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# More canonical forms and dense free subsets

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## Abstract

Assuming the existence of  $\omega$  compact cardinals in a model on GCH, we prove the consistency of some new canonization properties on  $\aleph_\omega$ . Our aim is to get as dense patterns in the distribution of indiscernibles as possible. We prove Theorem 2.1.

**Theorem 2.1.** *Suppose the consistency of “ZFC + GCH + there are infinitely many compact cardinals”. Then the following is consistent: ZFC + GCH + and for every family  $(f_n)_{0 < n < \omega}$  of functions on  $\aleph_\omega$  such that  $f_n$  is  $n$ -ary and regressive, there are sets  $S_n, 0 < n < \omega$ , such that for all  $0 < n < \omega, S_n \subseteq [\aleph_n, \aleph_{n+1}, |S_n| \geq \aleph_{n-1}$ , and  $f_n \upharpoonright (\prod_{i=1}^n S_i)$  is constant.*

We generalize this to higher arities, and find that the following is consistent relatively to the same large cardinal assumptions: Given a family of regressive functions  $(f_n)_{0 < n < \omega}$  on  $\aleph_\omega$  and a function  $r: \omega \rightarrow \omega$ , there is a family of sets  $(S_n)_{0 < n < \omega}$  that have a certain size and that are indiscernible for values under  $f_n$  for all  $0 < n < \omega$  simultaneously, if  $f_n$  picks  $r(m)$  increasing arguments from  $S_m$  for  $0 < m \leq n$ . We determine the locations of the sets  $S_n$  in  $\aleph_\omega$ . This, together with some additional work on indiscernibility over as many smaller parameters as possible, yields the consistency of the existence of free subsets with at least one point in every infinite cardinal interval of  $\aleph_\omega$ .

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## 0. Introduction

This work continues Devlin [1], Shelah [9–11], Koepke [6,7], our own [8], and is motivated by a long-standing question of Komjáth on the location of members of free subsets on the ordinal line.

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In this work, we strengthen a generalization of Shelah's result of [10] by inserting more sets of indiscernibles. This will be Theorem 1.2. It includes reworking Shelah's [10] and filling in numerous details, in particular, separating the various meanings of  $n$  and assigning different names. In Theorem 1.9 we shall prove the consistency of a different canonization property, which works with a polarized partition property instead of the Erdős Rado theorem.

In Section 2, we continue proving consistency results for canonical forms. We work with not so strictly separated Levy collapses, which necessitates numerous cut-offs and naming conditions. We shall show that the following is consistent: For suitable regressive functions there is a sequence  $\langle S_n : n \geq 1 \rangle$  of  $\langle 1 \rangle^{(\omega)}$ -indiscernibles such that  $S_n \subseteq [\aleph_n, \aleph_{n+1})$  and  $|S_n| \geq \aleph_{n-1}$ . This result is sufficient to get the dense free subsets up to one minor blemish: there is no member of a free subset in the interval  $[\aleph_0, \aleph_1)$ . Then we prove two versions with higher arities. Working with indiscernibles for expressions that allow all ordinals before the cardinal predecessor of the smallest variable as constants allows us to remedy the mentioned flaw and also to get dense free subsets from indiscernibles with higher arities.

In Section 3, we show how to get free subsets with specified locations from the canonization properties. We recall the known transition and then apply it to our canonization theorems. Then we improve the transition procedure in order to show how to get free subsets with one point in each interval from the higher-dimensional canonization theorems.

*Notation:* Natural numbers are denoted by  $j, k, \ell, m, n, r$  ordinals by  $\alpha, \beta, \gamma, \xi, \zeta, \eta, \nu$ , cardinals by  $\lambda, \kappa, \mu, \chi$ . We define  $\beth_0(\lambda) = \lambda$ , and  $\beth_\alpha(\lambda) = \sum_{\beta < \alpha} 2^{\beth_\beta(\lambda)}$  for  $\alpha > 0$ . Let  $\lambda^{<\mu} = \sum_{\kappa < \mu} \lambda^\kappa$ . For a linear order  $<$  on  $A$  and  $B, C \subseteq A$  and  $a \in A$ , we write  $a < B$  if  $(\forall x \in B)(a < x)$  and  $B < C$  if  $(\forall b \in B)(\forall c \in C)(b < c)$ . For a finite sequence  $\bar{\gamma}$  of ordinals  $\bar{\gamma} < \alpha$  means that all entries of the sequence are smaller than  $\alpha$ .  $\langle 1 \rangle^{(r)}$  denotes the sequence  $\langle 1, \dots, 1 \rangle$  of length  $r$ . For forcing conditions  $p$  and  $q$ ,  $p \leq q$  means that  $q$  is the stronger condition. So, we follow the Israeli convention. But we also try to stick to the alphabetical convention (see [3]) that the condition named by a later letter will be the stronger one. For a regular cardinal  $\mu$ , we say that a forcing notion  $P$  is  $\mu$ -complete, if every ascending chain of length  $< \mu$  has an upper bound. We say that a forcing notion  $P$  is  $\mu$ -directly closed, if every directed subset of size  $< \mu$  has an upper bound. A filter  $D$  is  $\mu$ -complete, if every subset of it of size  $< \mu$  has an intersection in  $D$ . "Countably complete" means  $\aleph_1$ -complete. Filters may be improper, i.e. contain the empty set, but when we say "ultrafilter" we mean a proper ultrafilter. A compact cardinal is a regular cardinal  $\kappa$  such that for every  $\mu \geq \kappa$  for every  $\kappa$ -complete proper filter  $F$  on  $\mu$  there is a  $\kappa$ -complete ultrafilter above it. Indeed, we use this only for  $\mu < \kappa^{+\omega}$ . We also use a  $\kappa$ -complete normal ultrafilter over  $\kappa$  (see [5]).

## 1. Canonization theorems

The following definition is a generalization of Erdős' and Rado's polarized partition properties to products of infinitely many factors and infinitely many functions working on suitable parts of the products. The arity of each single function is finite, but the

homogeneous sets will be the same for many functions, that can be seen together, so that the union of their domains contains some  $[A]^{<\omega}$  and such that there is a homogenized subset of this which contains numerous finite compatible sequences, and is a large part of some  $[B]^{<\omega}$  for some  $B$  that lies cofinal in  $\aleph_\omega$ .

*History:* Shelah introduced and worked with the canonization properties from Definition 1.1 and related notions in [9–11]. We do not change the content of this pattern with several parameters, but write the tuples in slightly different form and add some explanations. We will work with instances of the definition whose consistency is new.

Let  $\langle \lambda(\xi) : \xi < \theta \rangle$  and  $\langle \kappa(\xi) : \xi < \theta \rangle$  be two increasing sequences of length  $\theta \in \text{On}$  of cardinals. We assume that for every  $\xi$ ,  $\lambda(\xi) \geq \kappa(\xi)$ . The picture is: For  $\xi \in \theta$  we are given a set  $A_\xi$  of size  $\lambda(\xi)$ . We assume that  $A_\xi \subseteq \text{On}$  and that for  $\xi < \xi' < \theta$ ,  $A_\xi < A_{\xi'}$ . We want to find  $S_\xi \subseteq A_\xi$ ,  $\xi < \theta$ , such that  $|S_\xi| \geq \kappa(\xi)$  and such that all tuples from a certain combination of the  $S_\xi$ 's are indiscernible under the functions  $f_i$  simultaneously for  $i < \alpha$ .

First we describe the arity of each  $f_i$ , its range, and the number of coordinates which will not influence its value if  $f_i$  is restricted to a certain combination of the  $S_\xi$ 's. So we assign to  $f_i$  a quadruple  $(\bar{r}(i), \ell(i), \chi(i), \bar{\xi}(i))$  such that  $\chi(i)$  is a non-zero cardinal, and  $\bar{r}(i) = \langle r_1(i), \dots, r_k(i) \rangle$ ,  $r_j(i)$ ,  $1 \leq j \leq k$ , and  $\ell(i)$  are natural numbers, and for each  $\bar{r} = \langle r_1, \dots, r_k \rangle$  we denote  $w(\bar{r}) = \sum_{j=1}^{k(i)} r_j$  (think of weight),  $k(\bar{r}) = k$ ,  $\ell(i) \leq w(\bar{r}(i))$ .  $f_i$  will take an increasing argument of arity  $w(\bar{r}(i))$ , which is subdivided into arities  $r_j(i)$ ,  $j = 1, \dots, k(i)$ . Also  $\bar{\xi}(i) = \langle \xi_1(i), \dots, \xi_{k(i)}(i) \rangle$  is a strictly increasing sequence of elements of  $\theta$ . The argument of  $f_i$  in the  $j$ -block ranges over  $[A_{\xi_j(i)}]^{r_j(i)}$ , and under certain premises  $f_i$  will be independent of the arguments in the  $j$ th block when they range over  $[S_{\xi_j(i)}]^{r_j(i)}$ . In Section 1 and in the first part of Section 2  $i$  will be a pair  $(n, m)$ ,  $1 \leq n \leq m < \theta = \omega$ , and in the last part of Section 2  $i$  will consist of tuples, because this allows an easy description where and how  $f_i$  is working.

We shall ask for indiscernibility only in the last  $\ell(i)$  arguments, starting from the last block of arguments and going to earlier arguments until  $\ell(i)$  arguments are named. Of course one could think of a tuple of numbers that describe the indiscernible arguments in each block of  $\bar{r}(i)$  separately, but this imagined property can probably be proved only in situations where the property with fewer parameters (i.e. those in Definition 1.1, and the “fewer” is not a joke) is true.

**Definition 1.1.**  $\langle \lambda(\xi) : \xi < \theta \rangle$  has  $\langle \kappa(\xi) : \xi < \theta \rangle$ -canonical forms for  $\Gamma = \langle (\bar{r}(i), \ell(i), \chi(i), \bar{\xi}(i)) : i < \alpha \rangle$  if for every sequence  $\langle A_\xi : \xi < \theta \rangle$  such that  $|A_\xi| = \lambda(\xi)$  and  $A_\xi \subseteq \text{On}$  and  $A_\xi < A_\eta$  for  $\xi < \eta < \theta$  and every sequence  $\langle f_i : i < \alpha \rangle$ ,

$$f_i : \prod_{j=1}^{k(i)} [A_{\xi_j(i)}]^{r_j(i)} \rightarrow \chi(i),$$

there are  $S_\xi \subseteq A_\xi$  such that  $|S_\xi| = \kappa(\xi)$  such that for every  $i < \alpha$ ,  $f_i$  is  $(\bar{r}(i), \ell(i), \chi(i), \bar{\xi}(i))$ -canonical on  $\langle S_\xi : \xi < \theta \rangle$ . This means for every  $i < \alpha$ ,  $\bar{a}$ , if

$$\begin{aligned} \bar{\xi}(i) &= \langle \xi_j(i) : j < k(i) \rangle, \\ \xi_1(i) &< \dots < \xi_{k(i)}(i) < \theta, \\ a_1 &< \dots < a_{r_1(i)} \in S_{\xi_1(i)}, \end{aligned}$$

$$\begin{aligned} a_{r_1(i)+1} &< \cdots < a_{r_1(i)+r_2(i)} \in S_{\xi_2(i)}, \\ &\vdots \\ a_{\left(\sum_{j \leq k(i)-1} r_j(i)\right)+1} &< \cdots < a_{\sum_{j \leq k(i)} r_j(i)} \in S_{\xi_{k(i)}(i)}. \end{aligned}$$

then  $f_i(a_1, \dots, a_{w(\bar{r}(i))})$  depends only on  $\xi_1(i), \dots, \xi_{k(i)}(i), a_1, \dots, a_{w(\bar{r}(i))-\ell(i)}$  and not on  $a_{w(\bar{r}(i))-\ell(i)+1}, \dots, a_{w(\bar{r}(i))}$ . In case  $\ell(i) = w(\bar{r}(i))$  the independence means that  $f_i \upharpoonright \prod_{j=1}^{k(i)} [S_{\xi_j(i)}]^{r_j(i)}$  is constant.

**Remark.** In our setting, we wrote  $\overline{\xi(i)}$  as a parameter. In Shelah's work, it is often quantified over all  $\overline{\xi}$  such that

$$\chi(i) < \kappa(\xi_1(i)) \tag{1.1}$$

or something similar, containing exponentiations. Since we do not want to assume that  $r$  is an increasing function, we specified on which cells  $S_{\xi_j(i)}$ ,  $j=0, \dots, k(i)-1$ , the function  $f_i$  works, and in which arities it works. In some work by Shelah [10], the range over which  $\overline{\xi(i)}$  runs is not mentioned explicitly, but tacitly taken from [9]. As in [10], also in this work  $\ell(i)$  will always be maximal, that is  $\ell(i) = w(\bar{r}(i))$ .

In our applications  $\alpha$  will not be quite an ordinal: In Section 1 and in the first part of Section 2,  $i$  will range over  $I = \{i = (n, m) : 1 \leq m \leq n < \omega\}$ , and there will be one function  $\langle r(n) : 1 \leq n < \omega \rangle$  such that for all  $i = (n, m)$ ,  $\bar{r}(i) = (r(n), \dots, r(m))$ .  $\theta$  will be  $\omega$  and  $A_n = [\aleph_{k_1(n)-1}, \aleph_{k_1(n)})$ , and  $\langle k_1(n) : 1 \leq n < \omega \rangle$  will be chosen possibly very small, depending on  $r$  and on the difference between  $k_2(n)$  and  $k_1(n)$ .

We set  $\overline{\xi(i)} = \langle n, \dots, m \rangle$ . So the aim is to find large subsets  $S_n$  of the  $A_n$  such that  $1 \leq n \leq m < \omega$ ,  $f_{(n,m)} \upharpoonright [S_n]^{r(n)} \times \cdots \times [S_m]^{r(m)}$  be constant.

In this situation, we call  $\langle S_n : 1 \leq n < \omega \rangle$   $\langle r_n : 1 \leq n < \omega \rangle$ -indiscernibles for  $f_i$ ,  $i \in I$ .

A word on consistency strengths: The sets of indiscernibles for  $f_{(1,m)}$ ,  $m \geq 1$ , will be used in the application to free subsets and the running  $m$  are one be the source for the strength of the instances of canonization we look at. For  $r(n) = 1$  for all but finitely many  $n$ , the known lower bound on the strength of the results of all our theorems is one measurable [6], for larger  $r$  the best-known lower bound is  $o^{\text{long}}$  [7], and this is derived already from  $r(n) = 2$  and  $|S_n| \geq 3$  for infinitely many  $n$ .

For the last theorem of Section 2,  $i$  will even range over some triples. Combinatorially this is not much harder, however, it is better suitable for finding free subsets (see Theorem 3.7).

