CARDINAL INVARIANTS DISTINGUISHING PERMUTATION GROUPS

TARAS BANAKH AND HEIKE MILDENBERGER

Abstract. We prove that for infinite cardinals $\kappa < \lambda$ the alternating group $\mathrm{Alt}(\lambda)$ (of even permutations) of λ is not embeddable into the symmetric group $\mathrm{Sym}(\kappa)$ (of all permutations) of κ. To prove this fact we introduce and study several monotone cardinal group invariants which take value κ on the groups $\mathrm{Alt}(\kappa)$ and $\mathrm{Sym}(\kappa)$.

By Cayley's classical theorem [11, 1.6.8], each group G embeds into the group $\mathrm{Sym}(|G|)$ of all bijective transformations of the cardinal $|G|$. Observe that for the symmetric group $G = \mathrm{Sym}(\kappa)$ on an infinite cardinal κ Cayley's Theorem can be improved: the group $G = \mathrm{Sym}(\kappa)$ embeds into itself and $\kappa < |G| = 2^\kappa$. This suggests the following question: *can each infinite group G be embedded into the symmetric group $\mathrm{Sym}(\kappa)$ such that $|G| = 2^\kappa$?* Another question of the same flavor asks: *can the symmetric group $\mathrm{Sym}(\kappa)$ on an infinite cardinal κ be embedded into the symmetric group $\mathrm{Sym}(\lambda)$ on a smaller cardinal $\lambda < \kappa$?* In this paper we shall give negative answers to both questions. First, we need to introduce some notation.

Let κ be a (finite or infinite) cardinal. By $\mathrm{Sym}(\kappa)$ we denote the set of bijective functions from κ to κ, also called the *permutations* of κ. The set $\mathrm{Sym}(\kappa)$ endowed with the operation of composition of permutations is a group called the *symmetric group* on κ. This group contains a normal subgroup $\mathrm{Sym}_{\mathrm{fin}}(\kappa)$ consisting of permutations $f : \kappa \to \kappa$ with finite support $\mathrm{supp}(f) = \{x \in \kappa : f(x) \neq x\}$. A permutation $f : \kappa \to \kappa$ with two-element support $\mathrm{supp}(f)$ is called a *transposition* of κ. It is well-known that each finitely supported permutation can be written as a finite composition of transpositions. A permutation $f : \kappa \to \kappa$ is called *even* if it can be written as the composition of an even number of transpositions. The even permutations form a subgroup $\mathrm{Alt}(\kappa)$ of $\mathrm{Sym}(\kappa)$ called the *alternating group* on κ. It is a normal subgroup of index 2 in $\mathrm{Sym}_{\mathrm{fin}}(\kappa)$. So, we get the inclusions

$$\mathrm{Alt}(\kappa) \subset \mathrm{Sym}_{\mathrm{fin}}(\kappa) \subset \mathrm{Sym}(\kappa).$$

For an infinite cardinal κ these groups have cardinalities $|\mathrm{Alt}(\kappa)| = |\mathrm{Sym}_{\mathrm{fin}}(\kappa)| = \kappa$ and $|\mathrm{Sym}(\kappa)| = 2^\kappa$. More information on permutation groups can be found in the books [4], [5], [6].

The following theorem answers in negative the questions posed at the beginning of the paper.

Theorem 1. *Let $\kappa < \lambda$ be two infinite cardinals. Then there is no embedding of $\mathrm{Alt}(\lambda)$ into $\mathrm{Sym}(\kappa)$.*

The idea of the proof of this theorem is rather natural: find a cardinal characteristic $\varphi(G)$ of a group G which is invariant under isomorphisms of groups, is monotone under taking subgroups, and takes value κ on the groups $\mathrm{Alt}(\kappa)$ and $\mathrm{Sym}(\kappa)$. Then for any infinite cardinals $\kappa < \lambda$ we would have $\varphi(\mathrm{Alt}(\lambda)) = \lambda \not\leq \kappa = \varphi(\mathrm{Sym}(\kappa))$, and the monotonicity of φ would imply that $\mathrm{Alt}(\lambda)$ does not embed into $\mathrm{Sym}(\kappa)$. In the sequel, cardinal characteristics of groups which are invariant under isomorphisms of groups will be called *cardinal group invariants*. A cardinal group invariant φ is called *monotone* if $\varphi(H) \leq \varphi(G)$ for any subgroup H of a group G.

Many examples of monotone cardinal group invariants can be produced as minimizations of cardinal characteristics of (semi)topological groups over certain families of admissible topologies on a given group. Now we explain this approach in more details. First, we define four families $\mathcal{T}_s(G)$, $\mathcal{T}_c(G)$, $\mathcal{T}_g(G)$, and $\mathcal{T}_l(G)$ of admissible topologies on a group G.

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A topology τ on a group G is called *shift-invariant* if for every $a, b \in G$ the two-sided shift $s_{a,b} : G \to G$, $s_{a,b} : x \mapsto axb$, is a homeomorphism of the topological space (G, τ). This is equivalent to saying that the group multiplication $G \times G \to G$, $(x, y) \to xy$, is separately continuous. A group endowed with a shift-invariant topology is called a *semitopological group*, (see [1]). For a group G by $\mathcal{T}_s(G)$ we denote the family of all Hausdorff shift-invariant topologies on G.

We shall say that a topology τ on a group G has *separately continuous commutator* if the function $G \times G \to G$, $(x, y) \mapsto xyx^{-1}y^{-1}$, is separately continuous. By $\mathcal{T}_c(G)$ we denote the family of Hausdorff shift-invariant topologies on G having separately continuous commutator.

A topology τ on a group G is called a *group topology* if the function $G \times G \to G$, $(x, y) \mapsto xy^{-1}$, is jointly continuous. By $\mathcal{T}_g(G)$ we denote the family of Hausdorff group topologies on G.

A group topology τ on a group G is called *linear* if it has a neighborhood base at the unit 1_G of G consisting of τ-open subgroups of G. By $\mathcal{T}_l(G)$ we denote the family of linear Hausdorff group topologies on G.

It follows that $\mathcal{T}_s(G) \supset \mathcal{T}_c(G) \supset \mathcal{T}_g(G) \supset \mathcal{T}_l(G)$ for every group G.

By a *cardinal topological invariant of semitopological groups* we understand a function φ assigning to each semitopological group G some cardinal $\varphi(G)$ so that $\varphi(G) = \varphi(H)$ for any topologically isomorphic semitopological groups G, H. We shall say that the function φ is *monotone* if $\varphi(H) \le \varphi(G)$ for any subgroup H of a semitopological group G.

Any (monotone) cardinal topological invariant φ of semitopological groups induces four (monotone) cardinal group invariants φ_s, φ_c, φ_g, φ_l assigning to each group G the cardinals

- $\varphi_s(G) = \min\{\varphi(G, \tau) : \tau \in \mathcal{T}_s(G)\}$,
- $\varphi_c(G) = \min\{\varphi(G, \tau) : \tau \in \mathcal{T}_c(G)\}$,
- $\varphi_g(G) = \min\{\varphi(G, \tau) : \tau \in \mathcal{T}_g(G)\}$,
- $\varphi_l(G) = \min\{\varphi(G, \tau) : \tau \in \mathcal{T}_l(G)\}$.

