MATHIAS AND SILVER FORCING PARAMETRIZED BY DENSITY

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Abstract. We define and investigate versions of Silver and Mathias forcing with respect to lower and upper density. We focus on properness, Axiom A, chain conditions, preservation of cardinals and adding Cohen reals. We find rough forcings that collapse 2^ω to ω, while others are surprisingly gentle. We also study connections between regularity properties induced by these parametrized forcing notions and the Baire property.

1. Introduction

Forcings consisting of tree conditions are the *dramatis personae* extensively studied in descriptive set theory and set theory of the reals. The original interest on this type of forcings was mainly due to their applications in resolving problems of independence arisen from the study of cardinal characteristics, infinitary combinatorics and regularity properties. Through the years, the study of the combinatorial properties of tree-forcings has been intensified and has been interesting on its own.

From the forcing point of view, trees can be thought as conditions whose stem decides a finite fragment of the generic real, and the rest of the tree above describes the possible paths of the generic real. In this sense, Cohen forcing can be understood as the simplest tree-forcing notion, whose conditions decide a finite fragment of the generic and then leave all paths extending the stem possible. In general, the fatter the tree, the more freedom the path of the generic real; the slimmer the tree, the more restrictive the conditions on the path of the generic real. Under this point of view a Sacks tree can be seen as the other extreme, since in this case the tree can be shrunk as much as desired, and the only requirement is to keep perfectness.

Many other tree-forcings in between have been introduced and extensively studied, among them Silver and Mathias trees[1]. In this paper we focus on studying some variants where the set of nodes above the stem are governed by restrictions imposed on the density of the set of splitting nodes. Imposing fatness or slimness conditions though, may result in a non-proper forcing.

[1]The nodes of the Mathias (or Silver) tree p are all $t \in 2^{<\omega}$ that are initial segments of one of the possible generic branches that are compatible with a Mathias condition (s, A) (or a Silver condition f_p).

Each condition in Mathias forcing and in Silver forcing, when conceived as a tree, comes with an infinite set A_p such that any node t of p whose length is in A_p is a splitting node. In [19] Rosłanowski introduces a version of Sacks forcing in which conditions are just the Sacks trees p that come with an infinite set A_p of uniform splitting levels. A similar construction could be made for Miller forcing. In Bukovský-Namba forcing the conditions with uniform splitting levels are even dense, see e.g. [4, Theorem 2.2].

We let the A_p range over prescribed families that are sets of positive lower or upper density in ω. There is some work on forcings of this type. Grigorieff [9] parametrised Silver forcing by having the A_p range over a P-point and Halbeisen [10, Ch 24] generalised this to P-families. Mathias forcing with a Ramsey ultrafilter [15] is a versatile notion of forcing. Farah and Zapletal [8, Ch. 9] used the coideal of the density zero ideal as a reservoir for the infinite component of Mathias conditions.

In Sections 2 to 5 we investigate whether the variants we introduce of Silver and of Mathias forcing are proper at all. We show that Mathias with lower density $\geq \varepsilon$ is equivalent to a disjoint union of σ-centered forcings (see Proposition 3.6) and that Silver forcing with positive lower density collapses 2^ω to ω (see Theorem 5.1). We also show that the lower density is far from the notion of a measure (see Proposition 5.3). This result fits in the framework presented in [14].

In Section 6 and 7 we are concerned with regularity properties of these forcings. Regularity properties of tree-forcings by themselves are a well studied field in mathematics. The following table illustrates which of the most popular regularity properties of a subset of 2^ω correspond to which tree-forcing $\mathbb{P}$.

Tree-forcing $\mathbb{P}$	$\mathbb{P}$-measurable
Cohen $\mathbb{C}$	Baire property
Random $\mathbb{B}$	Lebesgue measurable
Silver $\mathbb{V}$	completely doughnut [16]
Sacks $\mathbb{S}$	Marczewski set [20]
Mathias $\mathbb{MA}$	completely Ramsey

In many cases adding a Cohen real is enough to establish a dependence between Baire property and the measurability given by the tree-forcing. This was made explicit in [13, Proposition 3.1.]. Proposition 7.1 is an improvement of this result.

In Section 8 we construct a model in which all On^ω-definable sets are $\mathbb{V}^+_\varepsilon$-measurable but $\Sigma^1_2(\mathbb{C})$ fails. In particular $\Sigma^1_2(\mathbb{V}^+_\varepsilon) \Rightarrow \Sigma^1_2(\mathbb{C})$ fails.

In the remainder of this introduction, we set up our notation.

Definition 1.1. Let X be non-empty.

(a) We let $X^{<\omega} = \{s : (\exists n < \omega)(s\colon n \to X)\}$. The set $X^{<\omega}$ is partially ordered by the initial segment relation $\trianglelefteq$, namely $s \trianglelefteq t$ if $s = t \restriction \mathrm{dom}(s)$. We use $\triangleleft$ for the strict relation.

(b) A set $p \subseteq X^{<\omega}$ is called a *tree* if it is closed under initial segments, i.e. $(t \in p \wedge s \trianglelefteq t) \to s \in p$. The elements of p are called *nodes*.

(c) A node is called a *splitting node* if it has at least two immediate successors in p. We write $\mathrm{Split}(p)$ for the set of splitting nodes of p.

(d) A tree p is called *perfect* if for every $s \in p$ there is a splitting node $t \trianglerighteq s$.

(e) For $t \in p$, we write $\mathrm{splsuc}(t)$ for the shortest splitting node extending t. When t is splitting, then $\mathrm{splsuc}(t) = t$.

(f) The *stem of* p, short $\mathrm{stem}(p)$, is the $\trianglelefteq$-least splitting node of p if it exists.

(g) For $n < \omega$ we let $\mathrm{Split}_n(p)$ consist of all splitting nodes t in p such that there are exactly n splitting nodes preceding t i.e., $\mathrm{stem}(p) = t_0 \triangleleft t_1 \triangleleft \cdots \triangleleft t_{n-1} \triangleleft t$, in particular $\mathrm{Split}_0(p) = \{\mathrm{stem}(p)\}$. Analogously, we define $\mathrm{Split}_{\leq n}(p)$.

(h) For $n < \omega$, let $\mathrm{Lev}_n(p) := \{t \in p : |t| = n\}$.

(i) For $t \in p$ we let $p \restriction t = \{s \in p : s \trianglelefteq t \vee t \triangleleft s\}$.

(j) Let $p \subseteq X^{<\omega}$ be a tree such that for any $t \in p$ and any n there is $s \in p$, $t \trianglelefteq s$ such that $|s| > n$. The body or rump of a tree p, short $[p]$, is the set $\{f \in X^\omega : (\forall n)(f \restriction n \in p)\}$.

Definition 1.2. A partial ordering $(\mathbb{P}, \leq)$ is called a *tree-forcing*, if there is a non-empty set X such that

- All conditions $p \in \mathbb{P}$ are perfect trees on $X^{<\omega}$.
- For all $p \in \mathbb{P}$ and $t \in p$ the restriction $p \restriction t$ is again a condition in $\mathbb{P}$.
- The partial order is the inclusion i.e., $q \leq p$ iff $q \subseteq p$.

We are interested in tree-forcings $\mathbb{P}$ defined over $2^{<\omega}$ or $\omega^{<\omega}$ with the following additional property.

Definition 1.3. Let p be a perfect tree defined over $2^{<\omega}$ or $\omega^{<\omega}$. The tree p is called *uniformly splitting*, if there is an infinite set $A_p \in [\omega]^\omega$ such that

$$\forall s \in p(|s| \in A_p \Leftrightarrow s \in \mathrm{Split}(p)).$$

A tree-forcing $\mathbb{P}$ is called *uniformly splitting tree-forcing*, if all conditions $p \in \mathbb{P}$ are uniformly splitting.

In this paper we study two well-known examples of uniformly splitting tree forcings: The Mathias forcing $\mathbb{MA}$ and the Silver forcing $\mathbb{V}$.

Example 1.4. (1) The Mathias forcing is given by conditions $(s, A) \in \mathbb{MA}$ if $s \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$ and $\max(s) < \min(A)$. $(t, B) \leq (s, A)$ if $t \setminus s \subseteq A$, $t \supseteq s$, and $B \subseteq A$. Now a Mathias tree $p = p(s, A)$ is given by $p \subseteq 2^{<\omega}$ and $t \in p$ if t is the characteristic function of $s \cup s'$ for some finite $s' \subseteq A$. Of course, now $A_p = A$, and stronger conditions correspond to subtrees. All trees are perfect and restrictions $p{\restriction}t$ are again conditions.

(2) A condition f is a Silver condition if there is an infinite set A such that $f : \omega \setminus A \to 2$. So we get a Silver tree $p \subseteq 2^{<\omega}$ by letting

$$t \in p \leftrightarrow (\forall n \in |t| \setminus A)(t(n) = f(n)).$$

Then $A_p = A$. Stronger conditions are extensions and again correspond to subtrees.

Definition 1.5. For a set $A \subseteq \omega$ we define the upper density $d^+(A)$ and lower density $d^-(A)$ of A via:

(1) $d^+(A) := \limsup\limits_{n\to\infty} \frac{|A\cap n|}{n}$,
(2) $d^-(A) := \liminf\limits_{n\to\infty} \frac{|A\cap n|}{n}$.

Definition 1.6. Let $\mathbb{P}$ be a uniformly splitting tree-forcing defined over $2^{<\omega}$ or $\omega^{<\omega}$. For $\varepsilon \in (0,1]$ we define two subforcings $\mathbb{P}^+_\varepsilon$ and $\mathbb{P}^-_\varepsilon$, and we define the upper and lower positive density versions:

(1) $p \in \mathbb{P}^+_\varepsilon$ if $p \in \mathbb{P}$ and A_p has upper density $\geq \varepsilon$.
(2) $p \in \mathbb{P}^-_\varepsilon$ if $p \in \mathbb{P}$ and A_p has lower density $\geq \varepsilon$.
(3) $p \in \mathbb{P}^+$ if $p \in \mathbb{P}$ and A_p has upper density > 0.
(4) $p \in \mathbb{P}^-$ if $p \in \mathbb{P}$ and A_p has lower density > 0.

In all forcing orders, a condition q is stronger than p iff q is a subset of p.

We focus our attention on $\mathbb{MA}^+_\varepsilon$, $\mathbb{MA}^-_\varepsilon$, $\mathbb{MA}^+$, $\mathbb{MA}^-$ and the same for Silver. We order our investigation now in pairs, according to the density requirement. Some steps work also for general $\mathbb{P}$.

2. Upper density $\geq \varepsilon$

Definition 2.1. A notion of forcing $(\mathbb{P}, \leq)$ has *Axiom A* if there are partial order relations $\langle \leq_n : n < \omega\rangle$ such that

(a) $q \leq_{n+1} p$ implies $q \leq_n p$, $q \leq_0 p$ implies $q \leq p$,

(b) If $\langle p_n : n < \omega\rangle$ is a fusion sequence, i.e., a sequence such that for any n, $p_{n+1} \leq_n p_n$, then there is a lower bound $p \in \mathbb{P}$, $p \leq_n p_n$.

(c) For any maximal antichain A in $\mathbb{P}$ and and $n \in \omega$ and any $p \in \mathbb{P}$ there is $q \leq_n p$ such that only countably many elements of A are compatible with q. Equivalently, for any open dense set D and any n, p, there is a countable set E_p of conditions in D and $q \leq_n p$ such that E_p is predense below q.

A notion of forcing $(\mathbb{P}, \leq)$ has *strong Axiom A* if the set of compatible elements in (c) is even finite.