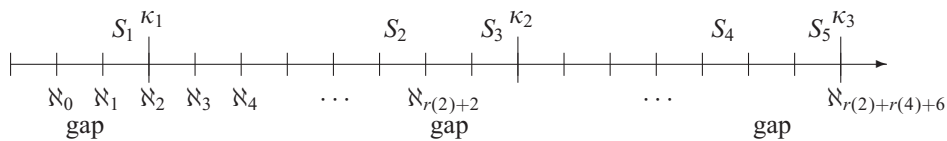
In order to avoid too many shifts, the indexing of sets to be thinned out starts only with 1. We will work with countably closed filters and thus the search for homogeneous sets starts only with a normal filter on the first compact cardinal  $\kappa_1$ , and we do not homogenize below  $\kappa_0 = \omega$ . Since for technical reasons all forcings have to be  $\omega_1$ -closed for our proof, the first part of the forcing will collapse all cardinals below the first compact cardinal to  $\aleph_1$  with countable conditions, and the first compact cardinal will hence be  $\aleph_2$  in the extension. Now, in Theorem 1.2, we will thin out starting with  $A_n$  and thin them out to  $\langle S_n : 0 < n < \omega \rangle$ . Shelah's original step was from the  $A_{2n}$  to the  $S_{2n}$  in our indexing, and there where no sets of indiscernibles with odd

indices. The generalization from  $n$ , the exponent in the  $n$ th block in Shelah’s work, to  $r(n)$  for an arbitrary function  $r : \omega \rightarrow \omega$  comes from separating the different meanings of  $n$ . Between any two sets  $S_{2n}, S_{2n+2}$  of indiscernibles there will be another type of set  $S_{2n+1}$  of indiscernibles. On  $S_{2n+1}$ , the arity for indiscernibility is  $r_{2n+1} = 1$  for each  $n$ .

**Theorem 1.2.** *Suppose the consistency of “ZFC + GCH + there are infinitely many compact cardinals”. Let  $r : \omega \setminus \{0\} \rightarrow \omega \setminus \{0\}$  such that for every  $n$ ,  $r(2n+1) = 1$ . Then the following is consistent: ZFC + GCH +  $\langle \aleph_{k_1(n)} : 0 < n < \omega \rangle$  has  $\langle \aleph_{k_2(n)} : 0 < n < \omega \rangle$ -canonical forms for*

$$\begin{aligned} \Gamma &= \langle \langle (r(n), \dots, r(m)), \ell(i), \aleph_{k_2(n)-1}, \langle n, n+1, \dots, m \rangle \rangle \\ &\quad i = (n, m), 1 \leq n \leq m < \omega \rangle, \\ \ell(i) &= r(n) + r(n+1) + \dots + r(m), \\ k_1(2n) &= \sum_{1 \leq j \leq n} (r(2j) + 2), \\ k_2(2n) &= \sum_{1 \leq j < n} (r(2j) + 2) + 3, \\ k_1(2n-1) &= k_1(2n-2) + 2, \quad k_1(0) = 0, \\ k_2(2n-1) &= k_1(2n-1) - 2. \end{aligned} \tag{1.2}$$

**Proof.** Since the indexing and the statement of the theorem are complex, we draw a picture which indicates the location of the  $\aleph_{k_1(n)}$ ’s and the  $S_n$ ’s.



We draw the  $S_{2n}$  towards the end of their interval. There are empty intervals which we do not call gaps because the  $S_{2n}$  are for arity  $r(2n)$  and hence under additional premises can be spread out over an interval of  $r(2n)$  cardinal steps. We will do more work on this in Section 3. Shelah’s work [10] gives a picture as the one above with gaps instead of  $S_{2n+1}$ ,  $n < \omega$ , and  $r(2n) = n$ .

In Shelah’s work,  $r(2\ell) = \ell$ , and there are no  $r(2\ell - 1)$ . For the proof, we first write some definitions and claims separately.

**Definition 1.3.** For cardinals  $\lambda, \mu, \chi$  and  $r < \omega$  let  $D_r(\lambda, \mu, \chi)$  be the following filter:

- (a) It is a filter over the set  $\text{Inc}(\lambda, \mu)$  of increasing sequences of length  $\mu$  of ordinals  $< \lambda$ .

(b) The filter is generated by the set of generators, where a generator is

$$\begin{aligned} Ge(F) &= Ge_r(F; \lambda, \mu, \chi) \\ &= \{\bar{a} \in \text{Inc}(\lambda, \mu) : (\exists \alpha \in \chi) \forall i(0) < \dots < i(r-1) < \mu, \\ &\quad F(a_{i(0)}, \dots, a_{i(r-1)}) = \alpha\}, \end{aligned}$$

for some  $F : [\lambda]^r \rightarrow \chi$ .

**Claim 1.4.** (1) If  $\chi = \chi^{<\kappa}$ , then the intersection of  $<\kappa$  generators of  $D_r(\lambda, \mu, \chi)$  is a generator; hence the filter  $D_r(\lambda, \mu, \chi)$  is  $\kappa$ -complete.

(2) If  $\lambda \rightarrow (\mu)_\chi^r$  (the usual partition relation) then  $D_r(\lambda, \mu, \chi)$  is a proper filter.

Let  $\ell \geq 1$ . The objects indexed by  $\ell$  will be used for finding  $S_{2\ell-1}$  and  $S_{2\ell}$ . Let  $E_\ell$  be a normal ultrafilter over  $\kappa_\ell$ . Let  $I_\ell = \text{Inc}(\kappa_\ell^{+(2\ell)}, \kappa_\ell^{+1})$  and  $J_\ell = \kappa_\ell \times I_\ell$ . Note that  $D_{r(2\ell)}(\kappa_\ell^{+(2\ell)}, \kappa_\ell^{+1}, \kappa_\ell)$  is a  $\kappa_\ell$ -complete proper filter, as the Erdős Rado Theorem  $(\square_{r(2\ell)-1}(\kappa_\ell))^+ \rightarrow (\kappa_\ell^+)^{r(2\ell)}_{\kappa_\ell}$  together with the GCH yield  $\kappa_\ell^{+(2\ell)} \rightarrow (\kappa_\ell^+)^{r(2\ell)}_{\kappa_\ell}$ . So, as  $\kappa_\ell$  is compact there is a  $\kappa_\ell$ -complete ultrafilter  $D_\ell^*$  over  $I_\ell$  extending  $D_{r(2\ell)}(\kappa_\ell^{+(2\ell)}, \kappa_\ell^{+1}, \kappa_\ell)$ . We set

$$F_\ell = E_\ell \times D_\ell^* = \{A \subseteq J_\ell : \{i < \kappa_\ell : \{t \in I_\ell : (i, t) \in A\} \in D_\ell^*\} \in E_\ell\}.$$

We call  $f : J_\ell \rightarrow \kappa_\ell$  regressive if  $(\forall 0 < \alpha < \kappa_\ell) f(\alpha, t) < \alpha$ . We call it regressive on  $A$  if  $f(\alpha, t) < \alpha$  for  $(\alpha, t) \in A$ , and almost regressive if it is regressive on some  $A \in F_\ell$ . Similarly we define when  $f$  is constant, constant on  $A$  and almost constant.

**Claim 1.5** (Shelah). *Every almost regressive function  $f : J_\ell \rightarrow \kappa_\ell$  is almost constant.*

**Proof.** Let  $f$  be regressive on  $B \in F_\ell$ . Let  $B_\alpha = \{t \in I_\ell : (\alpha, t) \in B\}$ . So, by the definition of  $F_\ell$ , there is some  $B' \in E_\ell$  such that for every  $\alpha \in B'$ ,  $B_\alpha \in D_\ell^*$ .

For each  $\alpha \in B'$ , we let  $A_\beta^\alpha = \{t \in B_\alpha : f(\alpha, t) = \beta\}$ , and thus get a partition  $\{A_\beta^\alpha : \beta < \alpha\}$  of  $B_\alpha$  into  $|\alpha| < \kappa_\ell$  parts. As  $D_\ell^*$  is  $\kappa_\ell$ -complete,  $B_\alpha \in D_\ell^*$ , there is for some  $\beta = h(\alpha) < \alpha$ , such that  $A_{h(\alpha)}^\alpha \in D_\ell^*$ .

So  $h$  is a regressive function on  $B'$ . Since  $B' \in E_\ell$  and  $E_\ell$  is normal, there is some  $\gamma < \kappa_\ell$  such that  $\{\alpha : h(\alpha) = \gamma\} \in E_\ell$ . By the definition of  $F_\ell$ , we have

$$\{(\alpha, t) : f(\alpha, t) = \gamma\} \in E_\ell \times D_\ell^* = F_\ell. \quad \square$$

**Continuation of the proof of Theorem 1.2.** Assume that GCH holds and that there are compact cardinals  $\kappa_0 = \omega < \kappa_1 < \kappa_2 < \dots$ .

Let  $P_\ell$  be the Levy collapse of  $\kappa_{\ell+1}$  to  $\kappa_\ell^{+(2\ell)+2}$ , i.e.,  $P_\ell$  collapses every  $\lambda$ ,  $\kappa_\ell^{+(2\ell)+2} < \lambda < \kappa_{\ell+1}$ , to  $\kappa_\ell^{+(2\ell)+1}$ , and each condition consists of  $\kappa_\ell^{+(2\ell)}$  atomic conditions of the form  $H_\lambda^\ell(\alpha) = \beta$  ( $\lambda$  as above,  $\alpha < \kappa_\ell^{+(2\ell)+1}$ ,  $\beta < \lambda$ ). See [4]. The order is inclusion. Let

$$p \upharpoonright \xi = \{H_\lambda^\ell(\alpha) = \beta : H_\lambda^\ell(\alpha) = \beta \in p, \lambda < \xi\}$$

and

$$\lambda(p) = \sup\{\lambda : (\exists \alpha, \beta) H_{\lambda}^{\ell}(\alpha) = \beta \in p\}.$$

Let  $P = \prod_{\ell < \omega} P_{\ell}$ . Let  $G \subseteq P$  be generic, and set  $G_{\ell} = G \cap P_{\ell}$ . Let  $\phi_{\ell} \in P_{\ell}$  be the empty condition. We identify  $\langle p_0, \dots, p_{\ell-1} \rangle \in \prod_{\ell' < \ell} P_{\ell'}$  with  $\langle p_0, \dots, p_{\ell-1}, \phi_{\ell}, \phi_{\ell+1}, \dots \rangle$  and  $p \in P_{\ell}$  with  $\langle \phi_0, \dots, \phi_{\ell-1}, p, \phi_{\ell+1}, \dots \rangle$ .

The first  $\omega$  cardinals in  $V[G]$  are  $\aleph_0 = \kappa_0, \kappa_0^{+1}, \kappa_1$ , (this differs from Shelah’s work)  $\dots, \kappa_1^{+r(2)+1}, \kappa_2, \dots, \kappa_{\ell}, \dots, \kappa_{\ell}^{+r(2\ell)+1}, \kappa_{\ell+1}, \dots$ . Also  $V[G]$  satisfies the GCH. Let for  $i = (n, m)$ , in  $V[G]$ ,

$$f_i : \prod_{j=n}^m [\aleph_{k_1(j)-1}, \aleph_{k_1(j)}]^{r(j)} \rightarrow \aleph_{k_2(j)-1}, \tag{1.3}$$

where the cardinal value of  $\kappa_{\ell}$  in  $V[G]$  is  $\aleph_{k_1(2\ell)}$ , for  $\ell \in \omega$ . W.l.o.g., from the value of  $f$  for  $\alpha_0, \dots, \alpha_k$  we can compute its value on any subsequence starting with  $\alpha_0$ . Now, in order to simplify the organization of the homogenization arguments, we take:

$$\begin{aligned} f_{(n,\cdot)} &= \langle f_{(n,m)} : m \geq n \rangle, \\ f(\alpha_0, \dots, \alpha_k) &= \langle f_{(n,\cdot)}(\alpha_0, \dots, \alpha_k) : n \text{ s.th. } \aleph_{k_2(n)} \leq \alpha_0 \rangle, \quad \text{for } \bar{\alpha} \in \sup_n \kappa_n. \end{aligned} \tag{1.4}$$

By the GCH we have that  $\aleph_{k_2(n)-1}^{\omega} = \aleph_{k_2(n)-1}$  and hence  $f_{(n,\cdot)}$  has still small range, which will be regarded as  $\aleph_{k_2(n)-1}$ . The value of  $f$  is a finite sequence of ordinals less than  $\aleph_{k_2(n)-1}$ , where  $n$  is the minimal index such that  $\alpha_0 \geq \aleph_{k_2(n)-1}$ , and hence the value can be seen as one ordinal less than  $\aleph_{k_2(n)-1}$ . So  $f$  is regressive in a strong sense that will be used in the pigeonhole arguments to come. The sets  $S_n$ , being indiscernible in the matching arities under  $f$  on each realm where  $f$  is regressive, also witness that there are canonical forms for  $f_{(n,m)}$  and  $\Gamma$  from Theorem 1.1, because of the connection given by Eq. (1.4).

**Claim 1.6** (Shelah). *If  $A \in F_{\ell+1}$ ,  $p_{(\alpha,t)} \in P_{\ell}$  for every  $(\alpha,t) \in A$ , then there is  $B \subseteq A$ ,  $B \in F_{\ell+1}$  and  $q \in P_{\ell}$  such that*

$$(\forall (\alpha,t) \in B) (p_{(\alpha,t)} \upharpoonright \alpha = q) \tag{1.5}$$

and hence

$$(\forall r) (q \leq r \in P_{\ell} \wedge \lambda(r) < \alpha \wedge (\alpha,t) \in B \rightarrow p_{(\alpha,t)} \not\leq r). \tag{1.6}$$

**Proof.** For  $(\alpha,t) \in A \cap \{(\gamma,t) \in J_{\ell+1} : \text{cf}(\gamma) > \kappa_{\ell}^{+r(2\ell)+2}\}$ ,  $(\alpha,t) \mapsto p_{(\alpha,t)} \upharpoonright \alpha \in \mathcal{P}_{\leq \kappa_{\ell}^{+r(2\ell)+1}}$  ( $\{H_{\lambda}^{\ell}(\gamma) = \beta : \lambda < \alpha, \gamma < \kappa_{\ell}^{+r(2\ell)+2}, \beta < \lambda\}$ ) can be coded as a regressive function, because, by GCH, the respective powers of cardinals are sufficiently small. Since  $E_{\ell+1}$  is a normal ultrafilter on  $\kappa_{\ell+1}$ , we have that  $(\forall \gamma < \kappa_{n+1}) (\{\alpha < \kappa_{n+1} : \text{cf}(\alpha) > \gamma\} \in E_{\ell+1})$ . (suppose otherwise:  $\{\alpha < \kappa_{\ell+1} : \text{cf}(\alpha) \leq \gamma\} \in E_{\ell+1}$  for some  $\gamma < \kappa_{\ell+1}$  Then, by  $< \kappa_{\ell+1}$ -completeness

of  $E_{\ell+1}$ , there is some  $\gamma' < \gamma$  such that  $C = \{\alpha < \kappa_{\ell+1} : \text{cf}(\alpha) = \gamma'\} \in E_{\ell+1}$ . For each  $\alpha \in C$  fix an cofinal sequence  $\langle x_\alpha(\beta) : \beta < \gamma' \rangle$ . For each  $\beta < \gamma'$  the function  $\alpha \mapsto x_\alpha(\beta)$  is regressive on  $C$ , and hence constant on a set  $C_\beta \in E_{\ell+1}$ . The intersection of all  $C_\beta$ ,  $\beta < \gamma'$ , is empty or a singleton, but should be a superset of some set in  $E_{\ell+1}$ . Contradiction.) So, by the previous claim there are some  $q$  and some  $B \in F_{\ell+1}$  such that  $B \subseteq A$  and  $(\forall(\alpha, q) \in B)(p_{(\alpha, t)} \upharpoonright \alpha = q)$ . For the second equation: If  $\lambda(r) < \alpha$ , then  $p_{(\alpha, t)} \setminus q$  has disjoint domain from  $r \setminus q$ .  $\square$

We add a lemma on compatible forcing conditions, that will be used to give the sets of 1-indiscernibles.