The inclusions $\mathcal{T}_s(G) \supset \mathcal{T}_c(G) \supset \mathcal{T}_g(G) \supset \mathcal{T}_l(G)$ imply the inequalities

$$\varphi_s(G) \le \varphi_c(G) \le \varphi_g(G) \le \varphi_l(G).$$

We shall apply this construction to three cardinal topological invariants of semitopological groups: the weight w, the spread s and the uniform spread u. A subset D of a semitopological group (G, τ) is called *uniformly discrete* if there exists a τ-open neighborhood $U \subset G$ of the unit 1_G such that $y \notin xU$ for any distinct points $x, y \in D$. For a semitopological group (G, τ) let

- $w(G, \tau) = \min\{|\mathcal{B}| : \mathcal{B} \subset \tau$ is a base of the topology $\tau\}$ be the *weight* of (G, τ);
- $s(G, \tau) = \sup\{|D| : D \subset G$ is a discrete subspace of $(G, \tau)\}$ be the *spread* of (G, τ) and
- $u(G, \tau) = \sup\{|D| : D \subset G$ is a uniformly discrete subset of $(G, \tau)\}$ be the *uniform spread* of (G, τ).

Observe that the weight and spread of (G, τ) depend only on the topology τ whereas the definition of the uniform spread involves both structures (algebraic and topological) of the semitopological group (G, τ). Taking into account that each uniformly discrete subset of a semitopological group (G, τ) is discrete, we conclude that $u(G) \le s(G) \le w(G)$. Theorem 5.5 of [7] implies that $w(G) \le 2^{2^{s(G)}}$. It is easy to see that u, s, w are monotone cardinal topological invariants of semitopological groups. Minimizing these cardinal functions over the families $\mathcal{T}_s(G)$, $\mathcal{T}_c(G)$, $\mathcal{T}_g(G)$ and $\mathcal{T}_l(G)$ we obtain 12 monotone cardinal group invariants that relate as follows. In the diagram an arrow $\varphi \to \psi$ between two cardinal group invariants φ, ψ indicates that $\varphi(G) \le \psi(G)$ for any group G.

$$\begin{tikzcd}
w_s \arrow[r] & w_c \arrow[r] & w_g \arrow[r] & w_l \arrow[r] & |\cdot| \\
s_s \arrow[r] \arrow[u] & s_c \arrow[r] \arrow[u] & s_g \arrow[r] \arrow[u] & s_l \arrow[u] & \\
u_s \arrow[r] \arrow[u] & u_c \arrow[r] \arrow[u] & u_g \arrow[r] \arrow[u] & u_l \arrow[u] &
\end{tikzcd}$$

Since we investigate the embeddility of groups independently of whether they are endowed with topologies, it is natural to look out for invariants that do not involve any topology. An example of such a combinatorial cardinal group invariant is the invariant $cn(G)$ that is called the *weak compatibility number* of a group G. It is defined as the smallest infinite cardinal κ for which the group G satisfies the *weak κ^+-compatibility condition*:

- for any finite group F and isomorphisms $f_i : F \to F_i$, $i < \kappa^+$, onto finite subgroups F_i of G there are two indices $i < j < \kappa^+$ and a homomorphism $\phi : \langle F_i \cup F_j \rangle \to F$ such that $\phi \circ f_i = \phi \circ f_j = \mathrm{id}_F$.

Here by κ^+ we denote the successor cardinal of κ and for a subset $A \subset G$ by $\langle A \rangle$ we denote the subgroup of G generated by A. The weak κ^+-compatibility condition is a weak version of a notion introduced in [10, Def. 1.9]. The definition of the weak compatibility number implies that it is a monotone cardinal group invariant. Observe that each torsion-free group G has $cn(G) = \omega$, so $cn(G)$ can be much smaller than the cardinal $s_s(G) \geq \log\log |G|$. The latter is just an equivalent form of Theorem 5.5 of [7] using the logarithm, where for a cardinal κ by $\log(\kappa) = \min\{\lambda : \kappa \leq 2^\lambda\}$ we denote the *logarithm* of κ.

In Proposition 3 we shall present an algebraic description of the linear weight w_l and using this description will prove that $cn(G) \leq w_l(G)$ for any group G. This inequality allows us to add the weak compatibility number cn to the diagram describing the relations between cardinal group invariants and obtain the diagram:

$$\begin{tikzcd}
w_s \arrow[r] & w_c \arrow[r] & w_g \arrow[r] & w_l & cn. \arrow[l] \\
s_s \arrow[r] \arrow[u] & s_c \arrow[r] \arrow[u] & s_g \arrow[r] \arrow[u] & s_l \arrow[u] & \\
u_s \arrow[r] \arrow[u] & u_c \arrow[r] \arrow[u] & u_g \arrow[r] \arrow[u] & u_l \arrow[u] &
\end{tikzcd}$$

The following theorem implies Theorem 1 and can be considered as a main result of this paper.

Theorem 2. *For any infinite cardinal κ and a group G with $\mathrm{Alt}(\kappa) \subset G \subset \mathrm{Sym}(\kappa)$ we get*

$$\kappa = s_s(G) = u_c(G) = w_l(G) = cn(G).$$

The proof of this theorem will be divided into three Lemmas 12, 15, 21. We start our proofs with an algebraic description of the linear weight $w_l(G)$ of a group G.

Proposition 3. *For an infinite group G its linear weight $w_l(G)$ is equal to the smallest cardinal κ for which there are subgroups G_i, $i \in \kappa$, of index $|G/G_i| \leq \kappa$ such that $\bigcap_{i\in\kappa} G_i$ coincides with the trivial subgroup $\{1_G\}$ of G.*

Proof. Let $w_l'(G)$ denote the smallest cardinal κ such that $\{1_G\} = \bigcap_{i\in\kappa} G_i$ for some subgroups G_i of index $|G/G_i| \leq \kappa$ in G. We need to prove that $w_l(G) = w_l'(G)$.

To prove that $w_l'(G) \leq w_l(G)$, use the definition of the linear weight $w_l(G)$ and find a linear group topology τ of weight $\kappa = w_l(G)$ on G. Let $\mathcal{B} \subset \tau$ be a base of the topology τ of cardinality $|\mathcal{B}| = \kappa$. Let $\mathcal{B}_1 = \{B \in \mathcal{B} : 1_G \in B\}$ be the neighborhood base at the unit 1_G of the group G. Since $|\mathcal{B}_1| \leq |\mathcal{B}| \leq \kappa$, the set $\mathcal{B}_1$ can be enumerated as $\mathcal{B}_1 = \{B_i\}_{i<\kappa}$. Since the topology is linear, each set $B_i \in \mathcal{B}_1$ contains an open subgroup H_i of G. Taking into account that the family $\{xH_i : x \in G\}$ is disjoint and each coset xH_i, $x \in G$, contains some basic set $U \in \mathcal{B}$, we conclude that $|\{xH_i : x \in G\}| \leq |\mathcal{B}| = \kappa$ and hence the subgroup H_i has index $\leq \kappa$ in G. The Hausdorff property of the topology τ guarantees that $\{1_G\} = \bigcap \mathcal{B}_1 = \bigcap_{i<\kappa} H_i$. So, the family $\{H_i\}_{i<\kappa}$ witnesses that $w_l'(G) \leq \kappa = w_l(G)$.

