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On the homogeneity property for certain quantifier logics

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Summary. The local homogeneity property is defined as in [Mak]. We show that $\mathscr{L}_{\omega\omega}(Q_1)$ and some related logics do not have the local homogeneity property, whereas cofinality logic $\mathscr{L}_{\omega\omega}(Q^{cf\,\omega})$ has the homogeneity property. Both proofs use forcing and absoluteness arguments.

1 Introduction

Our modeltheoretic notation is standard, see e.g. [Eb 85]. Let $\mathscr{L}$, $\mathscr{L}_1$, $\mathscr{L}_2$ denote regular logics. The *local homogeneity property* of a pair $(\mathscr{L}_1, \mathscr{L}_2)$ of logics gives an aspect of the strength of $\mathscr{L}_2$ with respect to $\mathscr{L}_1$: Suppose $(\mathfrak{A}, a_0) \equiv_{\mathscr{L}_2} (\mathfrak{A}, a_1)$. Has any $\phi \in \mathrm{Th}_{\mathscr{L}_1}(\mathfrak{A}, a_0, a_1)$ a model $(\mathfrak{B}, b_0, b_1)$ with an automorphism f of $\mathfrak{B}$ such that $f(b_0) = b_1$? If the answer is 'yes', $(\mathscr{L}_1, \mathscr{L}_2)$ is said to have the local homogeneity property [in short: $\mathrm{loc\,Hom}(\mathscr{L}_1, \mathscr{L}_2)$]. If there is even a $(\mathfrak{B}, b_0, b_1) \equiv_{\mathscr{L}_1} (\mathfrak{A}, a_0, a_1)$ with an automorphism of $\mathfrak{B}$ such that $f(b_0) = b_1$, then $(\mathscr{L}_1, \mathscr{L}_2)$ is said to have the *homogeneity property* [in short: $\mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2)$]. Of course, in the case of $\mathscr{L}_1$ being compact both notions coincide. $\mathscr{L}$ has the homogeneity property means that $(\mathscr{L}, \mathscr{L})$ has the homogeneity property.

The paper is organized as follows: In Sect. 2 counterexamples to the local homogeneity property of $\mathscr{L}_{\omega\omega}(Q_1)$ and $\mathscr{L}_{\infty\omega}$ are constructed via forcing. In the Sect. 3 we show that many cofinality logics have the homogeneity property. In the rest of the introduction we shall show that many familiar logics do not have the homogeneity property unless they are compact.

The homogeneity property together with a small occurrence number implies strong compactness properties. To be more precise: $[\kappa]$-compactness is relative κ-compactness, i.e. $\mathscr{L}$ is $[\kappa]$-*compact* iff for any sets Σ, Δ of $\mathscr{L}$-sentences with $|\Sigma| \leq \kappa$ the following holds: If for any subset Σ_0 of Σ with $|\Sigma_0| < \kappa$ the theory $\Sigma_0 \cup \Delta$ is satisfiable, then $\Sigma \cup \Delta$ is satisfiable. The occurrence number of $\mathscr{L}$, $o(\mathscr{L})$, is the smallest cardinal κ such that any $\phi \in \mathscr{L}[\tau]$ depends on less than κ symbols in τ [if such a cardinal exists, otherwise we write $o(\mathscr{L}) = \infty$]. Using techniques similar to those in the proof of the abstract amalgamation theorem [3.3.1 in Mak; 5.2 in Mak-Sh; II.2.8 in Sh 85] we show

Proposition 1.1. *Let κ be a regular cardinal. If* $\mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2)$ *and* $o(\mathscr{L}_2) \leqq \kappa$, *then* $\mathscr{L}_1$ *is* $[\kappa]$*-compact.*

As a corollary we get, e.g.,

Corollary 1.2. (i) $\neg\mathrm{Hom}(\mathscr{L}_{\omega\omega}(Q_1))$.

(ii) $\neg\mathrm{Hom}(\mathscr{L}_{\kappa\omega}, \mathscr{L}_{\kappa\kappa})$ *unless κ is compact (in which case Hodges and Shelah show* $\mathrm{Hom}(\mathscr{L}_{\kappa\omega}, \mathscr{L}_{\kappa\kappa})$, *see* [Ho-Sh]*).*

Proof of Proposition 1.1. We assume $o(\mathscr{L}_2) \leqq \kappa$ and that $\mathscr{L}_1$ is not $[\kappa]$-compact, and give a counterexample $(\mathfrak{A}, a_0, a_1)$ for $\mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2)$. κ is cofinally characterizable in $\mathscr{L}_1$, i.e. there is a τ-structure $\mathfrak{M} = (M, P^M, <^M, (c_\alpha^M)_{\alpha\in\kappa}, \ldots)$ such that for any $\mathfrak{N} \equiv_{\mathscr{L}_1} \mathfrak{M}$, the sequence $(c_\alpha^N)_{\alpha\in\kappa}$ is increasing and cofinal in the linear ordering $(P^N, <^N)$. Fix such an $\mathfrak{M}$, set $\mu = \mathrm{card}(M \setminus \{c_\alpha \mid \alpha \in \kappa\})$, and assume that $\tau_{\mathfrak{M}} \setminus \{c_\alpha \mid \alpha\in\kappa\}$ is relational. Then expand the branches of the partial order

$$\mu^* + (\{0,1\} \times {}^{\leqq\kappa}\mathbf{Z}, \{((i,f),(i,g)) \mid i=0,1,\ f,g \in {}^{\leqq\kappa}\mathbf{Z},\ f \text{ initial segment of } g\})$$

with copies of $\mathfrak{M}|\tau_{\mathfrak{M}} \setminus \{c_\alpha \mid \alpha\in\kappa\}$ such that

$$\mathfrak{A}_{i,f} := (\mu^* \cup (\{i\} \times \{f|\alpha \mid \alpha \in \kappa+1\}), (Q^{\mathfrak{A}_{i,f}})_{Q\in\tau_{\mathfrak{M}}} \setminus \{c\alpha | \alpha\in\kappa,\ (i, \mathrm{f}|\alpha)_{\alpha\in\kappa}) \cong \mathfrak{M}$$

via isomorphisms $h_{i,f}$ with the following properties:

For any $f, g : \kappa \to \mathbf{Z}$, $\alpha\in\kappa$, $y \in {}^{\leqq\kappa}\mathbf{Z}$, $z \in \mu^*$, $i=0,1$:

$$h_{0,f}(0,y) = h_{1,f}(1,y) \quad \text{and} \quad h_{0,f}(z) = h_{1,f}(z)$$

and, if $f|\alpha = g|\alpha$, then

$$(h_{i,f})^{-1}|\{x\in M \mid x <^{\mathfrak{M}} \alpha\} = (h_{i,g})^{-1}|\{x \in M \mid x <^{\mathfrak{M}} \alpha\}.$$

We let $\mathfrak{A}$ contain this expansion to a $\tau_{\mathfrak{M}} \setminus \{c_\alpha \mid \alpha\in\kappa\}$-structure and interpret additional relations and constants

$$S^{\mathfrak{A}} := \{((1-i,f),(i,g)) \mid i=0,1,\ f,g:\kappa\to\mathbf{Z},\ \{\alpha \mid f(\alpha) \neq g(\alpha)\} \text{ is finite and } \textstyle\sum(g(\alpha)-f(\alpha)) = i \bmod(2)\},$$

$$a_i := c_i^{\mathfrak{A}} := (i, f_0), \quad \text{where} \quad f_0 = (0,0,\ldots) \in {}^{\kappa}\mathbf{Z}, \quad i=0,1,$$

$$R_\alpha^{\mathfrak{A}} := \{(0, f_0|\alpha), (1, f_0|\alpha)\}, \quad \text{for} \quad \alpha\in\kappa.$$

