ON SPLITTING TREES

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Abstract. We investigate two variants of splitting tree forcing, their ideals and regularity properties. We prove connections with other well-known notions, such as Lebesgue measurablility, Baire- and Doughnut-property and the Marczewski field. Moreover, we prove that any *absolute* amoeba forcing for splitting trees necessarily adds a dominating real, providing more support to Hein's and Spinas' conjecture that $\mathrm{add}(\mathcal{I}_{\mathbb{SP}}) \leq \mathfrak{b}$.

1. Introduction

Trees and their associated forcing notions have been a crucial ingredient in set theory of the reals, specifically in questions concerning cardinal characteristics and regularity properties. The most popular such forcings are certainly $\mathbb{S}$, $\mathbb{M}$, $\mathbb{V}$, $\mathbb{L}$ and $\mathbb{MA}$ (Sacks, Miller, Silver, Laver and Mathias), but also other notions have played an important role; among them, there is some tradition in studying tree-forcing adding an *ω-splitting real*[1]. Spinas [14] introduced splitting tree forcing $\mathbb{SP}$, which has recently been studied also in [6]. We also investigate another form of splitting-tree forcing (called $\mathbb{FSP}$, Definition 1.3) somehow related to Spinas' one.

A notion of an ideal of *small sets* can be introduced when dealing with any tree-forcing notion, as specified in 3.1. For any such ideal one can associate the common cardinal characteristics, namely the covering, the additivity, the cofinality and the uniformity number. Furthermore, a notion of measurability (and weak-measurability) generalizing the well-known Lebesgue-measurability and Baire property can be established when dealing with any type of tree-forcing notion (Definition 3.1). It is well-known that in Solovay model any subset of the real line is $\mathbb{P}$-measurable, for a large variety of tree-forcing notions $\mathbb{P}$, including $\mathbb{SP}$ and $\mathbb{FSP}$.

In the analysis of the associated additivity numbers a crucial role is played by the so-called amoeba forcings, which are posets adding *generic trees*, see Definition 3.6. In our paper we address a question raised in [6] by Hein and Spinas related to the additivity number of the ideal $\mathcal{I}_{\mathbb{SP}}$ and the amoeba forcing for $\mathbb{SP}$. Even if we do not obtain a complete answer to the conjecture posed by Hein and Spinas, in Section 3, Proposition 3.7 and in Remark 3.9 we give more evidence to support such a conjecture, by showing that not only the natural amoeba for $\mathbb{SP}$ adds a dominating real, but that any *absolute* amoeba necessarily adds a dominating real.

Section 4 contains a brief digression on Silver forcing, and connections between Silver-amobea and Cohen reals, in line with [15].

In Section 5, we show some differences between the ideal $\mathcal{I}_{\mathbb{FSP}}$ and the ideal of null sets.

[1] An ω-splitting real is a real x in the forcing extension such that for any set $\{r_n : n < \omega\}$ in the ground model that contains infinite sets r_n, for each n we have that $r_n \cap x$ and $r_n \cap x^c$ are both infinite.

Our results in section 6 pertain only to the fat splitting forcing. We show that for any f-slalom in the ground model there is an $\mathbb{FSP}$-name that evades it. This is a strong negation of the Sacks property.

We prove in section 7 that actually the weak form of $\mathbb{SP}$- and $\mathbb{FSP}$-measurability for all sets of reals can be reached in a much simpler model, namely the $L(\mathbb{R})$ of the forcing extension obtained by a countable support iteration of Cohen forcing. We conclude with an application of Shelah's amalgamation of forcing method to fat splitting trees. Using evasion for slaloms of width $n \mapsto 2^{kn}$, $k \geq 1$ we separate the regularity properties of $\mathbb{FSP}$ from others.

In the remainder of this introduction, we set up our notation and end with one property of fat splitting.

Definition 1.1.

(a) Let X be a non-empty set. We let $X^{<\omega} = \{s \,:\, (\exists n \in \omega)(s\colon n \to X)\}$. The set $X^{<\omega}$ is partially ordered by the initial segment relation $\trianglelefteq$, namely $s \trianglelefteq t$ if $s = t{\restriction}\operatorname{dom}(s)$. We use $\triangleleft$ for the strict relation. For $s \in X^{<\omega}$ we let $\operatorname{dom}(s) = |s|$ be its domain.

(b) A set $p \subseteq 2^{<\omega}$ is called a *tree* if it is closed under initial segments, i.e. $t \in p \wedge s \trianglelefteq t \to s \in p$. The elements of p are called *nodes*.

(c) For a node $s \in p$ we let $\operatorname{Suc}_p(s) = \{t \in p \,:\, (s \triangleleft t \wedge |t| = |s| + 1)\}$ be the set of immediate successor nodes. A node is called a *splitting node* if it has two immediate successors in p.

(d) A tree p is called *perfect* if for every $s \in p$ there is a splitting node $t \trianglerighteq s$.

(e) We write $\operatorname{Split}(p)$ for the set of splitting nodes of p.

(f) For $t \in p$, we write $\operatorname{splsuc}(t)$ for the shortest splitting node extending t. When t is splitting, then $\operatorname{splsuc}(t) = t$.

(g) The *stem of* p, short $\operatorname{stem}(p)$, is the $\trianglelefteq$-least splitting node of p.

(h) We define the splitting degree $\operatorname{ot}^p\colon \operatorname{Split}(p) \to \omega$ recursively as follows:
- $\operatorname{ot}^p(\operatorname{stem}(p)) = 0$
- for every $i \in \{0,1\}$ and $t \in \operatorname{Split}(p)$ with $\operatorname{ot}^p(t) = n$, put $\operatorname{ot}^p(\operatorname{splsuc}(t^\frown i)) = n + 1$.

(i) For $n \in \omega$ let $\operatorname{Split}_n(p) = \{t \in \operatorname{Split}(p) \,:\, \operatorname{ot}^p(t) = n\}$ and $\operatorname{Split}_{\leq n}(p) = \{t \in \operatorname{Split}(p) \,:\, \operatorname{ot}^p(t) \leq n\}$.

(j) For $n \in \omega$, let $\operatorname{Lev}_n(p) := \{t \in p \,:\, |t| = n\}$.

(k) For $t \in p$ we let $p{\restriction}t = \{s \in p \,:\, s \trianglelefteq t \vee t \triangleleft s\}$.

(l) For $F \subseteq p$ we let $p{\restriction}F = \{s \in p \,:\, (\exists t \in F)(s \trianglelefteq t \vee t \triangleleft s)\}$.

(m) For $F \subseteq p$ we let $\operatorname{ter}(F) = \{s \in F \,:\, (\neg\exists t \in F)(s \triangleleft t)\}$. This is the set of terminal nodes of F.

(n) For each perfect tree $p \subseteq 2^{<\omega}$ we have a canonical splitting and lexicographical order preserving homomorphism

$$h\colon \operatorname{Split}(p) \to 2^{<\omega}$$

that is defined by induction on n for arguments in $\operatorname{Split}_n(p)$ as follows

$$h(\operatorname{stem}(p)) = \emptyset$$

$$h(\operatorname{splsuc}_p(t^\frown i)) = h(t)^\frown i \text{ for } t \in \operatorname{Split}(p), i \in 2.$$

(o) We let $\bar{H}$ denote h^{-1} and we let its lifting $H\colon 2^\omega \to [p]$ be defined by $H(f) = \bigcup\{\bar{H}(f{\restriction}n) \,:\, n < \omega\}$.

(p) The body or rump of a tree $p \subseteq 2^{<\omega}$, short $[p]$, is the set $\{f \in 2^\omega \,:\, (\forall n)(f{\restriction}n \in p)\}$.

Spinas [14] introduced splitting trees in order to analyse analytic splitting families.

Definition 1.2. A tree $p \subseteq 2^{<\omega}$ is called *splitting tree*, short $p \in \mathbb{SP}$, if for every $t \in p$ there is $k \in \omega$ such that for every $n \geq k$ and every $i \in \{0,1\}$ there is $t' \in p$, $t \trianglelefteq t'$ such that $t'(n) = i$. We denote the smallest such k by $K_p(t)$. The set $\mathbb{SP}$ is partially ordered by $q \leq_{\mathbb{SP}} p$ iff $q \subseteq p$.

Of course, every splitting tree is perfect.

Now we introduce a relative of splitting tree forcing that has stronger splitting properties.

Definition 1.3. A perfect tree $p \subseteq 2^{<\omega}$ is called *a fat splitting tree* ($p \in \mathbb{FSP}$) iff for every $t \in p$ there is $k \in \omega$ such that for every $n \geq k-1$ there is $t' \in \mathrm{Lev}_n(p{\restriction}t)$ such that $t' \in \mathrm{Split}(p)$. We denote the smallest such k by $K_p(t)$. Again subtrees are stronger, i.e., smaller, conditions.