Now we check that $w_l(G) \leq w_l'(G)$. By the definition of the cardinal $\kappa = w_l'(G)$, there exists a family $\mathcal{H}$ of subgroups of index $\leq \kappa$ in G such that $|\mathcal{H}| \leq \kappa$ and $\{1_G\} = \bigcap \mathcal{H}$. Observe that for any subgroup $H \in \mathcal{H}$, any $x \in G$ and $y \in xH$, we get $xHx^{-1} = yHy^{-1}$, which implies that the family $\{xHx^{-1} : x \in G\}$ has cardinality $\leq |G/H| \leq \kappa$. Then the family

$$\mathcal{U} = \Big\{ \bigcap_{i=1}^{n} x_i H_i x_i^{-1} : H_1, \dots, H_n \in \mathcal{H},\ x_1, \dots, x_n \in G \Big\}$$

has cardinality $|\mathcal{U}| \le \kappa$ and consists of subgroups of index $\le \kappa$ in G. Now consider the topology τ on G consisting of sets $U \subset G$ such that for every point $x \in U$ there is a subgroup $H \in \mathcal{U}$ such that $xH \subset U$. As in Theorem 1.3.2 in [1] it is shown the group (G, τ) is a linear topological group of weight $\le \kappa$: The topology is a group topology since $\mathcal{U}$ is closed under intersection with conjugates, τ is Hausdorff since $\bigcap H = \{1_G\}$, is it linear since $\mathcal{U} \subset \tau$ and it is of weight $\le \kappa$ because $\{xH : H \in \mathcal{U}, x \in G\}$ is a basis. Thus $w_l(G) \le \kappa = w_l'(G)$. □

Proposition 4. *Each infinite group G has $w_g(G) \le w_l(G) \le w_g(G)^\omega$.*

Proof. The inequality $w_g(G) \le w_l(G)$ is trivial. To prove that $w_l(G) \le w_g(G)^\omega$, fix a Hausdorff group topology τ on G such that $w(G, \tau) = w_g(G)$. Let $\kappa = w_g(G) = w(G, \tau)$ and fix a neighborhood base $\{U_\alpha\}_{\alpha \in \kappa} \subset \tau$ at the unit 1_G of the group G. For every $\alpha \in \kappa$ put $U_{\alpha,0} = U_\alpha$ and for every $n \in \mathbb{N}$ choose a symmetric neighborhood $U_{\alpha,n} = U_{\alpha,n}^{-1} \in \tau$ of the unit 1_G such that $U_{\alpha,n} U_{\alpha,n} \subset U_{\alpha,n-1}$. It is easy to see that the intersection $H_\alpha = \bigcap_{n \in \mathbb{N}} U_{\alpha,n}$ is a subgroup of G. We claim that this subgroup has index $|G/H_\alpha| \le \kappa^\omega$ in G. Let $q : G \to G/H_\alpha := \{xH_\alpha : x \in G\}$, $q : x \mapsto xH_\alpha$, be the quotient map and $s : G/H_\alpha \to G$ be any function such that $q \circ s = \mathrm{id}$. For every $n \in \mathbb{N}$ choose a maximal subset $K_n \subset G$ such that $xU_{\alpha,n} \cap yU_{\alpha,n} = \emptyset$ for any distinct points $x, y \in K_n$, and observe that K_n is a discrete subspace of (G, τ), which implies that $|K_n| \le w(G, \tau) = w_g(G) = \kappa$. By the maximality of K_n, for every $x \in G$ there is a point $f_n(x) \in K_n$ such that $xU_{\alpha,n} \cap f_n(x)U_{\alpha,n} \ne \emptyset$ and hence $x \in f_n(x)U_{\alpha,n}U_{\alpha,n}^{-1} \subset f_n(x)U_{\alpha,n-1}$. The functions f_n, $n \in \mathbb{N}$, form the function $f = (f_n)_{n=1}^\infty : G \to \prod_{n=1}^\infty K_n$. We claim that the composition $f \circ s : G/H_\alpha \to \prod_{n \in \mathbb{N}} K_n$ is injective. Given any cosets $X, Y \in G/H_\alpha$ with $f \circ s(X) = f \circ s(Y)$, consider the points $x = s(X)$ and $y = s(Y)$. The equality $f(x) = f \circ s(X) = f \circ s(Y) = f(y)$ implies that $f_n(x) = f_n(y)$ for all $n \in \mathbb{N}$ and hence $x, y \in f_n(x)U_{\alpha,n-1}$, which implies that $x^{-1}y \in U_{\alpha,n-1}^{-1} U_{\alpha,n-1} \subset U_{\alpha,n-2}$ for every $n \ge 2$. Then $x^{-1}y \in \bigcap_{n=2}^\infty U_{\alpha,n-2} = H_\alpha$ and hence $X = xH_\alpha = yH_\alpha = Y$, witnessing that the function $f \circ s : G/H_\alpha \to \prod_{n \in \mathbb{N}} K_n$ is injective and hence $|G/H_\alpha| \le \left|\prod_{n \in \mathbb{N}} K_n\right| \le \kappa^\omega$. Since $1_G \in \bigcap_{\alpha \in \kappa} H_\alpha \subset \bigcap_{\alpha \in \kappa} U_\alpha = \{1_G\}$, we can apply Proposition 3 and conclude that $w_l(G) \le \kappa^\omega = w_g(G)^\omega$. □

Proposition 3 has the following corollary.

Corollary 5. *Each proper subgroup H of an infinite simple group G has index $|G/H| \ge w_l(G)$ in G.*

Proof. If H is a proper subgroup of G, then the family $\mathcal{H} = \{xHx^{-1} : x \in G\}$ consists of subgroups of index $|G/H|$ and has cardinality $|\mathcal{H}| \le |G/H|$. The latter inequality follows from the equality $xHx^{-1} = yHy^{-1}$ holding for any points $x, y \in G$ with $xH = yH$. Since the group G is simple, the normal subgroup $\bigcap \mathcal{H} = \bigcap_{x \in G} xHx^{-1}$ of G is trivial. By Proposition 3, $w_l(G) \le |G/H|$. □

Modifying the proofs of Propositions 3 and 4 we can prove the following two facts about the cardinal group invariant u_l.

Proposition 6. *For an infinite group G its linear uniform spread $u_l(G)$ is equal to the smallest cardinal κ such that $\{1_G\} = \bigcap \mathcal{H}$ for some family $\mathcal{H}$ of subgroups of index $\le \kappa$ in G.*

Proposition 7. *Each infinite group G has $u_g(G) \le u_l(G) \le u_g(G)^\omega$.*

Both inequalities in Propositions 4 and 7 can be strict.

