm{range}(\bar{\eta})$ witnesses that $\delta = \sup\{\delta_i : i < \mathrm{cf}(\lambda)\} \in S \cap C$. Moreover, the approachability witness C_δ is closed in δ. □

3. The Case of κ being Weakly Mahlo

Let $\mathbb{Q}$ be one of our for tree forcings. In this section we name a combinatorial principle $\boxplus_{\kappa,S}$ for κ being weakly Mahlo and show that for stationary sets $S \subseteq \kappa$, the principle allows to define a $\mathbb{Q}$-name for a $\diamondsuit_\kappa(S)$-sequence. We show that $\diamondsuit_\kappa$ in the forcing extension $V[\mathbb{Q}]$ leads to Corollary 1.5.

Definition 3.1. Let δ be an ordinal of unbountable cofinality. Let $S \subseteq \delta$ be stationary in δ.

(1) The quantifier $\forall^{\mathrm{club}} \alpha \in S, \varphi(\alpha)$ says that there is a club C in δ such that $S_\varphi = \{\alpha \in S : \varphi(\alpha)\} \supseteq S \cap C$.
(2) We define the quantifier $\exists^{\mathrm{stat}} \alpha \in S, \varphi(\alpha)$ as $S_\varphi = \{\alpha \in S : \varphi(\alpha)\}$ is a stationary subset of δ.

The combinatorial principle $\boxplus_{\kappa,S}$ asserts that there are stationarily many $\delta \in S$ for which δ can be partitioned into δ-many parts such that each of them is stationary in δ, via a partition that does not depend on δ.

Definition 3.2. Let κ be a weakly Mahlo cardinal and let $S \subseteq \{\delta \in \kappa : \delta \text{ is an uncountable regular limit cardinal}\}$.

$\boxplus_{\kappa,S}$ is the following statement: There is a function $f \colon \kappa \to \kappa$ such that for any $\alpha < \kappa$, $f(\alpha) < \min(\alpha, 1)$ and

$$\begin{aligned} &(\exists^{\mathrm{stat}} \delta \in S)(\forall \beta < \delta) \\ &(S_{\delta,\beta} := \{\gamma \in \delta : f(\gamma) = \beta\} \text{ is stationary in } \delta). \end{aligned} \tag{3.1}$$

Now the proof of Theorem 1.1 consists of Lemma 3.3 and Lemma 3.6.

Lemma 3.3. *If κ is weakly Mahlo and $S \subseteq \{\delta < \kappa : \delta$ is a regular limit cardinal$\}$ then $\boxplus_{\kappa,S}$.*

Proof. We let

$$f(\gamma) = \begin{cases} \beta, & \text{if } \mathrm{cf}(\gamma) = \aleph_{\beta+1}; \\ 0, & \text{else.} \end{cases} \tag{3.2}$$

Then we have

$$(\forall \delta \in S)(\forall \beta < \delta)(\{\gamma < \delta : f(\gamma) = \beta\} \text{ is stationary in } \delta).$$

The latter is a slightly stronger version of statement (3.1). □

We use $\boxplus_{\kappa,S}$ and Lemma 3.3 for stationary sets S.

Definition 3.4. For $E \subseteq \kappa$ we write $\mathrm{acc}^+(E) = \{\alpha \in \kappa : \alpha = \sup(E \cap \alpha)\}$ and $\mathrm{acc}(E) = E \cap \mathrm{acc}^+(E)$.

Definition 3.5. Let $\mathbf{G}$ be a $\mathbb{Q}$-generic filter over V and assume that $\mathbb{Q}$ is one of our named forcings. The following function $\eta \colon \kappa \to \kappa$ is called *the generic branch*: $\eta = \bigcup\{\mathrm{stem}(p) : p \in \mathbf{G}\}$. We let $\underset{\sim}{\mathbf{G}}$ be a name of the generic filter. We let $\underset{\sim}{\eta} = \{(s,p) : \exists p \in \underset{\sim}{\mathbf{G}}, s \trianglelefteq \mathrm{tr}(p)\}$.

Since all our forcings are $(<\kappa)$-closed and contain trunk lengthenings, by [14, Proposition 1.2] we have $G = \{p : \eta \in [p]\}$, i.e., the generic branch determines the generic filter.

We state the following lemma for Sacks forcing $\mathbb{Q}^{\mathrm{Sacks}}_\kappa$. It holds for any of the four types of tree forcings. For Miller forcing and for Laver forcing, we work with one fixed partition of κ into two stationary sets T_0, T_1. This partition is used to define the trunk lengthenings: For $j = 0,1$, $\underset{\sim}{\eta}(\varepsilon) = j$ in Equation (3.3), in Clause $\circledast_3$(e), and in Equations (3.6), (3.8) is replaced by $\underset{\sim}{\eta}(\varepsilon) \in T_j$.

Lemma 3.6. *Let κ be a regular uncountable limit ordinal and let $S \subseteq \{\delta \in \kappa : \delta$ is an uncountable regular limit cardinal$\}$ be stationary. If $\boxplus_{\kappa,S}$ holds, then $\mathbb{Q}^{\mathrm{Sacks}}_\kappa \Vdash \diamondsuit_\kappa(S)$.*

Proof. Let f be a function witnessing $\boxplus_{\kappa,S}$. Then

$$S' := \{\delta \in C \cap S : \forall \beta < \delta, S_{\delta,\beta} = \{\gamma < \delta : f(\beta) = \gamma\} \text{ is stationary in } \delta\}$$

is a stationary subset of S. We show $\diamondsuit_\kappa(S')$.

We define the name $\langle \underset{\sim}{\nu}_\delta : \delta \in S' \rangle$ for a sequence by letting for $\delta \in S'$, $\beta < \delta$, $j = 0,1$,

$$\mathbb{Q}^{\mathrm{Sacks}}_\kappa \Vdash \text{“}\underset{\sim}{\nu}_\delta(\beta) = j \ \leftrightarrow\ (\forall^{\mathrm{club}} \varepsilon \in S_{\delta,\beta})(\underset{\sim}{\eta}(\varepsilon) = j)\text{”}. \tag{3.3}$$

We show:

$$\mathbb{Q}^{\mathrm{Sacks}}_\kappa \Vdash \text{“}\langle \underset{\sim}{\nu}_\delta : \delta \in S' \rangle \text{ is a } \diamondsuit_\kappa(S')\text{-sequence.”}$$

Towards this suppose that

$$p \Vdash \text{“}\underset{\sim}{x} \in {}^\kappa 2, \text{ and } \underset{\sim}{D} \text{ is a club subset of } \kappa.\text{”}$$

We show that there are some $q \geq p$ and $\delta \in S'$ such that q forces $\delta \in \underset{\sim}{D} \cap S'$ and $\underset{\sim}{x} \restriction \delta = \underset{\sim}{\nu}_\delta$.

We let $\chi = (\beth_\omega(\kappa))^+$ and let $<^*_\chi$ be a well-ordering of $H(\chi)$. We choose a κ-approximating sequence $\langle N_\varepsilon : \varepsilon < \kappa \rangle$ in $H(\chi)$ (see Definition 2.16) with

$$\mathbf{c} = (\kappa, p, \bar{\underset{\sim}{\nu}}, \underset{\sim}{x}, \underset{\sim}{D}, S) \in N_0. \tag{3.4}$$

We let $E = \{\alpha < \kappa : N_\alpha \cap \kappa = \alpha\}$. Since $\langle N_\varepsilon : \varepsilon < \kappa \rangle$ is continuous, the set E is a club. We pick any δ with $\delta \in S' \cap E$.

We show that there is $q \geq p$ such that $q \Vdash \delta \in \underset{\sim}{D} \wedge \underset{\sim}{\nu}_\delta = \underset{\sim}{x} \restriction \delta$. Such a q will be gotten as the limit of an ascending sequence $\langle (p_\varepsilon, x_\varepsilon, \gamma_\varepsilon, \delta_\varepsilon, \kappa_\varepsilon) : \varepsilon < \delta \rangle$.

Since $\delta \notin N_\delta$, of course, the whole sequence is not an element of N_δ. One of the difficulties in the construction is that N_δ is not closed under sequences of length $< \delta$ (unless δ is strongly inaccessible). However, we shall see that the regularity of δ and the fact that $\kappa \cap N_\delta = \delta$ will suffice thanks to first order definability. [1]

By induction on $\varepsilon < \delta$ we choose $(p_\varepsilon, x_\varepsilon, \gamma_\varepsilon, \delta_\varepsilon, \kappa_\varepsilon)$ with the following properties: For technical reasons we let $\kappa_{-1} = \delta_{-1} = 0$.

$(\circledast)_1$ For successors $\varepsilon = \zeta + 1$ and for $\varepsilon = 0$, $(p_\varepsilon, x_\varepsilon, \gamma_\varepsilon, \delta_\varepsilon, \kappa_\varepsilon)$ is the $<^*_\chi$-least element of N_δ with (a) to (g) where
- (a) $p_\varepsilon \geq p$.
- (b) $p_\varepsilon \geq p_\zeta$ for $\zeta < \varepsilon$.
- (c) p_ε forces values to $\underset{\sim}{x} \restriction \kappa_\zeta$, $\min(\underset{\sim}{D} \setminus (\kappa_\zeta + 1))$ call them x_ε, γ_ε respectively.
- (d) $\lg(\mathrm{tr}(p_\varepsilon)) \geq \gamma_\varepsilon > \kappa_\zeta \geq \delta_\zeta \geq \zeta$.
- (e) If ζ is a limit ordinal, then $\mathrm{dom}(\mathrm{tr}(p_\varepsilon)) \geq \kappa_\zeta$ and $\mathrm{tr}(p_\varepsilon)(\kappa_\zeta) = x_\zeta(f(\kappa_\zeta))$.
- (f) δ_ε least such that $\langle (p_\xi, x_\xi, \gamma_\xi, \delta_\xi, \kappa_\xi) : \xi < \varepsilon \rangle \cup \{(p_\varepsilon, x_\varepsilon, \gamma_\varepsilon)\} \in N_{\delta_\varepsilon + 1}$.
- (g) $\kappa_\varepsilon = \kappa \cap N_{\delta_\varepsilon + 1}$.