Definition 1.4. Let $X, S \in [\omega]^\omega$. We say S *splits* X if $S \cap X$ and $X \setminus S$ are both infinite.

Proposition 1.5. *The forcing $\mathbb{FSP}$ adds a generic real*

$$x_G = \bigcup\{\mathrm{stem}(p) \,:\, p \in G\}$$

such that for any infinite set $\{\{n_i, n_i+1\} \,:\, i < \omega\}$ in the ground model there are infinitely many i such that

$$|x_G \cap \{n_i, n_i+1\}| = 1.$$

Proof. We prove only the latter part. Let $p \in \mathbb{FSP}$ and $\{\{n_i, n_i+1\} \,:\, i < \omega\}$ and $k \in \omega$ be given. After possibly strengthening p we can assume $|\mathrm{stem}(p)| > k$. We take i such that $n_i \geq K_p(\mathrm{stem}(p)) - 1 \geq k$. Then there is $s \in \mathrm{Split}(p)$ such that $|s| = n_i + 1$. Assume that $s(n_i) = 0$. There is $s^\frown 1 \in \mathrm{Suc}_p(s)$. We let $q = p{\restriction}s^\frown 1$. The other case is symmetric. Since k was arbitrary, we have $p \Vdash (\exists^\infty i)(|x_G \cap \{n_i, n_i+1\}| = 1)$, as claimed. □

We do not know whether $\mathbb{SP}$ has the same property.

Question 1.6. *Does $\mathbb{SP}$ add a real $\dot{x} \subseteq \omega$ such that for any infinite set $\{\{n_i, n_i+1\} : i < \omega\}$ in the ground model there are infinitely many i such that $|\dot{x} \cap \{n_i, n_i+1\}| = 1$?*

Remark 1.7. (1) Any fat splitting tree is a splitting tree, and the function K_p in the sense of the fat splitting is an upper bound to a function witnessing splitting. We use the same function symbol K_p for the forcing orders $\mathbb{SP}$ and $\mathbb{FSP}$, although the interpretation of the symbol depends on the underlying forcing order. The technical treatment of the K_p is the same in both interpretations.

(2) For $\mathbb{P} \in \{\mathbb{SP}, \mathbb{FSP}\}$, K_p in the respective meaning, $K_p(s) \leq K_p(t)$ for $s \trianglelefteq t$.

(3) For $\mathbb{P} \in \{\mathbb{SP}, \mathbb{FSP}\}$ we have that $K_p(t) = K_p(\mathrm{splsuc}(t))$ for every condition $p \in \mathbb{P}$ and node $t \in p$.

(4) For $\mathbb{P} \in \{\mathbb{SP}, \mathbb{FSP}\}$, $K_p(t) \leq K_q(t)$ for $q \leq p \in \mathbb{P}$ and $t \in q$.

Proof. We prove (2).

$\mathbb{P} = \mathbb{FSP}$ This is seen as follows: Let $s \trianglelefteq t \in p$. We choose witnesses for $K_p(t)$, i.e., a sequence $\bar{t} = \langle t_n : n \in [K_p(t) - 1, \omega)\rangle$ such that $t \trianglelefteq t_n, t_n \in \mathrm{Lev}_n(p) \cap \mathrm{Split}(p)$. Then, by definition this sequence $\bar{t}$ witnesses $K_p(s) \leq K_p(t)$.

$\mathbb{P} = \mathbb{SP}$ This time we take a sequence of tuples $\bar{t} = \langle (t_{n,0}, t_{n,1}) : n \in [K_p, \omega)\rangle$ such that $t_{n,i} \trianglerighteq t, t_{n,i} \in \mathrm{Lev}_n(p)$ and $t_{n,i}(n-1) = i$ for $i \in 2$. As in the item above the sequence $\bar{t}$ witnesses $K_p(s) \leq K_p(t)$.

Item (4) follows analogously because a witness $\bar{t}$ for $K_q(t)$ witnesses $K_q(t) \geq K_p(t)$. □

2. Axiom A and dividing a condition into two

We provide some basic properties of $\mathbb{FSP}$ whose analogues for $\mathbb{SP}$ were proved by Hein and Spinas [6]. In the beginning of the section we show that $\mathbb{FSP}$ has strong Axiom A and that we can arrange lower bounds for the K_p values. We build on Hein's and Spinas' techniques for splitting trees and develop them further, both for $\mathbb{SP}$ and for $\mathbb{FSP}$.

Definition 2.1. A notion of forcing $(\mathbb{P}, \leq)$ has *Axiom A* if there are partial order relations $\langle \leq_n : n < \omega\rangle$ such that

(a) $q \leq_{n+1} p$ implies $q \leq_n p$, $q \leq_0 p$ implies $q \leq p$,

(b) If $\langle p_n : n < \omega\rangle$ is a fusion sequence, i.e., a sequence such that for any n, $p_{n+1} \leq_n p_n$, then there is a lower bound $p \in \mathbb{P}$, $p \leq_n p_n$.

(c) For any maximal antichain A in $\mathbb{P}$ and and $n \in \omega$ and any $p \in \mathbb{P}$ there is $q \leq_n p$ such that only countably many elements of A are compatible with q. Equivalently, for any open dense set D and any n, p, there is a countable set E_p of conditions in D and $q \leq_n p$ such that E_p is predense below q.

A notion of forcing $(\mathbb{P}, \leq)$ has *strong Axiom A* if the set of compatible elements in (c) is even finite.

Axiom A entails properness and strong Axiom A implies ${}^{\omega}\omega$-bounding (see, e.g., [11, Theorem 2.1.4, Cor 2.1.12]).

Definition 2.2. For $\mathbb{P} = \mathbb{FSP}$, we define a decreasing sequence of partial orderings $\langle \leq_n : n \in \omega\rangle$ by $q \leq_n p$ if

$$q \leq p \wedge (\mathrm{Split}_{\leq n}(p) = \mathrm{Split}_{\leq n}(q) \wedge (\forall t \in \mathrm{Split}_{\leq n}(p))(K_q(t) = K_p(t)).$$

Hein and Spinas [6, Lemma 3.9] introduce a countable notion of forcing:

Definition 2.3. Let $\mathbb{P} \in \{\mathbb{SP}, \mathbb{FSP}\}$ and let $p \in \mathbb{P}$. We define $\mathbb{P}_p$: Conditions in $\mathbb{P}_p$ are finite trees $F \subseteq p$ such that there is $g_F \in \omega$ such that $\mathrm{ter}(F) \subseteq 2^{g_F}$.

We let $F' \leq_{\mathbb{P}_p} F$ if

(a) $F' \supseteq F$ and

(b) $\forall s \in F$, $K_{p\restriction F'}(s) = K_{p\restriction F}(s)$.

By Lemma 2.4, the forcing $\mathbb{P}$ is atomless. Hence also $\mathbb{P}_p$ is atomless and equivalent to Cohen forcing. The union of a $\mathbb{P}_p$-generic condition is a condition p_G in $\mathbb{P}$ again. Our Lemma 2.4 is based on Hein's and Spinas' proof of [6, Lemma 3.9].

Lemma 2.4. *Let $k \in \omega$, $p \in \mathbb{FSP}$, $m \in \omega$, D open dense in $\mathbb{FSP}$. Then for $t \in \mathrm{Split}_k(p)$, $j = 0, 1$ there is $p_{t,j}$ and a finite set $E_{t,j}$ with the following properties:*

(1)

$$p_{t,j} \leq_0 p{\restriction}t \wedge E_{t,j} \subseteq D \wedge E_{t,j} \text{ is predense below } p_{t,j} \wedge$$
$$K_{p_{t,j}}(t) = K_p(t) \wedge \bigwedge_{j=0,1} K_{p_{t,j}}(t^\frown 1) > K_{p_{t,j}}(t^\frown 0) \geq m$$

(2) For $j = 0, 1$ we let

$$p_j := \bigcup \{p_{t,j} \,:\, t \in \mathrm{Split}_k(p)\}.$$

Then we have $p_j \leq_k p$ and $p_0 \perp p_1$ (even $p_0 \cap p_1$ is finite) and $p_0 \cup p_1 \leq_k p$. There is a finite set of strengthenings of p_j that is a subset of D and predense below p_j.

(3) For each $j = 0, 1$ separately, the

$$\{K_{p_j}(\mathrm{splsuc}(t^\frown i)) \,:\, t \in \mathrm{Split}_k(p_j) = \mathrm{Split}_k(p), i = 0, 1\}$$

are ordered lexicographically according to the splitting preserving homomorphic image of $\{t^\frown i \,:\, t \in \mathrm{Split}_k(p), i = 0, 1\}$ in $(2^{k+1}, \leq_{\mathrm{lex}})$.