Axiom A entails properness and strong Axiom A implies ${}^\omega\omega$-bounding (see, e.g., [18, Theorem 2.1.4, Cor 2.1.12]).

Remark 2.2. *Let $p \in \mathbb{P}^+_\varepsilon$. We define an increasing sequence $\langle k^p_n \in \omega : n \in \omega\rangle$ as follows:*

$n = 0:$ *We put $k^p_0 := \min(A_p)$,*

$n > 0:$ *We let $k^p_n := \min\{k \in \omega : (k > k^p_{n-1} \wedge |A_p \cap k| \geq k(\varepsilon - 2^{-n}))\}$.*

Such a sequence has the following property for $n \in \omega$:

$$\frac{|A_p \cap k^p_n|}{k^p_n} \geq \varepsilon - 2^{-n},$$

and therefore witnesses $d^+(A_p) \geq \varepsilon$.

We use the sequences $\langle k^p_n \in \omega : n \in \omega\rangle$ to define a *stronger-n*-relation $\leq_n$ on $\mathbb{P}^+_\varepsilon$.

Definition 2.3. We define a decreasing sequence of partial order relations $\langle \leq_n : n \in \omega\rangle$ on $\mathbb{P}^+_\varepsilon$ as follows:

$$q \leq_n p \Leftrightarrow q \leq p \wedge k^q_n = k^p_n \wedge A_q \cap k^q_n = A_p \cap k^p_n.$$

Observe that given two conditions satisfying $q \leq_n p$ we must have $k^q_i = k^p_i$ for $i \leq n$.

Fact 2.4. *Let $\mathbb{P} \in \{\mathbb{MA}, \mathbb{V}\}$. Let $\langle q_n : n \in \omega\rangle$ be a fusion sequence in $\mathbb{P}^+_\varepsilon$. Then, fusions exist. Especially, $q = \bigcap q_n$ is a condition in $\mathbb{P}^+_\varepsilon$.*

2.1. **Silver forcing with upper density $\varepsilon > 0$.** We quickly establish that the Silver forcing with positive upper density ε has strong Axiom A and thus is a proper forcing that does not add unbounded reals. The proof is a straightforward generalization of the standard case.
For the next proof we introduce the following notation: Let $q \leq p \in \mathbb{V}$ be two Silver trees and $n < |\operatorname{stem}(q)|$ a natural number. We prune the tree p above level n in such a way that we only choose nodes in p that copy some node in q above level n. We denote the resulting tree with $\operatorname{copy}(p, q, n)$. More precisely $\operatorname{copy}(p, q, n)$ is the following set of nodes:

$$\operatorname{Lev}_{\leq n}(p) \cup \{s \in p : |s| > n \wedge \exists t \in \operatorname{Lev}_{|s|}(q)\ \forall m \in [n, |s|)\ (s(m) = t(m))\}.$$

Note that $\operatorname{copy}(p, q, n)$ is in fact a Silver tree and in case $p, q \in \mathbb{V}^+_\varepsilon$ the upper density of the corresponding set of splitting levels of $\operatorname{copy}(p, q, n)$ and q coincide and thus $\operatorname{copy}(p, q, n) \in \mathbb{V}^+_\varepsilon$ as well.

Lemma 2.5. *The forcing $\mathbb{V}^+_\varepsilon$ has strong Axiom A.*

Proof. We take the partial order relations $\leq_n$ as defined in Definition 2.3. It is easy to see that the requirements (a) and (b) from Definition 2.1 are fulfilled. We make sure that the strong version of (c) can be fulfilled as well. So, fix an open dense set $D \subseteq \mathbb{V}^+_\varepsilon$, $n \in \omega$ and a condition $p \in \mathbb{V}^+_\varepsilon$. We have to find a finite set $E_p \subseteq D$ and a condition $q \leq_n p$ such that E_p is predense below q.
To this end, let $\langle k^p_m : m < \omega\rangle$ be the sequence associated to the condition p as given in Remark 2.2. Let $k := |A_p \cap k^p_n|$ denote the number of splitting levels

up to k_n^p and $\{t_i \in \mathrm{Split}(p) : i < 2^k\}$ enumerate all splitting nodes of length exactly k_n^p. We construct a decreasing sequence $p =: q_0 \geq q_1 \geq \cdots \geq q_{2^k} =: q$ such that $q_{i+1}{\restriction}t_i \in D, i < 2^k$. Fix $i < 2^k$. Now, pick $p_i \leq q_i{\restriction}t_i$ in D and copy p_i into q_i above level k_n^p, i.e., we let $q_{i+1} := \mathrm{copy}(q_i, p_i, k_n^p)$.
It is clear that the resulting tree q satisfies $q{\restriction}t \in D$, whenever $t \in q$ is of length at least k_n^p. Moreover, since we have pruned the tree q only above k_n^p we also made sure that

$$k_n^p = k_n^q \wedge A_p \cap k_n^p = A_q \cap k_n^p,$$

especially $q \leq_n p$. For the finite set E_p we can simply put $E_p := \{q{\restriction}t : t \in \mathrm{Lev}_{k_n^p}(q)\}$. This completes the proof. □

Corollary 2.6. *The forcing $\mathbb{V}_\varepsilon^+$ is proper and ω^ω-bounding for each $\varepsilon \in (0, 1]$.*

2.2. **Mathias forcing with upper density $\varepsilon > 0$.** In the following, we investigate the differences between $\mathbb{MA}_\varepsilon^+$ and the classical Mathias forcing $\mathbb{MA}$ and Mathias forcing $\mathbb{MA}(\mathcal{F})$ with respect to a filter $\mathcal{F}$. The decisive difference to the classical forcing is that $\mathbb{MA}_\varepsilon^+$ adds Cohens. From $\mathbb{MA}(\mathcal{F})$ it already differs by the fact that the set $\{A \subseteq \omega : d^+(A) \geq \varepsilon\}$ is not closed under finite intersection and thus not a filter. Note that this does not yet imply that the two forcing are not forcing equivalent.

Definition 2.7. Let $\mathcal{F}$ be a filter over ω. The partial order $\mathbb{MA}(\mathcal{F})$ consists of all $p \in \mathbb{MA}$ such that the set of splitting levels A_p is an element of the filter $\mathcal{F}$. $\mathbb{MA}(\mathcal{F})$ is ordered by inclusion.

Remember that a filter $\mathcal{F}$ over ω is called *Canjar*, if the corresponding Mathias forcing $\mathbb{MA}(\mathcal{F})$ does not add dominating reals. See for instance [5], where Michael Canjar constructed an ultrafilter $\mathcal{U}$ under the assumption $\mathfrak{d} = \mathfrak{c}$ such that $\mathbb{MA}(\mathcal{U})$ does not add dominating reals. The following lemma shows that $\mathbb{MA}_\varepsilon^+$ cannot be equivalent to $\mathbb{MA}(\mathcal{F})$, for $\mathcal{F}$ Canjar.

Lemma 2.8. $\mathbb{MA}_\varepsilon^+$ *adds dominating reals.*

Proof. Let G be $\mathbb{MA}_\varepsilon^+$-generic over V and let $x_G = \bigcup\{s : \exists A((s, A) \in G)\}$ denote the the $\mathbb{MA}_\varepsilon^+$-generic real. Observe that x_G has upper density ε in the generic extension $V[G]$. Therefore, the following function is well defined in $V[G]$:

$$f(n) := \min\left\{k : \frac{|x_G \cap k|}{k} \geq \varepsilon - 2^{-n}\right\}.$$

Remember that in Remark 2.2 we assigned to each condition $p \in \mathbb{MA}_\varepsilon^+$ a sequence $\langle k_n^p \in \omega : n < \omega\rangle$. That f is dominating now follows from the following two facts:

(i) $\forall p \in \mathbb{MA}_\varepsilon^+, p \Vdash \forall n < \omega(f(n) \geq k_n^p)$,
(ii) $\forall g \in \omega^\omega \forall p \in \mathbb{MA}_\varepsilon^+ \exists q \leq p, q \Vdash \forall^\infty n(k_n^q \geq g(n))$.

□

There is a more general reason why $\mathbb{MA}^+_\varepsilon$ and $\mathbb{MA}(\mathcal{F})$ for any filter $\mathcal{F}$ cannot be forcing equivalent. Since $\mathcal{F}$ is a filter, any two conditions $p, q \in \mathbb{MA}(\mathcal{F})$ with the same stem are compatible and in particular $\mathbb{MA}(\mathcal{F})$ is σ-centered.
The following lemma follows from the well-known fact that there is an almost disjoint family of size $\mathfrak{c}$ of sets of upper density 1.

Lemma 2.9. $\mathbb{MA}^+_\varepsilon$ *does not satisfy the countable chain condition.*

Now we turn our attention to a comparison with the classical Mathias forcing $\mathbb{MA}$. Usually the first step in showing that a given forcing satisfies *pure decision* (compare [1, Lemma 7.4.5]) is to show that the forcing satisfies *quasi pure decision* (compare [1, Lemma 7.4.6]). We will see that although $\mathbb{MA}^+_\varepsilon$ fails to satisfy pure decision it still satisfies quasi pure decision.

Lemma 2.10. $\mathbb{MA}^+_\varepsilon$ *satisfies quasi pure decision i.e., given a condition* $p = (s, A) \in \mathbb{MA}^+_\varepsilon$ *and an open dense set* $D \subseteq \mathbb{MA}^+_\varepsilon$ *there is* $B \subseteq A$ *such that the following holds:*

$$\text{If there is } (t, C) \leq (s, B) \text{ and } (t, C) \in D, \text{ then } (t, B \setminus (\max(t) + 1)) \in D.$$

Proof. The proof is a straightforward generalization of [1, Lemma 7.4.5.]. To make sure that the final set B has upper density $\geq \varepsilon$ we have to use finite sets b_n instead of singletons.
We construct $B = \bigcup\{b_n \in [\omega]^{<\omega} : n \in \omega\}$ together with a decreasing sequence $\{B_n \subseteq A : n \in \omega\}$. We start with $B_0 = A$ and $b_0 = \min(A)$. So, assume we have constructed all sets up to B_n and b_n. Let $\{t_0, \dots, t_{k-1}\}$ enumerate all subsets of $\bigcup\{b_0, \dots, b_n\}$. We construct B_{n+1} as a decreasing sequence $B_n =: B^0_{n+1} \supseteq \cdots \supseteq B^k_{n+1} =: B_{n+1}$. Let $j < k$ be given.

Case 1: There exists $C \subseteq B^j_{n+1} \setminus (\max(t_j) + 1)$ such that $(s \cup t_j, C) \in D$. Then put $B^{j+1}_{n+1} := C$.
Case 2: Otherwise put $B^{j+1}_{n+1} := B^j_{n+1}$.