Example 8. *The discrete group $G = \mathbb{Z}$ of integers has $u_g(G) = u_l(G) = w_g(G) = w_l(G) = \omega < \omega^\omega$.*

Example 9. *Let M be the unit interval or the unit circle. The homeomorphism group G of M has $u_g(G) = w_g(G) = \omega$ and $u_l(G) = w_l(G) = \mathfrak{c}$.*

Proof. The equality $w_g(G) = \omega$ follows from the fact that the compact-open topology on the homeomorphism group G is metrizable and separable. The equality $u_l(G) = \mathfrak{c}$ follows from Proposition 6 and Theorem 6 of [12] saying that the subgroup $G_+ \subset G$ of orientation preserving homeomorphisms of M is the only subgroup of index $< \mathfrak{c}$ in G. □

The following proposition gives an upper bound on the weak compatibility number cn.

Proposition 10. *Each infinite group G has $cn(G) \le w_l(G)$.*

Proof. It suffices to prove that G satisfies the weak κ^+-compatibility condition for $\kappa = w_l(G)$. By Proposition 3, there is a family $\mathcal{H}$ of subgroups of index $\le \kappa$ in G such that $|\mathcal{H}| \le \kappa$ and $\{1_G\} = \bigcap \mathcal{H}$. Replacing $\mathcal{H}$ by a larger family, we can assume that for any subgroups $H_1, \dots, H_n \in \mathcal{H}$ and any points $x_1, \dots, x_n \in G$ the subgroup $\bigcap_{i=1}^n x_i H_i x_i^{-1}$ belongs to the family $\mathcal{H}$.

To show that the group G satisfies the weak κ^+-compatibility condition, fix a finite group F and isomorphisms $f_i : F \to F_i$, $i < \kappa^+$, onto finite subgroups $F_i \subset G$. Since the family $\mathcal{H}$ is closed under finite intersections, for every $i < \kappa^+$ we can choose a subgroup $H_i \in \mathcal{H}$ such that $F_i \cap H_i = \{1_G\}$. Replacing H_i by the subgroup $\bigcap_{x \in F_i} x H_i x^{-1}$, we can assume that $x H_i x^{-1} = H_i$ for all points $x \in F_i$. Since $|\mathcal{H}| \le \kappa < \kappa^+$, for some subgroup $H \in \mathcal{H}$ the set $I = \{i \in \kappa^+ : H_i = H\}$ has cardinality $|I| = \kappa^+$. Consider the family of left cosets $G/H = \{xH : x \in G\}$ and the quotient map $q : G \to G/H$, $q : x \mapsto xH$. Since $|(G/H)^F| \le \kappa < \kappa^+$, there are two indices $i < j$ in I such that $q \circ f_i = q \circ f_j$. Now consider the subgroup $F_{ij} = \langle F_i \cup F_j \rangle$ generated by the set $F_i \cup F_j$. Taking into account that $xHx^{-1} = H$ for all $x \in F_i \cup F_j$, we conclude that $xHx^{-1} = H$ for all $x \in F_{ij}$, which implies that $L = F_{ij} \cdot H = \{xy : x \in F_{ij},\ y \in H\}$ is a subgroup of G and H is a normal subgroup in L. Consequently, the subspace $L/H = \{xH : x \in L\}$ is a group and the restriction $q|L : L \to L/H$ is a group homomorphism. The equality $q \circ f_i = q \circ f_j$ implies that $L/H = q(F_{ij}) = q(F_i) = q(F_j)$. Since $H \cap F_i = \{1_G\}$, the restriction $\varphi = q|F_i : F_i \to L/H$, being injective and surjective, is an isomorphism. Then $\psi = f_i^{-1} \circ \varphi^{-1} : L/H \to F$ is an isomorphism too. It can be shown that the homomorphism $\phi = \psi \circ q|F_{ij} : F_{ij} \to F$ has the required property: $\phi \circ f_j = \phi \circ f_i = \mathrm{id}_F$. □

Question 11. *Is $cn(G) \le w_g(G)$ for any group G?*

Now we are able to prove two equalities of Theorem 2.

Lemma 12. *For every infinite cardinal κ and group G with $\mathrm{Alt}(\kappa) \subset G \subset \mathrm{Sym}(\kappa)$ we get*

$$\kappa = cn(G) = w_l(G).$$

Proof. By Proposition 10, $cn(G) \le w_l(G)$. Since the cardinal group invariants cn and w_l are monotone, it suffices to prove that $w_l(\mathrm{Sym}(\kappa)) \le \kappa$ and $cn(\mathrm{Alt}(\kappa)) \ge \kappa$.

To see that $w_l(\mathrm{Sym}(\kappa)) \le \kappa$, for every $i \in \kappa$ consider the subgroup $G_i = \{f \in \mathrm{Sym}(\kappa) : f(i) = i\}$ and observe that the index of this subgroup in $\mathrm{Sym}(\kappa)$ is equal to κ. Taking into account that $\bigcap_{i \in \kappa} G_i = \{\mathrm{id}\}$ is the trivial subgroup of $\mathrm{Sym}(\kappa)$, and applying Proposition 3, we conclude that $w_l(\mathrm{Sym}(\kappa)) \le \kappa$.

To see that $cn(\mathrm{Alt}(\kappa)) \ge \kappa$, it suffices to check that for every $\lambda < \kappa$ the group $\mathrm{Alt}(\kappa)$ does not satisfy the weak λ^+-compatibility condition. For every ordinal $3 \le i < \lambda^+ \le \kappa$ consider the 4-element subset $K_i = \{0, 1, 2, i\}$ and its alternating group $\mathrm{Alt}(K_i) \subset \mathrm{Alt}(\kappa)$. Fix an isomorphism $f_i : \mathrm{Alt}(4) \to \mathrm{Alt}(K_i)$ and observe that for any $3 \le i < j < \lambda^+$ the subgroup $\langle \mathrm{Alt}(K_i) \cup \mathrm{Alt}(K_j) \rangle$ is equal to $\mathrm{Alt}(K_i \cup K_j)$ and is isomorphic to the alternating group $\mathrm{Alt}(5)$. Since $\mathrm{Alt}(5)$ is a simple group there does not exist a surjective homomorphism from $\langle \mathrm{Alt}(K_i) \cup \mathrm{Alt}(K_j) \rangle$ to $\mathrm{Alt}(4)$, which implies that the group $\mathrm{Alt}(\kappa)$ does not satisfy the weak λ^+-compatibility condition. □

Since for an infinite cardinal κ the alternating group $\mathrm{Alt}(\kappa)$ is simple, Lemma 12 and Corollary 5 imply the following well-known folklore result (which can be also proved by applying Jordan-Wielandt Theorem, see [5, 6.1]).

Corollary 13. *For an infinite cardinal κ the alternating group $\mathrm{Alt}(\kappa)$ contains no proper subgroup of index $< \kappa$.*

We say that a topological space X is *σ-discrete* if X can be written as a countable union of discrete subspaces. By [2] or [3, 6.1], for every cardinal κ the group $\mathrm{Sym}_{\mathrm{fin}}(\kappa)$ is σ-discrete in each shift-invariant Hausdorff topology on $\mathrm{Sym}_{\mathrm{fin}}(\kappa)$. The same fact is true for the alternating group $\mathrm{Alt}(\kappa)$.