$o(\mathscr{L}_2) \leqq \kappa$ and an automorphism argument yield: $(\mathfrak{A}, a_0) \equiv_{\mathscr{L}_2} (\mathfrak{A}, a_1)$. Since $\mathfrak{M}$ cofinally characterizes κ in $\mathscr{L}_1$, for any $(\mathfrak{B}, b_0, b_1) \equiv_{\mathscr{L}_1} (\mathfrak{A}, a_0, a_1)$ and for any automorphism f of $\mathfrak{B}$ we have $f(b_0) = b_1$ iff $f(b_1) = b_0$. But as $(\mathfrak{B}, b_0, b_1)$ and $(\mathfrak{A}, a_0, a_1)$ satisfy $\neg Sc_0c_1 \wedge Sc_1c_0$, there is no automorphism f of $\mathfrak{B}$ with $f(b_0) = b_1$. $\square$

Proposition 1.1 together with the finite dependence theorem [Theorem 4.3 in Mak-Sh] and with the properties of the least cardinal κ such that $\mathscr{L}$ is $[\kappa]$-compact of [Theorem 1.5 in Mak] lead to

Proposition 1.3. $\mathrm{Hom}(\mathscr{L})$ *and* $o(\mathscr{L}) <$ *the least measurable cardinal (if there exists one, otherwise* $o(\mathscr{L}) \in \mathrm{CARD}$ *is enough) imply the compactness of* $\mathscr{L}$.

In what follows, we investigate the local homogeneity property.

2 Some counterexamples

Proposition 2.1 collects some easy counterexamples. They are based on the existence of $EC_{\mathscr{L}}$-classes of rigid structures which contain a structure with two elements of the same $\mathscr{L}$-type. The main part of this section, however, deals with $\mathscr{L}_{\omega\omega}(Q_1)$ and infinitary logics for which this method of providing counterexamples does not work.

Proposition 2.1. *The following logics do not have the local homogeneity property:*

(i) *stationary logic* $\mathscr{L}_{\omega\omega}(\mathbf{aa})$ *(see* [Ba-Ka-Ma]*);*

(ii) *under CH the Magidor-Malitz logic* $\mathscr{L}_{\omega\omega}(Q_1^2)$ *("1" means the* $\aleph_1$*-interpretation, "2" means the arity of the quantifier, see* [Mag-Mal]*);*

(iii) $\mathscr{L}_{\omega\omega}(Q_1^4)$*;*

(iv) $(\mathscr{L}_{\mu\omega}, \mathscr{L}_{\mu\mu})$ *unless* μ *is strongly inaccessible.*

Proof. (i) Otto [Ot] gives an $\mathscr{L}_{\omega\omega}(\mathbf{aa})$-sentence ϕ that has only rigid models and that has models of arbitrarily large cardinality.

(ii) [Ot; Mil] give $\mathscr{L}_{\omega\omega}(Q_1^2)$-sentences ϕ which are satisfiable under CH and have only rigid models. A sentence that says "ε is extensional and $\mathfrak{A} \models \phi$" and that is true in $(\mathscr{P}(\mathscr{P}(A)); \mathscr{P}(A); \mathfrak{A}, \varepsilon)$ has a model with two elements of the same type and has only rigid models, too.

(iii) [Ot, Lemma 6.6(2)].

(iv) Easy, with the same method as in (ii). $\square$

Ebbinghaus [Eb 71] shows that there are no $\mathscr{L}_{\omega\omega}(Q_1)$-sentences having infinite but only rigid models. Nevertheless, $\mathscr{L}_{\omega\omega}(Q_1)$ does not have the local homogeneity property. This fact and some results on $\mathscr{L}_{\infty\omega}$ will be proved in the remainder of the section. The counterexamples are provided by a modification and expansion of a forcing construction given in [Claim 3.5 in Sh 85].

Theorem 2.2. $(\mathscr{L}_{\omega\omega}(Q_1), \mathscr{L}_{\omega\omega}(\mathbf{aa}))$ *does not have the local homogeneity property.*

The proof of 2.2 will be finished with 2.10. We give an overview: Definition 2.3 and Lemma 2.4 describe a sentence $\phi_{c_0 \not\mapsto c_1} \in \mathscr{L}_{\omega\omega}(Q_1)$ that forbids automorphisms f with $f(c_0)=c_1$. 2.5 up to 2.10 deal with a forcing notion $\mathbf{P}$ such that for $\mathbf{P}$-generic G, in $V[G]$ there is a model $(\mathfrak{A}, c_0^{\mathfrak{A}}, c_1^{\mathfrak{A}})$ of $\phi_{c_0 \not\mapsto c_1}$ with $(\mathfrak{A}, c_0^{\mathfrak{A}}) \equiv_{\mathscr{L}_2} (\mathfrak{A}, c_1^{\mathfrak{A}})$ for certain logics $\mathscr{L}_2$. If $\mathrm{Mod}(\phi_{c_0 \not\mapsto c_1})$ is a PC-class in $\mathscr{L}_1$ with symbols of $\tau_{\mathfrak{A}}$, then for suitable $\mathscr{L}_2$ we have the consistency of $\neg\mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2)$. If additionally satisfiability of $\mathscr{L}_2$-theories is absolute $\neg\mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2)$ is true in V.

We consider a class $\mathbf{K}$ of structures $(A, P, U, \leqq, f, c_0, c_1)$, where $A = P \dot\cup U \dot\cup \{c_0, c_1\}$, $\leqq$ is a linear order on P, P is uncountable, U is countable, $f: P \times P \to U$ satisfies for any a, b, c, d: if $f(a,c) = f(b,d)$, then $a \leqq b$ iff $c \leqq d$. This is nearly the same class Shelah used in Claim 3.5 in [Sh 85]. The square of a linear order $(A, \leqq)$ is the structure $(A \times A, \leqq^2)$, where $(a_1, a_2) \leqq^2 (b_1, b_2)$ iff $a_1 \leqq b_1$ and $a_2 \leqq b_2$. The restriction to P of the $\{\leqq\}$-reduct of any member of $\mathbf{K}$ is an order, whose square is the union of countably many chains, and hence does not have any anti-automorphism. We introduce a 3-ary relation R and a 4-ary relation F, expand the members of $\mathbf{K}$ and forget $\leqq$, such that any anti-automorphism of the unchanged structure corresponds to an automorphism of the new structure that carries c_0 to c_1. The resulting class can be described as the model class of some $\phi \in \mathscr{L}_{\omega\omega}(Q_1)$. To be precise we fix such a sentence.

Definition 2.3. Let U, R, F be relation symbols of arity 1, 3, 4, respectively and c_0, c_1 be symbols for constants. $\sigma := \{U, R, F, c_0, c_1\}$. Let $\phi_{c_0 \not\mapsto c_1}$ denote the $\mathscr{L}_{\omega\omega}(Q_1)[\sigma]$-

sentence $\phi_{c_0 \not\leftrightarrow c_1} = \bigwedge_{i<5} \phi_i$, where

$$\phi_0 := \neg Q_1 x U x;$$

$$\phi_1 := Q_1 x x = x;$$

ϕ_2 says, that the relation $R(c_0, \cdot, \cdot)$ is a linear ordering, whose field is the universe with the exception of the $U \cup \{c_0, c_1\}$-part;

$$\phi_3 := \forall xy(R(c_0, x, y) \leftrightarrow R(c_1, y, x));$$

ϕ_4 says that F is the graph of a function

$$f: \{(c_i, a, b) \mid i = 0, 1,\ R(c_i, a, b)\} \to U,$$

such that

$$\forall xy(R(c_0, x, y) \to f(c_0, x, y) = f(c_1, y, x)) \wedge \forall x_0 x_1 y_0 y_1 (f(c_0, x_0, x_1) = f(c_0, y_0, y_1) \to ((R(c_0, x_0, y_0) \vee x_0 = y_0) \leftrightarrow (R(c_0, x_1, y_1) \vee x_1 = y_1))).$$