Finally put $B_{n+1} := B^k_{n+1}$. Since B_{n+1} has upper density $\geq \varepsilon$ we can find $k_{n+1} \in \omega$ such that

$$\frac{|(\bigcup_{j\leq n} b_j \cup B_{n+1}) \cap k_{n+1}|}{k_{n+1}} \geq \varepsilon - 2^{-n-1}$$

and set $b_{n+1} := B_{n+1} \cap k_{n+1}$. $\square$

Corollary 2.11. *The forcing* $\mathbb{MA}^+_\varepsilon$ *has Axiom A.*

Proof. We take the partial order relations $\leq_n$ as defined in Definition 2.3 and also recall the sequences $\langle k^p_n : n < \omega\rangle$ from Remark 2.2. The crucial part is to make sure that the requirement (c) from Axiom A (Definition 2.1) is fulfilled. So, fix an open dense set $D \subseteq \mathbb{MA}^+_\varepsilon, n < \omega$ and a condition $p = (s, A_p) \in \mathbb{MA}^+_\varepsilon$. Let N be big enough and $\{t_i \in [\omega]^{<\omega} : i < N\}$

enumerate all subsets of $A_p \cap k_n^p$. We define a decreasing sequence $A_p \supseteq B_0 \supseteq B_1 \supseteq, \dots, \supseteq B_N$ such that for $i < N$:

$$\forall (t, C) \in D((t, C) \leq (s \cup t_i, B_i) \to (t, B_{i+1} \setminus (\max(t) + 1)) \in D).$$

We start with $B_0 := A_p \setminus (\max(t_0) + 1)$. To get the sets B_i for $i > 0$, simply apply Lemma 2.10 to the condition $(s \cup t_{i-1}, B_{i-1})$ and the open dense set D.

Finally, we set $B := (A \cap k_n^p) \cup B_N$ and
$E_p := \{(t, B \setminus (\max(t) + 1)) : \exists C \in [\omega]^\omega((t, C) \leq (s, B) \wedge (t, C) \in D)\}$. Then, $(s, B) \leq_n (s, A)$ and E_p is a countable predense set below (s, B). $\square$

Proposition 2.12. $\mathbb{MA}_\varepsilon^+$ *adds a Cohen real.*

Proof. Take $N \in \omega$ such that $1/N < \varepsilon$. Divide ω into $N + 1$ disjoint sets $a_i \subseteq \omega, i < N + 1$ of density $1/(N + 1)$ i.e., $d^+(a_i) = d^-(a_i) = 1/(N + 1), i < (N + 1)$. Then each set $A \subseteq \omega$ with upper density $\geq 1/N$ cannot be completely contained in a single set a_i. Furthermore there is at least one $i < N + 1$ such that both $A \cap a_i$ and $A \setminus a_i$ are infinite. Let x_G be the canonical name for the $\mathbb{MA}_\varepsilon^+$-generic real and $\langle n_k : k \in \omega \rangle$ enumerate all integers n such that $x_G(n) = 1$. We define a $\mathbb{MA}_\varepsilon^+$-name $\dot{c}$ via:

$$\dot{c}(k) := \begin{cases} 0, & \exists i < (N + 1)\{n_{2k}, n_{2k+1}\} \subseteq a_i \\ 1, & \text{else.} \end{cases}$$

Below any $\mathbb{MA}_\varepsilon^+$-condition there are two incompatible conditions who differ in the decision of $\dot{c}(k)$ for at least one k. Hence $\dot{c}$ is not in the ground model. We claim that $\dot{c}$ is Cohen. For this purpose, fix $t \in 2^{<\omega}, (s, A) \in \mathbb{MA}_\varepsilon^+$. W.l.o.g. we can assume that $|s^{-1}(\{1\})|$ is even. Let $r \in 2^{<\omega}$ maximal such that $(s, A) \Vdash r \trianglelefteq \dot{c}$ (such r exists since $\dot{c}$ is not in the ground model). We have to find $(s', A') \leq (s, A)$ with the property $(s', A') \Vdash r^\frown t \trianglelefteq \dot{c}$. We construct (s', A') as a decreasing sequence $(s, A) =: (s_0, A_0) \geq (s_1, A_1) \geq \cdots \geq (s_{|t|}, A_{|t|}) = (s', A')$ such that $|{s_i}^{-1}(\{1\})| + 2 = |{s_{i+1}}^{-1}(\{1\})|, i < |t|$ and $(s_i, A_i) \Vdash r^\frown t{\restriction}i \trianglelefteq \dot{c}$. We only carry out the first step of the construction. Take $i < (N + 1)$ such that both $A_0 \cap a_i$ and $A_0 \setminus a_i$ are infinite. Put $m := \min(A_0 \cap a_i)$. There are two cases:

$t(0) = 0$: Define $M := \min((A_0 \cap a_i) \setminus (m + 1))$ and put $s_1 := {s_0}^\frown m^\frown M$, $A_1 := A_0 \setminus (M + 1)$. Then $(s_1, A_1) \Vdash r^\frown t(0) \trianglelefteq \dot{c}$.

$t(0) = 1$: Define $M := \min(A_0 \setminus ((m + 1) \cup a_i))$ and put $s_1 := {s_0}^\frown m^\frown M$, $A_1 := A_0 \setminus (M + 1)$. Then $(s_1, A_1) \Vdash r^\frown t(0) \trianglelefteq \dot{c}$.

The rest of the construction is carried out analogously. $\square$

Thus, in contrast to $\mathbb{MA}$ we get.

Corollary 2.13. $\mathbb{MA}_\varepsilon^+$ *does not satisfy pure decision.*

3. Lower density $\geq \varepsilon$

In this section we investigate the Mathias forcing with lower density $\geq \varepsilon$. The first observation is.

Observation 3.1. *The family of sets* $\mathcal{F}_1 := \{A \subseteq \omega \ : \ d^-(A) = 1\}$ *is a filter.*

So, $\mathbb{MA}_1^-$ is in fact equivalent to Mathias forcing $\mathbb{MA}(\mathcal{F}_1)$ with respect to the filter $\mathcal{F}_1$.

Thus, we get:

Fact 3.2. $\mathbb{MA}_1^-$ *satisfies the countable chain condition.*

Now, we will see that the forcing $\mathbb{MA}_\varepsilon^-$ is a disjoint union of σ-centred forcings.

Definition 3.3. Let $A, B \subseteq \omega$. The lower density of B with respect to A is defined by:

$$d_A^-(B) := \liminf_{n\to\infty} \frac{|A \cap B \cap n|}{|A \cap n|}.$$

Lemma 3.4. *Let* $B \subseteq A \subseteq \omega, d^-(A) > 0$ *and* $d_A^-(B) < 1$.
Then, $d^-(B) < d^-(A)$.

Proof. Let $B \subseteq A \subseteq \omega$ be as in the lemma. Then, $d_A^-(B) < 1$ implies

$$(\exists n_0 < \omega)(\forall k < \omega)(\exists m \geq k)\left(\frac{|B \cap m|}{|A \cap m|} < 1 - 2^{-n_0}\right).$$

So, we in fact get

$$(\exists^\infty m)\left(|B \cap m| < (1 - 2^{-n_0}) \cdot |A \cap m|\right).$$

$\square$

Corollary 3.5. *If* $d^-(A) = \varepsilon$*, then* $\mathcal{F}(A) := \{B \subseteq A \ : \ d^-(B) = \varepsilon\}$ *is a filter.*

Proposition 3.6. *Let* $\varepsilon \in (0, 1]$*.*

The forcing notion $\mathbb{MA}_\varepsilon^-$ *is equivalent to a disjoint union of* σ*-centred forcings.*

Proof. First, note that $\mathcal{D} := \{p = (s, A_p) \in \mathbb{MA}_\varepsilon^- \ : \ d^-(A_p) = \varepsilon\}$ is an open dense subset of $\mathbb{MA}_\varepsilon^-$. Fix a maximal antichain $\mathcal{A} \subseteq \mathcal{D}$. For each $p \in \mathcal{A}$ the restriction of $\mathbb{MA}_\varepsilon^-$ to p is denoted by $\mathbb{MA}_\varepsilon^- {\restriction} p = \{q \in \mathbb{MA}_\varepsilon^- \ : \ q \leq p\}$. Then, each $\mathbb{MA}_\varepsilon^- {\restriction} p$ has the countable chain condition. Indeed, fix $p = (s, A_p) \in \mathcal{A}$. By Corollary 3.5 we know that the set $\{B \subseteq A_p \ : \ d^-(B) = \varepsilon\}$ is closed under finite intersections. Thus, any two conditions $q_0, q_1 \leq p$ which have the same stem, are compatible.

It follows that $\bigcup_{p\in\mathcal{A}} \mathbb{MA}_\varepsilon^- {\restriction} p$ is a dense subset of $\mathbb{MA}_\varepsilon^-$. $\square$

Corollary 3.7. $\mathbb{MA}_\varepsilon^-$ *is proper for each* $\varepsilon \in (0, 1)$*.*

Next, we show that $\mathbb{MA}_\varepsilon^-$ has antichains of size the continuum, whenever $\varepsilon < 1$.

Lemma 3.8. *Let $\varepsilon \in (0,1)$. There is a family of sets $\{A_f : f \in I\}$ such that*

(1) $I \subseteq 2^\omega$ has size continuum,
(2) $d^-(A_f) \geq \varepsilon$, for all $f \in I$,
(3) $d^-(A_f \cap A_g) < \varepsilon$, for all $f \neq g$ in I.

Proof. Let $\varepsilon \in (0,1)$. First, we fix a suitable set of indices $I \subseteq 2^\omega$. To this end, let I be any family of functions of size continuum with the property

$$\forall f, g \in I (f \neq g \to \exists^\infty i, j (f(i) = 1 > g(i) \wedge f(j) = 0 < g(j))). \tag{3.1}$$

Next, we define for $n \in \omega$ the interval $I_n := [2^{n+1}, 2^{n+2}) \subseteq \omega$. Then, all intervals are pairwise disjoint and each interval I_n has size 2^{n+1}. Now, we define for each $f \in I$ a function $F_f \in \omega^\omega$ via

$$F_f(n) := \sum_{i=0}^{n} f(i) \cdot 2^{n-i}.$$

Observe, that the functions F_f have the property that for $n \in \omega(F_f(n) \in [0, 2^{n+1}))$, in other words there are 2^{n+1} possible values for $F_f(n)$.
We define the sets A_f of lower density $\geq \varepsilon$ recursively over n as a disjoint union $A_f = \bigcup_n A_f^n$, such that $A_f^n \subseteq I_n$. To this end, fix n and let k be the closest natural number $\leq \varepsilon \cdot 2^{n+1}$, so $k := \max\{j \in \omega : j \leq \varepsilon \cdot 2^{n+1}\}$. We differentiate two cases:
If $2^{n+1} + F_f(n) + k - 1 < 2^{n+2}$: We set

$$A_f^n := \{2^{n+1} + F_f(n), 2^{n+1} + F_f(n) + 1, \ldots, 2^{n+1} + F_f(n) + k - 1\}$$

Otherwise, there is $j \leq k - 1$ such that $2^{n+1} + F_f(n) + j = 2^{n+2} - 1$ and we can set

$$A_f^n := \{2^{n+1}, \ldots, 2^{n+1} + k - 1 - j\} \cup \{2^{n+1} + F_f(n), \ldots, 2^{n+1} + F_f(n) + j\}$$

Finally we set $A_f := \bigcup A_f^n$. It follows directly from the definition that each set A_f has density ε and for $n \in \omega$ we get $(A_f \cap I_n = A_f^n)$.
It is left to show that for $f \neq g$ the set $A_f \cap A_g$ has lower density $< \varepsilon$. So, fix $f, g \in I$ and let i_0 be minimal such that $f(i_0) \neq g(i_0)$. W.l.o.g assume $f(i_0) = 1$ and $g(i_0) = 0$, so in particular $F_f(n) > F_g(n)$ for all $n \geq i_0$. By property (3.1) of I there is $i_1 > i_0$ such that $f(i_1) = 1$ and $g(i_1) = 0$. Then, for all $n > i_1$ we have

$$|F_f(n) - F_g(n)| \geq 2^{n-i_0} + 2^{n-i_1} - (\sum_{i>i_0}^{i_1-1} 2^{n-i} + 2^{n-i_1} - 1) > 2^{n-i_1+1}.$$

Claim 3.9. *There is a strictly positive constant $c \in (0,1)$ such that for all $n \in \omega$*

$$\frac{|A_f \cap A_g \cap I_n|}{|I_n|} \leq \varepsilon - c.$$

Clearly, the claim above implies $d^-(A_f \cap A_g) < \varepsilon$. So, fix $n \in \omega$. Since we are only interested in the size of $A_f \cap A_g \cap I_n$, we might assume, by a transformation argument, that $g(i) = 0, i < n$ and $f(0) = 0$. Thus, A_g^n consists of the first k elements of I_n i.e., $\{2^{n+1}, \ldots, 2^{n+1} + k - 1\}$ and $F_f(n) - F_g(n) = F_f(n) < 2^n$. Again, we have to differentiate two cases:
If $F_f(n) + 2^{n+1} + k - 1 < 2^{n+2}$: Then,

$$\frac{|A_f^n \cap A_g^n \cap I_n|}{2^{n+1}} \leq \frac{k - 2^{n-i_1+1}}{2^{n+1}} \leq \varepsilon - \frac{1}{2^{i_1}}.$$

Otherwise $F_f(n) + 2^{n+1} + k - 1 \geq 2^{n+2}$. In this case we must have $\varepsilon > 1/2$ and we get:

$$\frac{|A_f^n \cap A_g^n \cap I_n|}{2^{n+1}} \leq \frac{2^{n+1} - 2(2^{n+1} - k)}{2^{n+1}} \leq 2 \cdot \varepsilon - 1.$$

We can set $c := \min\{2^{-i_1}, (1 - \varepsilon)\}$. □

Corollary 3.10. $\mathbb{MA}_\varepsilon^-$ *has antichains of size continuum for* $\varepsilon \in (0, 1)$.