Theorem 14. *The alternating group $\mathrm{Alt}(\kappa)$ on a cardinal κ is σ-discrete in any Hausdorff shift-invariant topology τ on $\mathrm{Alt}(\kappa)$.*

Proof. If the cardinal κ is finite, then the alternating group $\mathrm{Alt}(\kappa)$ is finite and hence discrete in the topology τ. So, we assume that κ is infinite. To prove the theorem, it suffices to check that for every $n \in \omega$ the subspace $\mathrm{Alt}_n(\kappa) = \{f \in \mathrm{Alt}(\kappa) : |\mathrm{supp}(f)| = n\}$ of $\mathrm{Alt}(\kappa)$ is discrete. Given any permutation $f \in \mathrm{Alt}_n(\kappa)$ we shall construct an open set $O_f \in \tau$ such that $f \in O_f \cap \mathrm{Alt}_n(\kappa) \subset \{g \in \mathrm{Alt}(\kappa) : \mathrm{supp}(g) = \mathrm{supp}(f)\}$.

Choose two disjoint subsets $A, B \subset \kappa \setminus \mathrm{supp}(f)$ of cardinality $|A| = |B| = n + 1$ and for any points $x \in \mathrm{supp}(f)$ and $a \in A$, $b \in B$ consider the even permutation $\pi_{x,a,b} \in \mathrm{Alt}_3(\kappa)$ with support $\mathrm{supp}(\pi_{x,a,b}) = \{x, a, b\}$ such that $\pi_{x,a,b}(x) = a$, $\pi_{x,a,b}(a) = b$ and $\pi_{x,a,b}(b) = x$. Since the topology τ on $\mathrm{Alt}(\kappa)$ is shift-invariant and Hausdorff, by [3, 4.1(2)] the set $O_{x,a,b} = \{g \in \mathrm{Alt}(\kappa) : \pi_{x,a,b} \circ g \neq g \circ \pi_{x,a,b}\}$ is τ-open. Taking into account that $\pi_{x,a,b} \circ f(x) = f(x) \neq f(a) = f \circ \pi_{x,a,b}(x)$, we conclude that the permutation f belongs to the τ-open set $O_{x,a,b}$ and $O_f = \bigcap_{x \in \mathrm{supp}(f)} \bigcap_{a \in A} \bigcap_{b \in B} O_{x,a,b}$ is a τ-open neighborhood of f. We claim that the open set O_f has the desired property: $O_f \cap \mathrm{Alt}_n(\kappa) \subset \{g \in \mathrm{Alt}(\kappa) : \mathrm{supp}(g) = \mathrm{supp}(f)\}$. Take any permutation $g \in O_f \cap \mathrm{Alt}_n(\kappa)$. Assuming that $\mathrm{supp}(f) \neq \mathrm{supp}(g)$ and taking into account that $|\mathrm{supp}(g)| = |\mathrm{supp}(f)| = n$, we conclude that $\mathrm{supp}(f) \setminus \mathrm{supp}(g)$ contains some point x. Since $|A| = |B| > |\mathrm{supp}(g)|$, we can choose points $a \in A \setminus \mathrm{supp}(g)$ and $b \in B \setminus \mathrm{supp}(g)$. Then $\mathrm{supp}(\pi_{x,a,b}) \cap \mathrm{supp}(g) = \{x, a, b\} \cap \mathrm{supp}(g) = \emptyset$ and hence $\pi_{x,a,b} \circ g = g \circ \pi_{x,a,b}$, which contradicts the inclusion $g \in O_f$. This contradiction shows that the subspace $O_f \cap \mathrm{Alt}_n(\kappa) \subset \{g \in \mathrm{Alt}(\kappa) : \mathrm{supp}(g) = \mathrm{supp}(f)\}$ is finite and hence discrete. Then the point f is isolated in the discrete τ-open subset $O_f \cap \mathrm{Alt}_n(\kappa)$ of $\mathrm{Alt}_n(\kappa)$ and hence is isolated in $\mathrm{Alt}_n(\kappa)$, witnessing that the subspace $\mathrm{Alt}_n(\kappa)$ of the topological space $(\mathrm{Alt}(\kappa), \tau)$ is discrete. □

In the following lemma we prove another equality of Theorem 2.

Lemma 15. *For any infinite cardinal κ and any group G with $\mathrm{Alt}(\kappa) \subset G \subset \mathrm{Sym}(\kappa)$ we get $\kappa = s_s(G)$.*

Proof. Lemma 12 and obvious inequalities between the cardinal group invariants imply $s_s(G) \leq w_l(G) = \kappa$. It remains to prove that $s_s(G) \geq \kappa$. Assuming that $s_s(G) < \kappa$ we could find a Hausdorff shift-invariant topology τ with spread $s(G, \tau) < \kappa$. By Theorem 14, the subgroup $\mathrm{Alt}(\kappa)$ of G is σ-discrete in the topology τ. Consequently, $\mathrm{Alt}(\kappa) = \bigcup_{i \in \omega} D_i$ where each subspace D_i is discrete in (G, τ). Since $|\mathrm{Alt}(\kappa)| = \kappa > s(G, \tau)$, some set D_i has cardinality $|D_i| > s(G, \tau)$, which contradicts the definition of the spread $s(G, \tau)$. This contradiction shows that $s_s(G) \geq \kappa$. □

Establishing the equality $\kappa = u_c(G)$ in Theorem 2 is the most difficult part of the proof, which requires some preparatory work.

For a cardinal κ and a subgroup $G \subset \mathrm{Sym}(\kappa)$ by τ_p we denote the topology of pointwise convergence on G. This topology is generated by the subbase consisting of the sets

$$G(a, b) = \{g \in G : g(a) = b\} \quad \text{where} \quad a, b \in \kappa.$$

By Theorem 2.1 of [3], on any group G with $\mathrm{Sym}_{\mathrm{fin}}(\kappa) \subset G \subset \mathrm{Sym}(\kappa)$, the topology τ_p coincides with the *restricted Zariski topology* $\mathfrak{Z}'_G$ on G, generated by the subbase consisting of the sets

$$G \setminus \{a\}, \ \ \{x \in G : xbx^{-1} \neq aba^{-1}\} \ \text{ and } \ \{x \in G : (xcx^{-1})b(xcx^{-1})^{-1} \neq b\}$$

where $a, b, c \in G$ and $b^2 = c^2 = 1_G$. It is easy to check that the topology $\mathfrak{Z}'_G$ is shift-invariant. The following theorem generalizes Theorem 2.1 of [3].

Theorem 16. *Let κ be a cardinal. For any group G with $\mathrm{Alt}(\kappa) \subset G \subset \mathrm{Sym}(\kappa)$ the topology τ_p of pointwise convergence on G coincides with the restricted Zariski topology $\mathfrak{Z}'_G$.*

Proof. The proof of this theorem is just a suitable modification of the proof of Theorem 2.1 [3]. Fix a cardinal κ and a subgroup $G \subset \mathrm{Sym}(\kappa)$ containing the alternating group $\mathrm{Alt}(\kappa)$. If the cardinal κ is finite, then so is the group G. In this case the T_1-topologies τ_p and $\mathfrak{Z}'_G$ coincide with the discrete topology on G. So, we assume that the cardinal κ is infinite.