$(\circledast)_2$ For limits $\varepsilon \leq \delta$, we take $p_\varepsilon = \bigcap\{p_\xi : \xi < \varepsilon\}$, γ_ε, δ_ε, κ_ε be the supremum of their respective predecessors, $x_\varepsilon = \bigcup\{x_\xi : \xi < \varepsilon\}$ and γ_ε is p_ε-forced to be in $\underset{\sim}{D}$, since $\underset{\sim}{D}$ is p_0-forced to be club.

Along the induction we verify:
- $\mathbf{m}_\varepsilon = \langle (p_\zeta, x_\zeta, \gamma_\zeta, \delta_\zeta, \kappa_\zeta) : \zeta < \varepsilon \rangle$ is defined in N_δ by a formula $\varphi = \varphi(x, \bar{y})$ with x for $\mathbf{m}_\varepsilon$ and $\bar{y} = (y_0, y_1)$ with $y_0 = \bar{N} \restriction \kappa_\varepsilon$ and $y_1 = \mathbf{c}$ from (3.4).
- Since $\mathbf{m}_\varepsilon$ is defined in N_δ, there is a minimal $\zeta < \delta$, $\zeta > \varepsilon$, such that $\mathbf{m}_\varepsilon \in N_\zeta$.
- For limit $\varepsilon < \mathrm{cf}(\delta)$, by definition of an approximating sequence we have $\langle N_{\delta_\zeta} : \zeta < \varepsilon \rangle \in N_{\sup\{\delta_\zeta : \zeta < \varepsilon\} + 1}$. For limit ε, we have $\kappa_\varepsilon = \sup\{\kappa \cap N_{\delta_\xi} : \xi < \varepsilon\} = N_{\delta_\varepsilon} \cap \kappa$.

Beginning: We are given $\kappa_{-1} = 0$, and choose $p_0 \geq p$ and γ_0 such that $p_0 \Vdash \gamma_0 \in \underset{\sim}{D}$. We choose $\delta_0 < \delta$ such that $(p_0, \gamma_0) \in N_{\delta_0 + 1}$ and κ_0 (g).

Successor step $\varepsilon = \zeta + 1$:
Now for $\varepsilon = \zeta + 1$ we let p_ε and x_ε and γ_ε as in (c). We consider the slightly harder case that ζ is a limit ordinal.

We show that the crucial clause $(\circledast)$(e) does not cause problems. So let ζ be a limit ordinal. By induction hypothesis we have $\kappa_\zeta = \sup\{\kappa_\xi : \xi < \zeta\}$. By $(< \kappa)$-completeness $p_\varepsilon = \bigcap_{\zeta < \varepsilon} p_\zeta$ in $\mathbb{Q}^{\mathrm{Sacks}}_\kappa$. Also for $\xi < \varepsilon$, $\lg(\mathrm{tr}(p_\xi)) \geq \kappa_\xi \in N_{\kappa_\xi + 1}$. Hence $\lg(\mathrm{tr}(p_\xi)) \in [\kappa_\xi, \kappa_{\xi+1})$ for $\xi < \zeta$. The induction hypothesis and Definition 2.7 clause number (2) imply

$$(\forall \xi \leq \zeta < \varepsilon)(\bigcup\{\mathrm{tr}(p_\varrho) : \varrho < \zeta\} \in \mathrm{split}(p_\xi)).$$

[1]In all our later theorems, regularity of δ will be replaced by approachability.

Hence

$\bigcup\{\mathrm{tr}(p_\xi) : \xi < \zeta\} \in \bigcap\{\mathrm{split}(p_\xi) : \xi < \zeta\} = \mathrm{split}(\bigcap\{p_\xi : \xi < \zeta\} = \mathrm{split}(p_\zeta),$

and so $\mathrm{tr}(p_\zeta) = \bigcup\{\mathrm{tr}(p_\xi) : \xi < \zeta\}$ and $\mathrm{lg}(\mathrm{tr}(p_\zeta)) = \kappa_\zeta$.

Moreover $p_\zeta \Vdash x_\zeta = \underset{\sim}{x} \restriction \kappa_\zeta$. We can compute $\mathrm{tr}(p_\zeta)$, κ_ζ and $f(\kappa_\zeta)$ from $(\bar{N} \restriction \kappa_\zeta, \mathbf{c})$. Now

$$p_\zeta \Vdash x_\zeta = \underset{\sim}{x} \restriction \kappa_\zeta \wedge \kappa_\zeta \in \underset{\sim}{D}. \tag{3.5}$$

by the induction hypothesis.

The trunk $\mathrm{tr}(p_\zeta)$ has two immediate successors of length $\kappa_\zeta + 1$ in p_ζ and we can let $p_\varepsilon \geq p_\zeta$ be such that

$$\mathrm{tr}(p_\varepsilon)(\kappa_\zeta) = x_\zeta(f(\kappa_\zeta)). \tag{3.6}$$

This is clause $(\circledast)_1$(e) that we have to fulfil. Obviously there are δ_ε as in (f) and κ_ε as in (g).

Clearly such a $(p_\varepsilon, x_\varepsilon, \gamma_\varepsilon, \delta_\varepsilon, \kappa_\varepsilon)$ exists and hence one of them must be $<^*_\chi$-least. As each element of $H(\chi)$ mentioned above is computable from $\bar{N} \restriction \kappa_\zeta + 1$, it is an element of N_δ.

Limit step: let $\varepsilon < \delta$ be a limit ordinal. The sequence $\mathbf{m}_\varepsilon$ is definable and hence an element of N_δ. The element $\langle p_\varepsilon, x_\varepsilon, \gamma_\varepsilon, \delta_\varepsilon, \kappa_\varepsilon\rangle$ is definable from it and hence also an element of N_δ. For $\varepsilon = \delta$, $\langle p_\varepsilon, x_\varepsilon, \gamma_\varepsilon, \delta_\varepsilon, \kappa_\varepsilon\rangle$ exists in N_κ.

Now we carried the induction, each step in N_δ but the last one $\varepsilon = \delta$. We let $q = \bigcap_{\varepsilon<\delta} p_\varepsilon$. The increasing sequences $\langle \kappa_\varepsilon : \varepsilon < \delta\rangle$, $\langle \delta_\varepsilon : \varepsilon < \delta\rangle$ both converge to δ.

We show that

$$q \Vdash \underset{\sim}{x} \restriction \delta = \underset{\sim}{\nu}_\delta \wedge \delta \in \underset{\sim}{D}. \tag{3.7}$$

Equation (3.5) implies at the limit δ: $q \Vdash \underset{\sim}{x} \restriction \delta = \bigcup\{x_\varepsilon : \varepsilon < \delta\}$.

We fix $\beta < \delta$. We verify that for club many ε in the stationary set $S_{\delta,\beta}$ we have

$$q \Vdash (\kappa_\varepsilon = \varepsilon \wedge f(\varepsilon) = \beta) \rightarrow \eta(\varepsilon) = \underset{\sim}{x}(\beta) = x_\varepsilon(\beta). \tag{3.8}$$

Equations (3.5) and (3.6) hold at $\{\kappa_\varepsilon : \varepsilon < \delta, \varepsilon \text{ limit ordinal}\}$. Thus Equation (3.8) holds at at the club $\{\varepsilon < \delta : \varepsilon = \kappa_\varepsilon : \varepsilon < \delta, \varepsilon \text{ limit ordinal}\}$ and entails Equation (3.7). □

Now we turn to Corollary 1.5.
We call the first iterand $\mathbb{P}_1$.

Lemma 3.7. *Let $\mathbb{P}$ be a $\leq \kappa$-supported iteration of iterands of $\mathbb{Q}^{\mathrm{Sacks}}_\kappa$. If $\mathbb{P}_1 \Vdash \diamondsuit_\kappa$ and the forcing $\mathbb{P}_1$ does not collapse κ^+, then $\mathbb{P}$ is as in Corollary 1.5.*

Proof. If $\kappa > \aleph_1$ is a successor cardinal, [22] gives the diamond in $\mathbf{V}$. Now let κ be a regular limit cardinal. Let $\mathbf{G}$ be $\mathbb{P}_1$ generic over $\mathbf{V}$. In $\mathbf{V}[\mathbf{G}]$ we apply Theorem 1.4 to the $(\leq \kappa)$-support iteration $\langle \mathbb{P}_\alpha/\mathbf{G}, \underset{\sim}{\mathbb{Q}}_\beta/\mathbf{G} : \alpha \in [1, \delta], \beta \in [1, \delta)\rangle$. □

For defining fusion sequences, we use a well known notion of layered splitting fronts.

Definition 3.8. We assume $\kappa = \kappa^{<\kappa}$. We conceive a forcing notion as a tree $p \subseteq \kappa^{<\kappa}$ or $\subseteq 2^{<\kappa}$. Recall, for the κ-version of Laver and Miller forcings splitting means splitting into a club. For $\alpha < \kappa$ we let

$$\mathrm{spl}_\alpha(p) = \big\{t \in \mathrm{split}(p) : \mathrm{ot}(\{s \subsetneq t : s \in \mathrm{split}(p)\}) = \alpha\big\}.$$

$$\mathrm{cl}_\alpha(p) = \{s \in p : (\exists t \in \mathrm{spl}_\alpha(p))(s \subseteq t)\}.$$

We let $p \leq_\alpha q$ if $p \leq q$ and $\mathrm{spl}_\alpha(p) = \mathrm{spl}_\alpha(q)$.