Proof. Take the family of sets $\{A_f \, : \, f \in I\}$ as in the lemma above. Then, $\{(\langle\rangle, A_f) \, : \, f \in I\} \subseteq \mathbb{MA}_\varepsilon^-$ is an antichain of size continuum. □

Corollary 3.11. *There is no filter* $\mathcal{F}$ *such that* $\mathbb{MA}(\mathcal{F})$ *and* $\mathbb{MA}_\varepsilon^-$ *are forcing equivalent.*

Proof. This follows from the previous corollary, together with the fact, that $\mathbb{MA}(\mathcal{F})$ is a σ-centered forcing for each filter. □

Analog to the upper density case $\mathbb{MA}_\varepsilon^+$ we get that $\mathbb{MA}_\varepsilon^-$ adds Cohen reals.

Proposition 3.12. $\mathbb{MA}_\varepsilon^-$ *adds Cohen reals.*

Proof. We can repeat the proof of Proposition 2.12. Again we divide ω into $N + 1$ disjoint sets $a_i \subseteq \omega, i < N + 1$ of density $1/(N + 1)$, where N is such that $1/N < \varepsilon$. Now to carry out the rest of the construction it enough to see that any set of lower density ε intersects at least two of the sets a_i infinitely often. □

Let x_G be the generic real added by $\mathbb{MA}_\varepsilon^-$. Then, x_G has lower density 0 and upper density ε in the generic extension. So one might try to use the same recipe as in Lemma 2.8, where it was proven that $\mathbb{MA}_\varepsilon^+$ adds dominating reals. However, condition (ii) from the proof is not satisfied and the following question remains open.

Question 3.13. *Does* $\mathbb{MA}_\varepsilon^-$ *add dominating reals?*

Note, a negative answer to this question for $\varepsilon = 1$ would be a positive answer to [17][Question 38] from Raghavan.

4. Upper density > 0

In this section we investigate $\mathbb{MA}^+$ i.e., the forcing consisting of Mathias conditions $p \subseteq 2^{<\omega}$ such that the corresponding set of splitting levels A_p has strictly positive upper density.

Definition 4.1. The density zero ideal $\mathcal{Z}$ is defined by:

$$\mathcal{Z} := \{A \subseteq \omega \,:\, d^+(A) = 0)\},$$

and the corresponding coideal is denoted by $\mathcal{Z}^+ = \{A \subseteq \omega \,:\, d^+(A) > 0\}$.

Observe that for $p \in \mathbb{MA}$ we have $p \in \mathbb{MA}^+$ iff $A_p \in \mathcal{Z}^+$.

One might be tempted to think that the proof of Axiom A for $\mathbb{MA}^+_\varepsilon$ generalizes to $\mathbb{MA}^+$. The problem is to find the appropriate partial orders $\leq_n$. If we simply use Definition 2.3 we can construct a decreasing sequence $p_0 \geq_0 p_1 \geq_1 \dots$ such that the set of splitting levels A_p of the "fusion" $p = \bigcap_n p_n$ has density 0.

Question 4.2. *Does $\mathbb{MA}^+$ satisfy Axiom A?*

We show that the forcing $\mathbb{MA}^+$ is proper. In order to do this, we need the following Lemma (compare [8, Lemma 9.6.]).

Lemma 4.3. *$\mathbb{MA}^+$ is forcing equivalent to the two step iteration of $\mathcal{P}(\omega)/\mathcal{Z}^+ * \mathbb{MA}(\dot{\mathcal{F}})$, where $\dot{\mathcal{F}}$ is a $\mathcal{P}(\omega)/\mathcal{Z}^+$-generic filter.*

For sake of completeness we sketch a proof.

Proof. We define a map $i : \mathbb{MA}^+ \longrightarrow P(\omega)/\mathcal{Z}^+ * \mathbb{MA}(\dot{\mathcal{F}})$ via $i((s,A)) := ([A]_{\mathcal{Z}^+}, (s,A))$, where $[A]_{\mathcal{Z}^+} = \{B \subseteq \omega \,:\, A\Delta B \in \mathcal{Z}\}$ is the corresponding equivalence class. It is not hard to see that i preserves being stronger and being incompatible. We show that the range of i is dense. To this end, fix a condition $([A]_{\mathcal{Z}^+}, (s, \dot{C}))$. Then $\dot{C}$ is a $P(\omega)/\mathcal{Z}^+$-name for an infinite subset of ω such that $[A]_{\mathcal{Z}^+} \Vdash \dot{C} \in \dot{\mathcal{F}}$. This implies $[A]_{\mathcal{Z}^+} \Vdash A \setminus \dot{C} \in \mathcal{Z}$. So we get $i((s,A)) = ([A]_{\mathcal{Z}^+}, (s,A)) \leq ([A]_{\mathcal{Z}^+}, (s, \dot{C}))$. $\square$

Corollary 4.4. *$\mathbb{MA}^+$ is proper.*

Proof. Let Fin denote the ideal of finite subsets of ω. In [7, Theorem 1.3.] Farah proved that $P(\omega)/\mathcal{Z}^+$ is forcing equivalent to the two step iteration of $\mathcal{P}(\omega)/\mathsf{Fin}$ and a measure algebra of Maharam character $\mathfrak{c}$ and therefore is proper. So, $\mathbb{MA}^+$ is a finite iteration of proper forcings. $\square$

Theorem 4.5. *$\mathbb{MA}^+$ adds Cohen reals.*

The Theorem follows from [8, Lemma 9.8.] and the fact that the coideal $\mathcal{Z}^+$ is not semiselective (compare [6, Definition 2.1.]). However, since we will make use of the explicit construction of the Cohen real, we also give a proof. We freely identify sequences $x \in 2^{\leq\omega}$ with their corresponding sets of natural numbers $x \in [\omega]^{\leq\omega}$ via $x \mapsto \{n \,:\, x(n) = 1\}$.

Proof. We define maximal antichains $\mathcal{A}_n$ in $(\mathcal{Z}^+, \subseteq^*)$ as follows: For $n \in \omega$ let $A_n^i := \{k \in \omega : k = i \mod 2^{(n+1)}\}$ and put $\mathcal{A}_n := \{A_n^i : i < 2^{(n+1)}\}$, e.g. $\mathcal{A}_0$ consists of the even and odd numbers. Let $x_G \in 2^\omega$ be a $\mathbb{MA}^+$-generic real. Then, in the generic extension $V[x_G]$, there is for each $n \in \omega$ exactly one $i_n < 2^{(n+1)}$ such that $x_G \subseteq^* A_n^{i_n}$. To simplify notations we denote with A_n this unique $A_n^{i_n}$. We define two sequences $\langle n_i : i \in \omega \rangle$ and $\langle m_i : i \in \omega \setminus \{0\} \rangle$ as follows: We start with $n_0 := \min(x_G)$. When n_i is known, we put $m_{i+1} := \min(x_G \setminus (n_i + 1))$. To define n_{i+1} we differentiate two cases: If $x_G \setminus (m_{i+1} + 1) \subseteq A_{m_{i+1}}$ we put $n_{i+1} := m_{i+1}$. Otherwise we put $n_{i+1} := \min(x_G \setminus ((m_{i+1} + 1) \cup A_{m_{i+1}}))$. Observe that by definition we have $m_i \leq n_i$. We put

$$c(i) := \begin{cases} 0, & \text{if } x_G \setminus (m_{i+1} + 1) \subseteq A_{m_{i+1}} \\ 1, & \text{else.} \end{cases}$$

Observe that no condition $(s, A) \in \mathbb{MA}^+$ can decide all values of $c(i)$. Because otherwise it would also decide all antichains $\mathcal{A}_n$ but then the set A would have density zero. Which is a contradiction. Hence c is not in the ground model. We show that c is Cohen. For this purpose, fix a condition $(s, A) \in \mathbb{MA}^+$. By density we can assume that there is $i < \omega$ such that m_i, n_i and $c \restriction i$ are decided by (s, A) but none of the values of m_{i+1}, n_{i+1} nor $c(i)$. We must have $\max(s) = n_i$. Fix $j \in 2$. It is enough to find $(t, B) \leq (s, A)$ such that $(t, B) \Vdash c(i) = j$ and in addition (t, B) does not decide m_{i+2}, n_{i+2} or $c(i + 1)$. Depending on the value of j we distinguish two cases:

- $j = 0$: Let $m := \min(A), t := s \cup \{m\}$ and $B := A_m \cap A \setminus (m + 1)$.
- $j = 1$: Find $m \in A$ such that $A \not\subseteq A_m$ (such an m always exists since A has positive upper density and A_m has density $2^{-(m+1)}$). Next, pick $n \in A \setminus A_m$ and put $t := s \cup \{m\} \cup \{n\}$. Finally, define $B := A \setminus (n+1)$.

In both cases we have $\max(t) = n_{i+1}, (t, B)$ decides the value of $c(i)$ to be j but neither does it decide m_{i+2}, n_{i+2} and nor $c(i + 1)$. □

Next, we show that $\mathbb{MA}^+$ adds dominating reals.

Theorem 4.6. *$\mathbb{MA}^+$ adds dominating reals.*

Proof. Let $\mathcal{Z}^* = \{A \subseteq \omega : \omega \setminus A \in \mathcal{Z}\}$ be the dual filter of the density zero ideal. In [11, Corollary 3] Hrušak and Minami showed that $\mathbb{MA}(\mathcal{F})$ adds dominating reals, whenever $\mathcal{F}$ is a filter extending $\mathcal{Z}^*$. We use Lemma 4.3. Let $\dot{\mathcal{F}}$ denote the $P(\omega)/\mathcal{Z}^+$-generic filter. Then, in the generic extension $V[\dot{\mathcal{F}}]$ the filter $\dot{\mathcal{F}}$ extends $(\mathcal{Z}^*)^V$ and therefore $\mathbb{MA}(\dot{\mathcal{F}})$ adds dominating reals over V. Specifically, $\mathbb{MA}^+$ adds dominating reals over V. □

5. Positive lower density

In this section we show that $\mathbb{V}^-$ collapses the continuum to ω. We also construct an uncountable antichain in the partial order consisting of sets with strictly positive lower density ordered by inclusion.