For two distinct elements $x, y \in \kappa$ consider the unique transposition $t_{x,y}$ with support $\mathrm{supp}(t_{x,y}) = \{x, y\}$. The transposition $t_{x,y}$ exchanges x and y by their places and does not move other points of

κ. For a subset $A \subset \kappa$ consider the subgroups $G(A) = \{g \in G : \mathrm{supp}(g) \subset A\}$ and $G_A = \{g \in G : \mathrm{supp}(g) \cap A = \emptyset\} = \{g \in G : g|A = \mathrm{id}|A\}$.

Lemma 17. *For any 6-element subset $A \subset \kappa$ the subgroup G_A is $\mathfrak{Z}'_G$-closed in G.*

Proof. Given any permutation $f \in G \setminus G_A$ find a point $a \in A$ with $f(a) \neq a$. Next, choose a point $b \in A \setminus \{a, f(a)\}$ and two distinct points $c, d \in A \setminus \{a, b, f(a), f(b)\}$. Consider the even permutation $t = t_{a,b} \circ t_{c,d} \in \mathrm{Alt}(\kappa) \subset G$ and observe that $t \circ f(a) = f(a) \neq f(b) = f \circ t(a)$. Since $\mathrm{supp}(t) = \{a, b, c, d\} \subset A$, the transposition t commutes with all permutations $g \in G_A$, which implies that

$$O_f = \{g \in G : t \circ g \neq g \circ t\} = \{g \in G : gtg^{-1} \neq t\}$$

is a $\mathfrak{Z}'_G$-open neighborhood of f, disjoint with the subgroup G_A. □

Lemma 18. *For each 6-element subset $A \subset \kappa$ the subgroup G_A is $\mathfrak{Z}'_G$-open in G.*

Proof. To derive a contradiction, assume that for some 6-element set $A' \subset \kappa$ the subgroup $G_{A'}$ is not $\mathfrak{Z}'_G$-open. Being $\mathfrak{Z}'_G$-closed, this subgroup is nowhere dense in the semitopological group $(G, \mathfrak{Z}'_G)$. Observe that for any 6-element subset $A \subset \kappa$ and any even permutation $f \in \mathrm{Alt}(G)$ with $f(A) = A'$, we get $G_A = f^{-1} \circ G_{A'} \circ f$, which implies that the subgroup G_A is closed and nowhere dense in $(G, \mathfrak{Z}'_G)$.

We claim that for every 6-element set $A \subset \kappa$ and any finite subset $B \subset \kappa$ the set $G(A, B) = \{g \in G : g(A) \subset B\}$ is nowhere dense in $(G, \mathfrak{Z}'_G)$. Since A and B are finite, we can choose a finite set $F \subset G$ such that for any $g \in G(A, B)$ there exists $f \in F$ with $g|A = f|A$. Then $f^{-1} \circ g|A = \mathrm{id}_A$ which implies that $G(A, B) = \bigcup_{f \in F} f \circ G_A$ is nowhere dense in $(G, \mathfrak{Z}'_G)$.

Now fix any four pairwise disjoint 6-element subsets $A_1, A_2, A_3, A_4 \subset \kappa$ and consider their union $A = \bigcup_{i=1}^4 A_i$. Consider the finite subset $T = \{t_{a_1,a_2} \circ t_{a_3,a_4} : (a_i)_{i=1}^4 \in \prod_{i=1}^4 A_i\} \subset \mathrm{Alt}(\kappa) \subset G$. For any permutations $t, s \in T$ with $t \circ s \neq s \circ t$ the set

$$U_{t,s} = \{u \in G : (usu^{-1})t(usu^{-1})^{-1} \neq t\}$$

is a $\mathfrak{Z}'_G$-open neighborhood of 1_G by the definition of the topology $\mathfrak{Z}'_G$. Since T is finite, the intersection

$$U = \bigcap \{U_{t,s} : t, s \in T,\ ts \neq st\}$$

is a $\mathfrak{Z}'_G$-open neighborhood of 1_G. Choose a permutation $u \in U$ which does not belong to the nowhere dense subset $\bigcup_{i=1}^4 G(A_i, A)$. For every $i \in \{1, 2, 3, 4\}$ there is a point $a_i \in A_i$ such that $u(a_i) \notin A$. Choose any point $a'_2 \in A_2 \setminus \{a_2\}$ and consider the non-commuting permutations $t = t_{a_1,a'_2} \circ t_{a_3,a_4}$ and $s = t_{a_1,a_2} \circ t_{a_3,a_4}$ in T. It follows from $u \in U$ that the permutation $v = usu^{-1}$ does not commute with the permutation t. On the other hand, the support $\mathrm{supp}(v) = \mathrm{supp}(usu^{-1}) = u(\{a_1, a_2, a_3, a_4\})$ does not intersect the set $A \supset \{a_1, a'_2, a_3, a_4\} = \mathrm{supp}(t)$, which implies that $vt = tv$. This contradiction completes the proof of Lemma 18. □

Now we able to prove that $\mathfrak{Z}'_G = \tau_p$. Taking into account that τ_p is a Hausdorff group topology on G, we conclude that $\mathfrak{Z}'_G \subset \tau_p$. To prove the reverse inclusion, it suffices to check that for every $a, b \in \kappa$ the subbasic τ_p-open set $G(a, b) = \{g \in G : g(a) = b\}$ is $\mathfrak{Z}'_G$-open. Choose any 6-element subset $A \subset \kappa$ containing the point a. By Lemma 18 the subgroup G_A is $\mathfrak{Z}'_G$-open and hence for any $g \in G(a, b)$ the set $g \circ G_A \subset G(a, b)$ is a $\mathfrak{Z}'_G$-open neighborhood of g, witnessing that the set $G(a, b)$ is $\mathfrak{Z}'_G$-open. □

The following simple fact was proved in Proposition 4.1(1) of [3]. Its short proof is included for the convenience of the reader.

Proposition 19. *If τ is a T_1-topology on G and (G, τ) is a semitopological group with separately continuous commutator then $\mathfrak{Z}'_G \subset \tau$.*

Proof. We fix elements $a, b, c \in G$ such that $b^2 = c^2 = 1_G$. The function $\gamma_b \colon G \to G$ with $\gamma_b(x) = xbx^{-1}$ is τ-continuous since τ is a semitopological group topology with separately continuous commutator. Since τ is T_1, singletons are closed and hence

$$\gamma_b^{-1}(G \setminus \{aba^{-1}\}) = \{x \in G : xbx^{-1} \neq aba^{-1}\} \in \mathfrak{Z}'_G$$

is τ-open. For $a = 1_G$, in particular the set $U = \{x \in G : xbx^{-1} \neq b\}$ is τ-open. Also the set

$$\gamma_c^{-1}(U) = \gamma_c^{-1}(\{y \in G : yby^{-1} \neq b\}) = \{x \in G : (xcx^{-1})b(xcx^{-1})^{-1} \neq b\} \in \mathfrak{Z}'_G$$

is τ-open. $\square$

Now we combine Theorem 16 with Proposition 19 and get the following generalization of Theorem 4.2 of [3].