**Lemma 1.7.** *If  $A \in F_{\ell+1}$ ,  $q_{(\alpha, t)} \in P_\ell$  for every  $(\alpha, t) \in A$ ,  $\mu < \kappa_\ell^{+r(2\ell)+2}$ ,  $\mu$  regular, then there are  $C \subseteq \kappa_{\ell+1}$ ,  $|C| = \mu$  and  $t \in \bigcap \{\{t : (\gamma, t) \in A\} : \gamma \in C\}$  such that*

$$(\forall \gamma, \delta \in C)(q_{(\gamma, t)} \perp q_{(\delta, t)}) \quad (1.7)$$

and hence, since  $P_\ell$  is  $\kappa_\ell^{+r(2\ell)+2}$ -directed closed,

$$\bigcup_{\gamma \in C} q_{(\gamma, t)} \in P_\ell. \quad (1.8)$$

**Proof.** First we take  $B \in F_{\ell+1}$ ,  $B \subseteq A$ , as in Claim 1.6. Then we take the first  $\mu$  members of  $\{\alpha < \kappa_{\ell+1} : \{t : (\alpha, t) \in B\} \in D_{\ell+1}^*\}$ , say they are  $\{\alpha_i : i < \mu\}$ . We take  $t \in \bigcap_{i < \mu} \{t : (\alpha_i, t) \in B\} \in D_{\ell+1}^*$ . Then we take a subsequence  $\alpha_{i_j}$  of the  $\alpha_i$  by  $i_{j+1} > i$  is such that  $\lambda(p_{(\alpha_{i_j}, t)}) < \alpha_{i_{j+1}}$  and  $i_\delta = \sup\{i_j : \gamma < \delta\}$  at limit  $\delta$ . Now Eq. (1.6) of Claim 1.6 gives the pairwise compatibility.  $\square$

Now we continue the proof of the theorem.

Let, for  $n \geq 1$ ,  $A_n \subseteq \aleph_{k_1(n)}$  be cofinal. In order to avoid clumsy notation, we identify  $A_n$  with  $[\aleph_{k_1(n)-1}, \aleph_{k_1(n)})$ .

We prove that there are sets  $S_n$ ,  $n > 0$ ,  $S_n \subseteq A_n$ ,  $|S_n| = \aleph_{k_2(n)}$ , for all  $n \geq 1$ ,  $k \geq 0$ , for all  $\alpha_n, \bar{\beta}_n, \alpha'_n, \bar{\beta}'_n, \dots, \alpha_{n+k}, \bar{\beta}_{n+k}, \alpha'_{n+k}, \bar{\beta}'_{n+k}$  if  $\alpha_{n+i}, \alpha'_{n+i} \in S_{2(n+i)-1}$ ,  $\bar{\beta}_{n+i}, \bar{\beta}'_{n+i} \in [S_{2(n+i)}]^{r(2(n+i))}$  ( $0 \leq i \leq k$ ), then

$$\begin{aligned} f_{2n-1, 2n+2k-1}(\alpha_n, \bar{\beta}_n, \dots, \alpha_{n+k}, \bar{\beta}_{n+k}) &= f_{2n-1, 2n+2k-1}(\alpha'_n, \bar{\beta}'_n, \dots, \alpha'_{n+k}, \bar{\beta}'_{n+k}), \\ f_{2n, 2n+2k-1}(\bar{\beta}_n, \dots, \alpha_{n+k}, \bar{\beta}_{n+k}) &= f_{2n, 2n+2k-1}(\bar{\beta}'_n, \dots, \alpha'_{n+k}, \bar{\beta}'_{n+k}). \end{aligned}$$

**Remark 1.8.** In the following, we treat explicitly only the first kind of these equations. This will be enough. For the even first index, we can work with a dummy first variable such that the function is regressive even together with this first variable (it is not total, though, but defined on a set in the ultrafilter). Then we invoke Claim 1.5. Also, we do not need to deal with arguments ending with a variable of kind  $\alpha$ , because we will get indiscernibility for an extended sequence of arguments.



Since each  $P_\ell$  is  $\kappa_\ell^{+(2\ell)+1}$ -complete, we can find  $\bar{p}^0 = \langle p_0^0, p_1^0, \dots \rangle$ ,  $\bar{p} \leq \bar{p}^0$  such that for each  $\ell$ :

- (0)  $\bar{p}^0 \Vdash_P$  “ $f \upharpoonright \kappa_\ell^{+(2\ell)}$  is determined by forcing with  $\prod_{j < \ell} P_j$ ”. So, for some  $\prod_{j < \ell} P_j$ -name  $f_{\sim \ell}^0$ ,  $\bar{p}^0 \Vdash f \upharpoonright \kappa_\ell^{+(2\ell)} = f_{\sim \ell}^0$ .

This is proved as in [11] property (5) there.

Now we define by induction on  $k$ , a condition  $\bar{p}^k = \langle p_0^k, p_1^k, \dots \rangle$  and sets  $A_\ell^k \in F_\ell$  ( $0 < \ell < \omega$ ) and conditions  $q_{(\alpha,t)}^k \in P_{\ell-1}$  for  $(\alpha, t) \in A_\ell^k$ ,  $0 < \ell < \omega$ , such that for all  $\ell \geq 0$ :

- (1)  $p_\ell^k \leq p_\ell^{k+1}$  in  $P_\ell$ ,  $A_{\ell+1}^{k+1} \subseteq A_{\ell+1}^k$  and  $(\alpha, t) \in A_{\ell+1}^0 \rightarrow \kappa_\ell < \alpha$ ;
- (2)  $q_{(\alpha,t)}^k \leq q_{(\alpha,t)}^{k+1}$  for  $(\alpha, t) \in A_{\ell+1}^{k+1}$ ;
- (3)  $p_\ell^k = q_{(\alpha,t)}^k \upharpoonright \alpha$  for  $(\alpha, t) \in A_{\ell+1}^k$ ;
- (4') for every  $\ell \geq 0$ ,  $k \geq 1$  for some  $\prod_{j < \ell} P_j$ -name  $f_{\sim \ell}^k$  for any  $(\alpha_{\ell+1}, t_{\ell+1}) \in A_{\ell+1}^k$ ,  $(\alpha_{\ell+2}, t_{\ell+2}) \in A_{\ell+2}^k, \dots, (\alpha_{\ell+k}, t_{\ell+k}) \in A_{\ell+k}^k$  and increasing sequences  $\bar{\beta}_{\ell+j}$  from range( $t_{\ell+j}$ ) of length  $r(2(\ell+j))$  for  $j = 1, \dots, k$ ,

$$\bar{p}^k \cup \bigcup_{j=1}^k q_{(\alpha_{\ell+j}, t_{\ell+j})}^k \Vdash \text{“for any increasing sequence } \bar{\gamma} \text{ from } \kappa_\ell^{+(2\ell)} \text{ } f_{\sim \ell}^k(\bar{\gamma}, \alpha_{\ell+1}, \bar{\beta}_{\ell+1}, \dots, \alpha_{\ell+k}, \bar{\beta}_{\ell+k}) = f_{\sim \ell}^k(\bar{\gamma})\text{”}.$$

For  $k = 0$ .  $\bar{p}^0$  is already defined. For  $\ell \geq 0$ , let  $A_{\ell+1}^0 = \{(\alpha, t) \in J_\ell : \alpha > \kappa_\ell\}$ , and for  $(\alpha, t) \in A_{\ell+1}^0$ , set  $q_{(\alpha,t)}^0 = p_\ell^0$ . Property (4') does not speak about  $k = 0$ .

For  $k + 1$ . For  $\ell < \omega$ , the inductive hypothesis for  $k = 0$  and  $\ell + 1$  according to (0), and for  $k > 0$  according to (4) for  $k$  and  $\ell + 1$ , says, that there is a  $\prod_{j < \ell+1} P_j$ -name  $f_{\sim \ell+1}^k$  of a function whose domain are the increasing finite sequences from  $\kappa_{\ell+1}^{+(2(\ell+1))}$  and whose range is  $\kappa_{\ell+1}$ . Property (4') for  $\ell + 1$  and  $k$  says that  $\bar{p}^k \cup \bigcup_{j=1}^k q_{(\alpha_{\ell+1+j}, t_{\ell+1+j})}^k$  forces the  $f$ -indiscernibility of the  $\alpha_{\ell+1+1}, \bar{\beta}_{\ell+1+1}, \alpha_{\ell+1+2}, \bar{\beta}_{\ell+1+2}, \dots, \alpha_{\ell+1+k}, \bar{\beta}_{\ell+1+k}$  if  $(\alpha_{\ell+1+j}, t_{\ell+1+j}) \in A_{\ell+1+j}^k$  and  $\bar{\beta}_{\ell+1+j} \in t_{\ell+1+j}$  for  $j = 1, \dots, k$  and that  $f_{\sim \ell+1}^k$  describes  $f_{\sim \ell+1}$  on this set.  $\bar{\gamma}$  in (4) for  $\ell + 1$  is a sequence of length  $\sum_{1 \leq j \leq \ell+1} (r(2j) + 1)$ , but indeed, without any harm could have any finite length, and now our aim is to get indiscernibility in the uppermost block of  $1 + r(2(\ell + 1))$  coordinates. So in (4) with  $\ell + 1$  we have  $\bar{\gamma} = \bar{\gamma}' \hat{\ } \alpha_{\ell+1} \hat{\ } \bar{\beta}_{\ell+1}$ , with  $\alpha_{\ell+1} < \kappa_{\ell+1}$  and  $\bar{\beta}_{\ell+1} = \langle \beta_{\ell+1,0}, \dots, \beta_{\ell+1,r(2(\ell+1))} \rangle < \kappa_{\ell+1}^{+(2(\ell+1))}$ . Remember that the GCH holds and that each  $\kappa_\ell$  is regular and  $\prod_{j \leq \ell} P_j$  satisfies the  $\kappa_{\ell+1}$ -chain condition.

For each sequence  $\alpha_{\ell+1} \hat{\ } \bar{\beta}_{\ell+1}$  there are conditions  $r_i^{\alpha_{\ell+1} \hat{\ } \bar{\beta}_{\ell+1}} \in \prod_{j \leq \ell} P_j$ ,  $i < i(\alpha_{\ell+1} \hat{\ } \bar{\beta}_{\ell+1})$ , and a set  $\{(r_i^{\alpha_{\ell+1} \hat{\ } \bar{\beta}_{\ell+1}}, \gamma_i^{\alpha_{\ell+1} \hat{\ } \bar{\beta}_{\ell+1}}) : i < i(\alpha_{\ell+1} \hat{\ } \bar{\beta}_{\ell+1})\}$  such that  $\gamma_i^{\alpha_{\ell+1} \hat{\ } \bar{\beta}_{\ell+1}}$  is a name for a function  $h_{\sim \ell+1}^k(\alpha_{\ell+1} \hat{\ } \bar{\beta}_{\ell+1}) := \{\langle \bar{\gamma}', f_{\sim \ell+1}^k(\bar{\gamma}', \alpha_{\ell+1} \hat{\ } \bar{\beta}_{\ell+1}) \rangle : \bar{\gamma}' < \kappa_\ell^{+(2\ell)}\}$ , and such that  $\{r_i^{\alpha_{\ell+1} \hat{\ } \bar{\beta}_{\ell+1}} : i < i(\alpha_{\ell+1} \hat{\ } \bar{\beta}_{\ell+1})\}$  is a maximal antichain of  $\prod_{j \leq \ell} P_j$  above  $\bar{p}^k$  and  $r_i^{\alpha_{\ell+1} \hat{\ } \bar{\beta}_{\ell+1}} \Vdash h_{\sim \ell+1}^k(\alpha_{\ell+1} \hat{\ } \bar{\beta}_{\ell+1}) = \gamma_i^{\alpha_{\ell+1} \hat{\ } \bar{\beta}_{\ell+1}}$ . We define for  $\alpha_{\ell+1} \in \kappa_{\ell+1}$  an  $r(2(\ell + 1))$ -place function

$G_\ell^{k, \alpha_{\ell+1}}$  on  $\kappa_{\ell+1}^{+r(2(\ell+1))}$ :

$$G_\ell^{k, \alpha_{\ell+1}}(\bar{\beta}_{\ell+1}) = \{(r_i^{\alpha_{\ell+1} \hat{\beta}_{\ell+1}}, \gamma_i^{\alpha_{\ell+1} \hat{\beta}_{\ell+1}}) : i < i(\alpha_{\ell+1} \hat{\beta}_{\ell+1})\}.$$

The range of  $G_\ell^{k, \alpha_{\ell+1}}$  has cardinality less or equal  $\kappa_{\ell+1}$ , as  $i(\alpha_{\ell+1} \hat{\beta}_{\ell+1}) < \kappa_{\ell+1}$  because  $\prod_{j \leq \ell} P_j$  satisfies the  $\kappa_{\ell+1}$ -chain condition, and  $r_i^{\alpha_{\ell+1} \hat{\beta}_{\ell+1}} \in \prod_{j \leq \ell} P_j$ ,  $|\prod_{j \leq \ell} P_j| \leq \kappa_{\ell+1}$ , and for each  $\bar{\gamma}', f_{\sim_{\ell+1}}^k(\bar{\gamma}', \alpha_{\ell+1} \hat{\beta}_{\ell+1})$  is a  $\prod_{j \leq \ell} P_j$ -name of an ordinal less than  $\kappa_{\ell+1}$ , and hence the number of possible  $h_{\sim_{\ell+1}}^k$  can also without loss be bounded by  $\kappa_{\ell+1}^{<\kappa_{\ell+1}} = \kappa_{\ell+1}$ .