Theorem 5.1. *The forcing $\mathbb{V}^-$ collapses the continuum to ω.*

We first prove a lemma:

Lemma 5.2. *For each natural numbers $k, k_0 < \omega$, $k > 0$ there is a strictly increasing function $\ell\colon \omega \cup \{-1\} \to \omega$ and a $\mathbb{V}^-$-name $\dot{F}$ for real $F \in 2^\omega$ in $V[G]$ such that for all reals $\varrho \in 2^\omega$ and all conditions $p \in \mathbb{V}^-$ in the ground model the following holds:*

$$\left(\left(d^-(A_p) \geq \frac{2}{k} \wedge \forall n \geq k_0 \frac{|A_p \cap [\ell(n), \ell(n+1))|}{\ell(n+1) - \ell(n)} \geq \frac{1}{k}\right) \to \right. \tag{5.1}$$

$$\left. \exists q_\varrho \leq p \left(d^-(A_{q_\varrho}) \geq \frac{1}{4 \cdot k} \wedge q_\varrho \Vdash (\forall n \in \omega) \dot{F}(n) = \varrho(n)\right)\right) \tag{5.2}$$

Proof. Let $k, k_0 < \omega$ be natural numbers and $k_0 > 0$. The function ℓ is defined recursively by

$$\ell(n) = \begin{cases} 0, & \text{if } n = -1, \\ 3k \cdot (k_0 + 1), & \text{if } n = 0, \\ 4k \cdot \ell(n-1), & \text{if } n > 0. \end{cases}$$

We let x_G be the generic branch and define

$$f(n) = \text{ the closest natural number to } \frac{|x_G^{-1}[\{1\}] \cap \ell(n)|}{\ell(n)} \cdot \frac{3k}{4}.$$

We define in $V[G]$ the following function:

$$F(n) = \begin{cases} 0, & \text{if } f(n) \text{ is even;} \\ 1, & \text{else.} \end{cases}$$

So, assume that we are given a condition $p \in \mathbb{V}^-$ that meets the premise of the implication (5.1) for k and k_0 i.e.,

$$d^-(A_p) \geq \frac{2}{k} \text{ and } (\forall n \geq k_0) \frac{|A_p \cap [\ell(n), \ell(n+1))|}{\ell(n+1) - \ell(n)} \geq \frac{1}{k}.$$

Let $\varrho \in 2^\omega$ be given. Recall any Silver condition p is uniquely described by a function $f_p\colon \omega \setminus A_p \to 2$. By induction on $n < \omega$ we define an increasing sequence of partial functions $f_p = f_{q_{-1}} \subseteq f_{q_0} \subseteq f_{q_1} \subseteq \dots$ such that for all $n \geq 0$

(1) $f_{q_n} \restriction \ell(n-1) = f_{q_{n-1}} \restriction \ell(n-1)$,
(2) $f_{q_n} \restriction [\ell(n-1), \ell(n)) \supseteq f_{q_{n-1}} \restriction [\ell(n-1), \ell(n))$,
(3) $f_{q_n} \restriction [\ell(n), \infty) = f_p \restriction [\ell(n), \infty)$,

From the three conditions above it already follows that each corresponding tree q_n will be a member of $\mathbb{V}^-$. Additionally, we make sure that the following holds as well for $n \geq 0$:

(4) $\frac{|A_{q_n} \cap [\ell(n-1), \ell(n))|}{\ell(n) - \ell(n-1)} \geq \frac{1}{4k}$
(5) $q_n \Vdash \forall m \leq n F(m) = \varrho(m)$.

The conditions (1) – (4) together make sure that the decreasing sequence $\langle q_n : n < \omega \rangle$ has a lower bound in $\mathbb{V}^-$, namely $q := \bigcap_n q_n$. Condition (5) ensures that $q \Vdash \forall n \in \omega F(n) = \varrho(n)$. Thus, we can set $q_\varrho := q$ and are done.

Now for the step from $n-1$ to n: We have

$$\frac{|A_p \cap [\ell(n-1), \ell(n))|}{\ell(n) - \ell(n-1)} \geq \frac{1}{4}.$$

This means we can add at most $\frac{3}{4} \cdot \frac{1}{k} \cdot (\ell(n) - \ell(n-1))$ elements of $[\ell(n-1), \ell(n)) \cap A_p$ to $\mathrm{dom}(f_{q_n})$ and still make sure that condition (4) is met. We will later choose which of them are mapped to 0 and which are mapped to 1 by f_{q_n}.

We define two approximations to f and F respectively, which do not depend on the generic element x_G. We let

$$f(n,p) = \text{ the closest natural number to } \frac{|f_p^{-1}[\{1\}] \cap \ell(n)|}{\ell(n)} \cdot \frac{3k}{2},$$

and

$$F(n,p) = \begin{cases} 0, & \text{if } f(n,p) \text{ is even;} \\ 1, & \text{else.} \end{cases}$$

Now at most $\frac{2}{3} \cdot \frac{\ell(n)}{k}$ many new values are needed to change $f(n,p)$ to $f(n,q)$ such that the quotient

$$\frac{|f_p^{-1}[\{1\}] \cap \ell(n)|}{\ell(n)} \cdot \frac{3k}{2} \in \omega$$

and such that $F(n,q)$ coincides with $\varrho(n)$. However, we have to be careful not to contradict condition (4). Especially, the following must hold:

$$\frac{2}{3} \cdot \frac{\ell(n)}{k} \leq \frac{3}{4} \cdot \frac{1}{k} \cdot (\ell(n) - \ell(n-1)) = \frac{3}{4} \cdot \frac{(4k-1)}{4k} \cdot \frac{\ell(n)}{k}.$$

On the other hand, we need to ensure that q_n decides $F(n)$ to be $F(n, q_n)$. By construction and in particular condition (4), we get that the amount of digits that are in $\ell(n) \setminus \mathrm{dom}(f_{q_n})$ is $\frac{\ell(n)}{4k}$. Hence we need

$$\frac{2}{3} \cdot \frac{\ell(n)}{k} > 2 \cdot \frac{\ell(n)}{4k}$$

Both inequalities are true since:

$$\frac{1}{2} < \frac{2}{3} \leq \frac{3}{4} \cdot \frac{15}{16}.$$

□

Proof of the Theorem. In the generic extension $V[G]$ we construct a function $H : \omega^3 \to 2$ as follows: For two natural numbers k, k_0 with $k_0 > 0$ let $F_{k_0}^k$ denote the real $F \in 2^\omega \cap V[G]$ from the lemma. We put

$$H(k, k_0, n) = \begin{cases} 0, \text{ if } k_0 = 0 \\ F_{k_0}^k(n), \text{ if } k_0 > 0 \end{cases}$$

Now it is easy to see that

$$\Vdash_{\mathbb{V}^-} (\forall \varrho \in V \cap 2^\omega)(\exists k > 0)(\exists k_0)(\forall n H(k, k_0, n) = \varrho(n)).$$

Simply fix p and ϱ. Compute k and k_0 for p such that the prerequirement of the implication (5.1) are fulfilled. This is always possible since $d^-(A_p) > 0$. Then construct $q_\varrho \leq p$ as in the lemma above.

Hence, $\Vdash_{\mathbb{V}^-} (k, k_0) \mapsto H(k, k_0, \cdot)$ is a surjection from $\omega \times \omega$ onto $2^\omega \cap V$. □

5.1. $\mathbb{MA}^-$ **has large antichains.** The following lemma establishes that below any condition $p \in \mathbb{MA}^-$ there is an antichain of size continuum.

Proposition 5.3. *There is a family of sets $\{A_f \subseteq \omega : f \in I\}$ such that*

(1) $I \subseteq 3^\omega$ has size continuum,
(2) $d^-(A_f) \geq 1/2$, for all $f \in I$.
(3) $d^-(A_f \cap A_g) = 0$, for all $f \neq g$ in I.

Proof. First, we fix a suitable set of indices $I \subseteq 3^\omega$. Therefore pick any family of functions I of size continuum with the property, that for any two different functions $f, g \in I$ there are infinitely many $n \in \omega$ $(f(n) \neq g(n))$. In order to define the sets A_f we will define three auxiliary sets $B_0, B_1, B_2 \subseteq \omega$ and a sequence $\langle k_j^n : n \in \omega, j \in 3 \cup \{-1\}\rangle$ such that

i) $k_{-1}^n < k_0^n < k_1^n < k_2^n = k_{-1}^{n+1}$, for $n < \omega$,
ii) $\frac{k_{j-1}^n}{k_j^n} < 2^{-n}$, for $j < 3, n < \omega$,
iii) $d^-(B_i \cup B_j) \geq 1/2$, for $i \neq j$,
iv) $d^-(B_i) = 0$, for $i < 3$.

We construct the sets B_i recursively over $n \in \omega$ as a disjoint union of sets $\{B_i^n\}_n$ such that each set B_i^n is a subset of $[k_{-1}^n, k_2^n)$. Start by defining $B_i^0 := \emptyset, i < 3$ and $k_j^0 := j + 1, j \in 3 \cup \{-1\}$. Now assume we have constructed B_i^m and k_j^m for $m < n, i < 3$ and $j \in 3 \cup \{-1\}$. We start with k_j^n. Let $k_{-1}^n := k_2^{n-1}$ and choose $k_j^n, j < 3$ big enough such that conditions $i)$ and $ii)$ are fulfilled and additionally the difference $k_j^n - k_{j-1}^n$ is an even natural number for each $j < 3$. We perform three construction steps to define the sets B_i^n:

a) Divide the interval $[k_{-1}^n, k_0^n)$ evenly between the two sets B_1^n and B_2^n and avoid the set B_0^n entirely i.e.,

$$[k_{-1}^n, k_0^n) \cap B_0^n := \emptyset,$$
$$[k_{-1}^n, k_0^n) \cap B_1^n := \{k_{-1}^n, k_{-1}^n + 2, \ldots, k_0^n - 2\},$$
$$[k_{-1}^n, k_0^n) \cap B_2^n := \{k_{-1}^n + 1, k_{-1}^n + 3, \ldots, k_0^n - 1\}.$$

b) Divide the interval $[k_0^n, k_1^n)$ evenly between the two sets B_0^n and B_2^n and avoid the set B_1^n entirely i.e.,

$$\begin{aligned}
&[k_0^n, k_1^n) \cap B_1^n := \emptyset,\\
&[k_0^n, k_1^n) \cap B_0^n := \{k_0^n, k_0^n + 2, \dots, k_1^n - 2\},\\
&[k_0^n, k_1^n) \cap B_2^n := \{k_0^n + 1, k_0^n + 3, \dots, k_1^n - 1\}.
\end{aligned}$$

c) Divide the interval $[k_1^n, k_2^n)$ evenly between the two sets B_0^n and B_1^n and avoid the set B_2^n entirely i.e.,

$$\begin{aligned}
&[k_1^n, k_2^n) \cap B_2^n := \emptyset,\\
&[k_1^n, k_2^n) \cap B_0^n := \{k_1^n, k_1^n + 2, \dots, k_2^n - 2\},\\
&[k_1^n, k_2^n) \cap B_1^n := \{k_1^n + 1, k_1^n + 3, \dots, k_2^n - 1\}.
\end{aligned}$$

This completes the construction of the sets B_i^n and we can put $B_i := \bigcup_n B_i^n$. We check that conditions iii) and iv) are fulfilled. Let $i \neq j$ be given. By construction steps $a) - c)$, we know that $B_i \cup B_j$ selects at least each second natural number of each interval $[k_{-1}^n, k_2^n)$. Since the intervals $[k_{-1}^n, k_2^n)$ partition ω we get iii). Condition iv) follows from

$$\frac{|B_i \cap k_i^n|}{k_i^n} \leq \frac{k_{i-1}^n}{k_i^n} < 2^{-n}.$$

Now, we are in a position to define the sets A_f, $f \in I$. For $n \in \omega$ and $i \in 3$, we let $A_i^n := B_{i_0}^n \cup B_{i_1}^n$, where i_0 and i_1 are chosen such that $\{i, i_0, i_1\} = 3$. Then, we set $A_f := \bigcup_n A_{f(n)}^n$. We have to verify that the sets A_f satisfy conditions (2) and (3). That each set A_f has lower density $\geq 1/2$ follows directly from condition iii) for the sets B_i. So let $f, g \in I$ be two different functions and take n such that $f(n) \neq g(n)$. W.l.o.g. assume $f(n) = 0$ and $g(n) = 1$. Then, from construction step c) it follows that $B_0 \cap B_2 \cap [k_1^n, k_2^n) = \emptyset$ and thus

$$\frac{|A_f \cap A_g \cap k_2^n|}{k_2^n} \leq \frac{k_1^n}{k_2^n} < 2^{-n}.$$

Since f and g differ on infinitely many n we get $d^-(A_f \cap A_g) = 0$. □

The results of the previous sections are summarized in the table below.