Theorem 20. *Let κ be an infinite cardinal and G be a group such that $\mathrm{Alt}(\kappa) \subset G \subset \mathrm{Sym}(\kappa)$. Each shift-invariant T_1-topology τ with separately continuous commutator on G contains the topology τ_p of pointwise convergence on G.*

Now we are able to prove the final equality of Theorem 2.

Lemma 21. *For any infinite cardinal κ and any group G with $\mathrm{Alt}(\kappa) \subset G \subset \mathrm{Sym}(\kappa)$ we get $\kappa = u_c(G)$.*

Proof. It follows from Lemma 12 that $u_c(G) \leq w_l(G) = \kappa$. So, it remains to prove that $u_c(G) \geq \kappa$. To derive a contradiction, assume that $u_c(G) < \kappa$ and find a topology $\tau \in \mathcal{T}_c(G)$ such that $u(G, \tau) < \kappa$. By Theorem 20, $\tau_p \subset \tau$, which implies that the subgroup $G_0 = \{g \in G : g(0) = 0\}$ is τ-open. For every $i \in \kappa \setminus \{0\}$ fix a permutation $g_i \in \mathrm{Alt}(\kappa)$ such that $g_i(0) = i$ and observe that for every $0 < i \neq j < \kappa$ we get $g_j \notin g_i \circ G_0$, which means that the set $D = \{g_i\}_{0<i<\kappa}$ is uniformly discrete in (G, τ). So, $u(G, \tau) \geq \kappa$, which is a desired contradiction. $\square$

Problem 22. *Let κ be a cardinal and G be a group with $\mathrm{Alt}(\kappa) \subset G \subset \mathrm{Sym}(\kappa)$. Calculate the value of the cardinal invariant $u_s(G)$. Is $u_s(G) = \kappa$?*

The equalities in Theorem 2 hold also for some other groups, in particular for the additive group $\mathbb{R}$ of real numbers.

Example 23. *The group $\mathbb{R}$ does admit a linear group topology with countable weight and hence has $u_s(\mathbb{R}) = w_l(\mathbb{R}) = cn(\mathbb{R}) = \omega$.*

The equalities in this example follow from our next theorem evaluating the cardinal group invariants of infinite abelian groups.

Theorem 24. *Each infinite abelian group G has*

$$\log\log|G| \leq s_s(G) \leq w_s(G) = w_l(G) = \log|G| \quad \text{and} \quad u_s(G) = u_l(G) = cn(G) = \omega.$$

Proof. The inequalities $s_s(G) \leq w_s(G) \leq w_l(G)$ hold for any infinite group G. The lower bounds $\log\log|G| \leq s(G)$ and $\log|G| \leq w(G)$ follow from the inequalities $|X| \leq 2^{2^{s(X,\tau)}}$ and $|X| \leq 2^{w(X,\tau)}$ holding for any Hausdorff topological space (X, τ) (see Theorems 5.5 and 3.1 in [7]).

Now we prove that $w_l(G) \leq \log|G|$. Since each infinite abelian group embeds into a divisible abelian group of the same cardinality [11, 4.1.6], we can assume that G is divisible (which means that for every $x \in G$ and $n \in \mathbb{N}$ there exists $y \in G$ such that $y^n = x$). It is clear that the additive group $\mathbb{Q}_0$ of rational numbers is divisible and so is the quasi-cyclic p-group $\mathbb{Q}_p = \{z \in \mathbb{C} : \exists k \in \mathbb{N}\ z^{p^k} = 1\}$ for any prime number p. Let $\mathbb{P}$ denote the set of all prime numbers and $\mathbb{P}_0 = \{0\} \cup \mathbb{P}$. Endow the countable groups $\mathbb{Q}_p$, $p \in \mathbb{P}_0$, with the discrete topologies and for the cardinal $\kappa = \log|G|$ consider the Tychonoff product $L = \prod_{p\in\mathbb{P}_0} \mathbb{Q}_p^\kappa$. Taking into account that L is a linear topological group of weight κ, we conclude that $w_l(L) \leq \kappa$. It remains to prove that the divisible group G embeds into L.

For a cardinal λ and an abelian group A by $A^{(\lambda)}$ we denote the direct sum of λ many copies of the groups A. By [11, 4.1.5], the group G, being divisible, is isomorphic ($\approx$) to the direct sum $\bigoplus_{p\in\mathbb{P}_0} \mathbb{Q}_p^{(k_p)}$ for some cardinals $k_p \leq |G|$, $p \in \mathbb{P}_0$. By the same reason, for every $p \in \mathbb{P}_0$ the κ-th power $\mathbb{Q}_p^\kappa$ of $\mathbb{Q}_p$ is isomorphic to the direct sum $(\mathbb{Q}_0 \oplus \mathbb{Q}_p)^{(2^\kappa)}$ of 2^κ many copies of the groups $\mathbb{Q}_0 \oplus \mathbb{Q}_p$. Since $k_p \leq |G| \leq 2^\kappa$ for all $p \in \mathbb{P}_0$, the group $G \approx \bigoplus_{p\in\mathbb{P}_0} \mathbb{Q}_p^{(k_p)}$ embeds into the direct sum $\bigoplus_{p\in\mathbb{P}_0} \mathbb{Q}_p^{(2^\kappa)} \approx L$. Now the monotonicity of the cardinal group invariant w_l ensures that $w_l(G) \leq w_l(L) \leq \kappa = \log|G|$. Combining this inequality with $w_s(G) \geq \log|G|$, we get the equalities $w_s(G) = w_l(G) = \log|G|$.

Next, we prove that $u_s(G) = u_l(G) = \omega$. Since $\omega \le u_s(G) \le u_l(G)$, it suffices to show that $u_l(G) \le \omega$. Since the cardinal group invariant u_l is monotone, we can assume that the group G is divisible and hence is isomorphic to the direct sum of countable abelian groups. This implies that the trivial subgroup of G can be written as the intersection of subgroups of at most countable index in G. By Proposition 6, $u_l(G) \le \omega$.