Lemma 2.4. *Let $(\mathfrak{B}, c_0^{\mathfrak{B}}, c_1^{\mathfrak{B}}) \models \phi_{c_0 \not\leftrightarrow c_1}$. Then the square of $(B, R^{\mathfrak{B}}(c_0^{\mathfrak{B}}, \cdot, \cdot))$ is the union of countably many chains. Hence there is no automorphism g of $\mathfrak{B}$ with $g(c_0^{\mathfrak{B}}) = c_1^{\mathfrak{B}}$.*

In [Sh 76] on the base of **ZFC** Shelah constructs orders whose squares are the unions of countably many chains. In [Sh 85], he gives a rather brief sketch of a partial order that forces, for generic G, the existence of such orders in $V[G]$. We use the expansion of a suitable c.c.c.-suborder of that partial order. In a generic $V[G]$ this gives us a τ-structure $\mathfrak{A} \models \phi_{c_0 \not\leftrightarrow c_1}$ such that for certain logics $\mathscr{L}_2 \geqq \mathscr{L}_1$ the $\mathscr{L}_2$-theory of $(\mathfrak{A}, a_0)$ equals the $\mathscr{L}_2$-theory of $(\mathfrak{A}, a_1)$. The following definition collects some properties of a logic $\mathscr{L}_2$ that are suitable for the intended forcing.

Definition 2.5. $(\mathscr{L}, \models_{\mathscr{L}})$ is an *$\aleph_0$-definable logic with small syntax* iff there are a parameter $c \subsetneqq \omega$ and formulas $\psi_{\text{syn}}(x, y, z), \psi_{\text{sem}}(x, y, z) \in \mathscr{L}_{\omega\omega}[\{\varepsilon\}]$ such that for any pair of transitive models $\mathfrak{M} = (M, \varepsilon) \subseteqq \mathfrak{N} = (N, \varepsilon)$ of **ZFC** with $c \in M$ the following holds:

(i) For any χ, $\tau \in M$, $\theta \in N$:

$$(\mathfrak{M} \models \chi \in \mathscr{L}[\tau]) \Leftrightarrow \mathfrak{M} \models \psi_{\text{syn}}[\chi, \tau, c],$$

and $((\mathfrak{M} \models \psi_{\text{syn}}[\theta, \tau, c]$ and $\theta \in M) \Leftrightarrow \mathfrak{N} \models \psi_{\text{syn}}[\theta, \tau, c])$.

(ii) For any χ, $\tau \in M$ and any τ-structure $\mathfrak{C}$ in M:

$$\mathfrak{M} \models ((\mathfrak{C} \models_{\mathscr{L}} \chi) \leftrightarrow (\psi_{\text{syn}}[\chi, \tau, c] \wedge \psi_{\text{sem}}[\chi, \mathfrak{C}, c])).$$

That means: $\chi \in \mathscr{L}[\tau]$ and $\mathfrak{C} \models_{\mathscr{L}} \chi$ are definable with parameters $c \subsetneqq \omega$ in set theory. The relation $\chi \in \mathscr{L}[\tau]$ satisfies the strong absoluteness property (i): for any universe M, even in larger universes there are no new $\mathscr{L}[\tau]$-sentences for $\tau \in M$. The relation $\mathfrak{C} \models_{\mathscr{L}} \chi$ may be not absolute.

For technical reasons we take $\tau = \sigma \cup \{H\} \cup \{\underline{n} \mid n \in \omega\}$, where the arity of H is 3. The consideration of τ-structures allows us to use the following main theorem also for logics that do not contain $\phi_{c_0 \not\leftrightarrow c_1}$, e.g. for infinitary logics in Theorem 2.11.

Theorem 2.6. *Let $\mathscr{L}$ be an $\aleph_0$-definable logic with small syntax. Then there is a c.c.c. partial order $\mathbf{P} = (P, \leqq^P, 1^P)$, such that:*

$1^P \Vdash^{P}$ "*There is a τ-structure $(\mathfrak{A}, a_0, a_1)$, such that*

$$(\mathfrak{A}, a_0, a_1) \equiv_{\mathscr{L}} (\mathfrak{A}, a_1, a_0) \quad \textit{and}$$

$$(\mathfrak{A}, a_0, a_1) \models \phi_{c_0 \not\leftrightarrow c_1} \wedge \forall x \neq y \notin (U \cup \{c_0, c_1\}) \exists u \in U (H(x, u, \underline{0}) \wedge \neg H(y, u, \underline{0})) \wedge \forall x \Big(Ux \to \bigvee_{n \in \omega} x = \underline{n}\Big)\text{"}.$$

Proof. Let $\mathbf{P}=(P, \leqq^P, 1^P)$ be defined as follows:

$P:=\{p=(w_p, \leqq_p, f_p, h_p) \mid p$ satisfies the conditions $(P0)$ up to $(P5)\}$, where

$(P0)$ w_p is a finite subset of $\omega_1 \backslash \omega$.

$(P1)$ $\leqq_p$ is a nonstrict linear ordering on w_p.

$f_p: \{(\alpha, \beta) \in w_p \times w_p \mid \alpha <_p \beta\} \to \omega$ has the following properties:

$(P2)$ For any $(\alpha, \beta), (\alpha', \beta') \in \mathrm{dom}(f_p)$: If $f_p(\alpha, \beta) = f_p(\alpha', \beta')$, then (α, β) and (α', β') are comparable in $\leqq_p^2$.

For $\alpha, \beta \in w_p$ let $[\alpha, \beta]_p$ denote the set $\{\gamma \in w_p \mid \alpha \leqq_p \gamma \leqq_p \beta\}$.

For $\alpha \geqq \beta \in \omega_1$ let $\alpha - \beta$ be the γ with $\beta + \gamma = \alpha$, if $\beta > \alpha$ set $\alpha - \beta = 0$. The distance of α and β, $\mathrm{d}(\alpha, \beta)$, is the maximum of $\alpha - \beta$ and $\beta - \alpha$.

$(P3)$
$$\forall \alpha, \beta, \beta' \in w_p$$
$$((\alpha <_p \beta <_p \beta' \wedge f_p(\alpha, \beta) = f_p(\alpha, \beta')) \to \forall \delta \in [\beta, \beta']_p \alpha - \delta < \omega)$$

and
$$\forall \alpha, \alpha', \beta \in w_p$$
$$((\alpha <_p \alpha' <_p \beta \wedge f_p(\alpha, \beta) = f_p(\alpha', \beta)) \to \forall \delta \in [\alpha, \alpha']_p \beta - \delta < \omega).$$

$(P4)$
$$\forall \alpha, \alpha', \beta, \beta' \in w_p$$
$$((\alpha <_p \beta \wedge \alpha' <_p \beta' \wedge \alpha <_p \alpha' \wedge \beta <_p \beta' \wedge f_p(\alpha, \beta) = f_p(\alpha', \beta'))$$
$$\to (\forall \gamma \in [\alpha, \alpha']_p \exists \delta \in [\beta, \beta']_p \mathrm{d}(\gamma, \delta) < \omega \wedge \forall \gamma \in [\beta, \beta']_p \exists \delta \in [\alpha, \alpha']_p \mathrm{d}(\gamma, \delta) < \omega)).$$

$(P5)$
$$h_p: w_p \times \mathrm{rg}(f_p) \to \{0, 1\}.$$

For $p, q \in P$ let $q \leqq^P p$ iff
$$w_p \subseteqq w_q,$$
$$\leqq_p = \leqq_q \cap (w_p \times w_p),$$
$$f_p = f_q | w_p \times w_p,$$
$$h_p = h_q | w_p \times \mathrm{rg}(f_p).$$

$1^P := (\emptyset, \emptyset, \emptyset, \emptyset)$.