$\mathbb{P}$	$\mathbb{MA}_\varepsilon^+$	$\mathbb{V}_\varepsilon^+$	$\mathbb{MA}_1^-$	$\mathbb{MA}_\varepsilon^-$	$\mathbb{V}_\varepsilon^-$	$\mathbb{MA}^+$	$\mathbb{V}^+$	$\mathbb{MA}^-$	$\mathbb{V}^-$
proper	✓	✓	✓	✓		✓			✗
c.c.c	✗	✗	✓	✗	✗	✗	✗	✗	✗
Cohen	✓	✗	✓	✓		✓		✓	
dominating	✓	✗				✓			

6. Measurability

In this section we compare different notions of measurability. We first establish some notations.

Definition 6.1. Let $\mathcal{X}$ be a non-empty set and $\mathbb{P}$ be a tree-forcing defined over $\mathcal{X}^\omega$.

(1) A subset $X \subseteq \mathcal{X}^\omega$ is called $\mathbb{P}$-*nowhere dense,* if

$$(\forall p \in \mathbb{P})(\exists q \leq p)([q] \cap X = \emptyset).$$

We denote the ideal of $\mathbb{P}$-nowhere dense sets with $\mathcal{N}_\mathbb{P}$.

(2) A subset $X \subseteq \mathcal{X}^\omega$ is called $\mathbb{P}$-*meager* if it is included in a countable union of $\mathbb{P}$-nowhere dense sets. We denote the σ-ideal of $\mathbb{P}$-meager sets with $\mathcal{I}_\mathbb{P}$.

(3) A subset $X \subseteq \mathcal{X}^\omega$ is called $\mathbb{P}$-*measurable* if

$$(\forall p \in \mathbb{P})(\exists q \leq p)([q] \setminus X \in \mathcal{I}_\mathbb{P} \vee [q] \cap X \in \mathcal{I}_\mathbb{P}).$$

(4) A family $\Gamma \subseteq \mathcal{P}(\mathcal{X}^\omega)$ is called *well-sorted* if it is closed under continuous pre-images. We abbreviate the sentence "every set in Γ is $\mathbb{P}$-measurable" by $\Gamma(\mathbb{P})$.

We make two useful observations concerning the measurability of a set X.

Observation 6.2. *Let $\mathbb{P}$ be a tree-forcing and $X \subseteq \mathcal{X}^\omega$ a set of reals.*

(1) If H is $\mathbb{P}$-comeager, then X is $\mathbb{P}$-measurable if and only if $H \cap X$ is $\mathbb{P}$-measurable.

(2) If $\mathcal{I}_\mathbb{P} = \mathcal{N}_\mathbb{P}$, then X is $\mathbb{P}$-measurable if and only if

$$(\forall p \in \mathbb{P})(\exists q \leq p)([q] \subseteq X \vee [q] \cap X = \emptyset).$$

For Sacks, Laver, Miller, Silver and Mathias forcing we have $\mathcal{N}_\mathbb{P} = \mathcal{I}_\mathbb{P}$. In case of the Silver forcing the usual proof for $\mathcal{I}_\mathbb{V} = \mathcal{N}_\mathbb{V}$ also works for $\mathbb{V}_\varepsilon^+$, since it only makes use of fusion sequences. But it is unclear whether we can expect the same for the other versions of Silver forcing.
In case of the Mathias forcing however, we don't have an equality in none of the four versions of the forcing.

Theorem 6.3. *(1) $\mathcal{I}_{\mathbb{V}_\varepsilon^+} = \mathcal{N}_{\mathbb{V}_\varepsilon^+}$,*

(2) $\mathcal{I}_\mathbb{P} \neq \mathcal{N}_\mathbb{P}$ for $\mathbb{P} \in \{\mathbb{MA}_\varepsilon^+, \mathbb{MA}_\varepsilon^-, \mathbb{MA}^+, \mathbb{MA}^-\}$.

Proof. (1) As we mentioned above, the first part of the Theorem is a straightforward generalization of the proof for the usual Silver forcing. The only difference being that one has to use the partial orderings defined as in Definition 2.3 to ensure that fusions exist in $\mathbb{V}_\varepsilon^+$.
(2) We divide the proof into cases.

($\geq \varepsilon$) Let $\varepsilon \in (0,1]$ be given. The proof is closely intertwined with the fact that the forcings $\mathbb{MA}_\varepsilon^+$ and $\mathbb{MA}_\varepsilon^-$ add Cohen reals. We quickly recall the construction of the Cohen real from Lemma 2.12 and define a function φ such that the image of the generic real is a Cohen real. Fix $\varepsilon > 0, N \in \omega$ such that $1/N < \varepsilon$ and a partition $\{a_i\}_{i<N+1}$ of ω such that each set a_i has density $1/(N+1)$. Now let $H := \{x \in 2^\omega :$

$\exists^\infty i(x(i) = 1)\}$. For $x \in H$ let $\langle n^x_k : k < \omega\rangle$ enumerate all integers n such that $x(n) = 1$. We define a function $\varphi : H \to 2^\omega$ as follows:

$$\varphi(x)(k) := \begin{cases} 0, & \exists i < (N+1)\{n^x_{2k}, n^x_{2k+1}\} \subseteq a_i \\ 1, & \text{else.} \end{cases}$$

For $n \in \omega$ let $M_n := \{x \in H : \forall k \geq n(\varphi(x)(k) = 0)\}$. It is not hard to see that for $\mathbb{P} \in \{\mathbb{MA}^+_\varepsilon, \mathbb{MA}^-_\varepsilon\}$, and each n we have $M_n \in \mathcal{N}_\mathbb{P}$, but $M := \bigcup_n M_n \notin \mathcal{N}_\mathbb{P}$.

(> 0) In this case the meager set which is not nowhere dense lies directly at hand.

Claim 6.4. *The set $\mathcal{Z}^+$ is $\mathbb{P}$-meager but not $\mathbb{P}$-nowhere dense, for $\mathbb{P} \in \{\mathbb{MA}^+, \mathbb{MA}^-\}$.*

The proof works for both forcings analogously. We only check it for $\mathbb{MA}^+$ explicitly. For $n \in \omega$ we define the sets $N_n := \{x \in 2^\omega : d^+(x) \geq 1/n\}$. Let $n \in \omega$ and $p \in \mathbb{MA}^+$ be fixed. We can easily find $q \leq p$ such that $d^+(A_q) < 1/n$. Such a condition q satisfies the property $\forall x \in [q](d^+(x) < 1/n)$ and in particular $[q] \cap N_n = \emptyset$. This proves that each set N_n is $\mathbb{MA}^+$-nowhere dense and so $\mathcal{Z}^+ = \bigcup_n N_n$ is $\mathbb{MA}^+$-meager. To see that $\mathcal{Z}^+$ cannot be $\mathbb{MA}^+$-nowhere dense it is enough to check that each condition $p = (s, A_p) \in \mathbb{MA}^+$ contains a branch x of positive upper density. Clearly, the rightmost branch (i.e. $x(i) = 1 \Leftrightarrow i \in s \cup A_p$) fulfills this requirement.

$\square$

Since subsets of $\mathbb{P}$-meager sets are $\mathbb{P}$-meager as well, $\mathbb{P}$-meager and $\mathbb{P}$-comeager sets are $\mathbb{P}$-measurable we get the following.

Corollary 6.5. *The sets $\mathcal{Z}, \mathcal{Z}^*$ and $\mathcal{Z}^+$ are $\mathbb{P}$-measurable, $\mathbb{P} \in \{\mathbb{MA}^+, \mathbb{MA}^-\}$.*

Our next goal is to compare different notions of measurability. Let $\mathbb{P}$ be any tree-forcing. The statement

"$\Gamma(\mathbb{P}) \Rightarrow \Gamma(\mathbb{C})$, for each well-sorted family Γ"

is true for various tree-forcings adding Cohen reals e.g. Hechler forcing $\mathbb{D}$, Eventually different forcing $\mathbb{E}$, a Silver like version of Mathias forcing $\mathbb{T}$ introduced in [13, Definition 2.1.] and the full-splitting Miller forcing $\mathbb{FM}$ [12, Definition 1.1.]. In fact, it appears reasonable enough to ask.

Question 6.6. *Does each tree-forcing $\mathbb{P}$ adding a Cohen real, necessarily satisfy $\Gamma(\mathbb{P}) \Rightarrow \Gamma(\mathbb{C})$, for each well sorted family Γ?*

A partial answer to this question was given in [13, Proposition 3.1.]. We restate the proposition in a slightly modified version, that will allow us to generalize the result later on.

Proposition 6.7. *Let $\mathcal{X}$ be a set of size $\leq \omega$ and $\mathbb{P}$ be tree-forcing defined on $\mathcal{X}^{<\omega}$. Equip $\mathcal{X}$ with the discrete topology and $\mathcal{X}^\omega$ with the product topology.*

Let $\varphi^* : \mathcal{X}^{<\omega} \to 2^{<\omega}$ *and* $\varphi : \mathcal{X}^\omega \to 2^\omega$ *be two mappings that satisfy the following conditions:*

(1) φ^* *is order preserving and* $\varphi^*(\langle\rangle) = \langle\rangle$,
(2) φ *is continuous,*
(3) $(\forall p \in \mathbb{P})\varphi[p]$ *is open dense in* $[\varphi^*(\text{stem}(p))]$,
(4) $(\forall p \in \mathbb{P})\forall t \trianglerighteq \varphi^*(\text{stem}(p))\exists p' \leq p(\varphi^*(\text{stem}(p')) \trianglerighteq t)$.

Then $\Gamma(\mathbb{P}) \Rightarrow \Gamma(\mathbb{C})$*, for each well-sorted family.*

To understand why Proposition 6.7 is a partial answer to Question 6.6, it should be noted that given $\mathbb{P}, \varphi$ as in the proposition and x_G a $\mathbb{P}$-generic real, we have that $\varphi(x_G)$ is Cohen.