Lemma 3.9 (The Fusion Lemma)**.** *If $\delta \leq \kappa$ and $\langle p_\alpha : \alpha < \delta\rangle$ is a sequence with $p_\alpha \leq_\alpha p_\beta$ for $\alpha < \beta < \delta$, then $q = \bigcap\{p_\alpha : \alpha < \delta\} = \bigcup\{\mathrm{cl}_\beta(p_\beta) : \beta < \delta\}$ is a condition and for any $\beta < \delta$, $p_\beta \leq_\beta q$.*

Lemma 3.10. *Under $\kappa^{<\kappa} = \kappa$, the forcing $\mathbb{P}_1$ is κ-proper and hence does not collapse κ^+.*

Proof. Let $\chi > 2^\kappa$ is a regular cardinal. We first show κ-properness. Let $p \in \mathbb{P}_1$. We pick an $N \prec H(\theta)$ of size κ with ${}^{<\kappa}N \subseteq N$ κ, p, $\mathbb{P}_1 \in N$ and let $\langle I_\varepsilon : \varepsilon < \kappa\rangle$ list all open dense subseteq of $\mathbb{P}_1$ that are elements of N. Now by induction on $\varepsilon < \kappa$ we choose conditions p_ε, and sets $\{a_{s^\frown\langle i\rangle} : s \in \mathrm{spl}_\varepsilon(p_\varepsilon), i \in \mathrm{osucc}_{p_\varepsilon}(s)\} \subseteq {}^{(\varepsilon+1)}\kappa$ with the following properties:

(a) $p_\varepsilon \in N$.
(b) $p_0 = p$.
(c) If $\varepsilon < \delta$, then $p_\varepsilon \leq_\varepsilon p_\delta$.
(d) At limits ε, $p_\varepsilon = \bigcap\{p_\delta : \delta < \varepsilon\}$.
(e) if $s \in \mathrm{spl}_\varepsilon(p_\varepsilon)$, then for every $i \in \mathrm{osucc}_{p_\varepsilon}(s)$, the condition $p_{\varepsilon+1}^{\langle s^\frown\langle i\rangle\rangle} \in \bigcap_{\varepsilon' \leq \varepsilon} I_{\varepsilon'}$.

In the end the fusion $q = \bigcap\{p_\varepsilon : \varepsilon < \kappa\} = \bigcup\{\mathrm{cl}_\varepsilon(p_\varepsilon) : \varepsilon < \kappa\}$ is am N-generic condition, since it forces for any $\varepsilon < \kappa$ that one of the $q^{\langle s^\frown\langle i\rangle\rangle}$, $s \in \mathrm{spl}_\varepsilon(q) = \mathrm{spl}_\varepsilon(p_\varepsilon)$, $i \in \mathrm{osucc}_{p_\varepsilon}(s)$, is in $\mathbf{G} \cap I_\varepsilon \cap N$. For each $\varepsilon < \kappa$, we have for any $s \in \mathrm{spl}_\varepsilon(q) = \mathrm{spl}_\varepsilon(p_\varepsilon)$, $i \in \mathrm{osucc}_{p_\varepsilon}(s)$, $q^{\langle s^\frown\langle i\rangle\rangle} \geq p_{\varepsilon+1}^{\langle s^\frown\langle i\rangle\rangle}$.

Now we show that κ-properness entails the preservation of the cardinal κ^+. For this let $\underset{\sim}{\tau}$ be a name for function from κ into κ^+. We pick N as above with the additional property that $\tau \in N$. For $\varepsilon \in \kappa$ we define the open dense set $I_\varepsilon = \{q \in \mathbb{P}_1 : (\exists \alpha \in \kappa^+)(q \Vdash \tau(\varepsilon) = \alpha)\}$. We have $I_\varepsilon \in N$. By κ-properness, there is a $(N, \mathbb{P}_1)$-generic condition q. In particular, q forces for any ε, $\mathbf{G} \cap I_\varepsilon \cap N \neq \emptyset$. Now let r be such that $q \Vdash r \in \mathbf{G} \cap N \cap I_\varepsilon$. Then $r \in N$ and by elementarity, $N \models (r \Vdash (\exists \alpha < \kappa^+)(\tau(\varepsilon) = \alpha))$. Hence there is $s \leq r$, $s \in N$, and there is $\alpha \in N \cap \kappa^+$, $s \Vdash \tau(\varepsilon) = \alpha$. Thus q forces $\tau(\varepsilon) \in N \cap \kappa^+$. Since $N \cap \kappa^+ < \kappa^+$, we have that q forces that the range of τ is bounded in κ^+. □

We notice that Lemma 3.7 and Lemma 3.10 hold also for club Silver forcing and club Miller forcing. They could be mixed along an iteration. In the case of Laver conditions, κ-properness and the preservation of κ^+ follow from the fact that any two conditions with the same trunk are compatible. Hence each antichain has size at most κ and if it in an element of N then it is also a subset of N.

This concludes the proof of Corollary 1.5 in the weakly Mahlo case. In the general case of a κ that is not strongly inaccessible, we finish with Theorem 1.2.

Remark 3.11. In the tree forcings considered here, any stationary subset S of κ stays stationary in any $\mathbb{P}_1$-extension. This is so since $\mathbb{P}_1$ is strongly $(< \kappa)$-distributive, i.e., for any sequence $\langle D_\beta : \beta < \kappa \rangle$ of dense subsets of $\mathbb{P}_1$ and any $p \in \mathbb{P}_1$, there is a sequence $\langle p_\beta : \beta < \kappa \rangle$ such that for $\beta < \kappa$, $p_\beta \in D_\beta$ and $p_0 \leq p$, see [8, Lemma 3.8].

3.1. **Weakening $(< \kappa)$-Closure to a Strong Form of Strategic Closure.** A forcing $\mathbb{Q}$ is κ-strategically closed if the following holds: There is winning strategy in the game $G(\mathbb{Q}, \kappa)$ for player COM. The game is played as follows: Player COM starts with $p_0 = 1_{\mathbb{P}}$ and player INC plays in any round $q_\alpha \geq p_\alpha$. In successor rounds COM plays $p_{\alpha+1} \geq q_\alpha$. In limit rounds $\delta < \kappa$, Player COM plays $p_\delta \geq q_\alpha$ for $\alpha < \delta$. Player COM wins if p_δ exists for any $\delta \in \kappa$, otherwise player INC wins.

If σ is a winning strategy for COM and and COM modifies this strategy by first picking a move according to σ and thereafter strengthening it, then this is a winning strategy as well, since INC could have played this stengthening.

Under $\boxplus_{\kappa,S}$, we may consider the following property.

$\Pr(\kappa, S, \mathbb{Q})$: $\mathbb{Q}$ is a κ-strategically closed forcing and there is a name $\underset{\sim}{\tau} = \langle \underset{\sim}{\tau}_\varepsilon : \varepsilon < \kappa \rangle$, such that there is a winning strategy for COM in $G(\mathbb{Q}, \kappa)$ with the following property: In any play played according to this strategy: For a club C in κ for $\varepsilon \in C$ for $j = 0, 1$, there are upper bounds $r_{\varepsilon,0}$, $r_{\varepsilon,1}$ of $\langle p_\zeta, q_\zeta : \zeta < \varepsilon \rangle$ with $r_{\varepsilon,j} \Vdash \underset{\sim}{\tau}_\varepsilon = j$.

Theorem 3.12. *Suppose that κ is a weakly Mahlo cardinal, $\boxplus_{\kappa,S}$, $S \in \check{I}[\kappa]$ and that $\mathbb{Q}$ is a κ-strategically closed forcing with* $\Pr(\kappa, S, \mathbb{Q})$. *Then* $\mathbb{Q} \Vdash \diamondsuit_\kappa(S)$.

Proof. (Sketch) We modify the original proof by adding that the strategy is an element of N_0, the first element of an κ-approximating sequence.

We proceed as in the proof of Theorem 1.1. At successors of limit $\varepsilon = \zeta = 1$, clause $(\circledast)_1$(f) says $p_\varepsilon \Vdash \mathrm{tr}(p_\zeta)(\kappa_\zeta) = x_\varepsilon(f_\varepsilon(\kappa_\zeta))$. On the club set of these ζ player COM chooses $j \in 2$ such that that $p_\zeta = r_{\zeta,j} \Vdash \underset{\sim}{\tau}(\kappa_\zeta) = x_\varepsilon(f_{\kappa_\zeta}(\kappa_\zeta))$. $\square$

Remark 3.13. We wrote a "a strong form of strategic closure", since "$(< \kappa)$-strategically closed" means often that for each $\alpha < \kappa$, player COM has a winning strategy in $G(p, \alpha)$. A typical example is the forcing adding a $\square_{\aleph_1}$-sequence with $(< \aleph_2)$-sized closed initial segments for $\kappa = \aleph_2$. $(< \kappa)$-strategical completeness does not allow to carry out the above proof, since separate strategies for each $\delta < \kappa$ cannot be all contained as elements in an elementary submodel of size $< \kappa$.

4. Higher Miller Forcing with Splitting into a Club

Higher Miller forcing and higher Laver forcing are special among our forcings, since there is a name for a $\diamondsuit$-sequence in the respective forcing extensions that is much simpler than the other names for diamonds. Such a name will serve in the proof of Theorem 1.6. The proof works for either of these two forcings.

Proof. (Theorem 1.6) (1) Let κ be a regular uncountable cardinal. We give a $\mathbb{Q}_\kappa^{\mathrm{Miller}}$-name that witnesses $\mathbb{Q}_\kappa^{\mathrm{Miller}} \Vdash \diamondsuit_\kappa$. The same name works for Laver forcing, with literally the same proof.