Remark. Shelah's forcing just has properties $(P0)$, $(P1)$, and $(P2)$ except that the domain of his f_p is $w_p \times w_p$ rather than a proper subset. It collapses ω_1: Let G be generic with ordering $\leqq_G$ and function f_G. If $\alpha \neq \beta$, then for any $n \in \omega$ there is at most one γ such that α, $\beta <_G \gamma$, and $f_G(\beta, \gamma) = f_G(\alpha, \gamma) = n$. Put $h(\gamma) = n$. Then a denseness argument shows that h maps a subset of ω into a cofinal subset of ω_1.

Conditions $(P3)$ and $(P4)$ will be used to prove the c.c.c.

Claim 2.7. **P** *has the c.c.c.*

Proof. Let $P_0 \subseteqq P$ be uncountable. By the Δ-lemma and the pigeonhole principle there are finite sets $r \subset \omega_1 \backslash \omega$, $s \subset \omega$ and a limit ordinal $\xi \in \omega_1$ and an uncountable $P_1 \subseteqq P_0$, such that for any $p \neq q \in P_1$ the following is true: $r = w_p \cap w_q = \xi \cap w_p$ and $s = \mathrm{rg}(f_p)$ and
$$(w_p \cup \mathrm{rg}(f_p) \cup \{0, 1\}, <^{\omega_1} | (w_p \cup \mathrm{rg}(f_p) \cup \{0, 1\}), \leqq_p, f_p, h_p)$$
$$\cong_{\text{over } r \cup s \cup \{0,1\}} (w_q \cup \mathrm{rg}(f_q) \cup \{0, 1\}, <^{\omega_1} | (w_q \cup \mathrm{rg}(f_q) \cup \{0, 1\}), \leqq_q, f_q, h_q).$$

We shall show that any two elements of P_1 are compatible in **P**. Let r be $r_0 <^{\omega_1} r_1 <^{\omega_1} \ldots <^{\omega_1} r_{n-1}$. Given $p \neq q \in P_1$, we define $a(p, q)$ a sort of amalgam of p and q, which is in general not an element of P.

If $(w_p, <^{\omega_1}|w_p)=(r_0<^{\omega_1}r_1<^{\omega_1}\dots r_{n-1}<^{\omega_1}\varepsilon_0<^{\omega_1}\varepsilon_1<^{\omega_1}\dots\varepsilon_{m-1})$ and $(w_q, <^{\omega_1}|w_q)=(r_0<^{\omega_1}r_1<^{\omega_1}\dots r_{n-1}<^{\omega_1}\zeta_0<^{\omega_1}\zeta_1<^{\omega_1}\dots\zeta_{m-1})$, then $a(p,q):=(w_p\cup w_q, \leqq_a, f_a, h_a)$, where $\leqq_a$ is the transitive closure of $\leqq_p\cup\leqq_q\cup\leqq'$ with ($\varepsilon_i<'\zeta_k$ iff there is no $j<n$ with $\zeta_k\leqq_q r_j\leqq_p\varepsilon_i$). Let $f_a:=f_p\cup f_q\cup f'$, where f' has the appropriate domain and is an arbitrary injective function with $\mathrm{rg}(f')\cap\mathrm{rg}(f_p)=\emptyset$. For h_a take a continuation of $h_p\cup h_q$ with domain $(w_p\cup w_q)\times\mathrm{rg}(f_a)$.

$a(p,q)$ satisfies the defining properties of P except of $(P4)$. $(P0)$, $(P1)$, $(P5)$ are obvious; checking $(P2)$ and $(P3)$ is easy but tedious. Let us consider $(P2)$.

Let $f_a(\alpha,\beta)=f_a(\alpha',\beta')$. The only interesting case is $(\alpha,\beta)\in w_p\times w_p\backslash r\times r$ and $(\alpha',\beta')\in w_q\times w_q\backslash r\times r$. Let g be an isomorphism between p and q. Then

$$f_p(\alpha,\beta)=f_q(g(\alpha),g(\beta))=f_q(\alpha',\beta')=f_p(g^{-1}(\alpha'),g^{-1}(\beta')).$$

We prove by cases that (α,β) is $\leqq_a^2$-comparable to (α',β').

Case 1: $(\alpha,\beta)\leqq_p^2(g^{-1}(\alpha'),g^{-1}(\beta'))$.

The transitivity of $\leqq_a^2$ and $g(\gamma)\geqq_a\gamma$ yield $(\alpha,\beta)\leqq_a^2(\alpha',\beta')$.

Case 2: $(g^{-1}(\alpha'),g^{-1}(\beta'))\leqq_p^2(\alpha,\beta)$.

Subcase 2.1: $[g^{-1}(\alpha'),\alpha]_p\cap r\neq\emptyset$.

Here we have $\alpha'\leqq_a\alpha$. If $g^{-1}(\alpha')=\alpha$, then $\alpha'=\alpha$ and (α,β) is comparable to (α',β') in $\leqq_a^2$.

If $g^{-1}(\alpha')<_a\alpha$, then $[g^{-1}(\alpha'),\alpha]_p\cap r\neq\emptyset$ and $f_p(\alpha,\beta)=f_p(g^{-1}(\alpha'),g^{-1}(\beta'))$ and $(P3)$ [in case $g^{-1}(\beta')=\beta$] or $(P4)$ [in case $g^{-1}(\beta')<_p\beta$] show $[g^{-1}(\beta'),\beta]_p\cap r\neq\emptyset$, and hence $\beta'\leqq_a\beta$.

Subcase 2.2: $[g^{-1}(\alpha'),\alpha]_p\cap r=\emptyset$. The proof is similar to that of Subcase 2.1.

We make $a(p,q)$ "thicker" in order to get an $o(p,q)\leqq^P p,q$: $(\alpha,\beta)\in w_a\times w_a$ is said to be a *jump of* p,q iff β is a direct $<_a$-successor of α and not ($\alpha,\beta\in r$ or $\alpha,\beta\in w_p\backslash r$ or $\alpha,\beta\in w_q\backslash r$). For each $\gamma\in(w_p\cup w_q)\backslash r$ and each jump (α,β) of p,q we take a new countable ordinal $\delta=\delta(\gamma,\alpha,\beta)\notin w_p\cup w_q\cup\xi$, such that $\mathrm{d}(\gamma,\delta)<\omega$. Let $\leqq_{o(p,q)}=:\leqq_o$ be any linear ordering such that $\alpha<_o\delta(\gamma,\alpha,\beta)<_o\beta$ for all α,β,γ. Choose f_o,h_o in a similar manner as above. Then it is lengthy but easy verify that $o\in P$. Thus the claim is proved. $\square$

Let V be a countable transitive model of **ZFC**, **P** defined in V as above, G be **P**-generic over V. Since $D_\alpha:=\{p\in P\mid\alpha\in w_p\}$ is dense in P for all $\alpha\in\omega_1$, the structure $\bigcup G=(\bigcup\{w_p\mid p\in G\}, \bigcup\{\leqq_p\mid p\in G\},\dots)$ has support ω_1 $(=\omega_1^V)$. We denote it by $(\omega_1,\leqq_G,f_G,h_G)$. In $V[G]$, we define the τ-structure $(\mathfrak{A}_G,a_0,a_1)$ by $A_G:=\omega_1\cup\{-1,-2\}$,

$$R^{\mathfrak{A}_G}:=(\{-1\}\times\leqq_G)\cup(\{-2\}\times\geqq_G),$$

$$F^{\mathfrak{A}_G}:=(\{-1\}\times\mathrm{Graph}(f_G))$$
$$\cup(\{-2\}\times\{(\alpha,\beta,\gamma)\mid\beta<_G\alpha\text{ and }f_G(\beta,\alpha)=\gamma\}),$$

$$H^{\mathfrak{A}_G}:=\mathrm{graph}(h_G),\quad U^{\mathfrak{A}_G}:=\omega,\quad \underline{n}^{\mathfrak{A}_G}:=n,\quad n\in\omega,$$

$$c_i^{(\mathfrak{A}_G,a_0,a_1)}:=a_i:=-1-i\quad\text{for}\quad i=0,1.$$

$(\mathfrak{A}_G,a_0,a_1)\models\phi_{c_0\not\leftrightarrow c_1}$ follows from the definition of **P**. The existence of suitable dense subsets of **P** shows

$$(\mathfrak{A}_G,a_0,a_1)\models\forall x\neq y\notin(U\cup\{c_0,c_1\})\,\exists u\in U(H(x,u,\underline{0})\wedge\neg H(y,u,\underline{0})).$$