Moreover, for each $t \in \mathrm{Split}_k(p_j)$, $i = 0, 1$, $K_{p_j}(\mathrm{splsuc}(t^\frown i))$ is strictly larger than $\max\{K_{p_j}(s) \,:\, s \in \mathrm{Split}_{\leq k}(p_j)\}$.

Proof. We recall, h is defined in Def. 1.1(n). We go along the lexicographic order $\leq_{\mathrm{lex}}$ of $h(t)$ for $t \in \mathrm{Split}_k(p)$. Suppose the construction has already been performed for $t' \in \mathrm{Split}_k(p)$ such that $h(t') \leq_{\mathrm{lex}} h(t)$.

We assume that $m > \max\{K_{p_j}(s) \,:\, s \in \mathrm{Split}_{\leq k}(p_j)\}$.

Thinning out above $t^\frown 0$:

Let

$$K_0^t = \max(\{K_p(t'^\frown 0), K_p(t'^\frown 1) \,:\, t' \in \mathrm{Split}_k(p), h(t') \leq_{\mathrm{lex}} h(t)\} \cup \{m\}).$$

For any $s \in \mathrm{Lev}_{K_0^t}(p{\restriction}t^\frown 0)$, $j \in 2$, there is $p_{s,j}$ such that

$$p_{s,j} \leq p{\restriction}(\mathrm{splsuc}(s)^\frown j) \wedge p_{s,j} \in D. \tag{2.1}$$

Note that all the $p_{s,j}$ contain $\mathrm{splsuc}(s)^\frown j$ and do not contain $\mathrm{splsuc}(s)^\frown(1-j)$.

Thinning out above $t^\frown 1$:

Let

$$K_1^t = \max\{K_{p_{s,j}}(\mathrm{splsuc}(s)^\frown j) \,:\, s \in \mathrm{Lev}_{K_0^t}(p{\restriction}t^\frown 0), j \in 2\} + 1.$$

For any $s \in \mathrm{Lev}_{K_1^t}(p{\restriction}t^\frown 1)$ there is $p_{s,j}$ such that

$$p_{s,j} \leq p{\restriction}(\mathrm{splsuc}(s)^\frown j) \wedge p_{s,j} \in D. \tag{2.2}$$

Again all the $p_{s,j}$ contain $\mathrm{splsuc}(s)^\frown j$ and do not contain $\mathrm{splsuc}(s)^\frown(1-j)$. We let

$$p_{t,j} = \bigcup \{p_{s,j} \,:\, s \in \mathrm{Lev}_{K_0^t}(p{\restriction}(t^\frown 0))\} \cup \bigcup \{p_{s,j} \,:\, s \in \mathrm{Lev}_{K_1^t}(p{\restriction}(t^\frown 1))\}.$$

For $j = 0, 1$, a finite subset $E_{t,j}$ of D that is predense below $p_{t,j}$ is

$$E_{t,j} = \{p_{s,j} \,:\, s \in \mathrm{Lev}_{K_0^t}(p{\restriction}(t^\frown 0))\} \cup \{p_{s,j} \,:\, s \in \mathrm{Lev}_{K_1^t}(p{\restriction}(t^\frown 1))\}.$$

Then $p_j{\restriction}(t^\frown 0)$ (by reserving enough splitting in the interval above K_1^t) and $p_j{\restriction}(t^\frown 1)$ (for the interval below K_1^t) together witness that we have $K_{p_{t,j}}(t) = K_p(t)$. As stated, we let $p_j = \bigcup\{p_{t,j} \,:\, t \in \mathrm{Lev}_k(p)\}$. Thus we have $p_j \leq_k p$. By construction, for any $j = 0, 1$, $t \in \mathrm{Split}_k(p)$,

$$K_{p_{t,j}}(\mathrm{splsuc}(t^\frown 1)) \geq K_1^t > K_{p_{t,j}}(\mathrm{splsuc}(t^\frown 0)) \geq K_0^t \geq m$$

and the lexicographic order is carried on.

We turn to property (2). A finite subset of D that is predense below p_j is given by $\bigcup\{E_{t,j} : t \in \mathrm{Split}_k(p)\}$. Finally, properties (2.1) and (2.2) guarantee $p_0 \perp p_1$. □

Corollary 2.5. *For any $p \in \mathbb{FSP}$ there are 2^ω mutually incompatible conditions stronger than p.*

Proof. By in induction on $\mathrm{dom}(s)$ we construct for $j = 0, 1$, $p_{s^\frown j} \leq_{|s|} p_s$. The successor step is like the previous lemma with $p_s = p$ and $p_{s^\frown j} = p_j$ from there. Now that p_s for $s \in 2^{<\omega}$ are defined, we let for $b \in 2^\omega$, $p_b = \bigcap\{p_{b\restriction n} : n < \omega\}$. Since $p_{b\restriction n}$, $n < \omega$, is a fusion sequence, p_b is a condition. □

Corollary 2.6. *The set of fat splitting trees $p \in \mathbb{FSP}$ with the property such that there is a splitting preserving homomorphism h from $\mathrm{Split}(p)$ onto $2^{<\omega}$ such that for every s, t such that $|h(s)| = |h(t)|$ and $h(s) \leq_{\mathrm{lex}} h(t)$ we have*

$$K_p(s) < K_p(t)$$

is dense.

Proof. Let $p \in \mathbb{FSP}$. We construct a fusion sequence $\langle p_n : n \in \omega\rangle$ according to Lemma 2.4 by letting $j = 0$ all the time. □

Proposition 2.7. *The expanded fat splitting forcing $(\mathbb{FSP}, \leq, (\leq_n)_n)$ has strong Axiom A. The same holds for $\mathbb{SP}$.*

Proof. The result for $\mathbb{SP}$ is already known by work of Shelah and Spinas (see [14]). The fusion property follows from the definition of $\leq_n$. By Lemma 2.4, the forcing order $\mathbb{FSP}$ together with $\leq_n$ according to Definition 2.2 has strong Axiom A. □

3. Amoeba forcing and dominating reals

In the section we deal with a question addressed by Hein and Spinas, giving more evidence to support their conjecture.

Definition 3.1. Let $\mathbb{P}$ be a forcing whose conditions are perfect trees ordered by inclusion, in particular $\mathbb{P}$ could be $\mathbb{SP}$, $\mathbb{FSP}$.

(1) A subset $X \subseteq 2^\omega$ is called $\mathbb{P}$-*nowhere dense,* if

$$(\forall p \in \mathbb{P})(\exists q \leq p)([q] \cap X = \emptyset).$$

We denote the ideal of $\mathbb{P}$-nowhere dense sets with $\mathcal{N}_\mathbb{P}$.

(2) A subset $X \subseteq 2^\omega$ is called $\mathbb{P}$-*meager* if it is included in a countable union of $\mathbb{P}$-nowhere dense sets. We denote the σ-ideal of $\mathbb{P}$-meager sets with $\mathcal{I}_\mathbb{P}$.

(3) A subset $X \subseteq 2^\omega$ is called $\mathbb{P}$-*measurable* if

$$(\forall p \in \mathbb{P})(\exists q \leq p)([q] \setminus X \in \mathcal{I}_\mathbb{P} \vee [q] \cap X \in \mathcal{I}_\mathbb{P}).$$

(4) A subset $X \subseteq 2^\omega$ is called *weakly* $\mathbb{P}$-*measurable* if

$$(\exists q)([q] \setminus X \in \mathcal{I}_\mathbb{P} \vee [q] \cap X \in \mathcal{I}_\mathbb{P}).$$

Notice that these notions generalize some well-known ones:

$\mathbb{P}$	$\mathcal{I}_{\mathbb{P}}$	$\mathbb{P}$-measurable
$\mathbb{C}$	meager ideal	Baire property
$\mathbb{B}$	null ideal	measurable
$\mathbb{V}$	Doughnut null	Doughnut-property [5]
$\mathbb{S}$	Marczewski ideal [16]	Marczewski field

Remark 3.2. In general the two ideals do not coincide $\mathcal{I}_{\mathbb{P}} \neq \mathcal{N}_{\mathbb{P}}$, for instance when $\mathbb{P}$ is the Cohen forcing. In many other cases however, they do coincide $\mathcal{I}_{\mathbb{P}} = \mathcal{N}_{\mathbb{P}}$, for instance when $\mathbb{P} \in \{\mathbb{S}, \mathbb{L}, \mathbb{M}, \mathbb{V}, \mathbb{MA}\}$. When the latter occurs X is $\mathbb{P}$-measurable if and only if:

$$(\forall p \in \mathbb{P})(\exists q \leq p)([q] \subseteq X \vee [q] \cap X = \emptyset).$$

In our specific case for $\mathbb{P} \in \{\mathbb{SP}, \mathbb{FSP}\}$ the ideal of $\mathbb{P}$-nowhere dense sets is in fact a σ-ideal and so $\mathcal{I}_{\mathbb{P}} = \mathcal{N}_{\mathbb{P}}$. The proof is a fusion argument and is a consequence of Lemma 2.4. In fact, let $X \subseteq \bigcup_n X_n$ with $X_n \in \mathcal{N}_{\mathbb{P}}$, for $n \in \omega$. Fix a condition $p \in \mathbb{P}$ and recursively apply Lemma 2.4 with $D = \{q \in \mathbb{P} : [q] \cap X_n = \emptyset\}$ in order to construct a fusion sequence $p = p_0 \geq_0 p_1 \geq_1 \dots$ with the property that for every $n \in \omega$, $[p_n] \cap X_n = \emptyset$. And so, the fusion $q = \bigcap_n p_n$ is such that $[q] \cap X = \emptyset$.