Let $\langle S_\varepsilon : \varepsilon \in \kappa\rangle$ be a partition of κ into stationary sets. For each $\alpha < \kappa$, we let $\langle t_{\alpha,\varepsilon} : \varepsilon < \kappa\rangle$ be an enumeration of ${}^\alpha\kappa$. For $\alpha, i < \kappa$ we let $u_{\alpha,i} = t_{\alpha,\varepsilon}$ if $i \in S_\varepsilon$. Recall, $\underaccent{\tilde}{\eta}$ is a name of the generic branch. Now we give a name for a $\diamondsuit_\kappa(S)$-sequence:

$$\mathbb{Q}_\kappa^{\mathrm{Miller}} \Vdash \bar{\underaccent{\tilde}{d}} = \langle \underaccent{\tilde}{d}_\alpha : \alpha < \kappa\rangle \wedge (\forall \alpha < \kappa)(\underaccent{\tilde}{d}_\alpha = u_{\alpha,\underaccent{\tilde}{\eta}(\alpha)}).$$

We show

$$\mathbb{Q}_\kappa^{\mathrm{Miller}} \Vdash \text{“}\bar{\underaccent{\tilde}{d}} \text{ is a } \diamondsuit_\kappa\text{-sequence.”}$$

We assume $p \Vdash$ “$\underaccent{\tilde}{x} \in {}^\kappa\kappa \wedge \underaccent{\tilde}{C}$ is a club in κ”. We show that there are some $\alpha < \kappa$ and a stronger condition q that forces $\alpha \in \underaccent{\tilde}{C}$ and $\underaccent{\tilde}{x} \restriction \alpha = \underaccent{\tilde}{d}_\alpha$. By induction on $n < \omega$ we choose p_n and $\alpha_n \in \kappa$ such that

(a) $p_0 = p$,
(b) $p_n \le p_{n+1}$,
(c) $\alpha_n < \mathrm{dom}(\mathrm{tr}(p_n)) \le \alpha_{n+1}$,
(d) $p_n \Vdash \alpha_n \in \underaccent{\tilde}{C}$,
(e) For some $x_n \in \mathbf{V}$, $p_{n+1} \Vdash \underaccent{\tilde}{x} \restriction \alpha_n = x_n$.

Since the forcing $\mathbb{Q}_\kappa^{\mathrm{Miller}}$ is $(< \kappa)$-closed it does not add new elements to ${}^{\kappa>}\kappa$. Hence an x_n and a p_{n+1} as in (e) exist.

Once $\langle p_n : n \in \omega\rangle$ is chosen, again by $< \kappa$-closure of the forcing notion, the set $p_\omega = \bigcap\{p_n : n < \omega\}$ is a condition. We let $\alpha = \sup_n \alpha_n$. By clause (c), $\mathrm{dom}(\mathrm{tr}(p_\omega)) = \alpha$. We let $\bigcup\{x_n : n < \omega\} = x_\omega$ and notice $x_\omega \in {}^\alpha\kappa$. By construction,

$$p_\omega \Vdash \underaccent{\tilde}{x} \restriction \alpha = x_\omega \wedge \alpha \in \underaccent{\tilde}{C}.$$

Now we strengthen p_ω by a trunk lengthening: The set $\mathrm{osucc}_{p_\omega}(\mathrm{tr}(p_\omega))$ is a club subset of κ and thus has non-empty intersection with each S_ε, $\varepsilon < \kappa$. We choose ε to be an ε with $t_{\alpha,\varepsilon} = x_\omega$. We pick some $i \in S_\varepsilon \cap \mathrm{osucc}_{p_\omega}(\mathrm{tr}(p_\omega))$. Then $u_{\alpha,i} = t_{\alpha,\varepsilon}$. Now

$$p_\omega^{\langle \mathrm{tr}(p_\omega)^\frown\langle i\rangle\rangle} \Vdash \underaccent{\tilde}{\eta}(\alpha) = i \wedge \underaccent{\tilde}{d}_\alpha = u_{\alpha,i} = t_{\alpha,\varepsilon} = x_\omega = \underaccent{\tilde}{x} \restriction \alpha.$$

(2) Let a stationary set $S \subseteq \check{I}[\kappa]$ be given. We work with the same $\langle S_\varepsilon : \varepsilon < \kappa \rangle$, $\langle t_{\alpha,\varepsilon} : \varepsilon < \kappa \rangle$ and $u_{\alpha,i}$ for $i \in S_\varepsilon$ and $\alpha < \kappa$ as above. We pick some $\bar{a}$ that enumerates $[\kappa]^{<\kappa}$ such that $S \in \check{I}[\kappa](\bar{a})$, and we fix a club $C \subseteq \kappa$ and a sequence $\langle C_\delta : \delta \in S \cap C \rangle$ such that for $\delta \in S \cap C$, for any $\beta < \delta$, $C_\delta \cap \beta \in \{a_\alpha : \alpha < \delta\}$ and that C_δ is closed in δ. Again we let $E = \{\delta < \kappa : N_\delta \cap \kappa = \delta\}$.

We define the name $\langle \underset{\sim}{d}_\delta : \delta \in S \rangle$ for a sequence by letting for $\delta \in S$

$$\mathbb{Q}_\kappa^{\mathrm{Miller}} \Vdash \bar{\underset{\sim}{d}} = \langle \underset{\sim}{d}_\alpha : \alpha \in S \rangle \wedge (\forall \alpha \in S)(\underset{\sim}{d}_\alpha = u_{\alpha,\underset{\sim}{\eta}(\alpha)}).$$

We show:

$$\mathbb{Q}_\kappa^{\mathrm{Sacks}} \Vdash \text{“}\langle \underset{\sim}{d}_\delta : \delta \in S \rangle \text{ is a } \diamondsuit_\kappa(S)\text{-sequence.”}$$

Towards this suppose that

$$p \Vdash \text{“}\underset{\sim}{x} \in {}^\kappa 2, \text{ and } \underset{\sim}{D} \text{ is a club subset of } \kappa.\text{”}$$

We let $\chi = (\beth_\omega(\kappa))^+$ and let $<^*_\chi$ be a well-ordering of $H(\chi)$. We choose a κ-approximating sequence $\langle N_\varepsilon : \varepsilon < \kappa \rangle$ in $H(\chi)$ (see Definition 2.16) with

$$\mathbf{c} = (\kappa, \bar{a}, p, \bar{\underset{\sim}{\nu}}, \underset{\sim}{x}, \underset{\sim}{D}, S, \langle \bar{C}_\delta : \delta \in S \cap C \rangle) \in N_0. \tag{4.1}$$

We pick any δ with $\delta \in S \cap E \cap C$. Here C is the club from the definition of $S \in \check{I}[\kappa]$.

We show that there is $q \geq p$ such that $q \Vdash \delta \in \underset{\sim}{D} \wedge \underset{\sim}{\nu}_\delta = \underset{\sim}{x} \restriction \delta$. Such a q will be gotten as the limit of a sequence $\langle (p_\varepsilon, x_\varepsilon, \gamma_\varepsilon, \kappa_\varepsilon) : \varepsilon < \mathrm{cf}(\delta) \rangle$. We shall arrange $\lim \kappa_\varepsilon = \delta$. Since $\delta = \sup\{\kappa_\varepsilon : \varepsilon < \mathrm{cf}(\delta)\} \notin N_\delta$, of course, the whole sequence is not an element of N_δ.

We approach δ by an increasing sequence of length $\mathrm{cf}(\delta)$ in models of size $< \mathrm{cf}(\delta)$. These models, call them M_ε, $\varepsilon < \mathrm{cf}(\delta)$, will be gotten as Skolem hulls that are defined within N_δ. The whole construction reminds of [5, Claim 4.4].

As a guide to a sufficiently fast ascent, we take the approachability witness C_δ. Recall $\mathrm{ot}(C_\delta) = \mathrm{cf}(\delta)$. Let C_δ be increasingly enumerated as $\langle c_{\delta,\varepsilon} : \varepsilon < \mathrm{cf}(\delta) \rangle$. The point is that such a sequence can be found so that each of its strict initial segments $\bar{C}_\delta \restriction \varepsilon$, $\varepsilon < \mathrm{cf}(\delta)$, is an element of N_δ.

We let $\kappa_{-1} = 0$, $p_0 \geq p$ and $p_0 \Vdash \gamma_0 \in \underset{\sim}{D}$. We let $M_0 = \mathrm{Sk}^{N_\delta}(\{p_0, \gamma_0\})$ and $\kappa_0 = \sup(M_0 \cap \kappa)$.

By induction on $\varepsilon < \mathrm{cf}(\delta)$ we choose $(p_\varepsilon, x_\varepsilon, \gamma_\varepsilon, M_\varepsilon)$ with the following properties:

$(\oplus)_1$ For successors $\varepsilon = \zeta + 1$, $(p_\varepsilon, x_\varepsilon, \gamma_\varepsilon, M_\varepsilon)$ is the $<^*_\chi$-least element of N_δ such that
- (a) $p_\varepsilon \geq p$.
- (b) $p_\varepsilon \geq p_\zeta$ for $\zeta < \varepsilon$.
- (c) p_ε forces values to $\underset{\sim}{x} \restriction \kappa_\zeta$, $\min(\underset{\sim}{D} \setminus (\kappa_\zeta + 1))$ call them x_ε, γ_ε respectively.
- (d) $\lg(\mathrm{tr}(p_\varepsilon)) \geq \gamma_\varepsilon > \kappa_\zeta > c_{\delta,\zeta} \geq \zeta$.
- (e) We let $\mathbf{m}_\varepsilon = \langle (p_\zeta, x_\zeta, \gamma_\zeta, M_\zeta) : \zeta < \varepsilon \rangle$, $M_\varepsilon = \mathrm{Sk}^{N_\delta}(\bigcup\{M_\xi : \xi < \varepsilon\} \cup \{p_\varepsilon, x_\varepsilon, \gamma_\varepsilon\} \cup \{\langle c_{\delta,\xi} : \xi < \varepsilon \rangle, \mathbf{m}_\varepsilon\}))$.

(f) Then we let $\kappa_\varepsilon = \sup(\kappa \cap M_\varepsilon)$.

$(\oplus)_2$ For limits $\varepsilon \leq \mathrm{cf}(\delta)$, we take p_ε intersection, γ_ε being the union. Then (d) is fulfilled at ε and we define $\mathbf{m}_\varepsilon$ and M_ε and κ_ε also in the limit as in (e) and (f). We do not worry about continuity of the M_ε, κ_ε, $\varepsilon < \mathrm{cf}(\delta)$.