It remains to show that $(\mathfrak{A}_G,a_0)\equiv_{\mathscr{L}}(\mathfrak{A}_G,a_1)$; we shall even get $(\mathfrak{A}_G,a_0,a_1)\equiv_{\mathscr{L}}(\mathfrak{A}_G,a_1,a_0)$. Let $\mathscr{L}$, $\psi_{\mathrm{syn}}(x,y,z)$, $\psi_{\mathrm{sem}}(x,y,z)$, and $c\subsetneqq\omega$ be as in Definition 2.5. For $x\in V$ let $\check{x}$ be a canonical P-name for x, for $x\in V[G]$ let $\dot{x}$ be a P-name for x.

The names should be chosen in an obvious manner such that $(\mathfrak{A}_G, a_0, a_1)^{\cdot}$ is computable from $(\mathfrak{A}_G, a_1, a_0)^{\cdot}$. We suppress the dots in the following and show

$$1^P \Vdash^P \forall y(\psi_{\text{syn}}(y, \check{\tau}, \check{c}) \rightarrow (\psi_{\text{sem}}(y, (\mathfrak{A}_G, a_0, a_1), \check{c}) \leftrightarrow \psi_{\text{sem}}(y, (\mathfrak{A}_G, a_1, a_0), \check{c}))).$$

Let G be a **P**-generic filter over V, y be in $V[G]$ and $V[G] \models \psi_{\text{syn}}(y, \tau, c) \wedge \psi_{\text{sem}}(y, (\mathfrak{A}_G, a_0, a_1), c)$. Since the syntax of $\mathscr{L}$ is small, we have $y \in V$. We fix a $p \in G$ such that

$$p \Vdash^P \psi_{\text{syn}}(\check{y}, \check{\tau}, \check{c}) \wedge \psi_{\text{sem}}(\check{y}, (\mathfrak{A}, a_0, a_1), \check{c})$$

and show in 2.8 up to 2.10 that

$$D := \{q \in P \mid q \Vdash^P \psi_{\text{sem}}(\check{y}, (\mathfrak{A}, a_1, a_0), \check{c})\} \quad \text{is dense in } \mathbf{P} \text{ below } p.$$

The idea is that the order-reversed version of p forces $\psi_{\text{sem}}(\check{y}, (\mathfrak{A}, a_1, a_0), \check{c})$. Unfortunately, if $\operatorname{card}(w_p) \geqq 2$, its order-reversed version is incompatible with p and hence cannot be a member of D. So we introduce besides the order-reversing automorphisms of **P** automorphisms of **P** which shift the supports of the p's a little bit. Given $q \leqq^P p$, by reverting a suitably shifted version of p and joining it to q, we get an $r \in D$, $r \leqq^P q$.

Definition 2.8. (i) $\operatorname{Perm}(\omega_1, <\omega) := \{g : \omega_1 \rightarrow \omega_1 \mid g \text{ is bijective and } g|\omega = \operatorname{id}|\omega \text{ and } \operatorname{d}(\alpha, g(\alpha)) < \omega \text{ for all } \alpha \in \omega_1\}$.

(ii) Each $g \in \operatorname{Perm}(\omega_1, <\omega)$ induces a mapping $\operatorname{ind}(g) : P \rightarrow V$ via

$$\begin{aligned}\operatorname{ind}(g)(p) := (&g''(\omega_p), \{(\alpha, \beta) \in g''(\omega_p) \times g''(\omega_p) \mid g^{-1}(\alpha) \leqq_p g^{-1}(\beta)\},\\ &\{((\alpha, \beta), f_p(g^{-1}(\alpha), g^{-1}(\beta))) \mid (\alpha, \beta) \in g''(w_p) \times g''(\omega_p), g^{-1}(\alpha) <_p g^{-1}(\beta)\},\\ &\{((\alpha, \beta), h_p(g^{-1}(\alpha), \beta)) \mid \alpha \in g''(w_p), \beta \in \operatorname{rg}(f_p)\}).\end{aligned}$$

It is easy to see that for $g \in \operatorname{Perm}(\omega_1, <\omega)$ the function $\operatorname{ind}(g)$ is an automorphism of **P**. For $p \in P$ let $\operatorname{mirror}(p) = (w_p, \geqq_p, \{(\alpha, \beta, \gamma) \mid \alpha >_p \beta, f_p(\beta, \alpha) = \gamma\}, h_p)$, and for i any automorphism of **P** define $i^* : V^P \rightarrow V^P$, $i^*(\tau) = \{\langle i^*(\sigma), i(p)\rangle \mid \langle \sigma, p\rangle \in \tau\}$, see [VII.12 in Ku]. The next lemma will enable us to use the isomorphism lemma for forcing [VII.13.c in Ku] to find the required members of D.

Lemma 2.9. (i) *For* $g \in \operatorname{Perm}(\omega_1, <\omega)$ *we have:*

$$1^P \Vdash^P (g^{-1} \cup \{(a_0, a_0), (a_1, a_1)\})^{\vee} : (\mathfrak{A}_G, a_0, a_1) \cong \operatorname{ind}(g)^*((\mathfrak{A}_G, a_0, a_1)).$$

(ii) $$1^P \Vdash^P (\mathfrak{A}_G, a_1, a_0) = \operatorname{mirror}^*((\mathfrak{A}_G, a_0, a_1)).$$

Proof. (i) Each $p \in G$ gives a finite substructure p_τ of the generic structure $(\mathfrak{A}_G, a_0, a_1)$ by $p_\tau := (w_p \cup \operatorname{rg}(f_p) \cup \{0, 1\} \cup \{a_0, a_1\}, \ (\{a_0\} \times \leqq_p) \cup (\{a_1\} \times \geqq_p), (\{a_0\} \times \operatorname{Graph}(f_p)) \cup \{(a_1, \alpha, \beta, \gamma) \mid f_p(\beta, \alpha) = \gamma\}, \operatorname{graph}(h_p), a_0, a_1)$.

To prove (i), we show for **P**-generic G, that

$$(g^{-1} \cup \{(a_0, a_0), (a_1, a_1)\}) : (\mathfrak{A}_G, a_0, a_1) \cong (\operatorname{ind}(g)^*((\mathfrak{A}_G, a_0, a_1)))_G :$$

$$\begin{aligned}(\operatorname{ind}(g)^*((\mathfrak{A}_G, a_0, a_1)))_G &= (\operatorname{ind}(g)^*((\bigcup \{p_\tau \mid p \in G\})))_G\\ &= ((\bigcup \{p_\tau \mid \operatorname{ind}(g)(p) \in G\}))_G\\ &= \bigcup \{p_\tau \mid \operatorname{ind}(g)(p) \in G\} = \bigcup \{(\operatorname{ind}(g^{-1})(p))_\tau \mid p \in G\},\end{aligned}$$

and $g^{-1} \cup \{(a_0, a_0), (a_1, a_1)\} : p_\tau \cong (\operatorname{ind}(g^{-1})(p))_\tau$.

(ii) is proved in a similar way. □

Now let $q \leqq^P p$ be given. We show

1. For all $r \in P$, if there is a $g \in \mathrm{Perm}(\omega_1, <\omega)$ such that $r \leqq^P \mathrm{ind}(g)(\mathrm{mirror}(p))$, then $r \in D$.