Let  $B_{\alpha_{\ell+1}} = \{t \in I_{\ell+1} : G_\ell^{k, \alpha_{\ell+1}}$  has the same value, say  $h_{\ell+1}^k(\alpha_{\ell+1})$ , on all increasing sequences of length  $r(2(\ell + 1))$  from  $t\}$ . By definition

$$B_{\alpha_{\ell+1}} \in D_{\ell+1}(\kappa_{\ell+1}^{+r(2(\ell))}, \kappa_{\ell+1}^+, \kappa_{\ell+1}) \subseteq D_{\ell+1}^*.$$

Thus,  $B' = \{(\alpha, t) \in J_{\ell+1} : t \in B_\alpha\} \in F_{\ell+1}$ .

For every  $(\alpha, t) \in A_{\ell+1}^{k+1}$  choose an increasing sequence of length  $r(2(\ell + 1))$  from  $t$ , call it  $\bar{\beta}_{\ell+1}$ , and find  $q_{(\alpha, t)}^{k+1}$  such that  $q_{(\alpha, t)}^k \leq q_{(\alpha, t)}^{k+1} \in P_\ell$  and

$$\begin{aligned} q_{(\alpha, t)}^{k+1} \cup \bigcup_{j=1}^k q_{(\alpha_{\ell+j}, t_{\ell+j})}^k & \text{ forces} \\ \langle \bar{\gamma}, f_{\sim_{\ell+1}}^k(\bar{\gamma} \hat{\alpha} \hat{\beta}_{\ell+1}) \rangle & : \bar{\gamma} \text{ an increasing finite sequence from } \kappa_{\ell}^{r(2\ell)} \\ \text{of length } \sum_{1 \leq j < \ell} (r(2j) + 1) & \rangle = f_{\sim_{(\alpha, t)}}^k \end{aligned} \tag{1.9}$$

for some  $\prod_{j < \ell} P_j$ -name  $f_{\sim_{(\alpha, t)}}^k$  (possible as  $P_\ell$  is  $\kappa_{\ell}^{+r(2\ell)+2}$ -complete). If  $(\alpha, t) \in B'$  too, then the choice of  $\bar{\beta}_{\ell+1}$  is immaterial. Now by Claim 1.6 applied to  $q_{(\alpha, t)}^{k+1}$  and  $A_{\ell+1}^k \cap B'$ , we can find  $p_{\ell}^{k+1}$  and  $A_{\ell+1}^{k+1} \subseteq B' \cap A_{\ell+1}^k$  with property (3), and as the number of possible  $f_{\sim_{(\alpha, t)}}^k$  is  $\leq \kappa_{\ell}^{+r(2\ell)}$ , we can choose some  $f_{\sim_{\ell}}^{k+1}$  such that  $f_{\sim_{(\alpha, t)}}^k = f_{\sim_{\ell}}^{k+1}$  for every  $(\alpha, t) \in A_{\ell+1}^{k+1}$ .

By now,  $f_{(\alpha, t)}^k$  may depend, as written in Eq. (1.9), also on  $(\alpha_{\ell+2}, t_\ell), \dots, (\alpha_{\ell+k+1}, t_{\ell+k+1})$ . But the number of possible  $f_{(\alpha, t)}^k$  is less or equal  $\kappa_{\ell}^{+r(2\ell)+1}$  and  $F_{\ell+1}$  is  $\kappa_{\ell+1}$ -complete, and so we can assume that  $f_{(\alpha, t)}^k = f_{\ell}^{k+1}$  for every  $(\alpha, t) \in A_{\ell+1}^{k+1}$ . Furthermore, since  $F_{\ell+2}, \dots, F_{\ell+k+1}$  are all  $\kappa_{\ell+2}$ -complete, we can chose  $A_{\ell+j}^{k+1} \subseteq A_{\ell+j}^k$ ,  $A_{\ell+j}^{k+1} \in F_{\ell+j}$  such that  $f_{(\alpha, t)}^k$  is the same for all  $(\alpha_{\ell+2}, t_{\ell+2}) \in A_{\ell+2}^{k+1}, \dots, (\alpha_{\ell+k+1}, t_{\ell+k+1}) \in A_{\ell+k+1}^{k+1}$ . Hence, as required in (4'), there is indeed one name working for all conditions.

We define  $A_{\ell+1}^\omega = \bigcap_{k < \omega} A_{\ell+1}^k$ ,  $q_{(\alpha, t)}^\omega = \bigcup_{k < \omega} q_{(\alpha, t)}^k$  and  $p_\ell^\omega = \bigcup_{k < \omega} p_\ell^k$  for  $(\alpha, t) \in A_\ell^\omega$ . As each  $F_{\ell+1}$  is  $\kappa_{\ell+1}$ -complete,  $A_{\ell+1}^\omega \in F_{\ell+1}$ . It is also clear that  $p_\ell^\omega \in P_\ell$  and  $q_{(\alpha, t)}^\omega \in P_\ell$  for  $(\alpha, t) \in A_{\ell+1}^\omega$ .

Now choose for each  $\ell \geq 1$ ,  $\alpha_\ell^i$ ,  $i < \aleph_{k_2(2\ell-1)}$ , and  $t_\ell$  as in the proof of Lemma 1.7 such that  $q_{(\alpha_\ell^i, t_\ell)}^\omega$  are pairwise compatible. Now let

$$p^1 = \left\langle \bigcup_{i < \aleph_{k_2(1)}} q_{(\alpha_1^i, t_1)}^\omega, \bigcup_{i < \aleph_{k_2(3)}} q_{(\alpha_2^i, t_2)}^\omega, \dots \right\rangle$$

and  $S_{2\ell-1} = \{\alpha_\ell^i : i < \aleph_{k_2(2\ell-1)}\}$ ,  $S_{2\ell} = t_\ell$ . It is easy to check that they are as required.

□<sub>1,2</sub>

Now we present a canonization theorem which was for some time the one giving the densest free subsets. If Fact 3.3 is applied to it, it gives free subsets which contain one point in each cardinal interval for infinitely many blocks of  $r(n)$  adjacent intervals. Between two blocks there will be an interval of three cardinal steps. In the middle part of the interval there is one point of a free subset. The canonization theorem leaves it open whether the first and the third part of the three step interval can be hit by a free subset.

In comparison to Theorem 1.2, we change the homogeneous blocks: Instead of working in the  $(2\ell)$ th block  $S_{2\ell}$  with arity  $r(2\ell)$  we want to get  $r(2\ell)$  sets  $S_\ell^j$ ,  $0 \leq j < r(2\ell)$ , for arity 1, and (otherwise this would just be a weakening of the former)  $S_\ell^j \subseteq \aleph_{k_2(\ell)-r(\ell)+j}$ . We also changed the indexing. The former  $S_{2\ell-1}$  are now corresponding to  $S_{\ell,-1}$ ,  $r(2\ell)$  is now called  $r(\ell)$ , and  $S_{2\ell}$  corresponds to  $S_{\ell,0}, \dots, S_{\ell,r(\ell)-1}$ .

Strictly speaking we could now search simultaneously for indiscernibles for a family of functions

$$\begin{aligned} f_{(n,j_1),(m,j_2)} : \prod_{j=j_1}^{r(n)-1} [\aleph_{k_1(n,j)}, \aleph_{k_1(n,j+1)}] \times \prod_{\ell=n+1}^{m-1} \prod_{j=-1}^{r(\ell)-1} [\aleph_{k_1(\ell,j)}, \aleph_{k_1(\ell,j+1)}] \\ \times \prod_{j=-1}^{j_2-1} [\aleph_{k_1(m,j)}, \aleph_{k_1(m,j+1)}] \rightarrow \aleph_{k_2(n,j_1)} \end{aligned} \tag{1.10}$$

for  $1 \leq n \leq m < \omega$ ,  $j_1 \in \{-1, 0, \dots, r(n) - 1\}$ ,  $j_2 \in \{-1, 0, \dots, r(m) - 1\}$ . But we dispense with this and do it only for

$$f_{(n,m)} : \prod_{\ell=n}^m \prod_{j=-1}^{r(\ell)-1} [\aleph_{k_1(\ell,j)}, \aleph_{k_1(\ell,j+1)}] \rightarrow \aleph_{k_2(n,-1)} \tag{1.11}$$

for  $1 \leq n \leq m < \omega$ . This will again suffice for the purpose to find free subsets. Anyway, the coding of Eq. (1.4) is applied to the  $f_{(n,m)}$  so that we work with only one function. But we could have applied a coding to the numerous functions as in (1.10).

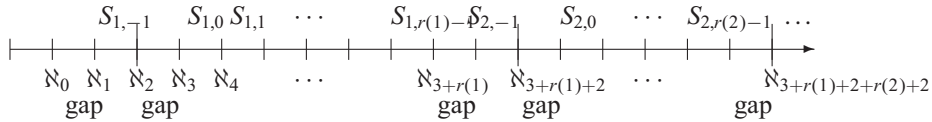
In Section 3, we will show that homogeneous sets in these locations given by  $\Gamma$  from Theorem 1.9 are particularly useful for our application to the free subset problem. For technical reasons another kind of cardinal gaps arises immediately after each collapsed  $\kappa_n$  from the kind of Ramsey theorem we use in 1.9. We write commata instead of concatenation symbols in the formulation of the next  $\Gamma$ .

**Theorem 1.9.** *Suppose the consistency of “ZFC + GCH + there are infinitely many compact cardinals”. Let  $r: \omega \setminus \{0\} \rightarrow \omega \setminus \{0\}$ . Then the following is consistent: ZFC + GCH +  $\langle \aleph_{k_1(n,j)} : 0 < n < \omega, -1 \leq j < r(n) \rangle$  has  $\langle \aleph_{k_2(n,j)} : 0 < n < \omega, -1 \leq j < r(n) \rangle$ -canonical forms for*

$$\begin{aligned} \Gamma = & \langle (\langle 1, \langle 1 \rangle^{(r(n))}, 1, \langle 1 \rangle^{(r(n+1))}, \dots, 1, \langle 1 \rangle^{(r(m))} \rangle, \ell(i), \aleph_{k_2(n,-1)}, \\ & \langle (n, -1), (n, 0), \dots, (n, r(n) - 1), \dots \\ & (m, -1), \dots, (m, r(m) - 1) \rangle) : i = (n, m), n \leq m < \omega \rangle, \\ \ell(i) = & 2(m - n + 1) + r(2n) + r(2n + 2) + \dots + r(2m), \end{aligned} \tag{1.12}$$

where for  $0 < n < \omega, -1 \leq j < r(n), k_1(n, j) = \left(\sum_{1 \leq j < n} r(j)\right) + 3n + j - 1$ ; and for  $0 < n < \omega, 0 \leq j < r(n) - 1, k_2(n, j) = \left(\sum_{1 \leq j < n} r(j)\right) + 3n - 1, k_2(n, r(n) - 1) = k_1(n, r(n) - 1)$ , and finally,  $k_2(n, -1) = \left(\sum_{1 \leq j < n} r(j)\right) + 3n - 4$ .

Again, we sketch a picture for indicating the locations but not the powers:



**Proof.** Assume that GCH holds and that there are compact cardinals  $\kappa_0 = \omega < \kappa_1 < \kappa_1 < \dots$ . We use the same technique as in the previous theorem, but with different filters, and there are now two kinds of gaps in the cardinal intervals where there are no indiscernibles, one old kind: the second but last interval before the collapse of  $\kappa_n$ , and one new kind: the interval starting with the collapsed  $\kappa_n$ .

**Definition 1.10.** For  $\lambda_k \geq \mu_k > \chi$ , let  $D(\langle \lambda_k, \mu_k : 0 \leq k < r \rangle, \chi)$  be the following filter:

- (a) It is a filter over the set  $\prod_{k < r} \text{Inc}(\lambda_k, \mu_k)$ .
- (b) The filter is generated by the set of generators, where a generator is

$$\begin{aligned} \text{Ge}(F) = & \text{Ge}(F; \langle \lambda_k, \mu_k : k < r \rangle, \chi) \\ = & \{ \bar{a} = \bar{a}_0 \hat{\ } \bar{a}_1 \hat{\ } \dots \hat{\ } \bar{a}_r : \bar{a}_k \in \text{Inc}(\lambda_k, \mu_k) \text{ for } 0 \leq k < r \text{ and} \\ & (\exists \alpha \in \chi) (\forall i(0 < \mu_0) (\forall i(1 < \mu(1)) \dots (\forall i(r - 1) < \mu_{r-1})) \\ & F(a_{i(0)}, \dots, a_{i(r-1)}) = \alpha \}, \end{aligned}$$

for some  $F: \lambda_0 \times \dots \times \lambda_{r-1} \rightarrow \chi$ .

**Claim 1.11.** (1) *If  $\chi = \chi^{< \kappa}$ , then the intersection of  $< \kappa$  generators of  $D(\langle \lambda_k, \mu_k : k < r \rangle, \chi)$  is a generator; hence the filter  $D(\langle \lambda_k, \mu_k : k < r \rangle, \chi)$  is  $\kappa$ -complete.*

(2) If

$$\begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{r-1} \end{pmatrix} \rightarrow \begin{pmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_{r-1} \end{pmatrix}_{\chi}^{(1)^{(r)}}$$

(the usual polarized partition relation, see [2]) then  $D(\langle \lambda_k, \mu_k : k < r \rangle, \chi)$  is a proper filter.

Let  $\ell \geq 1$  until the end of the next claim. Let  $E_\ell$  be a normal ultrafilter over  $\kappa_\ell$ . Let  $I_\ell = \prod_{k < r(2n)} \text{Inc}(\kappa_\ell^{+k+2}, \kappa_\ell)$  and  $J_\ell = \kappa_\ell \times I_\ell$ . Note that  $D(\langle \kappa_\ell^{+k+2}, \kappa_\ell : k < r - 1 \rangle \wedge \langle \kappa_{r-1}^{+\ell+2}, \kappa_{r-1}^{+\ell+2} \rangle, \kappa_\ell)$  is a  $\kappa_\ell$ -complete proper filter, as  $\kappa_\ell^{<\kappa_\ell} = \kappa_\ell$  (by the GCH) and because of the following result from [9],

$$\begin{pmatrix} (2^\kappa)^+ \\ (2^\kappa)^{++} \\ \vdots \\ (2^\kappa)^{+r} \end{pmatrix} \rightarrow \begin{pmatrix} \kappa \\ \vdots \\ \kappa \\ (2^\kappa)^{+r} \end{pmatrix}_{\chi}^{(1)^{(r)}}$$

for  $\chi = \kappa_\ell$  and  $\lambda_k = \kappa_\ell^{+k+2}$  and  $\mu_k = \kappa_\ell$ , ( $0 \leq k < r$ ), the premise of (2) in the above claim is true.