Along the induction we see: $\mathbf{m}_\varepsilon$ is defined in N_δ by a formula $\varphi = \varphi(x, \bar{y})$ with x for $\mathbf{m}_\varepsilon$ and $\bar{y} = (y_0, y_1)$ with $y_0 = \bar{N} \restriction \kappa_\varepsilon$ and $y_1 = \mathbf{c}$ from (4.1). The sequence $\bar{a}$ in $\mathbf{c}$ guarantees that initial segments are represented in the following sense: For any $\varepsilon < \mathrm{cf}(\delta)$, $\{c_{\delta,\xi} : \xi < \varepsilon\} = C_\delta \cap \alpha$ for some $\alpha < \delta$ and $C_\delta \cap \alpha = a_\beta \in N_\delta$ for some $\beta < \delta$ by the definition of the approachability N_0 and $\bar{N}$.

Successor step $\varepsilon = \zeta + 1$:
Now $(p_\varepsilon, x_\varepsilon, \gamma_\varepsilon, M_\varepsilon)$ is the $<^*_\chi$ first element of $\mathbb{Q}$ satisfying clauses $(\oplus)_1$(a) - (f). Since $\delta \in S \cap C$ is approachable by C_δ, we have for $C_\delta \cap \varepsilon \in N_\delta$ and hence $\langle c_{\delta,\xi} : \xi < \varepsilon \rangle \in N_\delta$ and $\mathbf{m}_\varepsilon \in N_\delta$. As each element of $H(\chi)$ mentioned above is computable from $\bar{N} \restriction \delta_\zeta + 1$, it is an element of N_δ.

Hence there is $(p_\varepsilon, x_\varepsilon, \gamma_\varepsilon, M_\varepsilon)$ as required (a) to (f) in $\in N_\delta$ and we can choose the $<^*_\chi$ first element.

Moreover $p_\zeta \Vdash x_{\kappa_\zeta} = \underset{\sim}{x} \restriction \kappa_\zeta$.

Limit step: let $\varepsilon < \mathrm{cf}(\delta)$ be a limit ordinal. Since $\underset{\sim}{D}$ is p_0 forced to be closed, γ_ε is p_ε-forced to be in $\underset{\sim}{D}$. The reason is that for any $\zeta < \varepsilon' < \varepsilon$,

$$\delta_{\varepsilon'} > \lg(\mathrm{tr}(p_{\varepsilon'})) \geq \gamma_{\varepsilon'} > \kappa_\zeta \geq \gamma_\zeta \geq \zeta.$$

We carried the induction. We let $q = p_{\mathrm{cf}(\delta)} = \bigcap\{p_\varepsilon : \varepsilon < \mathrm{cf}(\delta)\}$. As in the proof of Lemma 3.6 we have $\mathrm{tr}(p_{\mathrm{cf}(\delta)}) = \bigcup\{\mathrm{tr}(p_\varepsilon) : \varepsilon < \mathrm{cf}(\delta)\}$.

Finally, the limit of the κ_ε, $\varepsilon < \mathrm{cf}(\delta)$ is indeed δ itself, since $\sup(C_\delta) = \delta$ and since for any $\varepsilon < \mathrm{cf}(\delta)$, $\{c_{\delta,\xi} : \xi < \varepsilon\} = C_\delta \cap \alpha = a_\beta$ for some $\alpha, \beta < \delta$ and $C_\delta \cap \alpha \in N_\delta$ and $\mathrm{ot}(C_\delta) = \mathrm{cf}(\delta)$.

We write $x_{\mathrm{cf}(\delta)} = \bigcup\{x_\varepsilon : \varepsilon < \mathrm{cf}(\delta)\}$. By $(\oplus_1)$ and $(\oplus_2)$, we have

$$q \Vdash x_\delta = \underset{\sim}{x} \restriction \delta \wedge \delta \in \underset{\sim}{D} \tag{4.2}$$

and that $\mathrm{tr}(p_{\mathrm{cf}(\delta)}))$ has length δ.

We strengthen $p_{\mathrm{cf}(\delta)}$ by the following trunk lengthening: The set $\mathrm{osucc}_{p_{\mathrm{cf}(\delta)}}(\mathrm{tr}(p_{\mathrm{cf}(\delta)})$ is a club subset of κ and thus has non-empty intersection with each S_j, $j < \kappa$. We choose j to be an j with $t_{\delta,j} = x_{\mathrm{cf}(\delta)}$. We pick some $i \in S_j \cap \mathrm{osucc}_{p_{\mathrm{cf}(\delta)}}(\mathrm{tr}(p_{\mathrm{cf}(\delta)}))$. Then $u_{\delta,i} = t_{\delta,j}$. Now

$$p_{\mathrm{cf}(\delta)}^{\langle \mathrm{tr}(p_{\mathrm{cf}(\delta)})^\frown \langle i \rangle \rangle} \Vdash \underset{\sim}{\eta}(\delta) = i \wedge \underset{\sim}{d}_\delta = u_{\delta,i} = t_{\delta,j} = x_{\mathrm{cf}(\delta)} = \underset{\sim}{x} \restriction \delta.$$

□

This concludes the proof of Theorem 1.6.

Remark 4.1. Club κ-Miller and also club κ-Laver forcing adds a κ-Cohen real $\mathbb{C}_\kappa$. This is shown in [2]. There is a $\mathbb{C}_\kappa$-name of a diamond sequence in $V[\mathbb{C}_\kappa]$. Th pattern of the name is the same.

5. Successor Cardinals κ and the Accessible Limit Cardinals

Now we work with $\kappa^{<\kappa} = \kappa \geq \aleph_1$ with one of the two additional properties: κ is a regular limit cardinal that is not strongly inaccessible or κ is a successor cardinal. We present another type of name of a diamond that is based on Bernstein sets. [2] Again the approachability ideal is used, and as in Section 4, the guessing will be at approachable ordinals δ with approaching sequences of size $\mathrm{cf}(\delta)$.

Also based on Bernstein combinatorics, we show that under $\kappa^{<\kappa} > \kappa$ and additional hypotheses the forcing adds a collapse from $\kappa^{<\kappa}$ to κ. Our collapsing technique is different from [14, Section 4].

We first introduce a wider class of forcings.

Definition 5.1. Let $\kappa = \mathrm{cf}(\kappa) > \omega$ and let $W \subseteq \kappa$ be stationary in κ. We let $\mathbb{Q} = \mathbb{Q}^{\mathrm{Sacks}}_{(\kappa,W)}$ be the forcing notion that is defined as follows

(A) Conditions in the forcing order $\mathbb{Q}^{\mathrm{Sacks}}_{(\kappa,W)}$ are trees $p \subseteq {}^{\kappa>}2$ (see Definition 2.6) with the following additional properties:
  (1) (Perfectness) For any $s \in p$ there is an extension $t \trianglerighteq s$ in p such that t has two immediate successors.
  (2) (Closure of splitting in W) If δ is a limit ordinal and $\langle \eta_\varepsilon : \varepsilon < \delta\rangle$ is a $\triangleleft$-increasing sequence with $\bigwedge_{\varepsilon<\delta} \eta_\varepsilon \in \mathrm{split}(p)$ and $\bigcup_{\varepsilon<\delta} \mathrm{lg}(\eta_\varepsilon) \in W$, then $\bigcup_{\varepsilon<\delta} \eta_\varepsilon \in \mathrm{split}(p)$.
  (3) (Closure) The tree order on p is $(< \kappa)$-closed.

(B) $p \leq q$ if $p \supseteq q$.

Lemma 5.2. *Let $W \subseteq \kappa$ be stationary.*

1) The notion of forcing $\mathbb{Q}^{\mathrm{Sacks}}_{(\kappa,W)}$ is $< \kappa$-closed (see Definition 2.12).

2) Let $\mathop{\mathbf{G}}\limits_{\sim}{}_{\mathbb{Q}}$ be a name for the generic filter and let η be the generic branch (see Definition 3.5). Then $\mathbb{Q} \Vdash \mathop{\eta}\limits_{\sim} \in {}^{\kappa}2$ and $\mathbf{V}[\mathop{\eta}\limits_{\sim}] = \mathbf{V}[\mathop{\mathbf{G}}\limits_{\sim}{}_{\mathbb{Q}}]$.

Proof. 1) Let $\langle p_\alpha : \alpha < \gamma\rangle$ be an increasing sequence of conditions and $\gamma < \kappa$. We show that the intersection $q = \bigcap_{\alpha<\gamma} p_\alpha$ is a condition. We prove this by induction on γ. Since we go by induction, we can assume without loss of generality that the sequence is continuous. Since q is an intersection of trees, we have $\emptyset \in q$, and the closure properties (A)(2) and (A)(3) are obvious. We have to show that q is a perfect tree.

We let for $1 \leq \alpha \leq \gamma$, $q_\alpha = \bigcap\{p_\beta : \beta < \alpha\}$. Continuity entails that for limit ordinals α, $p_\alpha = q_\alpha$.

By induction on α we show for any $1 \leq \alpha \leq \gamma$:

$(1)_\alpha$ For any $s \in q_\alpha$, there is a branch b of q_α with $s \in b$ and such that for any $\beta < \alpha$, $U_\beta(b) = \{\xi : b \restriction \xi \in \mathrm{spl}(p_\beta), b \restriction \xi \trianglerighteq s\}$ is unbounded in κ.

$(2)_\alpha$ For any $s \in q_\alpha$ there is an extension $t \trianglerighteq s$, $t \in \mathrm{split}(q_\alpha)$.

[2] A basic form says: Given a regular cardinal τ and a set $\{A_\alpha : \alpha < \tau\}$ of sets $A_\alpha \in [\tau]^\tau$ there is a set $B \in [\tau]^\tau$ that meets each A_α and meets each $\tau \setminus A_\alpha$. Such a B is called a Bernstein set for $\{A_\alpha : \alpha < \tau\}$.

We first show: For any $1 \le \alpha \le \gamma$, $(1)_\alpha$ implies $(2)_\alpha$: For successor $\alpha = \beta+1$, $(2)_\alpha$ holds since p_β is a condition. For limit α, we use the stationarity of W in the following way. Given $s \in q_\alpha$, we pick a branch b for s as in $(1)_\alpha$. By $(1)_\alpha$, for $\beta < \alpha$, the set $\mathrm{acc}^+(U_\beta(b))$ is club in κ. Now $\bigcap\{\mathrm{acc}^+(U_\beta(b)) : \beta < \alpha\} \cap W \neq \emptyset$, and hence there is some $\delta \in \bigcap\{\mathrm{acc}^+(U_\beta(b)) : \beta < \alpha\} \cap W$. Now by the definition of $\mathbb{Q}^{\mathrm{Sacks}}_{(\kappa,W)}$, $b \restriction \delta \in \mathrm{spl}(p_\beta)$ for any $\beta < \alpha$ and $b \restriction \delta \in \mathrm{spl}(q_\alpha)$.