Theorem 6.8. $\Gamma(\mathbb{MA}^+_\varepsilon) \Rightarrow \Gamma(\mathbb{C})$*, for each well sorted family* Γ*.*

Proof. Our aim is to apply Proposition 6.7. To this end let $H^* := \{s \in 2^{<\omega} : |s^{-1}(\{1\})| \text{ is even }\}$ denote the set of all finite sequences with an even number of $1's$. For $s \in H^*$ let $\langle n_k : k < |s^{-1}(\{1\})|\rangle$ enumerate $s^{-1}(\{1\})$. Observe that H^* is dense in $2^{<\omega}$. Now fix $N \in \omega$ and a partition $\{a_i\}_{i<(N+1)}$ of ω such that $\varepsilon > 1/(N+1)$ and such that each set a_i has density $1/(N+1)$. We define the function φ^* as expected. Let $k \in \omega$ and $s \in H^*$ be such that $2k < |s^{-1}(\{1\})|$. We define:

$$\varphi^*(s)(k) := \begin{cases} 0, & \exists i < (N+1)(\{n_{2k}, n_{2k+1}\} \subseteq a_i) \\ 1, & \text{else.} \end{cases}$$

Let H and $\varphi : H \to 2^{<\omega}$ be defined as in the proof of Theorem 6.3 (2). Now, given $x \in H$ let $\langle n^x_k : k < \omega\rangle$ enumerate $x^{-1}\{1\}$. Then, $\varphi^*(x{\restriction}n^x_{2k})$ is defined for each k and $\varphi(x) = \bigcup_k \varphi^*(x{\restriction}n^x_{2k})$.
Note that the same argument used in the proof to show that $\mathbb{MA}^+_\varepsilon$ adds Cohen reals, gives us $\varphi[p] = [\varphi^*(\text{stem}(p))]$. Especially, condition (3) from the proposition is fulfilled. It is straightforward to check that the other three conditions are satisfied as well and since the set H is $\mathbb{MA}^+_\varepsilon$-comeager, we can apply the proposition. $\square$

7. $\mathbb{MA}^+$-MEASURABILITY

In this section we examine $\mathbb{MA}^+$-measurability and give a generalization of Proposition 6.7.
By Theorem 4.5 we know that $\mathbb{MA}^+$ adds Cohen reals. So, it seems reasonable enough to try the same method used in Section 6 to prove $\Gamma(\mathbb{MA}^+_\varepsilon) \Rightarrow \Gamma(\mathbb{C})$, with the forcing $\mathbb{MA}^+$. However, in doing so one encounters the following problem. The coding from Theorem 4.5 used to generate the Cohen real, uses information of the *whole* condition and does not depend solely on the stem. To make it clear what we mean, we quickly explain how using the proof of Theorem 4.5 one gets a coding function defined on a $\mathbb{MA}^+$-comeager set.

7.1. **Construction of the coding function φ.** Recall that we defined in the proof of Theorem 4.5 families $\mathcal{A}_n := \{A_n^i \;:\; i < 2^{(n+1)}\}$, where $A_n^i := \{k \in \omega \;:\; k = i \mod 2^{(n+1)}\}$. Each $\mathcal{A}_n$ is a maximal antichain in $(\mathcal{Z}^+, \subseteq^*)$ and each set A_n^i has density $2^{-(n+1)}$. Now we define $D_n := \{A \in [\omega]^\omega \;:\; \exists i < 2^{-(n+1)}(A \subseteq^* A_n^i)\}$. Then for each $n \in \omega$ and $(s, A) \in \mathbb{MA}(\mathcal{Z}^+)$ there is $B \in D_n$ such that $(s, B) \leq (s, A)$. This already implies that the set $D := \bigcap D_n$ is $\mathbb{MA}(\mathcal{Z}^+)$-comeager and thus it is enough to find a suitable coding function φ defined on D instead of 2^ω. First, note that for $x \in D$ we also must have the property $(\forall n \exists! i_n (x \subseteq^* A_n^{i_n}))$ and we abbreviate $A_n^{i_n}$ with A_n. So, we can define for $x \in D$ two sequences $\langle n_i \;:\; i < \omega\rangle$ and $\langle m_i \;:\; i < \omega \setminus \{0\}\rangle$ as in the proof of Theorem 4.5. Finally, $\varphi : D \to 2^\omega$ is defined as

$$\varphi(x)(i) := \begin{cases} 0, & \text{if } x \setminus (m_{i+1} + 1) \subseteq A_{m_{i+1}} \\ 1, & \text{else.} \end{cases}$$

This definition of the coding function φ seems promising. However, when one takes a close look at possible candidates for a corresponding φ^*, it becomes clear that φ^* cannot depend solely on the stem of a condition. Especially, we cannot apply Proposition 6.7 as it is stated in Section 6. We have to find a generalization like the following.

Proposition 7.1. *Let $\mathcal{X}, \mathcal{Y}$ be sets of size $\leq \omega$, $\mathbb{P}, \mathbb{Q}$ be tree-forcings defined on $\mathcal{X}^{<\omega}, \mathcal{Y}^{<\omega}$ respectively. Equip $\mathcal{X}, \mathcal{Y}$ with the discrete topology and $\mathcal{X}^\omega, \mathcal{Y}^\omega$ with the product topology. Let $\varphi^* : \mathbb{P} \to \mathbb{Q}$ and $\varphi : \mathcal{X}^\omega \to \mathcal{Y}^\omega$ be two mappings that satisfy the following conditions:*

(1) φ^* is order preserving and $\varphi^*(1_\mathbb{P}) = 1_\mathbb{Q}$,
(2) φ is continuous,
(3) $\forall p \in \mathbb{P} \varphi[p]$ is $\mathbb{Q}$-open dense in $[\varphi^*(p)]$,
(4) $\forall p \in \mathbb{P} \forall q \leq \varphi^*(p) \exists p' \leq p (\varphi^*(p') \leq q)$.

Then $\Gamma(\mathbb{P}) \Rightarrow \Gamma(\mathbb{Q})$, for each well-sorted family Γ.

The key difference is that φ^* is a map from $\mathbb{P}$ to $\mathbb{Q}$, instead of being defined for finite sequences.
Before we turn to the proof of the proposition we investigate further if we might apply it to $\mathbb{MA}^+$ and $\mathbb{C}$. To this end we already have defined a $\mathbb{MA}^+$-comeager set D and a coding function $\varphi : D \to 2^\omega$ satisfying $\varphi(x_G)$ is Cohen, where x_G is $\mathbb{MA}^+$-generic. We want to define $\varphi^* : \mathbb{MA}^+ \to \mathbb{C}$. Let $\dot{x}_G$ be the canonical name for the $\mathbb{MA}^+$-generic real. For $p \in \mathbb{MA}^+$ let $r_p \in 2^{<\omega}$ be maximal such that $p \Vdash \varphi(\dot{x}_G) \trianglerighteq r_p$ and put $\varphi^*(p) := r_p$. Observe that we have the following properties:

- φ^* is order preserving and $\varphi^*((\langle\rangle, \omega)) = \langle\rangle$,
- φ is not continuous,
- It follows from the proof that $\varphi(\dot{x}_G)$ is Cohen, that conditions (3) and (4) of Proposition 7.1 are satisfied.

This shows that we almost get $\Gamma(\mathbb{MA}^+) \Rightarrow \Gamma(\mathbb{C})$, for any well-sorted family Γ. In fact, the only time we need φ to be continuous is to ensure that the pre-image of a regular set $Y \in \Gamma$ is again regular. So, if we change the requirement of Γ of being well-sorted and instead assume that the family of sets Γ is closed under pre-images of the φ constructed above we get:

Corollary 7.2. *Let φ be defined as in the beginning of this section and Γ be a family of sets closed under pre-images of φ. Then $\Gamma(\mathbb{MA}^+) \Rightarrow \Gamma(\mathbb{C})$ holds.*

Now we prove Proposition 7.1. The key step is the following lemma.

Lemma 7.3. *Let $\mathbb{P}, \mathbb{Q}, \varphi, \varphi^*$ be as in the Proposition and $Y \subseteq \mathcal{Y}^\omega$. Define $X := \varphi^{-1}[Y]$. Assume there is $p \in \mathbb{P}$ such that $X \cap [p]$ is $\mathbb{P}$-comeager in $[p]$. Then $Y \cap [\varphi^*(p)]$ is $\mathbb{Q}$-comeager in $[\varphi^*(p)]$.*

Proof. We are assuming $X \cap [p]$ is $\mathbb{P}$-comeager, for some $p \in \mathbb{P}$. This implies that there is a collection $\{A_n : n < \omega \wedge A_n \text{ is } \mathbb{P}\text{-open dense in } [p]\}$ such that $\bigcap_n A_n \subseteq [p] \cap X$. W.l.o.g. assume $A_n \supseteq A_{n+1}$, for all n. Let $q = \varphi^*(p)$. We want to show that $\varphi[X] \cap [q] = Y \cap [q]$ is $\mathbb{Q}$-comeager in $[q]$ i.e., we want to find $\{B_n : n < \omega\}$ $\mathbb{Q}$-open dense sets in $[q]$ such that $\bigcap_n B_n \subseteq Y \cap [q]$. Given $\sigma \in \mathfrak{c}^{<\omega}$ we recursively define on the length of σ a set $\{p_\sigma : \sigma \in \mathfrak{c}^{<\omega}\} \subseteq \mathbb{P}$ with the following properties:

1.: $p_{\langle\rangle} = p$,
2.: $\forall \sigma \in \mathfrak{c}^{<\omega}$ $\bigcup_i [\varphi^*(p_{\sigma^\frown i})]$ is $\mathbb{Q}$-open dense in $[\varphi^*(p_\sigma)]$,
3.: $\forall \sigma \in \mathfrak{c}^{<\omega} \forall i \in \omega$ $([p_{\sigma^\frown i}] \subseteq \bigcap_{k \leq |\sigma|} A_k \wedge p_{\sigma^\frown i} \leq p_\sigma)$.

Assume we are at step n. Fix $\sigma \in \mathfrak{c}^n$ arbitrarily and then put $q_\sigma = \varphi^*(p_\sigma)$. We first make sure that 2. holds. For this purpose, let $\{q_i : i < \mathfrak{c}\}$ enumerate all conditions in $\mathbb{Q}$ below q_σ. By condition (4) from Proposition 7.1 we can find $p_i \leq p_\sigma$ such that $\varphi^*(p_i) \leq q_i$. Since each A_k is $\mathbb{P}$-open dense in $[p]$ we can find for each $i < \mathfrak{c}$ an extension $p_{\sigma^\frown i} \leq p_i$ such that $[p_{\sigma^\frown i}] \subseteq \bigcap_{k \leq n} A_k$. This ensures that also 3. holds. Finally, we put $B_n := \bigcup \{\varphi[[p_\sigma]] : \sigma \in \mathfrak{c}^n\}$. We have to check that each set B_n is $\mathbb{Q}$-open dense in $[q]$ and $\bigcap_n B_n \subseteq Y \cap [q]$. So fix $n \in \omega$ and $q' \leq q = \varphi^*(p)$. In the first construction step this q' was enumerated, say by $i < \mathfrak{c}$ so $q' = q_i$ and $\varphi^*(p_{\langle i \rangle}) \leq q'$. Especially, $\varphi^*(p_\sigma) \leq q'$, whenever $\sigma \in \mathfrak{c}^n, \sigma(0) = i$. By condition (3) from the Proposition $\varphi[[p_\sigma]]$ is $\mathbb{Q}$-open dense in $[\varphi^*(p_\sigma)]$. This proves that B_n is $\mathbb{Q}$-open dense in $[q]$. By construction of B_{n+1} we know $B_n \subseteq \varphi[\bigcap_{k \leq n+1} A_k]$ and hence

$$\bigcap B_n \subseteq \varphi[\bigcap_n A_n] \subseteq \varphi[[p] \cap X] \subseteq \varphi[p] \cap Y \subseteq [q] \cap Y.$$

□

Proof of the proposition. Let $Y \in \Gamma$ be given and put $X := \varphi^{-1}[Y]$. Then also $X \in \Gamma$, since Γ is well-sorted and φ is continuous. We now use the lemma to show that for every $q \in \mathbb{Q}$ there exists $q' \leq q$ such that $Y \cap [q']$ is $\mathbb{Q}$-meager or $Y \cap [q']$ is $\mathbb{Q}$-comeager.