Remark 3.3. Weak-$\mathbb{P}$-measurability is a weaker statement than $\mathbb{P}$-measurability, and if referred to a single set, it is not even a regularity property, as a given set can contain the branches through a tree $p \in \mathbb{P}$ but being very irregular outside of $[p]$. However classwise staments about weak measurability are in some cases sufficient to obtain measurability. More precisely, let Γ be a family of subsets of reals and

$$\Gamma(\mathbb{P}) := \text{“all sets in } \Gamma \text{ are } \mathbb{P}\text{-measurable”}$$

$$\Gamma_w(\mathbb{P}) := \text{“all sets in } \Gamma \text{ are weakly } \mathbb{P}\text{-measurable”}.$$

If Γ is closed under continuous pre-image and intersection with closed sets, for $\mathbb{P} \in \{\mathbb{S}, \mathbb{V}, \mathbb{M}, \mathbb{L}, \mathbb{MA}\}$ one has $\Gamma(\mathbb{P}) \Leftrightarrow \Gamma_w(\mathbb{P})$ (see [3, Lemma 2.1] and [4, Lemma 1.4]). Hence, we can obtain some straightforward implications, such as $\Gamma(\mathbb{L}) \Rightarrow \Gamma(\mathbb{M})$ and $\Gamma(\mathbb{V}) \Rightarrow \Gamma(\mathbb{S})$.

Definition 3.4. Let $\mathbb{P}$ be the splitting forcing or the fat splitting forcing, and let $K_p(t)$ be minimal with the properties in the definitions. We define $d\colon \mathbb{P} \to \omega^\omega$ as follows:

$$d_p(n) := \min\{K_p(t^\frown i) : i \in 2, t \in \mathrm{Split}_n(p)\}.$$

Note that the definition of the function d_p differs slightly from the one given in [6, Definition 3.11]. Nonetheless, the function d_p shares the same crucial properties, as the following lemma illustrates.

Lemma 3.5. *Let $\mathbb{P} \in \{\mathbb{SP}, \mathbb{FSP}\}$. The following holds.*

(1) For $q \leq p \in \mathbb{P}$ we have $d_q \geq d_p$.
(2) Given $f \in \omega^\omega$ and $p \in \mathbb{P}$, there exists $q \leq p$ such that $f \leq d_q$, i.e., $f(n) \leq d_q(n)$ holds for all $n < \omega$

Proof. A proof of (1) and (2) for $\mathbb{SP}$ can be found in [6, Lemma 3.12, Lemma 3.14] (keep in mind that they used a slightly different version of d_p).

The proofs can be translated to our setting without difficulties. As an example we prove (1) and (2) for $\mathbb{P} = \mathbb{FSP}$.

(1) So let $q \leq p$ be two fat splitting trees and $n < \omega$. We show that $d_q(n) \geq d_p(n)$. By definition we have

$$d_q(n) = \min\{K_q(t^\frown i) : i \in 2, t \in \mathrm{Split}_n(q)\}$$

So let $t \in \mathrm{Split}_n(q)$ and $i \in 2$ such that $K_q(t^\frown i) = d_q(n)$. Since $q \leq p$ there is $s \in \mathrm{Split}_n(p)$ such that $s \trianglelefteq t$. Choose $j \in 2$ such that $s^\frown j \trianglelefteq t^\frown i$. Then by items (2) and (4) from Remark 1.7 we get

$$K_p(s^\frown j) \leq K_q(s^\frown j) \leq K_q(t^\frown i).$$

This shows that $d_q(n) \geq d_p(n)$.

(2) Let $p \in \mathbb{FSP}$ and $f \in \omega^\omega$ be given. Using Lemma 2.4 we can build a fusion sequence $\langle q_n : n < \omega \rangle$ with the properties:

(1) $q_0 = p$
(2) $q_n \leq_n q_{n+1}$
(3) For all $t \in \mathrm{Split}_n(q)$, $K_{q_n}(t) \geq f(n-1)$, for $n > 0$

The fusion $q = \bigcap_n q_n$ satisfies $d_q(n) \geq f(n)$ for all $n < \omega$. □

Definition 3.6. We say that q is an *absolute $\mathbb{P}$-generic tree over V* iff $q \in \mathbb{P}$ and all its branches are $\mathbb{P}$-generic over V in any extension, more precisely such that for any ZF-model extension $N \supseteq V$ we have

$$N \models q \in \mathbb{P} \wedge \forall x \in [q](x \text{ is } \mathbb{P}\text{-generic over } V).$$

An *absolute amoeba forcing for* $\mathbb{P}$ is a poset adding an absolute $\mathbb{P}$-generic tree.

We remark that every amoeba forcing in literature, at least to our knowledge, satisfies this property including the natural amoeba for $\mathbb{SP}$ as defined in (see [6, Definition 3.15.]). Other examples would be the versions of amoeba for Laver and Miller (see [13, pp 709 and 714]) as proven in [13, Lemma 1.1.7, Lemma 1.1.8, Remark 1.1.10].

The main idea of using an amoeba forcing is to add a *large* set of *generic* reals. However, we must be careful that this notion is sufficiently absolute, otherwise we might end up with a useless amoeba. For example, if G is a Sacks-generic filter over V, then it is well-known that in $V[G]$ there is a perfect set P of Sacks-generic reals. But if we take H a Sacks-generic filter over $V[G]$, then in $V[G][H]$ the set P is no longer a perfect set of Sacks-generic reals. Moreover, in $V[G][H]$ the set of Sacks-generic reals over V is in the Marczewski ideal $\mathcal{I}_{\mathbb{S}}$.

The known application of an amoeba forcing to blow up the additivity number like in [13, Theorem 1.3.1] uses an absoluteness argument which justifies Definition 3.6. In light of that, the following proposition provides more support to Hein's and Spinas' conjecture that $\mathrm{add}(\mathcal{I}_{\mathbb{SP}}) \leq \mathfrak{b}$ and that any reasonable amoeba for $\mathbb{SP}$ adds a dominating real.

Proposition 3.7. *Let $\mathbb{P} \in \{\mathbb{SP}, \mathbb{FSP}\}$ and let $V \subseteq N$ be models of ZFC. If*

$$N \models \text{“There is an absolute } \mathbb{P}\text{-generic tree over } V\text{”}$$

then

$$N \models \text{“There is a dominating real over } V\text{”}.$$

Hence any absolute amoeba forcing for $\mathbb{P}$ adds a dominating real.

Proof. We construct an ω sized family $\{d_k : k \in \omega\}$ that dominates all $f \in V \cap \omega^\omega$. Note that this is enough since we can use a standard diagonal argument and define $x \in \omega^\omega$ as

$$x(n) := \sup\{d_k(n) : k \leq n\} + 1.$$

Then x almost dominates all $f \in V \cap \omega^\omega$.

Let $q \in N$ be the $\mathbb{P}$-generic tree over V.