2. There are $r \leqq^P q$ and g with these properties.

Ad1: By the isomorphism lemma for forcing we have:

$$\mathrm{ind}(g)(\mathrm{mirror}(p)) \Vdash^P \psi_{\mathrm{sem}}(\check{y}, \mathrm{ind}(g)^*(\mathrm{mirror}^*((\mathfrak{A}_G, a_0, a_1))), \check{c}). \qquad (+)$$

2.9(i) applied to $\mathrm{mirror}^*((\mathfrak{A}_G, a_0, a_1))$ yields

$$\begin{aligned} 1^P \Vdash^P (g^{-1} \cup \{(a_0, a_0), (a_1, a_1)\})^\vee : \mathrm{mirror}^*((\mathfrak{A}_G, a_0, a_1)) \\ \cong \mathrm{ind}(g)^*(\mathrm{mirror}^*((\mathfrak{A}_G, a_0, a_1))). \end{aligned}$$

Together with 2.9(ii) we get

$$1^P \Vdash^P (g^{-1} \cup \{(a_0, a_0), (a_1, a_1)\})^\vee : (\mathfrak{A}_G, a_1, a_0) \cong \mathrm{ind}(g)^*(\mathrm{mirror}^*((\mathfrak{A}_G, a_0, a_1))).$$

Hence $(+)$ and the isomorphism property of $\models_{\mathscr{L}}$ give

$$\mathrm{ind}(g)(\mathrm{mirror}(p)) \Vdash^P \psi_{\mathrm{sem}}(\check{y}, (\mathfrak{A}_G, a_1, a_0), \check{c}).$$

Ad2: Take a $g \in \mathrm{Perm}(\omega_1, <\omega)$ such that $g''(w_q) \cap w_q = \emptyset$. Let $w' = g''(w_q) \cup w_q$. Then fix an $h \in \mathrm{Perm}(\omega_1, <\omega)$, such that $h''(w') \cap w' = \emptyset$. Let $p' := \mathrm{ind}(g)(\mathrm{mirror}(p))$. Then $g''(w_p) = w_{p'}$. Now take an r with $w_r = w_q \cup h''(w_q) \cup g''(w_p)$; let $\leqq_h$ be some linear ordering on $h''(w_q)$ and $\leqq_r = \leqq_q \cup \leqq_h \cup \leqq_{p'} \cup (w_q \times (h''(w_q) \cup g''(w_p))) \cup (h''(w_q) \times g''(w_p))$; f_r be a prolongation of $f_q \cup f_{p'}$ to $w_r \times w_r$ with new and pairwise different values for the new arguments, $h_r \supseteqq h_q \cup h_{p'}$. It is easy to see that $r \in P$. Now 2.6 is proved. $\square$

As a corollary, we get

Corollary 2.10. *Let $\mathscr{L}_2$ be an $\aleph_0$-definable logic with small syntax, c as in 2.5 a fixed element of V. If $\phi_{c_0 \mapsto\!\!\!\!\not\;\; c_1} \in \mathscr{L}_1$, then*

(i) $\mathrm{Cons}(\neg \mathrm{loc\,Hom}(\mathscr{L}_1, \mathscr{L}_2))$.

(ii) *If additionally the satisfiability of*

$$T_{\mathscr{L}_2} := \{\phi_{c_0 \mapsto\!\!\!\!\not\;\; c_1}\} \cup \{\chi(c_0) \leftrightarrow \chi(c_1) \mid \chi \in \mathscr{L}_2[\sigma \setminus \{c_0, c_1\}]\}$$

is absolute for c.c.c. forcings (see [Vä]*) then* $\neg \mathrm{loc\,Hom}(\mathscr{L}_1, \mathscr{L}_2)$.

Proof. (i) In $V[G]$, $((\mathfrak{A}_G, a_0, a_1), \phi_{c_0 \mapsto\!\!\!\!\not\;\; c_1})$ is a counterexample.

(ii) The satisfiability of $T_{\mathscr{L}_2}$ in $V[G]$ implies the satisfiability in V. $\square$

(ii) finishes the proof of Theorem 2.2. The same forcing technique and a suitable expansion of $(\mathfrak{A}_G, a_0, a_1)$ provide an example for $\neg \mathrm{loc\,Hom}(\mathscr{L}_{\omega\omega}(Q^{\mathrm{dense}}))$, where $Q^{\mathrm{dense}} xy\phi(x, y)$ says "ϕ is a dense linear ordering with a countable dense subset", see [Mil].

The main step for the next theorem is to show the existence of a model $(\mathfrak{A}, (u)_{u \in U^{\mathfrak{A}}}, a_0, a_1) \models \phi'_{c_0 \mapsto\!\!\!\!\not\;\; c_1}$ such that $(\mathfrak{A}, a_0) \equiv_{\mathscr{L}_{\infty\omega}(\mathbf{aa})} (\mathfrak{A}, a_1)$, where $\phi'_{c_0 \mapsto\!\!\!\!\not\;\; c_1} \in \mathscr{L}_{\omega_2\omega}$ satisfies $\mathrm{Mod}(\phi'_{c_0 \mapsto\!\!\!\!\not\;\; c_1}) | \sigma \subsetneqq \mathrm{Mod}(\phi_{c_0 \mapsto\!\!\!\!\not\;\; c_1})$.

Theorem 2.11. $\neg \mathrm{loc\,Hom}(\mathscr{L}_{\omega_2\omega}, \mathscr{L}_{\infty\omega}(\mathbf{aa}))$, *and therefore* $\neg \mathrm{loc\,Hom}(\mathscr{L}_{\infty\omega}(Q^{cf\omega}))$, $\neg \mathrm{loc\,Hom}(\mathscr{L}_{\infty\omega})$ *etc.*

Proof. Set

$$\begin{aligned}
T := &\{\phi_{c_0 \mapsto\!\!\!\!/ c_1}\} \\
&\cup \{\text{“}H \text{ is the graph of a binary function } h:(U\cup\{c_0,c_1\})^c \times U \mapsto \{\underline{0},\underline{1}\}\text{”} \\
&\wedge \forall x, y \notin (U\cup\{c_0,c_1\})\, \exists u \in U(H(x,u,\underline{0}) \wedge \neg H(y,u,\underline{0}))\} \\
&\cup \Big\{\forall \overset{n}{x}\Big(\bigwedge_{i<n} Ux_i \to (\chi(c_0,c_1,\overset{n}{x}) \leftrightarrow \chi(c_1,c_0,\overset{n}{x}))\Big)\Big| \\
&\qquad \chi(c_0,c_1,\overset{n}{x}) \in \mathscr{L}_{\omega\omega}(\mathbf{aa})[\sigma\cup\{H,\underline{0},\underline{1}\}]\Big\}.
\end{aligned}$$

By the proof of 2.10, and by absoluteness of satisfiability of countable $\mathscr{L}_{\omega\omega}(\mathbf{aa})$-theories T has a model. Starting with such a model the use of back-and-forth techniques as in [Ca] and a compactness argument yield a $(\mathfrak{B},(u)_{u\in U^{\mathfrak{B}}},c_0^{\mathfrak{B}},c_1^{\mathfrak{B}}) \models T$ such that

$$(\mathfrak{B},(u)_{u\in U^{\mathfrak{B}}},c_0^{\mathfrak{B}},c_1^{\mathfrak{B}}) \equiv_{\mathscr{L}_{\infty\omega}(\mathbf{aa})} (\mathfrak{B},(u)_{u\in U^{\mathfrak{B}}},c_1^{\mathfrak{B}},c_0^{\mathfrak{B}}).$$