So, as  $\kappa_\ell$  is compact, there is a  $\kappa_\ell$ -complete ultrafilter  $D_\ell^*$  over  $I_n$  extending  $D(\langle \kappa_\ell^{+k+2}, \kappa_\ell : k < r - 1 \rangle \wedge \langle \kappa_{r-1}^{+\ell+2}, \kappa_{r-1}^{+\ell+2} \rangle, \kappa_\ell)$ . We set  $F_\ell = E_\ell \times D_\ell^*$ . Again we take  $P = \prod_{\ell \in \omega} P_\ell$  but this time  $P_\ell$  is the Levy collapse from  $\kappa_{\ell+1}$  to  $\kappa_\ell^{+r(2^\ell)+3}$ . Now we are at the stage where we can jump into the proof of Theorem 1.2 and continue as there.  $\square$

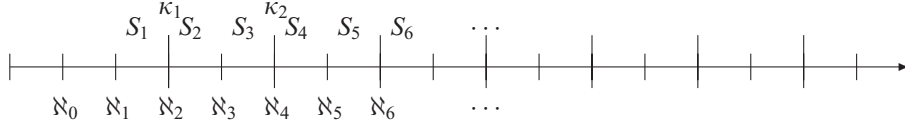
## 2. Canonization theorems with indiscernibles in every cardinal interval

In this section, we first prove a canonization theorem that is incompatible in strength with the former ones, but that is best suited for our application: to get free subsets which have one point in each cardinal interval. The aim, hitting each interval, is reached at the expense of working with sets of indiscernibles  $S_\ell$  of cardinal size one less than formerly. In the second part of this section, we will return to higher arities.

**Theorem 2.1.** *Suppose the consistency of “ZFC + GCH + there are infinitely many compact cardinals”. Then the following is consistent: ZFC + GCH +  $\langle \aleph_{n+1} : 0 < n < \omega \rangle$  has  $\langle \aleph_{n-1} : 0 < n < \omega \rangle$ -canonical forms for*

$$\begin{aligned} \Gamma = & \langle \langle \langle 1 \rangle^{(m-n+1)} \rangle, m - n + 1, \aleph_n, \langle n, \dots, m \rangle \rangle : \\ & i = (n, m), 1 \leq n \leq m < \omega \rangle. \end{aligned} \tag{2.1}$$

**Proof.** Now, for  $n = 1$  and  $m$  running we have a simple and beautiful picture:



Assume that GCH holds and that there are compact cardinals  $\kappa_0 = \omega < \kappa_1 < \kappa_2 < \dots$ .

We take the filters  $D_\ell(\lambda, \mu, \chi)$  be the same filters as in Definition 1.3, but this time  $\mu$  will not be fully used (or, alternatively, can be chosen one cardinal step smaller). We will get homogeneous sets of the cardinality  $\mu$  given by a suitable partition theorem, but at certain steps we shall carry onwards only the homogeneous part that is “known” by one condition, and this part’s size is usually the cardinal predecessor of  $\mu$ .

Again we suppose that  $\chi = \chi^{<\kappa}$ , and then the intersection of  $<\kappa$  generators of  $D_\ell(\lambda, \mu, \chi)$  is a generator; hence the filter  $D_\ell(\lambda, \mu, \chi)$  is  $\kappa$ -complete. If  $\lambda \rightarrow (\mu)^\chi_\chi$  (the usual partition relation) then  $D_\ell(\lambda, \mu, \chi)$  is a proper filter.

Let  $\kappa_0 = \omega$  and let  $\langle \kappa_\ell : 1 \leq \ell < \omega \rangle$  be a strictly increasing sequence of compact cardinals. Let  $E_\ell$  be a normal ultrafilter over  $\kappa_\ell$ . Let  $I_\ell^+ = \text{Inc}(\kappa_\ell^{+1}, \kappa_\ell^{+1})$  (or alternatively  $I_\ell^+ = \text{Inc}(\kappa_\ell^{+1}, \kappa_\ell)$ ) and  $J_\ell^+ = \kappa_\ell \times I_\ell^+$ . Note that  $D_1(\kappa_\ell^{+1}, \kappa_\ell^{+1}, \kappa_\ell)$  is a  $\kappa_\ell$ -complete proper filter, as  $\kappa_\ell^+ \rightarrow (\kappa_\ell^{+1})^1_{\kappa_\ell}$ . So, as  $\kappa_\ell$  is compact there is a  $\kappa_\ell$ -complete ultrafilter  $D_\ell^+$  over  $I_\ell^+$  extending  $D_1(\kappa_\ell^{+1}, \kappa_\ell^{+1}, \kappa_\ell)$ . We set

$$F_\ell^+ = E_\ell \times D_\ell^+ = \{A \subseteq J_\ell : \{i < \kappa_\ell : \{t \in I_\ell : (i, t) \in A\} \in D_\ell^+\} \in E_\ell\}.$$

Now we use the filters  $F_\ell^+$  to define similar notions as the ones in the previous section: We call  $f : J_\ell^+ \rightarrow \kappa_\ell$  regressive if  $(\forall \alpha < \kappa_\ell) f(\alpha, t) < \alpha$ . We call it regressive on  $A$  if  $f(\alpha, t) < \alpha$  for  $(\alpha, t) \in A$ , and almost regressive if it is regressive on some  $A \in F_\ell^+$ . Similarly we define when  $f$  is constant, constant on  $A$  and almost constant.

Since in Claim 1.5 we used only the normality of  $E_\ell$  and the  $\kappa_\ell$ -completeness of  $D_\ell^+$ , its adoption to the modified notions yields: Every almost regressive function  $f : J_\ell^+ \rightarrow \kappa_\ell$  is almost constant.

Now start with  $\ell = 0$ . Let  $P_\ell$  be the Levy collapse of  $\kappa_{\ell+1}$  to  $\kappa_\ell^{+2}$ , i.e.,  $P_\ell$  collapses every  $\lambda$ ,  $\kappa_\ell^{+1} < \lambda < \kappa_{\ell+1}$  to  $\kappa_\ell^{+1}$ , and each condition consists of  $\kappa_\ell$  atomic conditions of the form  $\tilde{H}_\lambda^\ell(\alpha) = \beta$  ( $\lambda$  as above,  $\alpha < \kappa_\ell^{+1}$ ,  $\beta < \lambda$ ). See [4]. The order is inclusion. Let  $p \upharpoonright \xi$  and  $\lambda(p)$  be defined as in the previous section. We have to find a replacement for the conditions enjoying property (0) from the proof of 1.2. For this purpose, we use the following parametrized form of Claim 1.6:

**Claim 2.2.** *If  $A \in F_{\ell+1}$ ,  $q_{(\alpha,t)}(\delta) \in P_\ell$  for every  $(\alpha, t) \in A$ ,  $\delta \in \kappa_\ell^+$ , are such that for each  $(\alpha, t)$ ,  $\langle q_{(\alpha,t)}(\delta) : \delta < \kappa_\ell^+ \rangle$  is increasing in  $\delta$ , then there are  $B \subseteq A$ ,  $B \in F_{\ell+1}$  and  $p(\delta) \in P_\ell$  such that*

$$\begin{aligned} &(\forall (\alpha, t) \in B)(q_{(\alpha,t)}(\delta) \upharpoonright \alpha = p(\delta)), \\ &p(\delta) \text{ are increasing with } \delta \end{aligned} \tag{2.2}$$

and hence

$$(\forall \delta)(\forall r)(p(\delta) \leq r \in P_n \wedge \lambda(r) < \alpha \wedge (\alpha, t) \in B \rightarrow q_{(\alpha,t)}(\delta) \not\leq r). \tag{2.3}$$

**Proof.** By induction on  $\delta$ , for each  $\delta$  we repeat the proof of Claim 1.6. Instead of starting with  $A$ , we start with  $\bigcap_{\gamma < \delta} B_\gamma$ . It is easy to see that  $q(\gamma) \leq q(\delta)$  is fulfilled automatically.  $\square$

Claim 2.2 will be applied only for one  $\delta$  in the end, but we have to work first with unboundedly in  $\kappa_\ell^+$  many  $\delta$ 's in order to homogenize.

Now we continue with the proof of Theorem 2.1: The first  $\omega$  cardinals in  $V[G]$  are  $\aleph_0 = \kappa_0, \kappa_0^+, \kappa_1, \kappa_1^+, \kappa_2, \dots$ . Also  $V[G]$  satisfies the GCH. Let for  $i = (n, m)$ , in  $V[G]$ ,

$$f_{(n,m)} : \prod_{\ell=n}^m [\aleph_\ell, \aleph_{\ell+1}) \rightarrow \aleph_n,$$

where the cardinal value of  $\kappa_\ell$  in  $V[G]$  is  $\aleph_{2\ell}$ , for  $r \in \omega$ . W.l.o.g. from the value of  $f$  for  $\alpha_0, \dots, \alpha_k$  we can compute its value on any subsequence starting with  $\alpha_0$ . Now, in order to simplify the organization of the homogenization arguments, we take  $f$  as in Eq. (1.4).

Let, for  $\ell \geq 1$ ,  $A_\ell \subseteq \aleph_{\ell+1}$  be cofinal,  $\langle A_\ell : 1 \leq n < \omega \rangle \in V[G]$ . In order to avoid clumsy notation, we identify  $A_{2\ell-1}$  with  $[\kappa_{\ell-1}^+, \kappa_\ell)$  and  $A_{2\ell}$  with  $[\kappa_\ell, \kappa_{\ell+1}^+)$  and do not write the bijections (that exist in  $V[G]$ ). Note, that as in 1.2 and in 1.9 also here we work with two kinds of  $A_\ell$ 's. The ones with odd index  $2\ell - 1$  (starting with  $A_1$ ) are the ones leading to the indiscernibles lying immediately before  $\kappa_\ell$ , the ones with even index, starting with 2, are the ones which come from the second factor in the ultrafilters.

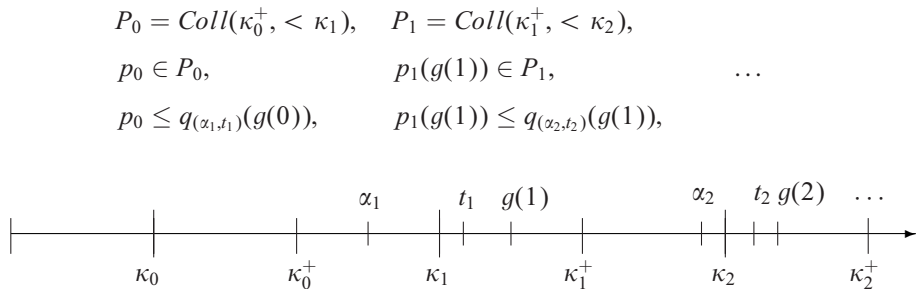
We have to prove that there are sets  $S_n$ ,  $n > 0$ ,  $S_n \subseteq A_n$ ,  $|S_n| = \aleph_{n-1}$ , such that for all  $n \geq 1$ ,  $k \geq 0$ , for all  $\beta_n, \beta'_n, \dots, \beta_{n+k}, \beta'_{n+k}$  if  $\beta_{n+i}, \beta'_{n+i} \in S_{n+i}$  ( $0 \leq i \leq k$ ), then

$$f_{(n,k)}(\beta_n, \dots, \beta_{n+k}) = f_{(n,k)}(\beta'_n, \dots, \beta'_{n+k}). \tag{2.4}$$

By Eq. (1.4) and Remark 1.8 it is enough to find indiscernibles for  $f$  and to think of Eq. (2.4) only for odd  $n$ .

The analogue of property (0) from the proof of Theorem 1.2 (and the other theorems in Section 1) does not hold, and we work with the following parametrized versions of this property. Let  $G$  be  $P$ -generic over  $V$ .

The following picture gives a rough sketch from where the variables are taken in  $V$ :



We claim that we can find for each function  $g \in \prod_{1 \leq \ell < \omega} \kappa_\ell^+$  conditions  $p_\ell^0(g(\ell)) \in P_\ell$ ,  $\ell > 0$ , such that  $\bar{p}^0(g) := \langle p_0^0, p_1^0(g(1)), p_2^0(g(2)) \dots \rangle$  (note that  $p_\ell^k(g(\ell))$  does not depend on the other values of  $g$ , though we like to write  $\bar{p}(g)$  as an

abbreviation sometimes),  $\bar{p} \leq \bar{p}^0(g)$ ,  $p_\ell^k(0) = p_\ell^k$ ,  $\bar{p}^0(g) \in G$ , and for each  $\ell < \omega$ ,  $\beta = g(\ell) \in \kappa_\ell^+$ :

- (0a)  $\langle \phi, \dots, \phi, p_\ell^0(g(\ell)), p_{\ell+1}^0, p_{\ell+2}^0 \dots \rangle \Vdash_P$  “ $f \upharpoonright g(\ell)$  is determined by forcing with  $\prod_{j < \ell} P_j$ ”. So, for some  $\prod_{j < \ell}$ -name  $\tilde{f}_\ell^0$ , it forces that  $\tilde{f} \upharpoonright g(\ell) = \tilde{f}_\ell^0$ .
- (0b)  $p_\ell^0(\alpha) \leq p_\ell^0(\beta)$  for  $\alpha < \beta \in \kappa_\ell^+$ .

The listed properties are a cut off version of [11] property (5) there, and the cuts are coherent by property (0b). We show how to find the  $\bar{p}^0(g)$ : We define by induction on  $j < \omega$ , conditions  $t_{\ell,j} \in P_\ell$  for  $\ell > j$  and  $t_{j,j}(\alpha) \in P_j$ ,  $\alpha \in [\kappa_j, \kappa_j^+)$ , increasing with  $j$  and at point  $(j, j)$  increasing in  $\alpha$ . For  $j = 0$ ,  $t_{\ell,j} = p_\ell$ . For  $j + 1$  we define  $t_{j+1,j+1}(\beta), t_{j+2,j+1}, t_{j+3,j+1}, \dots$  such that  $t_{j+1,j} \leq t_{j+1,j+1}(0), t_{j+2,j} \leq t_{j+2,j+1} \dots$  and for all  $\beta < \kappa_j^+$ :

If  $k, r < \omega$ ,  $1 \leq k + r \leq j + 1$ ,  $c < \beta$ ,  $\alpha(k) < \kappa_k, \beta(k) < \kappa_k^+, \dots, \alpha(k + r) < \kappa_{k+r}, \beta(k + r) < \beta$  and  $\bar{p}^* = \langle \emptyset_0, \dots, \emptyset_j, t_{j+1,j+1}(\beta), t_{j+2,j+1}, t_{j+3,j+1}, \dots \rangle$  and  $\bar{p}^* \leq \bar{p}'$  and  $\bar{p}' \Vdash_P f(\alpha(k), \beta(k), \dots, \alpha(k + r), \beta(k + r)) = c$ , then

$$\langle p'_0, \dots, p'_j, t_{j+1,j+1}(\beta), t_{j+2,j+1}, t_{j+3,j+1}, \dots \rangle \Vdash_P f(\alpha(k), \beta(k), \dots, \alpha(k + r), \beta(k + r)) = c.$$

This is possible as the number of possible  $\langle k, r, c, p'_0, \dots, p'_j, \alpha(k), \dots, \beta(k + r) \rangle$  is  $\leq \kappa_j$  and as  $\prod_{\ell \geq j} P_\ell$  is  $\kappa_j^+$ -complete and as  $\prod_{\ell \geq j+1} P_\ell$  is  $\kappa_{j+1}^+$ -complete. By density arguments, all conditions can be chosen in  $G$ . In the end we set  $t_{\ell,\ell}(g(\ell)) = p_\ell^0(g(\ell))$ . Then (0a) and (0b) are true.