Now we carry out the induction: Given $(2)_\beta$ for $1 \le \beta < \alpha$, we prove $(1)_\alpha$. The statement is obvious for $\alpha = 1$ and for $\alpha = \beta+1$ being a successor ordinal, since $q_{\beta+1} = p_\beta$ is a condition. Now let α be a limit ordinal. We establish $(1)_\alpha$ by going in κ-many steps of size α each: Let $s \in q_\alpha$ be given.

By induction on $k \in \kappa$, we define a sequence r_k, $k < \kappa$, of nodes in q_α such that the sequence is continuously increasing in $\trianglelefteq$ and such that $r_0 = s$ and for each $k \in \kappa$, in $\{t \in q_\alpha : r_k \trianglelefteq t \triangleleft r_{k+1}\}$, for any $\beta < \alpha$, there is a splitting node of p_β. We choose an increasing sequence $\langle \alpha_i : i < \mathrm{cf}(\alpha)\rangle$ with limit α.

We carry out the successor step of the induction: Let $r_k \in q_\alpha = \bigcap\{p_{\alpha_i} : i < \mathrm{cf}(\alpha)\}$ be given. We let s_{α_0} be the minimal splitting node above r_k in p_{α_0}. By $(2)_{\alpha_0}$ such an s_{α_0} exists. Suppose that s_{α_i}, $i < j$, is defined such that $s_{\alpha_i} \in \mathrm{split}(p_{\alpha_i})$ and such that for $i' < i < j$, $s_{\alpha_{i'}} \triangleleft s_{\alpha_i}$. If j is a successor, then we invoke $(2)_{\alpha_j}$. If j is a limit, then by closure $s'_{\alpha_j} = \bigcup\{s_{\alpha_i} : i < j\}$ is a node of each of the q_{α_i}, $i < j$. Hence it is a node of $q_{\alpha_j} = \bigcap\{q_{\alpha_i} : i < j\}$. The shortest splitting node in $p_{\alpha_j} = q_{\alpha_j}$ above s'_{α_j} serves as s_{α_j}. By induction hypothesis, such a node exists. Now the induction over $j < \mathrm{cf}(\alpha)$ is carried out. We let $r_{k+1} = \bigcup\{s_{\alpha_j} : j < \mathrm{cf}(\alpha)\}$. By (A)(3), $r_{k+1} \in p_{\alpha_j}$ for any $j < i$ and hence $r_{k+1} \in q_\alpha$. The increasing, not necessarily continuous sequence $\langle s_{\alpha_i} : i < \mathrm{cf}(\alpha)\rangle$ witnesses that for each $\beta < \alpha$ there is a splitting node of p_β between r_k and r_{k+1}.

For limits ordinals $k \in \kappa$, we let $r_k = \bigcup\{r_j : j < k\}$ and again invoke (A)(3). Now the induction on $k \in \kappa$ is performed. The branch $b = \bigcup\{r_k : k < \kappa\}$ is a witness for $(1)_\alpha$. This finished the limit step of the induction on α.

Now q fulfils (A)(1) because we have $(2)_\gamma$.

2) Let $\mathbf{G}$ be $\mathbb{Q}$-generic over $\mathbf{V}$. Then $\eta = \bigcup\{\mathrm{tr}(p) : p \in \mathbf{G}\}$ is a function from κ to 2, since for any $p \in \mathbb{Q}$ and any $t \in p$ also the subtree $p^{\langle t\rangle}$ is a condition, and if t and t' are incompatible nodes in p, the conditions $p^{\langle t\rangle}$ and $p^{\langle t'\rangle}$ are incompatible. For each $\alpha < \kappa$, the generic filter meets the dense set $D_\alpha = \{p \in \mathbb{Q}^{\mathrm{Sacks}}_{(\kappa,W)} : \mathrm{dom}(\mathrm{tr}(p)) \ge \alpha + 1\}$. The generic branch η contains the full information about $\mathbf{G}$ since for any generic filter $\mathbf{G}$ we have for any p: $p \in \mathbf{G}$ iff $\eta \in [p]$. For a detailed proof see [14, Proposition 1.2]. $\square$

Remark 5.3. If W is a non-stationary set, then $\mathbb{Q}^{\mathrm{Sacks}}_{(\kappa,W)}$ is not $< \kappa$-complete.

Proof. Let C be club in κ with $C \cap W = \emptyset$. Let $\langle c_\alpha : \alpha < \kappa \rangle$ be a continuous enumeration of C. Now let for $i < \omega$,

$$\begin{aligned} p_n = \{t \in {}^{\kappa>}2 : & (\forall \alpha \in \mathrm{dom}(t) \setminus C)(t(\alpha) = 0) \wedge \\ & (\forall \alpha \in \kappa)((c_\alpha \in \mathrm{dom}(t) \wedge (\exists \lambda < \kappa, \lambda \text{ limit})(\exists n \in \omega) \\ & \alpha = \lambda + i, i \leq n) \rightarrow t(c_\alpha) = 0)\}. \end{aligned}$$

Then for $n < \omega$, $p_n \in \mathbb{Q}^{\mathrm{Sacks}}_{(\kappa,W)}$ and $\bigcap\{p_n : n < \omega\} = \{b_0\}$ with $b_0(\alpha) = 0$ for $\alpha < \kappa$. □

Fact 5.4. *Assume $\kappa > \aleph_0$ is regular and $W \subseteq \kappa$ is stationary and $\mathbb{Q} = \mathbb{Q}^{\mathrm{Sacks}}_{(\kappa,W)}$. If $\kappa = \kappa^{<\kappa}$, then $\mathbb{Q}$ is κ-proper.*

Proof. The proof given in Lemma 3.10 applies also here, since it does not use the fact that the limit of splitting nodes is a splitting node but just fusion and the $< \kappa$-closure of $\mathbb{Q}^{\mathrm{Sacks}}_{(\kappa,W)}$. □

Now we introduce some Bernstein combinatorics on closed subtrees.

Definition 5.5. Suppose that $\delta \in \kappa$ and $\mathrm{cf}(\delta) = \mathrm{cf}(\sigma)$.

- (A) A function $f \colon {}^{\sigma>}2 \to {}^{\delta>}2$ is called a *σ-tree embedding of height δ*, if the following holds:
  - (a) for any $s, t \in {}^{\sigma>}2$, if $s \trianglelefteq t$, then $f(s) \trianglelefteq f(t)$.
  - (b) For any $b \in {}^{\sigma}2$, $\bigcup\{f(b \restriction i) : i < \sigma\} \in {}^{\delta}2$.
- (B) A σ-tree embedding of height δ is called *one-to-one* if for any $s, t \in {}^{\sigma>}2$, if $s \perp t$ then $f(s) \perp f(t)$. We write $s \perp t$, if s and t are incomparable (which is the same as incompatible) in $\trianglelefteq$.
- (C) Given a σ-tree embedding f of height δ, there is a lift to branches $\bar{f} \colon {}^{\sigma}2 \to {}^{\delta}2$ given by $\bar{f}(b) = \bigcup\{f(b \restriction i) : i < \sigma\}$.

Remark 5.6.

- (a) We do not require that the tree emdeddings fulfil $f(s \cap t) = f(s) \cap f(t)$. The righthand side might be longer. We let $\sigma = \lim\langle \sigma_i : i < \mathrm{cf}(\sigma)\rangle$ for an increasing continuous sequence.
- (b) By the rule (A)(a), for any sequence $\langle \sigma_i : i < \mathrm{cf}(\sigma)\rangle$ that converges to σ, the restriction $f \restriction \bigcup\{{}^{\sigma_i}2 : i < \mathrm{cf}(\sigma)\}$ determines the function f.

The following lemma is used for names of diamonds and for names of collapsing functions.

Lemma 5.7 (Bernstein Lemma)**.** *We assume that $\kappa = \kappa^{<\kappa} > \aleph_0$ and $2^\sigma = \kappa$ and $2^{<\sigma} < \kappa$. For each $\delta \in \kappa \cap \mathrm{cof}(\mathrm{cf}(\sigma))$ we let*

$$\begin{aligned} \mathcal{F}_{\sigma,\delta} = \{(f_1, f_2) : & f_1, f_2 \text{ are } \sigma\text{-tree embeddings} \\ & \text{of height } \delta \text{ and } f_1 \text{ is one-to-one}\}. \end{aligned}$$

Then there is some $h_\delta \colon {}^{\delta}2 \to {}^{\delta}2$ such that:

$$(\forall f_1, f_2 \in \mathcal{F}_{\sigma,\delta})(\exists \eta \in {}^{\sigma}2)(h_\delta(\bar{f}_1(\eta)) = \bar{f}_2(\eta)) \text{ and} \tag{5.1}$$

$$(\forall f_1 \in \mathcal{F}_{\sigma,\delta})(\forall \alpha \in {}^{\delta}2)(\exists \eta' \in {}^{\sigma}2)(h_\delta(\bar{f}_1(\eta')) = \alpha). \tag{5.2}$$

In Theorem 1.2 we use only (5.1). We use (5.2) at $\mu > \kappa$ for Proposition 5.9. In a diagonalisation of length κ, we can find some h_δ with both properties.

Proof. For $\delta \in \kappa \cap \mathrm{cof}(\mathrm{cf}(\sigma))$ we have $|\mathcal{F}_{\sigma,\delta}| \leq \kappa$. Since $2^{<\sigma} < \kappa$, $2^{<\delta} \leq \kappa$ and $\kappa^{2^{<\sigma}} \leq \kappa^{<\kappa} = \kappa$, we have $|\mathcal{F}_{\sigma,\delta}| \leq (2^{<\delta})^{(2^{<\sigma})} \leq \kappa$ and $2^\delta \leq \kappa$.