Now let $\bar{H} : 2^{<\omega} \to \mathrm{Split}(q)$ be as in Definition 1.1 (o), i.e., $\bar{H}(\emptyset) = \mathrm{stem}(q)$ and $\bar{H}(\sigma^\frown i) = \mathrm{splsuc}(\bar{H}(\sigma)^\frown i)$, for every $\sigma \in 2^{<\omega}$, $i \in \{0,1\}$; and $H\colon 2^\omega \to [q]$ be its natural associated extension. Then consider $\dot{c}$ the canonical $\mathbb{C}$-name for a Cohen real over N. Note that for every Cohen real c over N we have

$$N[c] \models H(c) \in [q] \wedge H(c) \text{ is } \mathbb{P}\text{-generic over } V.$$

Let G_c be the filter associated with $H(c)$ that is $\mathbb{P}$-generic over V, i.e., $N[c] \models H(c) = \bigcap\{[p] : p \in G_c\}$. Hence for any $\mathbb{P}$-open dense set $D \in V$ there is $p \in D \cap G_c$, and so $N[c] \models H(c) \in [p]$. This implies that there is $\sigma_{D,p} \in \mathbb{C}$ such that $\sigma_{D,p} \Vdash H(\dot{c}) \in [p]$

So we get the following:
for every $\mathbb{P}$-open dense D of $\mathbb{P}$ in V there exists $p \in D$ and $\sigma \in 2^{<\omega}$ such that

$$\begin{aligned} &\text{for each Cohen real } c \text{ extending } \sigma \\ &N[c] \models H(c) \in [p] \cap [q{\restriction}\bar{H}(\sigma)]. \end{aligned} \tag{3.1}$$

Claim 3.8. *From (3.1) it follows that $q{\restriction}\bar{H}(\sigma) \leq p$.*

To prove the claim we argue by contradiction, so assume $t \in q{\restriction}\bar{H}(\sigma) \setminus p$. Pick $\tau \trianglerighteq \sigma$ such that $\bar{H}(\tau) \trianglerighteq t$. Now fix any Cohen real c extending τ. Then $H(c)$ extends t and therefore $H(c) \in [q{\restriction}\bar{H}(\sigma)] \setminus [p]$. This contradicts 3.1.

Now we finish the proof of Proposition 3.7. Let $\{q_k : k \in \omega\}$ enumerate $\{q{\restriction}\bar{H}(\sigma) : \sigma \in \mathbb{C}\}$. We claim that $\{d_{q_k} : k \in \omega\}$ dominates all $f \in V \cap \omega^\omega$. In fact, fix $f \in V \cap \omega^\omega$. By Lemma 3.5, for every $f \in V \cap \omega^\omega$ we pick a $\mathbb{P}$-open dense $D_f \subseteq \mathbb{P}$ in V such that for every $p \in D_f$, $f \leq^* d_p$. Then pick $\sigma \in \mathbb{C}$ and $p \in D_f$ as in (3.1). Take $k \in \omega$ such that $q_k = q{\restriction}\bar{H}(\sigma)$. Note that by Claim 3.8 we have that $q_k \leq p$ and so for all but finitely many $n \in \omega$, $d_{q_k}(n) > f(n)$.

□

Remark 3.9. Recall that the bounding number, $\mathfrak{b}$, is the minimal size of an $\leq^*$-unbounded family. Beyond what we prove in Proposition 3.7, in [6] Hein and Spinas addressed also the following parallel question: is $\mathrm{add}(\mathcal{I}_{\mathbb{SP}}) \leq \mathfrak{b}$ provable in ZFC? In a previous version of the paper, we tried to use the method implemented in Proposition 3.7 in order to prove $\mathrm{add}(\mathcal{I}_{\mathbb{P}}) \leq \mathfrak{b}$, for $\mathbb{P} \in \{\mathbb{SP}, \mathbb{FSP}\}$. We deeply thank Otmar Spinas to find a gap in the argument and write us in a private communication. The point where our proof specifically works in this case is that the $\mathbb{P}$-generic tree q is covered by any $\mathbb{P}$-open dense from the ground model V, but when trying to use a similar argument for proving $\mathrm{add}(\mathcal{I}_{\mathbb{P}}) \leq \mathfrak{b}$ one needs a finer argument to obtain $H(c)$ be caught by a tree p in a given $\mathbb{P}$-open dense set in N.

4. A brief digression on the Silver amoeba and Cohen reals

Spinas [15] showed that the ideal of meager sets $\mathcal{M}$ Tukey-embeds into the σ-ideal $\mathcal{I}_{\mathbb{V}}$ corresponding to Silver forcing, in symbols $\mathcal{M} \leq_T \mathcal{I}_{\mathbb{V}}$. This result is surprising because it is in sharp contrast with the other popular non-ccc tree forcings: see [10] and [9] for Sacks, [13] for Laver and Miller, and for Mathias is folklore. Spinas' brilliant proof idea essentially involves two key steps: 1) a quite technically demanding investigation of the Silver antichain number; 2) a coding by Hamming weights of a Cohen real *inside* a Silver tree. This result concerning the existence of such a Tukey embedding is in parallel with the fact that any amoeba

for Silver necessarily adds Cohen reals. In fact, given any absolute amoeba $\mathbb{AV}$ for Silver and assume $\mathbb{AV}$ is proper. Then, a countable support iteration of length ω_2, blows up $\mathrm{add}(\mathcal{I}_{\mathbb{V}})$. Now using Spinas' result of the Tukey-reducibility one gets in the generic extension $\aleph_1 < \mathrm{add}(\mathcal{I}_{\mathbb{V}}) \leq \mathrm{add}(\mathcal{M})$ and thus one deduces that Cohen reals must have been added during the iteration. However, it remains unclear, whether a single step of the amoeba $\mathbb{AV}$ necessarily adds a Cohen. For instance, the following situation may occur: First $\mathbb{AV}$ adds half a Cohen but no Cohen reals and second $\mathbb{AV} * \mathbb{AV}$ adds a Cohen real. Forcings with these two properties exist (see [17, Theorem 1.3]).

In this section we give a direct proof that any absolute amoeba for Silver adds Cohen reals. We still use the coding from part 2), but replace the argument about Silver antichains with an alternative method based on Cohen forcing as in the proof of Proposition 3.7.

We do not go through the details of the coding (by Hamming weights) and we refer the reader to [15, Coding Lemma 6 and Thinning-Out Lemma 7] for the proofs. We only recall from [15] the definition and the main result we need in our proof below.

Definition 4.1. For $n, k < \omega$ we define $d(n,k) \in 2$ by letting $d(n,k) = 0$ iff the unique $j < \omega$ such that $n \in [j \cdot 2k, (j+1) \cdot 2k)$ is even. Let $e(n)$ be the minimal $k < \omega$ with $2k > n$. Let $c(n) = \langle d(n,k) : k < \omega \rangle$ and $c^*(n) = c(n) \restriction e(n)$.

For $\sigma \in 2^{<\omega}$ the *Hamming weight of* σ is defined as $HW(\sigma) := |\{i < |\sigma| : \sigma(i) = 1\}|$.

As usual we denote with $\mathbb{C} = (2^{<\omega}, \subseteq)$ the Cohen forcing.

Lemma 4.2. ([15, Thinning-Out-Lemma 7]) *Given $\{D_j : j < \omega\}$ a family of open dense sets $D_j \subseteq \mathbb{C}$ and $p \in \mathbb{V}$, there exists $q \in \mathbb{V}$ such that $q \leq p$ and for every $n < \omega$ and $\sigma \in \mathrm{Split}_{n+1}(q)$, if $m = |\tau|$ for any $\tau \in \mathrm{Split}_n(q)$ and $k = HW(\sigma)$, then $w^\frown c^*(k) \in D_j$ for every $w \in 2^{\leq m}$ and $j \leq n$.*

The property given in the Thinning-Out-Lemma is not only dense, but even open, i.e., for every $p \in \mathbb{V}$ there exists $q \leq p$ such that every $q' \leq q$ satisfies the property of the Thinning-Out-Lemma (see [15, Remark 8]).

Proposition 4.3. *Let $\mathbb{V}$ be the Silver forcing and let $V \subseteq N$ be models of ZFC. If*

$$N \models \text{“There is an absolute } \mathbb{V}\text{-generic tree over } V\text{”}$$

then

$$N \models \text{“There is a Cohen real over } V\text{”}.$$

Hence any absolute amoeba forcing for $\mathbb{V}$ adds a Cohen real.

Proof. The proof follows the line of that one for Proposition 3.7. First of all, given $E \subseteq \mathbb{C}$ open dense, we apply Spinas' Thinning-Out-Lemma with $\{D_j : j < \omega\}$ such that $D_j = E$, for all $j \in \omega$, in order to get a $\mathbb{V}$-open dense set $D_E \subseteq \mathbb{V}$ such that for every $q \in D_E$

$$(\forall n < \omega)(\forall w \in 2^{\leq m_n})(w^\frown \xi_n \in E), \tag{4.1}$$

where $\xi_n := c^*(HW(\sigma_n))$ with σ_n is the leftmost sequence in $\mathrm{Split}_{n+1}(q)$, and $m_n \in \omega$ is the length of any $\tau \in \mathrm{Split}_n(q)$.