There is an uncountable $F \subseteq {}^{U^{\mathfrak{B}}}\{0,1\}$ such that $(\mathfrak{B},(u)_{u\in U^{\mathfrak{B}}},c_0^{\mathfrak{B}},c_1^{\mathfrak{B}}) \models \phi'_{1,F}$, where $\phi'_{1,F} = \bigwedge_{f\in F} \exists x \notin U\cup\{c_0,c_1\} \bigwedge_{u\in U^{\mathfrak{B}}} H(x,\underline{u},\underline{f(u)})$. Let $\phi'_0 := \forall x\Big(Ux \to \bigvee_{u\in U^{\mathfrak{B}}} x=\underline{u}\Big)$ and $\phi'_{c_0 \mapsto\!\!\!\!/ c_1} := \phi'_0 \wedge \phi'_{1,F} \wedge \bigwedge_{i=2,3,4} \phi_i$. Then $\mathrm{Mod}(\phi'_{c_0 \mapsto\!\!\!\!/ c_1})|\sigma \subseteq \mathrm{Mod}(\phi_{c_0 \mapsto\!\!\!\!/ c_1})$, and hence $((\mathfrak{B},(u)_{u\in U^{\mathfrak{B}}},c_0^{\mathfrak{B}},c_1^{\mathfrak{B}}),\phi'_{c_0 \mapsto\!\!\!\!/ c_1})$ forms a counterexample. □

3 A positive result

Let Γ be a class of regular cardinals. $Q^{cf\Gamma}xy\phi(x,y,\bar{c})$ means $\phi(\cdot,\cdot,\bar{c})$ is a linear ordering whose right cofinality is in Γ. In this section the homogeneity property of the cofinality logic $\mathscr{L}_{\omega\omega}(Q^{cf\Gamma})$ is shown.

The proof proceeds in two steps. In the first one, for a given $(\mathfrak{A},a_0) \equiv_{\mathscr{L}_{\omega\omega}(Q^{cf\omega})} (\mathfrak{A},a_1)$ with $\tau_{\mathfrak{A}}$ countable, we give a notion of forcing such that for generic G, in $V[G]$ there is a model $(\mathfrak{B},b_0,b_1) \equiv_{\mathscr{L}_{\omega\omega}(Q^{cf\omega})} (\mathfrak{A},a_0,a_1)$ with $(\mathfrak{B},c_0^{\mathfrak{B}}) \equiv_{\mathscr{L}_{\omega\omega}(\mathbf{aa})} (\mathfrak{B},c_1^{\mathfrak{B}})$. Then we apply Shelah's result on $\mathrm{Hom}(\mathscr{L}_{\omega\omega}(Q^{cf\omega}),\mathscr{L}_{\omega\omega}(\mathbf{aa}))$ [Sh 85, Sects. 5, 6], compactness of $\mathscr{L}_{\omega\omega}(Q^{cf\Gamma})$ and the transfer theorems of [Sh 72] to get $\mathrm{Hom}(\mathscr{L}_{\omega\omega}(Q^{cf\Gamma}))$.

Theorem 3.1. *Let* $(\mathfrak{A},a_0) \equiv_{\mathscr{L}_{\omega\omega}(Q^{cf\omega})} (\mathfrak{A},a_1)$, $\tau_{\mathfrak{A}} = \tau$ *be countable. Then* $T := \mathrm{Th}_{\mathscr{L}_{\omega\omega}(Q^{cf\omega})}((\mathfrak{A},a_0,a_1)) \cup \{\phi(\underline{a_0}) \leftrightarrow \phi(\underline{a_1}) \mid \phi(x) \in \mathscr{L}_{\omega\omega}(\mathbf{aa})[\tau]\}$ *is satisfiable.*

Proof. The methods of Proposition 2.1 in [Me-Sh] provide an $\mathscr{L}_{\omega\omega}(Q^{cf\omega})$-$\omega$-homogeneous model $(\mathfrak{C},c_0,c_1) \equiv_{\mathscr{L}_{\omega\omega}(Q^{cf\omega})} (\mathfrak{A},a_0,a_1)$ such that for any $n\in\omega$ there are only countably many $\mathscr{L}_{\omega\omega}(Q^{cf\omega})[\tau]$-$n$-types over $\emptyset$ realized in $(\mathfrak{C},c_0,c_1)$. We fix such a $(\mathfrak{C},c_0,c_1)$. W.l.o.g. let τ contain an n-ary relation $R(\phi)$ for any $\mathscr{L}_{\omega\omega}(Q^{cf\omega})[\tau]$-definable relation $\phi(\overset{n}{x})$. There is a countable $\mathscr{L}_{\omega\omega}$-$\omega$-homogeneous substructure $\mathfrak{D} \prec_{\mathscr{L}_{\omega\omega}} \mathfrak{C}$ with the following properties:

1. All $\mathscr{L}_{\omega\omega}[\tau]$-types over $\emptyset$, that are realized in $\mathfrak{C}$, are realized in $\mathfrak{D}$, too.

2. For any $n\in\omega$, $\phi(\bar{z},x,y) \in \mathscr{L}_{\omega\omega}(Q^{cf\omega})$, $\overset{n}{c} \in D$: If $\mathfrak{C} \models R(Q^{cf\omega}xy\phi(\bar{z},x,y))[\overset{n}{c}]$, then $(R(\phi)[\overset{n}{c},\cdot,\cdot])^{\mathfrak{D}}$ is a cofinal suborder of $(R(\phi)[\overset{n}{c},\cdot,\cdot])^{\mathfrak{C}}$.

Taking such a $\mathfrak{D}$, we define a forcing $\mathbf{P} = (P, \leqq^P, 1^P)$ by

$\mathbf{P} = (P, \leqq^P, 1^P)$, where $P := \{\mathfrak{M} \mid \mathfrak{M}$ is a τ-structure, M is a countable limit ordinal and $\mathfrak{M} \cong \mathfrak{D}\} \cup \{1\}$;

$1^P := 1$;

for $\mathfrak{M},\mathfrak{N}\in P$ let $\mathfrak{M}\leqq^P\mathfrak{N}$ iff $\mathfrak{N}=1$ or $\mathfrak{N}=\mathfrak{M}$ or the following conditions are true: $\mathfrak{N}\prec_{\mathscr{L}_{\omega\omega}}\mathfrak{M}$ and for any $\bar{\alpha}\in N$, $\phi\in\mathscr{L}_{\omega\omega}(Q^{cf\omega})$ the conditions (i) and (ii) are equivalent, where:

(i) $(R(\phi)[\bar{\alpha},\cdot,\cdot])^{\mathfrak{N}}$ is a linear order without last element, and there is a $\gamma\in M\setminus N$, such that $(R(\phi)[\bar{\alpha},\beta,\gamma])^{\mathfrak{M}}$ for any β in the field of $(R(\phi)[\bar{\alpha},\cdot,\cdot])^{\mathfrak{N}}$.

(ii) $\mathfrak{N}\models R(\neg Q^{cf\omega}xy\phi(\bar{z},x,y)\wedge$ "$\phi(\bar{z},\cdot,\cdot)$ is a linear order without last element")$[\bar{\alpha}]$.

Let G be **P**-generic over V, and in $V[G]$ define $\mathfrak{B}_G:=\bigcup G$, the union over the semilattice of structures in G. We shall show that for suitable b_0,b_1, in $V[G]$, $(\mathfrak{B}_G,b_0,b_1)$ is a model with the desired properties. Since satisfiability of countable $\mathscr{L}_{\omega\omega}$(**aa**)-theories is absolute (we consider T as an $\mathscr{L}_{\omega\omega}$(**aa**)-theory, see [4.4 in Sh 85]), 3.1 will be proved.

Because of the density of $D_\alpha:=\{\mathfrak{M}\in P\mid\alpha\in M\}$, $\alpha\in\omega_1$, $\bigcup G$ has support ω_1, which is equal to $\omega_1^{V[G]}$, as **P** is ω-closed.