By (0b) and the properties of product forcing,  $\bar{p}(g), \bar{p}(g')$  are compatible in  $P$ .

Now we define by induction on  $k$ , for all  $g \in \prod_{1 \leq \ell < \omega} \kappa_\ell^+$  a condition  $\bar{p}^k(g) = \langle p_0^k, p_1^k(g(1)), \dots \rangle$  and sets  $A_\ell^k \in F_\ell$  ( $0 < \ell < \omega$ ) and conditions  $q_{(\alpha,t)}^k(g(\ell)) \in P_\ell$  for  $(\alpha, t) \in A_{\ell+1}^k$  (note that here the indices  $(\alpha, t)$  for conditions in  $P_\ell$  really stem from the cell number  $\ell + 1$ ),  $0 < \ell < \omega$ , such that for all  $\ell \geq 0$ ,  $\gamma \in [\kappa_\ell, \kappa_\ell^+)$  for  $\ell > 0$  and  $\gamma = 0$  for  $\ell = 0$ :

- (1)  $p_\ell^k(\gamma) \leq p_\ell^{k+1}(\gamma)$  in  $P_\ell$ ,  $A_{\ell+1}^{k+1} \subseteq A_{\ell+1}^k$  and  $(\alpha, t) \in A_{\ell+1}^0 \rightarrow \kappa_\ell < \alpha$ ; and the  $p_\ell^k(\gamma)$  are increasing in  $\gamma$ ;
- (2)  $q_{(\alpha,t)}^k(\gamma) \leq q_{(\alpha,t)}^{k+1}(\gamma)$  for  $(\alpha, t) \in A_{\ell+1}^{k+1}$ ; and the  $q_{(\alpha,t)}^k(\gamma)$  are increasing in  $\gamma$ ;
- (3)  $p_\ell^k(\gamma) = q_{(\alpha,t)}^k(\gamma) \upharpoonright \alpha$  for  $(\alpha, t) \in A_{\ell+1}^k$ ;
- (4'') for every  $\ell \geq 0$ ,  $k \geq 1$  for some  $\prod_{j < \ell} P_j$ -name  $\tilde{f}_\ell^k$  for any  $(\alpha_{\ell+1}, t_{\ell+1}) \in A_{\ell+1}^k$ ,  $(\alpha_{\ell+2}, t_{\ell+2}) \in A_{\ell+2}^k, \dots, (\alpha_{\ell+k}, t_{\ell+k}) \in A_{\ell+k}^k$  and points  $\beta_{\ell+j}$  from  $t_{\ell+j} \cap g(\ell + j)$  for  $j = 1, \dots, k$ , for any  $g$ ,

$$\bar{p}^k \cup \bigcup_{j=0}^{k-1} q_{(\alpha_{\ell+j+1}, t_{\ell+j+1})}^k(g(\ell + j)) \cup p_{\ell+k}^k(g(\ell + k)) \Vdash$$

“for any increasing sequence  $\bar{\gamma}$  from  $\kappa_\ell^+ \cap g(\ell)$

$$\tilde{f}_\ell^k(\bar{\gamma}, \alpha_{\ell+1}, \beta_{\ell+1}, \dots, \alpha_{\ell+k}, \beta_{\ell+k}) = \tilde{f}_\ell^k(\bar{\gamma})”.$$



For  $k=0$ .  $\bar{p}^0(g)$  is already defined. For  $\ell \geq 0$ , let  $A_{\ell+1}^0 = \{(\alpha, t) \in J_n : \alpha > \kappa_\ell\}$ , and for  $(\alpha, t) \in A_{\ell+1}^0$ , set  $q_{(\alpha,t)}^0(\delta) = p_\ell^0(\delta)$ . Property (4'') does not speak about  $k=0$ .

For  $k+1$ . For  $\ell < \omega$ , the inductive hypothesis for  $k=0$  and  $\ell+1$  according to (0), and for  $k > 0$  according to (4'') for  $k$  and  $\ell+1$ , says, that there is a  $\prod_{j < \ell+1} P_j$ -name  $f_{\sim \ell+1}^k$  of a function whose domain are the increasing finite sequences form  $\kappa_{\ell+1}^{+1}$  and whose range is  $\kappa_{\ell+1}$ . Property (4'') for  $\ell+1$  and  $k$  says that  $\bar{p}^k \cup \bigcup_{j=0}^{k-1} q_{(\alpha_{\ell+1+j+1}, t_{\ell+1+j+1})}^k (g(\ell+1+j)) \cup p_{\ell+1+k}^k (g(\ell+1+k))$  forces the  $f$ -indiscernibility of the  $\bar{\gamma}$ ,  $\alpha_{\ell+1+1}$ ,  $\beta_{\ell+1+1}$ ,  $\alpha_{\ell+1+2}$ ,  $\beta_{\ell+1+2}, \dots, \alpha_{\ell+1+k}$ ,  $\beta_{\ell+1+k}$  if  $(\alpha_{\ell+1+\ell}, t_{\ell+1+\ell}) \in A_{\ell+1+j}^k$  and  $\bar{\beta}_{\ell+1+j} \in t_{\ell+1+j} \cap g(\ell+j+1)$  for  $j=1, \dots, k$ ,  $\bar{\gamma} \in g(\ell)$ , and that  $f_{\sim \ell+1}^k$  describes  $f$  on this set.  $\bar{\gamma}$  in (4'') for  $\ell+1$  is a sequence of length  $2\ell$ , but indeed, without any harm could have any finite length, and now our aim is to get indiscernibility in the uppermost coordinate. So in (4'') with  $\ell+1$  we have  $\bar{\gamma} = \bar{\gamma}' \hat{\ } \alpha_{\ell+1} \hat{\ } \beta_{\ell+1}$ , with  $\alpha_{\ell+1} < \kappa_{\ell+1}$  and  $\beta_{\ell+1} < \kappa_{\ell+1}^{+1}$ . Remember that the GCH holds and that each  $\kappa_\ell$  is regular and  $\prod_{j \leq \ell} P_j$  satisfies the  $\kappa_{\ell+1}$ -chain condition.

For each triple  $\alpha_{\ell+1} < \kappa_{\ell+1}$ ,  $\beta_{\ell+1} < \kappa_{\ell+1}^{+1}$  and  $\delta = g(\ell) \in \kappa_\ell^+$  there are conditions  $r_{i,\delta}^{\alpha_{\ell+1} \hat{\ } \beta_{\ell+1}} \in \prod_{j \leq \ell} P_j$ ,  $i < i(\alpha_{\ell+1} \hat{\ } \beta_{\ell+1}, \delta)$ , and a set  $\{(r_{i,\delta}^{\alpha_{\ell+1} \hat{\ } \beta_{\ell+1}}, \gamma_{i,\delta}^{\alpha_{\ell+1} \hat{\ } \beta_{\ell+1}}) : i < i(\alpha_{\ell+1} \hat{\ } \beta_{\ell+1}, \delta)\}$  such that  $\gamma_{i,\delta}^{\alpha_{\ell+1} \hat{\ } \beta_{\ell+1}}$  is a name for a function  $h_{\ell+1}^k(\alpha_{\ell+1} \hat{\ } \beta_{\ell+1}) \upharpoonright \delta := \{\langle \bar{\gamma}', f_{\sim \ell+1}^k(\bar{\gamma}', \alpha_{\ell+1} \hat{\ } \beta_{\ell+1}) \rangle : \bar{\gamma}' < \kappa_\ell^{+1} \cap \delta\}$ , and such that  $\{r_{i,\delta}^{\alpha_{\ell+1} \hat{\ } \beta_{\ell+1}} : i < i(\alpha_{\ell+1} \hat{\ } \beta_{\ell+1}, \delta)\}$  is a maximal antichain of  $\prod_{j \leq \ell} P_j$  and  $\gamma_{i,\delta}^{\alpha_{\ell+1} \hat{\ } \beta_{\ell+1}}$  is a name of a function as above and

$$r_{i,\delta}^{\alpha_{\ell+1} \hat{\ } \beta_{\ell+1}} \cup \bigcup_{j=0}^{k-1} q_{(\alpha_{\ell+1+j+2}, t_{\ell+1+j+2})}^k (g(\ell+j+1)) \cup p_{\ell+k+1}^k (g(\ell+k+1)) \\ \Vdash h_{\ell+1}^k(\alpha_{\ell+1} \hat{\ } \beta_{\ell+1}) \upharpoonright \delta = \gamma_{i,\delta}^{\alpha_{\ell+1} \hat{\ } \beta_{\ell+1}}.$$

We define for  $\alpha_{\ell+1} < \kappa_{\ell+1}$  a unary function  $G_\ell^{k, \alpha_{\ell+1}}$  on  $\kappa_{\ell+1}^{+1}$ :

$$G_\ell^{k, \alpha_{\ell+1}}(\beta_{\ell+1}) = \{(r_{i,\delta}^{\alpha_{\ell+1} \hat{\ } \beta_{\ell+1}}, \gamma_{i,\delta}^{\alpha_{\ell+1} \hat{\ } \beta_{\ell+1}}) : i < i(\alpha_{\ell+1} \hat{\ } \beta_{\ell+1}, \delta), \delta < \kappa_\ell^+\}.$$

The range of  $G_\ell^{k, \alpha_{\ell+1}}$  has cardinality  $\leq \kappa_{\ell+1}$ , as  $i(\alpha_{\ell+1} \hat{\ } \beta_{\ell+1}) < \kappa_{\ell+1}$  because  $\prod_{j \leq \ell} P_j$  satisfies the  $\kappa_{\ell+1}$ -chain condition, and  $r_{i,\delta}^{\alpha_{\ell+1} \hat{\ } \beta_{\ell+1}} \in \prod_{j \leq \ell} P_j$ ,  $|\prod_{j \leq \ell} P_j| \leq \kappa_{\ell+1}$ , and for each  $\bar{\gamma}' \in f_{\sim \ell+1}^k(\bar{\gamma}', \alpha_{\ell+1} \hat{\ } \beta_{\ell+1})$  is a  $\prod_{j \leq \ell} P_j$ -name of an ordinal less than  $\kappa_{\ell+1}$ , and hence the number of possible  $h_{\ell+1}^k$  can also without loss be bounded by  $\kappa_{\ell+1}^{< \kappa_{\ell+1}} = \kappa_{\ell+1}$ .

Let  $B_{\alpha_{\ell+1}} = \{t \in I_{\ell+1} : G_\ell^{k, \alpha_{\ell+1}}$  has the same value, say  $h_{\ell+1}^k(\alpha_{\ell+1})$ , on all points from  $t\}$ . By definition

$$B_{\alpha_{\ell+1}} \in D_1(\kappa_{\ell+1}^{+1}, \kappa_{\ell+1}, \kappa_{\ell+1}) \subseteq D_{\ell+1}^+.$$

Thus  $B' = \{(\alpha, t) \in J_{\ell+1} : t \in B_\alpha\} \in F_{\ell+1}$ .

For every  $(\alpha, t) \in A_{\ell+1}^k$  choose a point from  $t$ , call it  $\beta_{\ell+1}$ , and find by induction on  $\delta < \kappa_\ell^+$  a condition  $q_{(\alpha,t)}^{k+1}(\delta)$  such that  $q_{(\alpha,t)}^k(\delta) \leq q_{(\alpha,t)}^{k+1}(\delta) \in P_\ell$  and such that  $q_{(\alpha,t)}^{k+1}(\delta) \geq q_{(\alpha,t)}^{k+1}(\delta')$  for  $\delta' < \delta$  and such that  $q_{(\alpha,t)}^{k+1}(\delta) \cup \bigcup_{j=0}^{k-1} q_{(\alpha_{\ell+1+j+2}, t_{\ell+1+j+2})}^k$

$(g(\ell + j) \cup p_{\ell+k+1}^k(g(\ell + k + 1)))$  forces for  $\delta < g(\ell)$  that  $\langle (\bar{y}, f_{\sim_{\ell+1}}^k(\bar{y} \hat{\wedge} \alpha \hat{\wedge} \beta_{\ell+1})) : \bar{y}$  an increasing finite sequence from  $\kappa_{\ell}^{+1} \cap \delta$  of length  $2\ell - 2$  be equal to some  $\prod_{j < \ell} P_j$ -name  $f_{\sim_{(\alpha,t)}}^k$  (possible as  $P_{\ell}$  is  $\kappa_{\ell}^{+1}$ -complete). Choose  $f_{\sim_{(\alpha,t)}}^k$  independent of  $\delta$ . If  $(\alpha, t) \in B'$  too, then the choice of  $\beta_{\ell+1}$  is immaterial. Now by Claim 2.1 applied to  $q_{(\alpha,t)}^{k+1}(\delta)$  and  $A_{\ell+1}^k \cap B'$ , we can find  $p_{\ell}^{k+1}(\delta)$  and  $A_{\ell+1}^{k+1} \subseteq B' \cap A_{\ell+1}^k$  with property (3), and as the number of possible  $f_{(\alpha,t)}^k$  is  $\leq \kappa_{\ell}^{+2}$  and as  $F_{\ell+1}$  is  $\kappa_{\ell+1}$ -complete, we can assume that  $f_{(\alpha,t)}^k = f_{\ell}^{k+1}$  for every  $(\alpha, t) \in A_{\ell+1}^{k+1}$ .

By now,  $f_{(\alpha,t)}^k$  may depend on  $(\alpha_{\ell+2}, t_{\ell+2}), \dots, (\alpha_{\ell+k+1}, t_{\ell+k+1})$ . But the number of possible  $f_{(\alpha,t)}^k$  is less or equal  $\kappa_{\ell}^{+2}$  and  $F_{\ell+1}$  is  $\kappa_{\ell+1}$ -complete, and so we can assume that  $f_{(\alpha,t)}^k = f_{\ell}^{k+1}$  for every  $(\alpha, t) \in A_{\ell+1}^{k+1}$ . Furthermore, since  $F_{\ell+2}, \dots, F_{\ell+k+1}$  are all  $\kappa_{\ell+2}$ -complete, we can choose  $A_{\ell+j}^{k+1} \subseteq A_{\ell+j}^k$ ,  $A_{\ell+j}^{k+1} \in F_{\ell+j}$  such that  $f_{(\alpha,t)}^k$  is the same for all  $(\alpha_{\ell+2}, t_{\ell+2}) \in A_{\ell+2}^{k+1}, \dots, (\alpha_{\ell+k+1}, t_{\ell+k+1}) \in A_{\ell+2}^{k+1}$ . Hence, as required in (4''), there is indeed one name  $f_{\ell}^{k+1}$  working for all conditions  $\bar{p}^{k+1}(g)$  for various  $g$ 's.