Let $q \in N$ be an absolute $\mathbb{V}$-generic tree over the ground model V. As in Definition 1.1 (o), let $\bar{H} : 2^{<\omega} \to \mathrm{Split}(q)$ be the $\trianglelefteq$-preserving function and H :

$2^\omega \to [q]$ its natural extension. Let c be a Cohen real over V. Like in the proof of Proposition 3.7, we conclude that for every $\mathbb{V}$-open dense set $D \in V$ there exists $p \in D$ such that $N[c] \models H(c) \in [p]$, and so there exists $\sigma \in \mathbb{C}$ such that $\sigma \Vdash H(c) \in [p]$. As before we get the following claim.

Claim 4.4. *For every $\mathbb{V}$-open dense set $D \in V$ there exists $\sigma \in \mathbb{C}$ and $p \in D$ such that $q{\restriction}\bar{H}(\sigma) \leq p$.*

So let $\{q_k : k \in \omega\}$ enumerate all $q{\restriction}\bar{H}(\sigma)$'s, and let m^k_n, ξ^k_n be associated with q_k as in (4.1) above. Then put

$$z := \xi^0_{n_0} {}^\frown \xi^1_{n_1} {}^\frown \ldots {}^\frown \xi^k_{n_k} {}^\frown \ldots,$$

where the n_k's are chosen recursively as follows: $n_0 = 0$ and for $k \geq 1$, n_k is such that $|\xi^0_{n_0} {}^\frown \xi^1_{n_1} {}^\frown \ldots {}^\frown \xi^{k-1}_{n_{k-1}}| \leq m^k_{n_k}$.

We aim at showing that z is Cohen over V. Let $E \subseteq \mathbb{C}$ be open dense in V. Let $D_E \subseteq \mathbb{V}$ be a $\mathbb{V}$-open dense set in V such that every $p \in D_E$ satisfies (4.1). By Claim 4.4 there exists $k \in \omega$ such that $q_k \leq p$, for some $p \in D_E$. By construction, for every $w \in 2^{\leq m^k_{n_k}}$ we have $w {}^\frown \xi^k_{n_k} \in E$. Since $|\xi^0_{n_0} {}^\frown \xi^1_{n_1} {}^\frown \ldots {}^\frown \xi^{k-1}_{n_{k-1}}| \leq m^k_{n_k}$, it follows $\xi^0_{n_0} {}^\frown \xi^1_{n_1} {}^\frown \ldots {}^\frown \xi^{k-1}_{n_{k-1}} {}^\frown \xi^k_{n_k} \in E$.

Hence, for every $E \in V$ open dense of $\mathbb{C}$, there exists $n \in \omega$ such that $z{\restriction}n \in E$, which means z is Cohen over V.

□

5. Fatness versus measure

Let μ denote the standard measure on 2^ω. Then, the Random forcing $\mathbb{B}$ consists of all perfect trees $p \subseteq 2^{<\omega}$ with positive measure, ordered by inclusion. We denote the σ-ideal of measure zero sets with $\mathcal{N}$.

In this section we compare $\mathbb{FSP}$ with $\mathbb{B}$. We show that each random condition is modulo a measure zero set equal to a fat splitting tree, but the converse does not hold. In fact, we show that the set of fat splitting trees with measure zero is dense in $\mathbb{FSP}$. We conclude this section by scrutinizing the differences between the two corresponding σ-ideals $\mathcal{I}_{\mathbb{FSP}}$ and $\mathcal{N}$.

Lemma 5.1. $\mathbb{B} \cap \mathbb{FSP}$ *is dense in* $\mathbb{B}$.

Proof. We call a level n of a tree $p \subseteq 2^{<\omega}$ *nowhere splitting* if $|\operatorname{Lev}_n(p)| = |\operatorname{Lev}_{n+1}(p)|$, i.e., if there is no splitting node $s \in \operatorname{Lev}_n(p)$. Let $p \in \mathbb{B}$ be given and put $N := \{t \in p : \mu([p{\restriction}t]) = 0\}$. This is a countable set and therefore $\mu(\bigcup_{t\in N}[p{\restriction}t]) = 0$. So, $q := p \setminus N$ is still a perfect tree with positive measure and the additional property that for each $t \in q$ $(\mu([q{\restriction}t]) > 0)$. We claim that q is a fat splitting tree. To reach a contradiction assume that there is a node $t \in q$ with no corresponding $K_q(t)$, i.e., there are infinitely many $n \in \omega$ such that $\operatorname{Lev}_n(q{\restriction}t)$ is nowhere splitting. We fix such a node $t \in q$ and an increasing enumeration $\langle n_i : i \in \omega\rangle$ of all nowhere splitting levels of $q{\restriction}t$. It is enough to show that $\mu([q{\restriction}t]) \leq 2^{-(i+1)}$ holds for each $i \in \omega$. Fix $i \in \omega$. Since for each $j < i$ we know that $\operatorname{Lev}_{n_j}(q{\restriction}t)$ is nowhere splitting, we get that $|\operatorname{Lev}_{n_i+1}(q{\restriction}t)| \leq 2^{n_i-i}$. Therefore we get

$$\mu([q{\restriction}t]) \leq \mu\left(\bigcup\{[s] : s \in \operatorname{Lev}_{n_i+1}(q{\restriction}t)\}\right) \leq 2^{n_i-i}\frac{1}{2^{n_i+1}} = 2^{-(i+1)}.$$

□

Lemma 5.2. *Below any fat splitting tree $p \in \mathbb{FSP}$ we can find an antichain $\{p^x \leq p : x \in 2^\omega\}$ such that $\mu([p^x]) = 0$ and $[p^x] \cap [p^y] = \emptyset$, whenever $x \neq y$.*

Proof. Let $p \in \mathbb{FSP}$ be given. By induction on $n \in \omega$ we construct a set of conditions $\{p^s_n : n \in \omega, s \in 2^n\}$ such that for any $n \in \omega$:

(1) $p^{\langle\rangle}_0 = p$,
(2) $p^s_n \geq_{n+1} p^{s^\frown i}_{n+1}, i \in 2$,
(3) $[p^{s^\frown 0}_{n+1}] \cap [p^{s^\frown 1}_{n+1}] = \emptyset$,
(4) $\mu([p^{s^\frown i}_{n+1}]) \leq \frac{1}{2}\mu([p^s_n])$.

Let $n \in \omega$ and $s \in 2^n$ be given. We apply Lemma 2.4 to the condition p^s_n to get two incompatible fat splitting trees $q^{s^\frown i}_{n+1}, i \in 2$ satisfying conditions (2) and (3). We have to make sure that also (4) holds. Therefore we compare the two measures of $[q^{s^\frown i}_{n+1}], i \in 2$. W.l.o.g. assume $\mu([q^{s^\frown 0}_{n+1}]) \leq \mu([q^{s^\frown 1}_{n+1}])$. Since the two trees $q^{s^\frown 0}_{n+1}, q^{s^\frown 1}_{n+1}$ have disjoint bodies, we must have $\mu([q^{s^\frown 0}_{n+1}]) \leq \frac{1}{2}\mu([p^s_n])$. Thus, we can set $p^{s^\frown 0}_{n+1} := q^{s^\frown 0}_{n+1}$. Now via the same argument as above this time for $q^{s^\frown 1}_{n+1}$ instead of p^s_n, we get a condition $p^{s^\frown 1}_{n+1} \leq_{n+1} q^{s^\frown 1}_{n+1}$ such that $\mu([p^{s^\frown 1}_{n+1}]) \leq \frac{1}{2}\mu([q^{s^\frown 1}_{n+1}])$. This completes the construction.
Now for each $x \in 2^\omega$, we define $p^x := \bigcap_{n\in\omega} p^{x\restriction n}_n$. This is a fat splitting tree by (2) and has measure zero by condition (4). Condition (3) ensures that for two different $x \neq y \in 2^\omega$ we have $[p^x] \cap [p^y] = \emptyset$. □

Corollary 5.3. *$\mathbb{FSP} \cap \mathcal{N}$ is dense in $\mathbb{FSP}$.*

Corollary 5.4. *$\mathcal{N} \setminus \mathcal{I}_{\mathbb{FSP}} \neq \emptyset$.*

Proof. Any $p \in \mathbb{FSP}$ with measure zero is a witness for $[p] \in \mathcal{N} \setminus \mathcal{I}_{\mathbb{FSP}}$. □

Proposition 5.5. *Assume $\mathsf{cov}(\mathcal{N}) = \mathfrak{c}$. Then $\mathcal{I}_{\mathbb{FSP}} \setminus \mathcal{N} \neq \emptyset$.*

Proof. The proof follows the idea in [1, 1.4]. Let $\langle p_\alpha \in \mathbb{FSP} : \alpha < \mathfrak{c}\rangle$ be an enumeration of all fat splitting trees with measure zero. By Corollary 5.3 this is a dense set in $\mathbb{FSP}$. Now fix an enumeration $\langle q_\alpha \in \mathbb{B} : \alpha < \mathfrak{c}\rangle$ of the random forcing. Our aim is to construct a sequence $\langle x_\alpha \in 2^\omega : \alpha < \mathfrak{c}\rangle$ such that

i) $x_\alpha \notin \bigcup_{\beta<\alpha}[p_\beta]$
ii) $x_\alpha \in [q_\alpha]$.