Finally, we prove that $cn(G) = \omega$. It suffices to prove that G satisfies the weak ω_1-compatibility condition. Fix a finite group F and isomorphisms $f_i : F \to F_i$, $i \in \omega_1$, onto subgroups $F_i \subset G$. The Δ-Lemma [8, 9.18] yields an uncountable subset $\Omega \subset \omega_1$ and a finite subgroup $D \subset G$ such that $F_i \cap F_j = D$ for any distinct indices $i, j \in \Omega$. By the Pigeonhole Principle, for some subgroup $E \subset F$ the set $\Omega_1 = \{i \in \Omega : f_i^{-1}(D) = E\}$ is uncountable. Since the set of isomorphisms from E to D is finite, for some isomorphism $f : E \to D$ the set $\Omega_2 = \{i \in \Omega_1 : f_i|E = f\}$ is uncountable. Now take any two ordinals $i < j$ in the set Ω_2 and consider the subgroup $F_{ij} = F_i + F_j$ of G. Define a homomorphism $\phi : F_{ij} \to F$ assigning to a point $x = x_i + x_j$ with $x_i \in F_i$ and $x_j \in F_j$ the point $\phi(x) = f_i^{-1}(x_i) + f_j^{-1}(x_j)$. Let us show that the map ϕ is well-defined. Assume that $x = x_i' + x_j'$ for some points $x_i' \in F_i$ and $x_j' \in F_j$. Then $0 = x - x = (x_i - x_i') + (x_j - x_j')$ implies that $x_i - x_i' = x_j' - x_j \in F_i \cap F_j = D$ and hence

$$f_i^{-1}(x_i) - f_i^{-1}(x_i') = f_i^{-1}(x_i - x_i') = f^{-1}(x_i - x_i') = f^{-1}(x_j' - x_j) = f_j^{-1}(x_j' - x_j) = f_j^{-1}(x_j') - f_j^{-1}(x_j)$$

and finally $f_i^{-1}(x_i) + f_j^{-1}(x_j) = f_i^{-1}(x_i') + f_j^{-1}(x_j')$. Therefore the map $\phi : F_{ij} \to F$ is a well-defined homomorphism such that $\phi \circ f_i = \phi \circ f_j = \mathrm{id}_F$, witnessing that the group G satisfies the weak ω_1-compatibility condition and $wc(G) = \omega$. □

Theorem 24 implies that for infinite abelian groups the diagram describing the relations between the cardinal group invariants takes the following form:

$$\begin{tikzcd}
 & w_s \arrow[r, equal] & w_c \arrow[r, equal] & w_g \arrow[r, equal] & w_l \arrow[r, equal] & \log|\cdot| \\
\log\log|\cdot| \arrow[r] & s_s \arrow[u] \arrow[r] & s_c \arrow[u] \arrow[r] & s_g \arrow[u] \arrow[r] & s_l \arrow[u] & \\
 & u_s \arrow[u] \arrow[r, equal] & u_c \arrow[u] \arrow[r, equal] & u_g \arrow[u] \arrow[r, equal] & u_l \arrow[u] \arrow[r, equal] & cn = \omega
\end{tikzcd}$$

Looking at this diagram it is natural to ask about the values of the cardinal invariants in the middle row. Are they equal to $\log\log|G|$ or to $\log|G|$? Surprisingly, the answer depends on additional set-theoretic assumptions.

Theorem 25. (1) *PFA implies that each group G of cardinality $|G| > \mathfrak{c}$ has $s_s(G) > \omega = \log\log \mathfrak{c}^+$.*

(2) *For any regular uncountable cardinals $\lambda \le \kappa$ it is consistent that $2^\omega = \lambda$, $2^{\omega_1} = \kappa$ and each infinite abelian group G of cardinality $|G| \le \kappa$ has linear spread $s_l(G) = \omega = \log\log|G|$.*

Proof. 1. The first statement follows from a result of Todorcevic [13, 8.12] saying that under PFA each Hausdorff space X of countable spread has cardinality $|X| \le \mathfrak{c}$.

2. By [9, 4.10] for any regular uncountable cardinals $\lambda \le \kappa$ it is consistent that $2^\omega = \lambda$, $2^{\omega_1} = \kappa$ and there exists a subspace $X \subset \{0,1\}^\lambda$ of cardinality $|X| = \kappa$ such that each finite power X^n is hereditarily separable and hence has countable spread. The following lemma completes the proof the statement (2). □

Lemma 26. *Let X be a zero-dimensional space such that each finite power X^n of X has countable spread. Then any infinite abelian group G of cardinality $|G| \le |X|$ has linear spread $s_l(G) = \omega$.*

Proof. Consider the space $C_p(X, \mathbb{Q}) \subset \mathbb{Q}^X$ of continuous functions from X to the discrete group $\mathbb{Q}$ of rational numbers, and let $C_pC_p(X, \mathbb{Q})$ be the space of continuous functions from $C_p(X, \mathbb{Q})$ to $\mathbb{Q}$. Observe that the topology on $C_pC_p(X, \mathbb{Q})$ inherited from the Tychonoff product $\mathbb{Q}^{C_p(X,\mathbb{Q})}$ is linear.

The zero-dimensionality of X can be used to prove that the map $\delta : X \to C_pC_p(X, \mathbb{Q})$ assigning to each $x \in X$ the Dirac measure $\delta_x : C_p(X, \mathbb{Q}) \to \mathbb{Q}$, $\delta_x : f \mapsto f(x)$, is a topological embedding. Let L be the $\mathbb{Q}$-linear hull of the set $\delta(X)$ in the $\mathbb{Q}$-linear space $C_pC_p(X, \mathbb{Q})$ and Z be the group hull of $\delta(X)$ in $C_pC_p(X, \mathbb{Q})$. It can be shown that Z is a closed subgroup of L, so we can consider the quotient group L/Z and notice that it is a divisible torsion group isomorphic to the direct sum $\bigoplus_{p\in\mathbb{P}} \mathbb{Q}_p^{(|X|)}$. Here $\mathbb{P}$ is the set of prime numbers and $\mathbb{Q}_p = \{z \in \mathbb{C} : \exists k \in \mathbb{N}\ z^{p^k} = 1\}$ for $p \in \mathbb{P}$. Consequently, the group $L \times (L/Z)$ is isomorphic to $\bigoplus_{p\in\mathbb{P}_0} \mathbb{Q}_p^{(|X|)}$ where $\mathbb{P}_0 = \mathbb{P} \cup \{0\}$ and $\mathbb{Q}_0 = \mathbb{Q}$. It can be shown that the space $L \times (L/Z)$ can be written as the countable union of spaces homeomorphic to continuous images of finite powers of X, which implies that the space $L \times (L/Z)$ has countable spread.

Since the group $L \times (L/Z)$ carries a linear Hausdorff group topology of countable spread, so does the group $\bigoplus_{p\in\mathbb{P}_0} \mathbb{Q}_p^{(|X|)}$. By [11, 4.1.5 and 4.1.6], every infinite abelian group G of cardinality $|G| \le |X|$ embeds into $\bigoplus_{p\in\mathbb{P}_0} \mathbb{Q}_p^{(|G|)} \subset \bigoplus_{p\in\mathbb{P}_0} \mathbb{Q}_p^{(|X|)}$ and hence G carries a linear Hausdorff group topology with countable spread, which implies that $s_l(G) = \omega$. □

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Taras Banakh: Jan Kochanowski University in Kielce (Poland) and Ivan Franko National University of Lviv (Ukraine)

E-mail address: t.o.banakh@gmail.com

Heike Mildenberger: Abteilung für Mathematische Logik, Mathematisches Institut, Universität Freiburg, Eckerstr. 1, 79104 Freiburg im Breisgau, Germany

E-mail address: heike.mildenberger@math.uni-freiburg.de