An induction over $\phi\in\mathscr{L}_{\omega\omega}(Q^{cf\omega})$ shows that for any $\bar{\alpha}\in\omega_1$ we have:

$$(\mathfrak{B}_G\models\phi[\bar{\alpha}])^{V[G]}$$
$$\text{iff there is an }\ \mathfrak{M}\in G,\ \text{ such that }\ \bar{\alpha}\in M\ \text{ and }\ \mathfrak{M}\models R(\phi)[\bar{\alpha}].$$

The crucial $Q^{cf\omega}$-step is based on the definition of $\leqq^P$.

There are $d_0,d_1\in D$ such that $\mathfrak{D}\models\{R(\phi)[d_0,d_1]\mid\phi$ such that $\mathfrak{A}\models\phi[a_0,a_1]\}$. Therefore, there are $b_0,b_1\in\omega_1$, such that $(\mathfrak{B}_G,b_0,b_1)\equiv_{\mathscr{L}_{\omega\omega}(Q^{cf\omega})}(\mathfrak{A}_G,a_0,a_1)$ in $V[G]$. For any such b_0,b_1 we have $(\mathfrak{B}_G,b_0)\equiv_{\mathscr{L}_{\omega\omega}(Q^{cf\omega})}(\mathfrak{B}_G,b_1)$. The fact $(\mathfrak{B}_G,b_0)\equiv_{\mathscr{L}_{\omega\omega}(\mathbf{aa})}(\mathfrak{B}_G,b_1)$ now follows from:

Lemma 3.2. *In $V[G]$, for any $\overline{\gamma_0},\overline{\gamma_1}\in\omega_1^{<\omega}$: If $(\mathfrak{B}_G,\overline{\gamma_0})\equiv_{\mathscr{L}_{\omega\omega}(Q^{cf\omega})}(\mathfrak{B}_G,\overline{\gamma_1})$ then $(\mathfrak{B}_G,\overline{\gamma_0})\equiv_{\mathscr{L}_{\omega\omega}(\mathbf{aa})}(\mathfrak{B}_G,\overline{\gamma_1})$.*

Proof. Let $(\mathfrak{B}_G,\overline{\gamma_0})\equiv_{\mathscr{L}_{\omega\omega}(Q^{cf\omega})}(\mathfrak{B}_G,\overline{\gamma_1})$. We take $\mathfrak{M}_1\in G$ such that $\mathfrak{M}_1\Vdash^{\underline{P}}(\dot{\mathfrak{B}}_G,\overline{\gamma_0})\equiv_{\mathscr{L}_{\omega\omega}(Q^{cf\omega})}(\dot{\mathfrak{B}}_G,\overline{\gamma_1})\wedge\mathfrak{B}_G\models\psi[\overline{\gamma_0}]$ and $\overline{\gamma_0},\overline{\gamma_1}\in M_1$. By the last lemma, $(\mathfrak{M}_1,\overline{\gamma_0})\equiv_{\mathscr{L}_{\omega\omega}}(\mathfrak{M}_1,\overline{\gamma_1})$, and because of the homogeneity, there is an h,

$$h:(\mathfrak{M}_1,\overline{\gamma_0})\cong(\mathfrak{M}_1,\overline{\gamma_1}).$$

h induces an automorphism of

$$P_1:=\{\mathfrak{M}\in P\mid\mathfrak{M}\leqq^P\mathfrak{M}_1\}$$

by stipulating $h_1=h\cup\mathrm{id}_{M\setminus M_1}$ and

$$\hat{h}(\mathfrak{M})\cong\mathfrak{M}\quad\text{via } h_1.$$

The second claim in 2.1 in [Me-Sh] yields for any $\phi\in\mathscr{L}_{\omega\omega}(\mathbf{aa})$ and $\mathfrak{M}\in P_1$:

$$\mathfrak{M}\Vdash^{\underline{P_1}}\dot{\mathfrak{B}}_G\models\phi[\bar{\gamma}]\Leftrightarrow\hat{h}(\mathfrak{M})\Vdash^{\underline{P_1}}\dot{\mathfrak{B}}_G\models\phi[h_1(\bar{\gamma})].$$

Since $\hat{h}(\mathfrak{M}_1)=\mathfrak{M}_1$ we get $\mathfrak{M}_1\Vdash^{\underline{P}}\dot{\mathfrak{B}}_G\models\psi[\overline{\gamma_1}]$ and $\mathfrak{B}_G\models\psi[\overline{\gamma_1}]$. $\square$

Theorem 3.3. $\mathrm{Hom}(\mathscr{L}_{\omega\omega}(Q^{cf\omega}),\mathscr{L}_{\omega\omega}(\mathbf{aa}))$.

Theorem 3.3 can be proved along the lines of [Sh 85, Sects. 5, 6]. A more detailed elaboration is given in [Mil]. Indeed, $(\mathscr{L}_{\omega\omega}(Q^{cf\omega}),\mathscr{L}_{\omega\omega}(\mathbf{aa}))$ has a stronger property: One may replace the points a_0,a_1 in the homogeneity property by arbitrarily long strings of relations $\bar{R},\bar{S}$ of the same arities: If $(\mathfrak{A},\bar{R}^{\mathfrak{A}})\equiv_{\mathscr{L}_{\omega\omega}(Q^{cf\omega})}(\mathfrak{A},\bar{S}^{\mathfrak{A}})$, then there is a $(\mathfrak{B},\bar{R}^{\mathfrak{B}},\bar{S}^{\mathfrak{B}})\equiv_{\mathscr{L}_{\omega\omega}(Q^{cf\omega})}(\mathfrak{A},\bar{R}^{\mathfrak{A}},\bar{S}^{\mathfrak{A}})$ with $(\mathfrak{B},\bar{R}^{\mathfrak{B}})\cong(\mathfrak{B},\bar{S}^{\mathfrak{B}})$. $\mathscr{L}_{\omega\omega}(Q^{cf\omega})$ does not have this stronger property, as can be shown by a modification of a counterexample to the Robinson property of $\mathscr{L}_{\omega\omega}(Q^{cf\omega})$.

Theorem 3.4. $\mathrm{Hom}(\mathscr{L}_{\omega\omega}(Q^{cf\Gamma}))$.

Proof. Let $(\mathfrak{A}, a_0) \equiv_{\mathscr{L}_{\omega\omega}(Q^{cf\Gamma})} (\mathfrak{A}, a_1)$. We have to show that

$$T := \mathrm{Th}_{\mathscr{L}_{\omega\omega}(Q^{cf\Gamma})}(\mathfrak{A}, a_0, a_1) \cup \{f(\underline{a_0}) = \underline{a_1}\} \cup \{\text{“}f \text{ is an automorphism”}\}$$

is satisfiable, where $f \notin \tau_{\mathfrak{A}}$ is a symbol for a unary function. By compactness it suffices to show that any countable subset T' of T has a model. Given such a T', we have $T' \subset \mathscr{L}_{\omega\omega}(Q^{cf\Gamma})[\sigma \cup \{f\}]$ for a suitable countable $\sigma \subsetneqq \tau$. We use the transfer theorem in [Sh 72] to get a model $(\mathfrak{C}, c_0, c_1)$ of $\mathrm{Th}_{\mathscr{L}_{\omega\omega}(Q^{cf\Gamma})}(\mathfrak{A}, a_0, a_1)$ in the ω-interpretation. By 3.2 and 3.3 we can assume that there is also a model $(\mathfrak{B}, b_0, b_1)$ of $\mathrm{Th}_{\mathscr{L}_{\omega\omega}(Q^{cf\omega})}((\mathfrak{C}, c_0, c_1)|\sigma)$ with an automorphism f of the required kind. A further application of the transfer theorem leads to a model of $\mathrm{Th}_{\mathscr{L}_{\omega\omega}(Q^{cf\omega})}((\mathfrak{B}, b_0, b_1, f))$ in the Γ-interpretation, i.e. to a model to T'. $\square$

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