We first check that we can indeed find such a sequence and then verify that the resulting set $X := \{x_\alpha : \alpha < \mathfrak{c}\}$ witnesses $\mathcal{I}_{\mathbb{FSP}} \setminus \mathcal{N} \neq \emptyset$. So fix $\alpha < \mathfrak{c}$. Our assumption $\mathsf{cov}(\mathcal{N}) = \mathfrak{c}$ implies that $\bigcup_{\beta<\alpha}[p_\beta]$ does not cover $[q_\alpha]$ and therefore we can pick x_α satisfying conditions i) and ii). Condition ii) ensures that $X \notin \mathcal{N}$ holds. So we are left to show that we can find for each $p \in \mathbb{FSP}$ a stronger condition $q \leq p$ such that $[q]$ and X are disjoint. Therefore, fix $p \in \mathbb{FSP}$ and pick $\alpha < \mathfrak{c}$ with $p_\alpha \leq p$. Now by Lemma 5.2 there is an antichain $\{p'_\beta \in \mathbb{FSP} : \beta < \mathfrak{c}\}$ below p_α satisfying $[p'_\beta] \cap [p'_\gamma] = \emptyset$, whenever $\beta \neq \gamma$. Condition i) implies that $|[p_\alpha] \cap X| < \mathfrak{c}$. In particular, we can find some β with $[p'_\beta] \cap X = \emptyset$. □

Question 5.6. *Does $\mathcal{I}_{\mathbb{FSP}} \setminus \mathcal{N} \neq \emptyset$ hold without the assumption $\mathsf{cov}(\mathcal{N}) = \mathfrak{c}$?*

6. The Sacks property

We show that $\mathbb{FSP}$ does not have the Sacks property.

Definition 6.1.

(1) $\langle S_n : n < \omega\rangle$ is called an *f-slalom* if $S_n \subseteq [\omega]^{f(n)}$.
(2) A function $f : \omega \to \omega \setminus \{\emptyset\}$ is called *diverging* if $\lim_n f(n) = \infty$.
(3) A forcing $\mathbb{P}$ has *the Sacks property* if for any diverging $f\colon \omega \to \omega \setminus \{\emptyset\}$ it has the following property:

> For any $\mathbb{P}$-name τ for a real and any condition p there is an f-slalom $\langle S_n : n < \omega\rangle$ and there is $q \leq p$ such that $q \Vdash (\forall^\infty n)(\tau(n) \in S_n)$.

We remark that in the definition of the Sacks property above, it is enough that the forcing $\mathbb{P}$ has the property for *some* diverging $f : \omega \to \omega \setminus \{\emptyset\}$.

Lemma 6.2. *For any function $f\colon \omega \to \omega \setminus \{\emptyset\}$ there is a $\mathbb{FSP}$-name τ such that for any f-slalom $\langle S_n : n < \omega\rangle \in V$*

$$\mathbb{FSP} \Vdash (\exists^\infty n)(\tau(n) \notin S_n).$$

Proof. Fix $f \in \omega^\omega \cap V$. We aim at finding $\dot h \in \omega^\omega \cap V^{\mathbb{FSP}}$ so that for every slalom $S \in ([\omega]^{<\omega})^\omega \cap V$, with $|S(n)| \leq f(n)$, we have h is not captured by S, i.e., there is $n \in \omega$ such that $h(n) \notin S(n)$. We let $\dot x$ be the name for the $\mathbb{FSP}$-generic real, i.e. the union $\bigcup\{\mathrm{stem}(p) : p \in G\}$.

Let code$\colon \{2^I : I \subseteq \omega, I \text{ finite}\} \to \omega$ be an injective function.

We fix a partition $\{I_n : n \in \omega\}$ of ω, where each I_n is an interval, $\max I_n < \min I_{n+1}$, and $|I_n| > f(n)$. We define

$$\dot h := \langle \mathrm{code}(\dot x{\restriction}I_n) : n \in \omega\rangle.$$

We aim to show that $\dot h$ cannot be captured by any f-slalom in the ground model. So fix an f-slalom $S \in V$ and a condition $p \in \mathbb{FSP}$. It is enough to find $n \in \omega$, $q \leq p$ such that $q \Vdash \dot h(n) \notin S(n)$. Pick $n \in \omega$ such that $\min(I_n) > K_p(\mathrm{stem}(p))$. Then for every $j \in I_n$, there is $t \in \mathrm{Split}(p)$ such that $|t| = j$. Hence on level $\max(I_n)$ of p we have at least $|I_n| > f(n)$ nodes that are pairwise different in I_n. Let $\{t_k : k \in |I_n|\}$ enumerate the first $|I_n|$ of them. Then

$$p{\restriction}t_k \Vdash \dot h(n) = \mathrm{code}(\dot x{\restriction}I_n) = \mathrm{code}(t_k{\restriction}I_n),$$

Since the $t_k{\restriction}I_n$ are pairwise different and code is injective there are at least $|I_n|$ possibilities for p to decide $h(n)$ and because $|S(n)| \leq f(n) < |I_n|$ there is some $k \in |I_n|$ such that

$$p{\restriction}t_k \Vdash \mathrm{code}(\dot x{\restriction}I_n) = \mathrm{code}(t_k{\restriction}I_n) = \dot h(n) \notin S(n).$$

So, $q := p{\restriction}t_k \leq p$ is the condition with the desired property. □

Corollary 6.3. *$\mathbb{FSP}$ does not satisfy the Sacks property.*

In the proof we made use of combinatorial properties that apply exclusively to $\mathbb{FSP}$. Hence, our proof does not work for $\mathbb{SP}$. However, Jonathan Schilhan found a totally different proof to show that $\mathbb{SP}$ does not have the Sacks property [12, Theorem 2.8].

Proposition 6.4 (Schilhan)**.** *$\mathbb{SP}$ does not have the Sacks property.*

7. Splitting-measurability

Here we investigate the complexity of $\mathbb{P}$-measurable sets. The first three results hold for fat splitting trees and splitting trees, while the proof of Theorem 7.4 specifically uses Lemma 6.2, which only applies to $\mathbb{FSP}$.

Note that if X is weakly $\mathbb{FSP}$-measurable, then it is also weakly $\mathbb{SP}$-measurable. In fact, let $X \subseteq 2^\omega$ be weakly $\mathbb{FSP}$-measurable. Then, there is a condition $p \in \mathbb{FSP}$ such that either $X \cap [p] \in \mathcal{I}_{\mathbb{FSP}}$ or $[p] \setminus X \in \mathcal{I}_{\mathbb{FSP}}$. As $\mathcal{I}_{\mathbb{FSP}} = \mathcal{N}_{\mathbb{FSP}}$ holds, Remark 3.2 applies and we can assume that $[p] \cap X = \emptyset$ or $[p] \subseteq X$. Now since $p \in \mathbb{SP}$ as well, it follows that X is weakly-$\mathbb{SP}$-measurable.

Proposition 7.1. *For every set $X \subseteq 2^\omega$ we have*

(1) X has the Baire property implies X is weakly $\mathbb{FSP}$-measurable.
(2) X is Lebesgue-measurable implies X is weakly $\mathbb{FSP}$-measurable.