We enumerate

$$\{(f_1, f_2, x) : (f_1, f_2) \in \mathcal{F}_{\sigma,\delta}, x \in {}^{\delta}2\}$$

as $\langle (f_1^\alpha, f_2^\alpha, x_\alpha) : \alpha < \kappa\rangle$ such that each triple appears κ often.

We define $\eta_\alpha, z_\alpha \in {}^{\delta}2$ and $h_\delta(\bar{f}_1^\alpha(\eta_\alpha)) := \bar{f}_2^\alpha(\eta_\alpha)$ and $h_\delta(\bar{f}_1^\alpha(z_\alpha)) := x_\alpha$ by induction on α. Suppose that $\langle (\eta_\beta, z_\beta, h_\delta(\bar{f}_1^\beta(\eta_\beta)), h_\delta(\bar{f}_1^\alpha(z_\beta)) : \beta < \alpha\rangle$ is defined. At step α we have to take care of (f_1^α, f_2^α) and we have to ensure that x_α gets into the range of $h_\delta \circ \bar{f}_1^\alpha$.

Since f_1^α is one-to-one, there is some $\eta = \eta_\alpha \in {}^{\sigma}2 \setminus \{\eta_\beta : \beta < \alpha\}$ such that $\bar{f}_1^\alpha(\eta_\alpha) \neq \bar{f}_1^\beta(\eta_\beta)$ and for each $\beta < \alpha$. We let $h_\delta(\bar{f}_1^\alpha(\eta_\alpha)) = \bar{f}_2^\alpha(\eta_\alpha)$ and we can pick some $z_\alpha \in {}^{\sigma}2 \setminus (\{\eta_\beta : \beta \leq \alpha\} \cup \{z_\beta : \beta < \alpha\})$ and let $h_\delta(\bar{f}_1^\alpha(z_\alpha)) = x_\alpha$. Here we again use that $\bar{f}_1^\alpha$ is one-to-one. Now the induction is carried out and we have defined a partial function, a subfunction of h_δ. If after the induction the domain of this part of h_δ is not yet the full set ${}^{\sigma}2$, we can define h_δ at the remaining arguments in an arbitrary manner. □

Proof of Theorem 1.2

Proof. Now pinning down names and appropriate strengthening of conditions is carried out for each branch of the full ${}^{\sigma>}2$-tree. Each single branch of this tree will support a construction similar to to $(\oplus)_1$ and $(\oplus)_2$ from Theorem 1.6. Moreover, different branches will lead to conditions with incompatible trunks. Each branch has initial segments that are elements of a small submodel M_ε of N_δ, $\varepsilon < \mathrm{cf}(\sigma)$ with $|M_\varepsilon| < \mathrm{cf}(\sigma)$. The role of the M_ε is similar to their role in the proof of Theorem 1.6(2). However, since a level of the tree ${}^{\sigma_\varepsilon\geq}2$ could be of size $\mathrm{cf}(\sigma)$ already, we just have that each branch separately allows to define a chain of models of size $< \mathrm{cf}(\sigma)$. Each initial segment ${}^{\sigma_\varepsilon\geq}2$ of the tree ${}^{\sigma>}2$ is an element of N_δ. The construction of the whole tree is carried with initial segments in N_δ.

We fix some $S \in \check{I}[\kappa]$ that is stationary in κ, $S \subseteq \kappa \cap \mathrm{cof}(\mathrm{cf}(\sigma))$. For $\delta \in S$ we let h_δ be as in the Lemma 5.7. We define a $\mathbb{Q}$-name $\bar{\underset{\sim}{\nu}} = \langle \underset{\sim}{\nu}_\delta : \delta \in S\rangle$ by

$$\mathbb{Q} \Vdash \underset{\sim}{\nu}_\delta = h_\delta(\underset{\sim}{\eta} \restriction \delta).$$

We show that $\mathbb{Q}$ forces that $\bar{\underset{\sim}{\nu}}$ is a $\diamondsuit(S)$-sequence. Let

$$p \Vdash \underset{\sim}{x} \in {}^{\kappa}2 \wedge \underset{\sim}{D} \text{ is a club in } \kappa.$$

We have to find a $\delta \in S$ and some $q \geq p$ such that

$$q \Vdash \delta \in \underset{\sim}{D} \wedge \underset{\sim}{\nu}_\delta = \underset{\sim}{x} \restriction \delta. \tag{5.3}$$

Let $\bar{a} = \langle a_\alpha : \alpha < \kappa\rangle$ be an enumeration of $[\kappa]^{<\kappa}$. As $S \in \check{I}[\kappa]$ and $S \subseteq \kappa \cap \mathrm{cof}(\mathrm{cf}(\sigma))$ there is a club C in κ and there is $\bar{C}$ such that $\bar{C} = \langle C_\alpha : \alpha \in S \cap C\rangle$, $C_\alpha \subseteq \alpha$ is club in α and $\mathrm{otp}(C_\alpha) = \mathrm{cf}(\delta)$ and for any $\beta < \alpha$, $C_\alpha \cap \beta \in \{a_\gamma : \gamma < \alpha\}$. We fix a continuously increasing sequence $\langle \sigma_\varepsilon : \varepsilon < \mathrm{cf}(\sigma)\rangle$ with limit $\bar{\sigma} = \sigma$.

We let $\chi = (\beth_\omega(\kappa))^+$ and let $<^*_\chi$ be a well-ordering of $H(\chi)$. We choose a κ-approximating sequence $\langle N_\varepsilon : \varepsilon < \kappa\rangle$ in $H(\chi)$ (see Definition 2.16) with

$$\mathbf{c} = (\kappa, \bar{a}, \bar{\sigma}, p, \underset{\sim}{\bar{\nu}}, \underset{\sim}{x}, \underset{\sim}{D}, S, \langle \bar{C}_\delta : \delta \in S \cap C\rangle) \in N_0. \tag{5.4}$$

and ${}^{\sigma >}2 \subseteq N_0$. We let $E = \{\alpha < \kappa : N_\alpha \cap \kappa = \alpha\}$. Since $\langle N_\varepsilon : \varepsilon < \kappa\rangle$ is continuous, E is a club. We pick any δ with $\delta \in S \cap E \cap C$. Note that $N_\delta \cap \kappa = \delta$.

We show that there is $q \geq p$ such that $q \Vdash \delta \in \underset{\sim}{D} \wedge \underset{\sim}{\nu}_\delta = \underset{\sim}{x} \restriction \delta$. For a suitable $\varrho \in {}^{\mathrm{cf}(\sigma)}2$, such a q will be gotten as the limit of a continuously ascending sequence $\langle (p_{\varepsilon,\varrho\restriction\varepsilon}, x_{\varepsilon,\varrho\restriction\varepsilon}, \gamma_{\varepsilon,\varrho\restriction\varepsilon}, \delta_{\varepsilon,\varrho\restriction\varepsilon}) : \varepsilon < \mathrm{cf}(\delta)\rangle$. The point is for each ϱ separately, that such a sequence can be found so that each of its strict initial segments is an element of N_δ.

For the application of h_δ to a suitable $\eta = \mathrm{tr}(p_{\mathrm{cf}(\delta),\varrho})$ as in (5.1) in the end, we have to choose ϱ only after the choice of the whole tree.

By induction on $\varepsilon \leq \mathrm{cf}(\sigma)$ we chose a four-tuple $(\bar{p}_\varepsilon, \bar{x}_\varepsilon, \bar{\gamma}_\varepsilon, \bar{\kappa}_\varepsilon)$ of the form $\bar{p}_\varepsilon = \langle p_{\varepsilon,\varrho} : \varrho \in {}^{\sigma_\varepsilon}2\rangle$ and so forth. We use ϱ as a variable for an element of ${}^{\sigma\geq}2$. Also for a fixed level $\langle \varrho : \varrho \in {}^{\sigma_\varepsilon}2\rangle$ the tuple $(\bar{p}_\varepsilon, \bar{x}_\varepsilon, \bar{\gamma}_\varepsilon, \bar{\kappa}_\varepsilon)$ is an element of N_δ. Again the definition in the successor steps is given separately from the limit steps. In the successor step $\varepsilon = \zeta + 1$, a thread $\varrho \in {}^{\sigma_\zeta}2$ is continued in t-threads for $t\colon [\sigma_\zeta, \sigma_{\zeta+1}) \to 2$ that have mutually incompatible trunk lengthenings. So for $\varepsilon = \zeta + 1$, $\varrho \in {}^{\zeta}2$, $t\colon [\sigma_\zeta, \sigma_{\zeta+1}) \to 2$ we let

$(\odot)_1$ $\langle (p_{\varepsilon,\varrho^\frown t}, x_{\varepsilon,\varrho^\frown t}, \gamma_{\varepsilon,\varrho^\frown t}, \kappa_{\varepsilon,\varrho^\frown t}) : \varrho^\frown t \in {}^{\sigma_\varepsilon}2\rangle$ is the $<^*_\chi$-least element of N_δ such that for each $\varrho^\frown t$:

(a) $p_{\varepsilon,\varrho^\frown t} \geq p$.

(b) $p_{\varepsilon,\varrho^\frown t} \geq p_{\zeta,\varrho}^{\langle \mathrm{tr}(p_{\zeta,\varrho})^\frown \mathrm{emb}(p_{\zeta,\varrho},t)\rangle}$. Here on the right side, $\mathrm{emb}(p,t)$ is defined by induction on the length of t in as going left or right in $p_{\zeta,\varrho}$ above the next splitting node and thus naturally defines a part of an injective tree embedding $\varrho^\frown t \mapsto \mathrm{tr}(p_{\varepsilon,\varrho^\frown t})$. For limit ordinals $\alpha \in [\sigma_\zeta, \sigma_{\zeta+1})$ and t of limit lengths we just take $\mathrm{emb}(t) = \bigcup\{\mathrm{emb}(t \restriction \beta) : \beta < \alpha\}$.