Observe that by conditions (1) and (4) we get $\varphi^*[\mathbb{P}]$ is dense in $\mathbb{Q}$. Now fix $q \in \mathbb{Q}$ arbitrarily and pick $p \in \mathbb{P}$ such that $\varphi^*(p) \leq q$. By assumption X is $\mathbb{P}$-measurable, and so:

- in case there exists $p' \leq p$ such that $X \cap [p']$ is $\mathbb{P}$-comeager; put $q' := \varphi^*(p')$. By the lemma above, $Y \cap [q']$ is $\mathbb{Q}$-comeager in $[q']$;
- in case there exists $p' \leq p$ such that $X \cap [p']$ is $\mathbb{P}$-meager, then apply the lemma above to the complement of X, in order to get $Y \cap [q']$ be meager in $[q']$, with $q' := \varphi^*(p')$.

□

Corollary 7.4. *Let $\mathbb{P}, \mathbb{Q}, \varphi^*$ and φ be as in proposition. Then $\varphi(x_G)$ is $\mathbb{Q}$-generic, where x_G is a $\mathbb{P}$-generic real.*

Proof of the Corollary. Fix an $\mathbb{Q}$-open dense set $D \subseteq \mathbb{Q}$. We want to show that the conditions $p \in \mathbb{P}$ such that $p \Vdash \exists q \in D(\varphi(x_G) \in [q])$ is dense in $\mathbb{P}$. To this end, fix a condition $p \in \mathbb{P}$. Then, since D is dense in $\mathbb{Q}$ there is $q' \in D$ below $\varphi^*(p)$. By condition (4) of Proposition 7.1 there is $p' \leq p$ such that $\varphi^*(p') \leq q'$. By condition (3) we know that each condition $r \in \mathbb{P}$ forces $\varphi(x_G)$ into $[\varphi^*(r)]$ and hence $p' \Vdash \varphi(x_G) \in [\varphi^*(p')] \subseteq [q']$.

□

In light of this one might also generalize Question 6.6 to the following.

Question 7.5. *Let $\mathbb{P}, \mathbb{Q}$ be two tree-forcings and assume $\mathbb{P}$ adds a $\mathbb{Q}$-generic. Does $\Gamma(\mathbb{P}) \Rightarrow \Gamma(\mathbb{Q})$ hold, for each well-sorted family Γ?*

8. A model for $\Sigma^1_2(\mathbb{V}^+_\varepsilon) \wedge \neg\Sigma^1_2(\mathbb{C})$

We construct a model in which the implication $\Gamma(\mathbb{V}^+_\varepsilon) \Rightarrow \Gamma(\mathbb{C})$ fails for $\Gamma = \Sigma^1_2$.
We recall that the shortest splitting node extending $s \in 2^{<\omega}$ is denoted by $\mathrm{splsuc}(s)$ (see Definition 1.1 (e)).

Lemma 8.1. *Let $\varepsilon \in (0,1]$ and $p \in \mathbb{V}^+_\varepsilon$. Let $\bar\varphi : \mathrm{Split}(p) \to 2^{<\omega}$ such that $\bar\varphi(\mathrm{stem}(p)) := \langle\rangle$ and for every $t \in \mathrm{Split}(p)$ and $j \in \{0,1\}$,*

$$\bar\varphi(\mathrm{splsuc}(t^\frown\langle j\rangle)) := \bar\varphi(t)^\frown\langle j\rangle.$$

Let $\varphi : [p] \to 2^\omega$ be the expansion of $\bar\varphi$, i.e. for every $x \in [p]$, $\varphi(x) := \bigcup_{n\in\omega} \bar\varphi(t_n)$, where $\langle t_n : n \in \omega\rangle$ is a $\trianglelefteq$-increasing sequence of splitting nodes in p such that $x = \bigcup_{n\in\omega} t_n$.

If c is Cohen generic over V, then

$$V[c] \models \exists p' \in \mathbb{V}^+_\varepsilon \wedge p' \subseteq p \wedge \forall x \in [p'](\varphi(x) \text{ is Cohen over } V).$$

Proof. For a finite tree $T \subseteq 2^{<\omega}$ we define the set of terminal nodes $\mathrm{Term}(T) := \{s \in T : \neg\exists t \in T(s \triangleleft t)\}$. Consider the following forcing $\mathbb{P}$ consisting of finite trees $T \subseteq 2^{<\omega}$ such that for all $s, t \in T$ the following holds:

(1) If $s, t \in \mathrm{Term}(T)$, then $|s| = |t|$.
(2) If $s, t \notin \mathrm{Term}(T)$ and $|s| = |t|$, then $s^\frown i \in T$ iff $t^\frown i \in T$, $i \in 2$.

The partial order $\mathbb{P}$ is ordered by end-extension: $T' \leq T$ iff $T' \supseteq T$ and $\forall t \in T' \setminus T \exists s \in \mathrm{Term}(T)(s \trianglelefteq t)$.

Note $\mathbb{P}$ is countable and non-trivial, thus it is equivalent to Cohen forcing $\mathbb{C}$. Let $p_G := \bigcup G$, where G is $\mathbb{P}$-generic over V. We claim that $p' := \bar{\varphi}^{-1}{}''p_G$ satisfies the required properties. It is left to show that:

(1) for every $x \in [p']$ one has $\varphi(x)$ is Cohen generic, i.e., every $y \in [p_G]$ is Cohen generic;
(2) $p' \in \mathbb{V}^+_\varepsilon$.

For proving (1), let D be an open dense subset of $\mathbb{C}$ and $T \in \mathbb{P}$. It is enough to find $T' \leq T$ such that every $t \in \mathrm{Term}(T')$ is a member of D.

Let $\{t_j : j < N\}$ enumerate all terminal nodes in T and pick r_N, so that for every $j < N$, $(t_j{}^\frown r_N \in D)$. Then put $T' := \{t \in 2^{<\omega} : \exists t_j \in \mathrm{Term}(T)(t \trianglelefteq t_j{}^\frown r_N\}$. Hence $T' \leq T$ and $T' \Vdash \forall y \in [p_G] \exists t \in (T' \cap D)(t \triangleleft y)$. Hence we have proven that

$$\Vdash_{\mathbb{C}} \forall y \in [p_G] \exists t \in D(t \triangleleft y),$$

which means every $y \in [p_G]$ is Cohen generic over V.

For proving (2), one has to verify that the resulting set of splitting levels $A_{p'}$ has upper density $\geq \varepsilon$. It is easy to see that given any condition $T \in \mathbb{P}$ and $n \in \omega$ one can always find an end-extension $T_n \leq T$ such that

$$T_n \Vdash \exists k < \omega \left(\frac{|A_{p'} \cap k|}{k} \geq \varepsilon - 2^{-n} \right).$$

□

Proposition 8.2. *Let $\mathbb{C}_{\omega_1}$ be an ω_1-product with finite support and let G be $\mathbb{C}_{\omega_1}$-generic over the constructible universe L. Then, for every $\varepsilon \in (0,1]$*

$$L(\mathbb{R})^{L[G]} \models \text{“All } On^\omega\text{-definable sets are } (\mathbb{V}^+_\varepsilon)\text{-measurable”} \wedge \neg\Sigma^1_2(\mathbb{C}).$$

Proof. The argument is the same as in the proof of [2, Proposition 3.7]. Fix $\varepsilon \in (0,1]$. Let for $\alpha \leq \omega_1$ $\mathbb{C}_\alpha$ denote the forcing adding α Cohen reals. Let G be $\mathbb{C}_{\omega_1}$-generic over L.

Let X be an On^ω-definable set of reals, i.e. $X := \{x \in 2^\omega : \psi(x,v)\}$ for a formula ψ with a parameter $v \in \mathrm{On}^\omega$, and let $p \in \mathbb{V}^+_\varepsilon$. We aim to find $q \leq p$ such that $[q] \subseteq X$ or $[q] \cap X = \emptyset$.

We can find $\alpha < \omega_1$ such that $v, p \in L[G{\restriction}\alpha]$. Let $\varphi : [p] \to 2^\omega$ be as in Lemma 8.1. Let $c = G(\alpha)$ be the next Cohen real and write $\mathbb{C}$ for the α-component of $\mathbb{C}_{\omega_1}$. We let

$$b_0 = \big[\!\big[[\![(\psi(\varphi^{-1}(c), v))]\!]_{\mathbb{C}_\alpha} = \mathbf{0} \big]\!\big]_{\mathbb{C}} \quad \text{and} \quad b_1 = \big[\!\big[[\![\psi(\varphi^{-1}(c), v))]\!]_{\mathbb{C}_\alpha} = \mathbf{1} \big]\!\big]_{\mathbb{C}}.$$

Then, by $\mathbb{C}$-homogeneity, $b_0 \wedge b_1 = \mathbf{0}$ and $b_0 \vee b_1 = \mathbf{1}$. Hence, by applying Lemma 8.1, one can then find $q \leq p$ such that $q \subseteq b_0$ or $q \subseteq b_1$ and for every $x \in [q]$, $\varphi(x)$ is Cohen over $L[G{\restriction}\alpha]$. We claim that q satisfies the required property.

- Case $q \subseteq b_1$: note for every $x \in [q]$, $\varphi(x)$ is Cohen over $L[G{\restriction}\alpha]$, and so $L[G{\restriction}\alpha][\varphi(x)] \models \psi(\varphi^{-1}(\varphi(x)), v)$. Hence $L[G] \models \forall x \in [q](\psi(x,v))$, which means $L[G] \models [q] \subseteq X$.

- Case $q \subseteq b_0$: we argue analogously and get $L[G] \models \forall x \in [q](\neg\psi(x, v))$, which means $L[G] \models [q] \cap X = \emptyset$.

Moreover in $L[G]$ it is well-known that $\mathbf{\Sigma}^1_2(\mathbb{C})$ fails (see [3, Theorem 5.8] and [1, 6.5.3, p. 313]). Hence in $L[G]$ all On^ω-definable sets are $\mathbb{V}^+_\varepsilon$-measurable, but there is a $\mathbf{\Sigma}^1_2$ set not satisying the Baire property. As a consequence, in particular we obtain $L(\mathbb{R})^{L[G]} \models \Sigma^1_2(\mathbb{V}^+_\varepsilon) \wedge \neg\Sigma^1_2(\mathbb{C})$. □

We conclude this section by summarizing our results from Sections 6,7 and 8.

$\mathbb{P}$	$\mathbb{MA}^+_\varepsilon$	$\mathbb{MA}^+$	$\mathbb{V}^+_\varepsilon$
$\mathcal{I}_\mathbb{P} = \mathcal{N}_\mathbb{P}$	✗	✗	✓
$\Gamma(\mathbb{P}) \Rightarrow \Gamma(\mathbb{C})$	for all Γ	$\Gamma = \mathcal{P}(\omega)$	$\Sigma^1_2(\mathbb{V}^+_\varepsilon) \not\Rightarrow \Sigma^1_2(\mathbb{C})$

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