We define  $A_{\ell+1}^{\omega} = \bigcap_{k < \omega} A_{\ell+1}^k$ ,  $q_{(\alpha,t)}^{\omega}(\delta) = \bigcup_{k < \omega} q_{(\alpha,t)}^k(\delta)$  and  $p_{\ell}^{\omega}(\delta) = \bigcup_{k < \omega} p_{\ell}^k(\delta)$  for  $(\alpha, t) \in A_{\ell}^{\omega}$ . As each  $F_{\ell+1}$  is  $\kappa_{\ell+1}$ -complete,  $A_{\ell+1}^{\omega} \in F_{\ell+1}$ . It is also clear that  $p_{\ell}^{\omega}(\delta) \in P_{\ell}$  and  $q_{(\alpha,t)}^{\omega}(\delta) \in P_{\ell}$  for  $(\alpha, t) \in A_{\ell+1}^{\omega}$ .

Now choose as in the proof of Lemma 1.7 applied to the results of Claim 2.2 by induction on  $\ell \geq 0$  ordinals  $g(\ell) \in \kappa_{\ell}^{+}$  and  $\alpha_{\ell+1}^i \in \kappa_{\ell+1}$ ,  $i < \aleph_{2\ell}$  (which is the predecessor of the predecessor of the collapsed  $\kappa_{\ell+1}$ ) out of the  $E_{\ell+1}$ -many  $\alpha$  such that

$$P(\alpha_{\ell+1}^i) := \{t_{\ell+1} \in I_{\ell+1} : (\alpha_{\ell+1}^i, t_{\ell+1}) \in A_{\ell+1}^{\omega}\} \in D_{\ell+1}^*$$

such that the  $q_{(\alpha_{\ell+1}^i, t_{\ell+1})}^{\omega}(g(\ell))$ ,  $i < \aleph_{2\ell}$  are all compatible.

We start with  $g(0) = \aleph_0$ .

In the step  $\ell$ , given  $g(\ell)$ , we choose  $\alpha_{\ell}^i \in \kappa_{\ell+1}$ ,  $i < \aleph_{2\ell}$ , out of the  $E_{\ell+1}$ -many  $\alpha$  such that

$$P(\alpha_{\ell+1}^i) := \{t_{\ell+1} \in I_{\ell+1} : (\alpha_{\ell+1}^i, t_{\ell+1}) \in A_{\ell+1}^{\omega}\} \in D_{\ell+1}^*$$

such that the  $q_{(\alpha_{\ell+1}^i, t_{\ell+1})}^{\omega}(g(\ell))$ ,  $i < \aleph_{2\ell+1}$  are all compatible. Now take  $t_{\ell+1} \in \bigcap_{i < \aleph_{2\ell+1}} P(\alpha_{\ell+1}^i)$ . Choose  $g(\ell + 1) < \kappa_{\ell+1}^{+}$  such that

$$|\text{range}(t_{\ell+1}) \cap g(\ell + 1)| \geq \kappa_{\ell+1}.$$

Now let for  $\ell < \omega$ ,

$$p^1 = \left\langle \bigcup_{i < \aleph_0} q_{(\alpha_1^i, t_1)}^{\omega}(\aleph_0), \bigcup_{i < \aleph_2} q_{(\alpha_2^i, t_2)}^{\omega}(g(1)), \bigcup_{i < \aleph_4} q_{(\alpha_3^i, t_3)}^{\omega}(g(2)) \dots \right\rangle$$

and  $S_{2\ell+1} = \{\alpha_{\ell+1}^i : i < \aleph_{2\ell+1}\}$ ,  $S_{2\ell+2} = t_{\ell+1}$ . It is easy to check, with the help of (4''), that they are as required.  $\square_{2.1}$

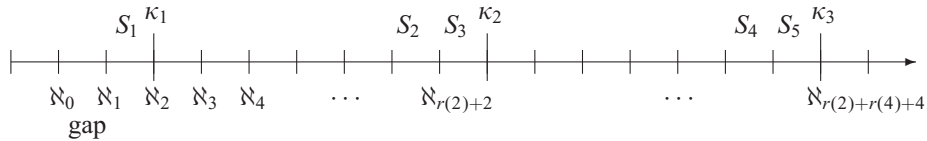
The following theorem is the generalization of Theorem 2.1 to arities as in Theorem 1.2:

**Theorem 2.3.** *Suppose the consistency of “ZFC + GCH + there are infinitely many compact cardinals”, and that  $r : \omega \setminus \{0\} \rightarrow \omega \setminus \{0\}$  with  $r(2n + 1) = 1$ . Then the following is consistent: ZFC + GCH +  $\langle \aleph_{k_1(n)} : 0 < n < \omega \rangle$  has  $\langle \aleph_{k_2(n)} : 0 < n < \omega \rangle$ -canonical forms for*

$$\Gamma = \left\langle \left( \langle \langle 1 \rangle^{r(n)}, \langle 1 \rangle^{r(n+1)} \dots, \sum_{i=n}^m r(i), \aleph_n, \langle n, n + 1, \dots, m \rangle \right) : i = (n, m), 1 \leq n \leq m < \omega \right\rangle$$

with  $k_1(2n) = 2 + r(2) + 1 + r(4) + 1 + \dots + 1 + r(2n)$ ,  $k_1(2n + 1) = 2 + r(2) + 1 + \dots + r(2n) + 1$ ,  $k_2(2n) = k_1(2n) - r(2n) + 1$ ,  $k_2(2n + 1) = k_1(2n + 1) - 2$ .

**Proof.** Again we sketch a picture of the locations:



We draw the  $S_{2n}$  towards the end of their interval, the collapsed  $[\aleph_n, \aleph_n^{+r(2n)})$ , which is  $[\aleph_{1+\sum_{i<n} (r(2i)+1)}, \aleph_{1+\sum_{i\leq n} (r(2i)+1)})$ . The proof is very similar to the one of Theorem 2.1, only that this time we carry the notation  $r(2n)$  for the lengths of the  $2n$ th tuples of indiscernibles all the time with us. The partition theorem invoked to show that the filters corresponding to the  $F_\ell^+$  in the proof of 2.1 are sufficiently complete is as in Theorem 1.2 the Erdős Rado Theorem.  $\square$

In contrast to Theorem 2.2, it is open whether Theorem 2.3 suffices to give dense free subsets. The  $S_{2n}$  need to give rise to sets that are spread over an interval of  $r(2n)$  cardinal steps and still have strong indiscernibility properties. Under additional premises one can establish a “spreading procedure”, see Theorem 3.7. The premise there is the following strengthening of Theorem 2.3, in which  $S_{2n}$  will be  $r(n) - o$  indiscernible for functions  $f_{n,m,o}$  with range  $\aleph_{k_2(n)-1+o}$  for  $o < r(n)$ ,  $1 \leq n \leq m < \omega$ .

**Theorem 2.4.** *Suppose the consistency of “ZFC + GCH + there are infinitely many compact cardinals”, and that  $r : \omega \setminus \{0\} \rightarrow \omega \setminus \{0\}$  with  $r(2n + 1) = 1$ . Then the following is consistent: ZFC + GCH +  $\langle \aleph_{k_1(n)} : 0 < n < \omega \rangle$  has  $\langle \aleph_{k_2(n)} : 0 < n < \omega \rangle$ -canonical forms for*

$$\Gamma = \left\{ \left( \langle \langle 1 \rangle^{r(n)-o}, \langle 1 \rangle, \langle 1 \rangle^{r(n+2)} \dots, \left( \sum_{i=n}^m r(i) \right) - o, \aleph_{k_2(n)-1+o}, \langle n, n + 1, \dots, m \rangle \right) : i = (n, m, o), 1 \leq n \leq m < \omega, o < r(n), n \text{ even} \right\}$$

$$\cup \left\{ \left( \langle \langle 1 \rangle, \langle 1 \rangle^{r(n+1)} \dots \rangle, \sum_{i=n}^m r(i), \aleph_{k_2(n)}, \langle n, n+1, \dots, m \rangle \right) \right. \\ \left. : i = (n, m), 1 \leq n \leq m < \omega, n \text{ odd} \right\}$$

with  $k_1(2n) = 2+r(2)+1+r(4)+1+\dots+1+r(2n)$ ,  $k_1(2n+1) = 2+r(2)+1+\dots+r(2n)+1$ ,  $k_2(2n) = k_1(2n) - r(2n) + 1$ ,  $k_2(2n+1) = k_1(2n+1) - 2$ .

**Proof.** So we need to work with an improved version of the filters from Definition 1.3 and simultaneous colouring theorems. Our filters will now describe a homogeneous set for functions with different arities and ranges at the same time: the higher the arity, the smaller the range. The highest arity was already used in the previous theorem.

**Definition 2.5.** For cardinals  $\lambda$ ,  $\mu$ ,  $\chi$  and  $r < \omega$  let  $D_r^*(\lambda, \mu, \chi)$  be the following filter:

- (a) It is a filter over the set  $\text{Inc}(\lambda, \mu)$  of increasing sequences of length  $\mu$  of ordinals  $< \lambda$ .
- (b) The filter is generated by the set of generators, where a generator is

$$\text{Ge}(F) = \text{Ge}_{r,o}(F; \lambda, \mu, \chi) \\ = \{ \bar{a} \in \text{Inc}(\lambda, \mu) : (\exists \alpha \in \chi) \forall i(0) < \dots < i(r-1-o) < \mu, \\ F(a_{i(0)}, \dots, a_{i(r-1-o)}) = \alpha \},$$

for some  $F : [\lambda]^{r-o} \rightarrow \chi^{+o}$  for some  $o < r$ .

**Claim 2.6.** (1) If  $\chi^{+o} = (\chi^{+o})^{< \kappa}$ , then the intersection of  $< \kappa$  generators of  $D_r(\lambda, \mu, \chi)$  is a generator; hence the filter  $D_r(\lambda, \mu, \chi)$  is  $\kappa$ -complete.

(2) If  $\lambda \rightarrow (\mu)_{\chi^{+o}}^{(r-o)}$  for all  $o < r$  (the usual partition relation) then  $D_r(\lambda, \mu, \chi)$  is a proper filter.

Let  $\ell \geq 1$ . The objects indexed by  $\ell$  will be used for finding  $S_{2\ell-1}$  and  $S_{2\ell}$ . Let  $E_\ell$  be a normal ultrafilter over  $\kappa_\ell$ . Let  $I_\ell = \text{Inc}(\kappa_\ell^{+r(2\ell)}, \kappa_\ell^{+1})$  and  $J_\ell = \kappa_\ell \times I_\ell$ . Note that  $D_{r(2\ell)}^*(\kappa_\ell^{+r(2\ell)}, \kappa_\ell^{+1}, \kappa_\ell)$  is a  $\kappa_\ell$ -complete proper filter, as the Erdős Rado Theorem together with the GCH yield

$$\kappa_\ell^{+r(2\ell)} = (\beth_{r(2\ell)-1}(\kappa_\ell))^+ = (\beth_{r(2\ell)-1-o}(\kappa_\ell^{+o}))^+ \rightarrow (\kappa_\ell^{+1+o})_{\kappa_\ell^{+o}}^{r(2\ell)-o}.$$

So, as  $\kappa_\ell$  is compact there is a  $\kappa_\ell$ -complete ultrafilter  $D_\ell^{**}$  over  $I_\ell$  extending  $D_{r(2\ell)}^*(\kappa_\ell^{+r(2\ell)}, \kappa_\ell^{+1}, \kappa_\ell)$ . We set

$$F_\ell^* = E_\ell \times D_\ell^{**} = \{ A \subseteq J_\ell : \{ i < \kappa_\ell : \{ t \in I_\ell : (i, t) \in A \} \in D_\ell^{**} \} \in E_\ell \}.$$

Again we call  $f : J_\ell \rightarrow \kappa_\ell$  almost regressive if it is regressive on some  $A \in F_\ell^*$ . Similarly we define when  $f$  is constant, constant on  $A$  and almost constant.

Now we can continue as in Theorems 2.2 and 2.3.  $\square_{2.4}$

### 3. Applications to free subsets

In this section we first recall some definitions, and then we recall how free subsets can be chosen from sequences of indiscernible sets. In the model from Theorem 2.2, there are free subsets with one point in every infinite cardinal interval in  $\aleph_\omega$  but  $[\aleph_0, \aleph_1)$ . So this almost solves the old problem. Then we give a rather combinatorial result, how to shift indiscernible sets that are indiscernible in a stronger sense, and still be able to build a free subset whose members are picked from the images of the shifts. Applying this result in a model of the canonization property from Theorem 2.4, we will show the existence of free subsets having one point in every infinite cardinal interval but  $[\aleph_0, \aleph_1)$ . Finally, our technique allows in a last step to add one point in  $[\aleph_0, \aleph_1)$  and preserve freeness.

**Definition 3.1.** Let  $\mathfrak{A}$  be a  $\tau$ -structure over  $A$ . A subset  $S \subseteq A$  is called free subset of  $\mathfrak{A}$  if for every  $s \in S$ ,  $s$  is not in the range of a composition of functions of  $\mathfrak{A}$  applied to constants and  $S \setminus \{s\}$ . A tuple is free if its range is free. Let  $\mu, \kappa, \lambda$  be cardinals. By  $\text{Fr}_\mu(\lambda, \kappa)$  we abbreviate the following: For every  $\tau$ -structure  $\mathfrak{A}$  with  $|\tau| \leq \mu$  and  $|A| \geq \lambda$  there is a free subset  $S$  of  $\mathfrak{A}$  of size at least  $\kappa$ .

Shelah [9] proved the consistency of  $\text{Fr}_\omega(\aleph_\omega, \omega)$  from countably many measurable cardinals, and Kunen proved that  $\text{Fr}_\omega(\aleph_\omega, \omega)$  implies that  $V \neq L$ . Later Koepke [6] improved both results by proving that  $\text{Fr}_\omega(\aleph_\omega, \omega)$  is equiconsistent with one measurable cardinal. Shelah's as well as Koepke's proofs use as an intermediate step the relative consistency of  $\text{ZFC} + \text{GCH} + \langle \aleph_{2n} : 0 < n < \omega \rangle$  has  $\langle \aleph_{2n} : 0 < n < \omega \rangle$ -canonical forms for

$$\Gamma = \{ \langle \langle \langle 1 \rangle^{(m-n+1)}, m-n+1, \aleph_{2n-1}, \langle n, n+1, \dots, m \rangle \rangle \} \\ : i = (n, m), 1 \leq n \leq m < \omega \}. \tag{3.1}$$

We recall how to get  $\text{Fr}_\omega(\aleph_\omega, \omega)$  from this. Suppose  $V$  is a model in which the latter canonization property holds and  $\mathfrak{A}$  is a structure on  $\aleph_\omega$  with countable signature  $\tau = \{g_i : i \in \omega\}$  such that  $\tau$  is closed under compositions of functions. Without loss of generality,  $\tau$  does not contain relation symbols, since relations of  $\mathfrak{A}$  do not have any influence on freeness.