(c) $p_{\varepsilon,\varrho^\frown t}$ forces values to $\underset{\sim}{x} \restriction \kappa_\zeta$, $\min(\underset{\sim}{D} \setminus (\kappa_\zeta + 1))$ call them $x_{\varepsilon,\varrho^\frown t}$, $\gamma_{\varepsilon,\varrho^\frown t}$ respectively. We assume that $\mathrm{dom}(p_{\varepsilon,\varrho^\frown t}) \geq \gamma_{\varepsilon,\varrho^\frown\langle j\rangle}$.

(d) $\mathrm{lg}(\mathrm{tr}(p_{\varepsilon,\varrho^\frown t})) \geq \gamma_{\varepsilon,\varrho^\frown t} > \kappa_{\zeta,\varrho} \geq \zeta$.

(e) We pick $\alpha_\varepsilon < \delta$ such that $\mathrm{ot}(C_\delta \cap \alpha_\varepsilon) = \varepsilon$. Now for a fixed $\varrho^\frown t$ we let

$$\begin{aligned} M_{\varepsilon,\varrho^\frown t} = &\mathrm{Sk}^{N_\delta}\Big(\bigcup\{M_{\xi,\varrho\restriction\sigma_\xi} : \xi < \varepsilon\} \cup \{p_{\varepsilon,\varrho^\frown t}, x_{\varepsilon,\varrho^\frown t}, \gamma_{\varepsilon,\varrho^\frown t}\} \\ &\cup \{C_\delta \cap \alpha_\varepsilon, \langle M_{\xi,\varrho\restriction\sigma_\xi} : \xi < \varepsilon\rangle\}\Big). \end{aligned}$$

Then we let $\kappa_{\varepsilon,\varrho^\frown t} = \sup(\kappa \cap M_{\varepsilon,\varrho^\frown t})$.

$(\odot)_2$ For limits $\varepsilon \leq \mathrm{cf}(\delta)$ and $\varrho \in {}^{\sigma_\varepsilon}2$, we take $p_{\varepsilon,\varrho} = \bigcap\{p_{\zeta,\varrho\restriction\sigma_\zeta} : \zeta < \varepsilon\}$, and let $x_{\varepsilon,\varrho} = \bigcup\{x_{\zeta,\varrho\restriction\sigma_\zeta} : \zeta < \varepsilon\}$, $\gamma_{\varepsilon,\varrho} = \bigcup\{\gamma_{\zeta,\varrho\restriction\sigma_\zeta} : \zeta < \varepsilon\}$. $\gamma_{\varepsilon,\varrho}$ is p_ε-forced to be in $\underset{\sim}{D}$, since $\underset{\sim}{D}$ is p_0-forced to be club and since C_δ is club in δ. If $\varepsilon < \mathrm{cf}(\delta)$, then $\langle (p_{\varepsilon,\varrho}, x_{\varepsilon,\varrho}, \gamma_{\varepsilon,\varrho}, \kappa_{\varepsilon,\varrho}) : \varrho \in {}^{\sigma_\varepsilon}2\rangle \in N_\delta$.

We can carry the induction since $\mathbb{Q}$ is $(<\kappa)$-complete. For $\varrho \in {}^{\sigma>}2$, the model $M_{\varepsilon,\varrho}$ is of size $< \mathrm{cf}(\sigma)$. This together with $2^{\sigma_i} \in N_\delta$ guarantees that all the $\kappa_{\varepsilon,\varrho}$ stay below δ.

For each $\varrho \in {}^\sigma 2$, the sequence $\langle \kappa_{\varepsilon,\varrho\restriction\sigma_\varepsilon} : \varepsilon < \delta\rangle$ converges to δ. This is because $\sup C_\delta = \delta$ and $\mathrm{ot}(C_\delta) = \mathrm{cf}(\sigma) = \mathrm{cf}(\delta)$. In step ε we include the first ε elements of C_δ in $M_{\varepsilon,\varrho}$ for any $\varrho \in {}^{\sigma_\varepsilon}2$. Hence by the definition of C_δ witnessing that $\delta \in S \cap C \in \check{I}[\kappa]$, for any $\varrho \in {}^\sigma 2$, $\lim_{\varepsilon\to\mathrm{cf}(\sigma)} \kappa_{\varepsilon,\varrho\restriction\sigma_\varepsilon} = \delta$.

For each $\varrho \in {}^\sigma 2$ Equation $\odot$ implies: $p_{\sigma,\varrho} \Vdash \underset{\sim}{x} \restriction \delta = \bigcup\{x_{\varepsilon,\varrho\restriction\sigma_\varepsilon} : \varepsilon < \mathrm{cf}(\delta)\}$.

Now by Lemma 5.7 applied to the pair (f_1, f_2) with $f_1(\varrho \restriction \sigma_\varepsilon) = \mathrm{tr}(p_{\varepsilon,\varrho\restriction\sigma_\varepsilon})$ and with $f_2(\varrho\restriction\sigma_\varepsilon) = x_{\varepsilon,\varrho\restriction\varepsilon}$ for $\varrho \in {}^\sigma 2$ and $\varepsilon < \mathrm{cf}(\delta)$, there is some $\varrho \in {}^\sigma 2$ with for any $i < \mathrm{cf}(\sigma)$, $\varrho \restriction \sigma_i \in N_\delta$ such that

$$p_{\sigma,\varrho} \Vdash \bar{f}_1(\varrho) = \underset{\sim}{\eta} \restriction \delta \wedge h_\delta(\bar{f}_1(\varrho)) = \underset{\sim}{\nu}_\delta = \bar{f}_2(\varrho) = x_{\sigma,\varrho} = \underset{\sim}{x} \restriction \delta \wedge \delta \in \underset{\sim}{D}.$$

For the first equality in the forcing statement, we use $(\odot)_1(d)$. For the very last equality we use $(\odot)_1$(c). So $q = p_{\sigma,\varrho}$ and δ are as in (5.3). □

Now Kanamori's premise on iterability is true in the one-step extension:

Corollary 5.8. *We assume $\aleph_1 \leq \kappa = \kappa^{<\kappa}$. Let $W \subseteq \kappa$ be stationary.*

(a) For $\kappa = \aleph_1$ for any stationary S, we have $\mathbb{Q}^{\mathrm{Sacks}}_{(\kappa,W)} \Vdash \diamondsuit_{\aleph_1}(S)$.

(b) For κ that is not a strong limit, $\mathbb{Q}^{\mathrm{Sacks}}_{(\kappa,W)} \Vdash \diamondsuit_\kappa$.

Proof. (a) Any stationary $S \subseteq \aleph_1$ is in the approachability ideal. (b) By Theorem 2.17, there is a stationary set $S \subseteq \kappa \cap \mathrm{cof}(\mathrm{cf}(\sigma))$ with $S \in \check{I}[\kappa]$. □

This concludes the proof of Corollary 1.5.

In the next proposition we show that the Bernstein technique Lemma 5.7 at $2^\sigma > \kappa$ may provide a name of a collapse of 2^σ to κ under additional hypotheses.

Proposition 5.9. *If there is a cardinal $\sigma < \kappa$ an such that $2^\sigma = 2^{<\kappa} = \mu > \kappa$, $2^{<\sigma} < \kappa$ and $\mu^{2^{<\sigma}} \leq \mu$, then $\mathbb{Q}^{\mathrm{Sacks}}_{(\kappa,W)}$ collapses μ to κ.*

Proof. By Theorem 2.17, there is a stationary $S \in \check{I}[\kappa]$, $S \subseteq \kappa \cap \mathrm{cof}(\mathrm{cf}(\sigma))$. The conditions on σ and $\delta < \kappa$ are as in Lemma 5.7 at the cardinal μ. Hence for $\delta \in \kappa \cap \mathrm{cof}(\mathrm{cf}(\sigma))$, we have $h_\delta \colon {}^\delta 2 \to {}^\delta 2$ such that

$$(\forall f_1 \in \mathcal{F}_{\sigma,\delta})(\forall \alpha \in {}^\delta 2)(\exists \varrho \in {}^\sigma 2)(h_\delta(\bar{f}_1(\varrho)) = \alpha).$$

as there. Moreover, for any $\delta \in \kappa \cap \mathrm{cof}(\mathrm{cf}(\sigma))$ we fix a bijection $h_\delta \colon {}^\delta 2 \to \mu$. We show that the function

$$\delta \mapsto b_\delta \circ h_\delta(\underset{\sim}{\eta} \restriction \delta)$$

with domain $\kappa \cap \operatorname{cof}(\operatorname{cf}(\sigma))$ is a name for a function that collapses μ to κ. Given p, we define N_0 and $\bar{N}$ as above and then define E as above. We pick $\delta \in S \cap C \cap E$ and then define an ascent to δ literally as above in $\odot$, with the simplification that we do not have to pin down initial segments of a name $\underset{\sim}{x}$. This time the tree $({}^{\delta>}2, \trianglelefteq)$ has μ many branches, and each of its initial levels has only $< \kappa$ many nodes.

Let $\xi \in \mu$ be given. We choose $\alpha \in {}^{\delta}2$ such that $b_\delta(\alpha) = \xi$ and we choose a function $f_1 \in \mathcal{F}_{\sigma,\delta}$ by letting for $\varrho \in {}^{\sigma>}2$, $f_1(\varrho) = \operatorname{tr}(p_{\operatorname{dom}(\varrho),\varrho})$. Then $\bar{f}_1 \colon {}^{\sigma}2 \to {}^{\delta}2$. Since h_δ is as above, for the given f_1, α there is some for $\varrho \in {}^{\sigma}2$ such

$$p_{\sigma,\varrho} \Vdash h_\delta(\bar{f}_1(\varrho)) = h_\delta(\underset{\sim}{\eta} \restriction \delta) = \alpha,$$

and the latter entails $p_{\sigma,\varrho} \Vdash b_\delta(h_\delta(\underset{\sim}{\eta} \restriction \delta)) = b_\delta(\alpha) = \xi$. □

Remark 5.10. Proposition 5.9 is proved differently for ordinary κ-Sacks in [14], where Solovay partitions of stationary sets in pairwise disjoint stationary sets are used and Clause (2) Definition 2.7 is used.

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