Let  $n(g_i)$  be the arity of  $g_i$ . For the  $f_{(1, \cdot)}$  from Eq. (1.4) we take the function  $f = f_{\mathfrak{A}}$  by

$$f : [\aleph_\omega]^{<\omega} \rightarrow 2, \\ f(\alpha_0, \dots, \alpha_{r-1}) = \begin{cases} 0, & \text{if } \{\alpha_0, \dots, \alpha_{r-1}\} \text{ is free in } \mathfrak{A}, \\ \min\{(i, \ell) : \alpha_\ell \in g_i[[\{\alpha_k : k \neq \ell\}]^{n(g_i)}]\}, & \text{else,} \end{cases} \tag{3.2}$$

and get by property (3.1) some  $\langle S_n : 1 \leq n < \omega \rangle$  that is  $\langle 1 \rangle^{(\omega)}$ -indiscernible for it. Then it is easy to see that for any  $i_1 < i_2 < \dots < i_r < \omega$  for any  $s_{i_k} \in S_{i_k}$ ,  $f(s_{i_0}, \dots, s_{i_r}) = 0$ . Hence, if we take one  $s_n$  out of each  $S_n$ , we get a free set  $\{s_n : n \in \omega\}$ .

So the free subsets witnessing  $\text{Fr}_\omega(\aleph_\omega, \omega)$  in the known models of (3.1) contain one point in every second cardinal interval and leave it open whether one could get free subsets witnessing  $\text{Fr}_\omega(\aleph_\omega, \omega)$  containing one point in each cardinal interval. By old results of Devlin et al. this would be optimal: Suppose that  $\mathfrak{A}$  is a  $\tau$ -structure on  $\aleph_\omega$ , and that  $|\tau| \leq \aleph_0$ .

**Choice 3.2.** For every  $\alpha < \aleph_\omega$ , let  $h_\alpha : \alpha \rightarrow |\alpha|$  be a bijection. We suppose that the binary partial function  $h \in \tau$ , interpreted by  $h^\mathfrak{A}(\alpha, \beta) = h_\alpha(\beta)$  for  $\beta < \alpha$  belongs to every structure  $\mathfrak{A}$  we consider. Moreover,  $\mathfrak{A}$  shall also contain the reverse function  $\hat{h}$  such that  $\hat{h}(h_\alpha(\beta), \alpha) = \beta$ .

We assume that  $h$  and  $\hat{h}$  belong to  $\mathfrak{A}$ . If we furthermore assume that each  $n \in \omega$  is a constant in  $\mathfrak{A}$ , then the structures built in [1] show, that no free subset of  $\mathfrak{A}$  can contain more than one point in each interval  $[\aleph_k, \aleph_{k+1})$ .

Given  $\mathfrak{A}$  we take  $f_\mathfrak{A}$  as in Eq. (3.2), then the same considerations show

**Fact 3.3.** If  $\langle S_n : 1 \leq n < \omega \rangle$  are  $\langle r(n) : 1 \leq n < \omega \rangle$ -indiscernibles for  $f_\mathfrak{A}$ , then picking at most  $r(n)$  points from each  $S_n$  and putting the points together into one set yields a free subset in  $\mathfrak{A}$ .

This fact can be used in the models of canonization properties from the previous sections, and thus shows that in these models there are new patterns of free subsets. Among these possibilities we name the one that motivated our present work:

**Theorem 3.4.** Assume the consistency of “ZFC + GCH + there are infinitely many compact cardinals”. Then the following is consistent: ZFC + GCH + every structure on  $\aleph_\omega$  with countable signature has a free subset that contains one point in  $[\aleph_n, \aleph_{n+1})$  for every  $n \geq 1$ .

**Proof.** We apply Fact 3.3 for  $r(n) = 1$  to the canonization property that holds in the model from Theorem 2.1.  $\square$

There is a minor blemish in this result: the interval  $[\aleph_0, \aleph_1)$  does not contain a member of a free subset so far, although this is not forbidden in the mentioned structures  $\mathfrak{A}$  given by Devlin et al. This will be remedied by a fact that comes out as a byproduct of our next steps.

Now we want to spread the sets  $S_n$  of  $(r(n) - o)$ -indiscernibles for function with ranges  $\aleph_{k_2(n)+o}$  downwards on the cardinal scale, and thereafter pick  $r(n)$  points, such that combining the choices made for every  $n \geq 1$  we get free subset. We will show how to get in this way a dense free subset from the canonization property of Theorem 2.4, even one have a point in  $[\aleph_0, \aleph_1)$ . For this  $f_\mathfrak{A}$  from Eq. (3.1) will be replaced in Definition 3.6 by functions describing more of  $\mathfrak{A}$ , which corresponds, roughly speaking,

to introducing constant names for all ordinals smaller than the cardinality of the smallest argument. To get such a suitable description we first define the dropping operations  $\text{drop}(\bar{\alpha}, k)$ ,  $\widehat{\text{drop}}(\bar{\alpha}, k)$ ,  $k < \text{lg}(\bar{\alpha})$ .

**Definition 3.5.** We fix  $h$  as in the Choice 3.2. For  $\bar{\alpha} = \langle \alpha^i : i < \text{lg}(\bar{\alpha}) \rangle$ , a strictly increasing sequence of ordinals, we define by induction on  $k \leq \text{lg}(\bar{\alpha})$ ,  $\text{drop}(\bar{\alpha}, k) = \langle \text{drop}(\bar{\alpha}, k, i) : i < \text{lg}(\bar{\alpha}) \rangle$ .

$$\begin{aligned} \text{drop}(\bar{\alpha}, 0) &= \bar{\alpha}, \\ \text{for } i \leq k, \widehat{\text{drop}}(\bar{\alpha}, k + 1, i) &= h_{\text{drop}(\bar{\alpha}, k + 1)}(\text{drop}(\bar{\alpha}, k, i)), \\ \text{for } i > k, \widehat{\text{drop}}(\bar{\alpha}, k + 1, i) &= \text{drop}(\bar{\alpha}, k, i), \\ \text{drop}(\bar{\alpha}, k + 1, i) &= \text{the increasing enumeration of } \widehat{\text{drop}}(\bar{\alpha}, k + 1, i). \end{aligned}$$

**Remark.** 1.  $\langle \text{drop}(\bar{\alpha}, k, i) : i < \text{lg} \alpha \rangle$ , is determined by  $h$  and  $\bar{\alpha}$ .

2. So, if we start with  $\bar{\alpha}$  all of whose  $\alpha_i$  are in the same  $[\aleph_{r-1}, \aleph_r)$  but pairwise different, then  $\text{drop}(\alpha, k, i) \in \aleph_{r-\text{lg}(\bar{\alpha})+i}$  for  $k \geq \text{lg}(\bar{\alpha}) - i$ ,  $0 \leq i < \text{lg}(\bar{\alpha})$ , so  $\text{drop}(\bar{\alpha}, \text{lg}(\bar{\alpha}) - 1)$  is spread out over the  $\text{lg}(\bar{\alpha})$  intervals whose highest one is the initial one.

3. There are  $\prod_{1 \leq i < \text{lg}(\bar{\alpha})} (i - 1)!$  possible permutations on the transitions from  $\bar{\alpha}$  to  $\text{drop}(\bar{\alpha}, k)$ . We get a colouring on the  $\text{lg}(\bar{\alpha})$ -tuples with these colours by taking the sequence of permutations that appeared as a colour. Then we may use the Erdős Rado theorem. We assume that  $[S_n]^{r(n)}$  is already monochromatic under this colouring. (This can be taken into the  $f_i$ ). And the monochromatic colour is the product of the identity permutation, if  $S_n$  is infinite, because of the well order.

4. The fifth highest entry of  $\text{drop}(\bar{\alpha}, k)$   $k \geq 5$ , depends only on the on the five highest entries of  $\bar{\alpha}$  and is not changed after the fifth step anymore, and so on, more formally:  $(\forall z \geq o)(\forall k, k' \geq \text{lg}(\bar{\alpha}) - z) \text{drop}(\langle \alpha_o, \dots, \alpha_{r(n)-1} \rangle, k, z) = \text{drop}(\langle \alpha_o, \dots, \alpha_{r(n)-1} \rangle, k', z)$ .

Now we shall apply the canonization property from Theorem 2.4 to the following functions:

**Definition 3.6.** Suppose we are given a  $\{g_i : i \in \omega\}$ -structure  $\mathfrak{A}$  on  $\aleph_\omega$ . For even  $n$ ,  $o < r(n)$ ,  $n \leq m < \omega$ , we let

$$\begin{aligned} f_{n,m,o} : A_n^{r(n)-o} \times A_{n+1} \times A_{n+2}^{r(n+2)} \times \dots \times A_{m-1} \times A_m^{r(m)} &\rightarrow \aleph_{k_2(n)+o} \\ f_{n,m,o}(\alpha_{n,o}, \dots, \alpha_{n,r(n)-1}, \overline{\alpha_{n+1}}, \dots, \overline{\alpha_m}) & \\ = \begin{cases} \min\{(j, \bar{\beta}) : \bar{\beta} \in \aleph_{k_2(n)+o}, j \in \omega \wedge \\ g_j(\bar{\beta} \widehat{\text{drop}}(\langle \alpha_{n,o+1}, \dots, \alpha_{n,r(n)-1} \rangle, r(n) - o) \\ \widehat{\text{drop}}(\overline{\alpha_{n+1}}, r(n+1)) \dots \widehat{\text{drop}}(\overline{\alpha_m}, r(m))) \\ = \text{drop}(\langle \alpha_{n,o}, \dots, \alpha_{n,r(n)-1} \rangle, r(n) - o, o) \} & \text{if nonempty} \\ 0 & \text{else.} \end{cases} \end{aligned}$$

For odd  $n$ , choose  $f_{n,m}$  analogously. The minimum is taken with respect to some well-order  $\leq$  on the class of finite sequences of ordinals. Strictly speaking, the range is  $\omega \times [\aleph_{k_2(n)+o}]^{<\omega}$ , but this is identified with  $\aleph_{k_2(n)+o}$ .

**Theorem 3.7.** *Let  $S_n \subseteq [\aleph_{k_1(n-1)}, \aleph_{k_1(n)}]$  witness the canonization property from Theorem 3.6 applied to the functions from Definition 3.6. Suppose that  $\bar{\alpha}_n \in [S_n]^{r(n)}$ . Then  $F = \{\text{drop}(\bar{\alpha}_n, r(n), i) : i < r(n), n < \omega\}$  is free.*

**Proof.** Suppose that there is a dependence, say there is some  $g_j \in \tau$  there are  $n, k$ ,  $\bar{\alpha}_{n+i} \in [S_{n+i}]^{r(n+i)}$ ,  $i < k$ , such that

$$\begin{aligned} g_j(\text{drop}(\bar{\alpha}_n, r(n)), \dots, \text{drop}(\bar{\alpha}_{n+i}, r(n+i), \delta), \dots, \text{drop}(\bar{\alpha}_{n+k}, r(n+k))) \\ = \text{drop}(\bar{\alpha}_{n+i}, r(n+i), o), \end{aligned} \quad (3.3)$$

where check means argument left out. Thus for some

$$(j', \bar{\beta}) \leq (j, \text{drop}(\bar{\alpha}_n, r(n)) \hat{\ } \dots \hat{\ } (\text{drop}(\bar{\alpha}_{n+1}, r(n+1)) \uparrow o)),$$

we have that

$$f_{n+i,n+k,o}(\alpha_{n+i,o}, \alpha_{n+i,o+1}, \dots, \alpha_{n+i,r(n+i)-1}, \bar{\alpha}_{n+i+1}, \dots, \bar{\alpha}_{n+k}) = (j', \bar{\beta}) \neq 0. \quad (3.4)$$

Then  $\alpha_{n+i,o}$  drops to  $\text{drop}(\bar{\alpha}_{n+i}, r(n), o)$  in  $r(n+i) - o$  steps (the order is never reversed in the fourth line of the description of drop) and  $S_{n+i}$  is homogenized for the dropping operation as in Remark 3. Then  $\text{drop}(\bar{\alpha}_{n+i}, u, o)$  is never the highest of  $\text{drop}(\bar{\alpha}_{n+i}, u, p)$ ,  $u \leq r(n+i)$ ,  $p > o$ .

Then by the indiscernibility of the  $S_n$ ,  $\alpha_{n+i,o}$  can range over the whole  $S_n$  below  $\alpha_{n+i,o+1}$ , and thus the same  $c$  together with the fixed higher  $\alpha_{n+i,o'} > o$  and  $\bar{\alpha}_{n+i+1}, \dots, \bar{\alpha}_{n+k}$  would also generate all pairwise different  $\text{drop}(\bar{\alpha}_{n+i}, r(n+i), o)$ 's, depending on  $\alpha_{n+i,o}$ , and on the same time would produce all the same  $\text{drop}(\bar{\alpha}_{n+i}, r(n+i), o')$ .  $o' > o$  and  $\text{drop}(\bar{\alpha}_{n+i+1}, r(n+i+1)), \dots, \text{drop}(\bar{\alpha}_{n+i+1}, r(n+i+k))$  by the indiscernibility property equation (3.4). Then (3.4) read for all its arguments but  $\alpha_{n+i,o}$  fixed, and  $\alpha_{n+i,o}$  ranging over two different values in  $S_{n+i}$  gives us just as many different right-hand sides in (3.3), that can be generated by one constant element and the function  $g_{j'}$ . Contradiction.  $\square$

Finally, how do we fill the interval  $[\aleph_0, \aleph_1]$ ? In the model gotten in Theorem 3.7 with the  $F$  from Theorem 3.7, we add a constant from  $[\aleph_0, \aleph_1] \setminus \{g_i(\bar{\alpha}) : \alpha \in F\}$  to the free set  $F$ . It stays free, because the proof of Theorem 3.7 shows that  $F$  was free even if every element of  $[\aleph_0, \aleph_1]$  has a constant name in  $\mathfrak{A}$ . So also the gap  $[\aleph_0, \aleph_1]$  is filled. Finitely many points could be added by stepping downward on the cardinal scale in this way.

**Conclusion 3.8.** *Assume the consistency of “ZFC + GCH + there are infinitely many compact cardinals”. Then the following is consistent: ZFC + GCH + every structure on  $\aleph_\omega$  with countable signature has a free subset that contains one point in  $[\aleph_n, \aleph_{n+1}]$  for every  $n \geq 0$ .*



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