Proof. (1) Recall that X has the Baire property implies that X either is meager or there is $s \in 2^{<\omega}$ such that $X \cap [s]$ is comeager in $[s]$. Therefore, it is enough to show that any for comeager set D there exists $p \in \mathbb{FSP}$ such that $[p] \subseteq D$. So fix a comeager set D and let $\{D_n : n \in \omega\}$ be a $\subseteq$-decreasing family of open dense subsets such that $\bigcap D_n \subseteq D$. We aim at finding $p \in \mathbb{FSP}$ such that $[p] \subseteq \bigcap_{n\in\omega} D_n$. We will do so by constructing an $\subseteq$-increasing family of finite trees $\{F_n \subseteq 2^{<\omega} : n \in \omega\}$ and taking $p := \bigcup_n F_n$. Consider the following recursive construction:

Step 0. For $i \in 2$, pick $t_i \trianglerighteq \langle i\rangle$ such that $[t_i] \subseteq D_0$, and $|t_1| > |t_0|$. For every $\langle 0\rangle \trianglelefteq s \triangleleft t_0$, let $s'_j := s^\frown j$, where $j \in \{0,1\}$ is chosen so that $s^\frown j \not\trianglelefteq t_0$. Then pick $t'_j \trianglerighteq s'_j$ such that $[t'_j] \subseteq D_0$ and $|t'_j| > |t_1|$. Let T_0 be the set containing t_0, t_1 and all such ${t_j}'$s and put $N_0 := \max\{|t| : t \in T_0\}$. Finally consider the set

$$F_0 := \{s \in 2^{\leq N_0} : \exists t \in T_0 (s \trianglelefteq t \vee t \trianglelefteq s)\}.$$

By construction, $[F_0] \subseteq D_0$ and

$$\forall n < N_0 \exists s \in \mathrm{Lev}_n(F_0)(s \text{ is splitting}).$$

$n+1$. Assume we already constructed F_n and N_n. Then, for every $t \in \mathrm{ter}(F_n)$, we can repeat the construction described at Step 0 starting with $t_i \trianglerighteq t^\frown i, (i \in 2)$ in order to construct F^t_{n+1} satisfying the following properties:
- $\forall t' \in F^t_{n+1}$ $(t \trianglelefteq t' \vee t' \trianglelefteq t)$,
- $[F^t_{n+1}] \subseteq D_{n+1}$.

Let $T_{n+1} := \bigcup\{\mathrm{ter}(F^t_{n+1}) : t \in \mathrm{ter}(F_n)\}$ and $N_{n+1} := \max\{|t| : t \in T_{n+1}\}$. We define

$$F_{n+1} := \{s \in 2^{\leq N_{n+1}} : \exists t \in T_{n+1}(s \trianglelefteq t \vee t \trianglelefteq s)\}.$$

Finally let $p := \bigcup_{n\in\omega} F_n$.
Using the sequence $\langle N_n : n < \omega\rangle$ it is not hard to see that $p \in \mathbb{FSP}$. In fact, let $t \in p$ be given. We check that $K_p(t)$ is defined. There is $n < \omega$ such that $|t| \in [N_n, N_{n+1})$. It follows from the construction that $K_p(t) \leq N_{n+1}$.
Since we also made sure that $[p] \subseteq \bigcap_{n\in\omega} D_n$ holds, we are done.

(2) Recall that if X is Lebesgue measurable then there is $p \in \mathbb{B}$ such that $[p] \subseteq X$ or $[p] \cap X = \emptyset$. By Lemma 5.1 there is $p' \subseteq p$ such that $p' \in \mathbb{FSP}$.

□

Proposition 7.2. *Let G be $\mathbb{C}_{\omega_1}$-generic over V. Then*

$$V[G] \models \textit{All } On^{\omega}\textit{-definable sets are weakly-}\mathbb{FSP}\textit{-measurable.}$$

In order to prove Proposition 7.2 we need the following result.

Lemma 7.3. *Cohen forcing adds a $\mathbb{FSP}$-tree consisting of Cohen branches, i.e. $\mathbb{C}$ adds a tree $q \in \mathbb{FSP}$ such that*

$$\Vdash_{\mathbb{C}} \forall x \in [q](x \textit{ is Cohen over } V).$$

Proof. First recall Definition 1.1 (l), where we defined $p\restriction F = \{s \in p : (\exists t \in F)(s \trianglelefteq t \vee t \triangleleft s)\}$. for any perfect tree p and any subset $F \subseteq p$.
Consider the forcing $\mathbb{P}$ defined as follows: $F \in \mathbb{P}$ iff

(1) $F \subseteq 2^{<\omega}$ is a finite tree,
(2) $\forall s, t \in \mathrm{ter}(F)(|s| = |t|)$.

The partial order on $\mathbb{P}$ is given by:

$$F' \leq F \leftrightarrow F \subseteq F' \wedge \forall t \in F' \setminus F \exists s \in \mathrm{ter}(F)(s \trianglelefteq t)$$
$$\wedge \forall s \in \mathrm{ter}(F) \forall n \in [\mathrm{ht}(F), \mathrm{ht}(F'))\exists t \in \mathrm{Lev}_n((p\restriction s)\restriction F')(t^\frown 0, t^\frown 1 \in F').$$

Given two conditions $F_1 \leq F_0$. Let n be maximal such that each $s \in \mathrm{ter}(F_0)$ has at least n splitting predecessors. The partial order $\leq$ satisfies the following: For $p_i := \{t \in 2^{<\omega} \,:\, \exists s \in \mathrm{ter}\,(F_i)(t \trianglelefteq s \vee s \trianglelefteq t)\}$, we have $p_0, p_1 \in \mathbb{FSP}$ and $p_1 \leq_n p_0$. Specifically, taking a $\mathbb{P}$-generic filter G and defining $p_G := \bigcup G$, we also get that p_G is a fat splitting tree (in the generic extension).
Note that $\mathbb{P}$ is a countable atomless forcing order and so equivalent to $\mathbb{C}$. In fact, to see that $\mathbb{P}$ is atomless let $F \in \mathbb{P}$ be given. We have to find two incompatible extensions F_0, F_1 of F. Let $n = \mathrm{ht}(F)$. We construct $F_i, i \in 2$ in three steps:

$$F_i' := F \cup \{t \in 2^{n+1} \,:\, \exists s \in \mathrm{ter}(F)(s \trianglelefteq t)\},$$
$$F_i'' := F_i' \cup \{t \in 2^{n+2} \,:\, \exists s \in \mathrm{ter}(F_i')(s \trianglelefteq t \wedge (s(n) = 0 \to t(n+1) = i))\}$$
$$F_i := F_i'' \cup \{t \in 2^{n+3} \,:\, \exists s \in \mathrm{ter}(F_i'')(s \trianglelefteq t \wedge (s(n) = 1 \to t(n+2) = i))\}.$$

By the above we make sure that the terminal nodes of F_0 and F_1 are disjoint and in particular they are incompatible in $\mathbb{P}$. This proves that $\mathbb{P}$ is atomless.
Let $p_G := \bigcup G$, where G is $\mathbb{P}$-generic over V. It is left to show that every branch in $[p_G]$ is Cohen.

So let D be an open dense subset of $\mathbb{C}$ and $F \in \mathbb{P}$. It is enough to find $F' \leq F$ such that every $t \in \mathrm{ter}(F')$ is a member of D.

Therefore fix arbitrarily $t \in \mathrm{ter}(F)$ and consider the following construction. Pick $t_0 \trianglerighteq t^\frown 0$ such that $t_0 \in D$ and put

$$F_0(t) := F \cup \{s \in 2^{<\omega} : t^\frown 0 \trianglelefteq s \trianglelefteq t_0\} \cup \{s \in 2^{<\omega} : t^\frown 1 \trianglelefteq s \wedge |s| \leq |t_0|\}.$$

Then for every $s \in \mathrm{ter}(F_0(t))$ with $s \trianglerighteq t^\frown 1$, pick $t_s \trianglerighteq s$ such that $t_s \in D$. Note that since D is open dense and we only deal with finitely many $s \in \mathrm{ter}(F_0(t))$, we can pick the t_s's with the same length, say N_t. We then define

$$F_1(t) := F \cup \{r \in 2^{<\omega} : t_0 \trianglelefteq r \wedge |r| \leq N_t\} \cup \{r \in 2^{<\omega} : \exists s \in \mathrm{ter}(F_0(t))(t^\frown 1 \trianglelefteq s \trianglelefteq r \trianglelefteq t_s)\}.$$

Now we let $F'' := \bigcup\{F_1(t) : t \in \mathrm{ter}(F)\}$. We are almost done, we only have to make sure that all terminal nodes are of the same length. Therefore let $N := \max\{N_t \,:\, t \in \mathrm{ter}(F)\}$ and define $F' := \{s \in 2^{\leq N} \,:\, \exists r \in \mathrm{ter}(F'')(r \trianglelefteq s)\}$. By construction, $F' \leq F$ and $\mathrm{ter}(F') \subseteq D$.

□

Proof of Proposition 7.2. The argument follows the line of the proof of [4, Proposition 3.7]. Let G be $\mathbb{C}_{\omega_1}$-generic over V. In $V[G]$, let X be an On^ω-definable set of reals, i.e. $X := \{x \in 2^\omega : \varphi(x, v)\}$ for a parameter $v \in \mathrm{On}^\omega$. We aim to find $p \in \mathbb{FSP}$ such that $[p] \subseteq X$ or $[p] \cap X = \emptyset$.

First note that we can absorb v in the ground model, i.e, we can find $\alpha < \omega_1$ such that $v \in V[G{\restriction}\alpha]$. So, without the loss of generality we can pretend that $\alpha = 0$, i.e., $v \in V$.

Let c be Cohen over V. Then there is $s_0 \in \mathbb{C}$ such that $s_0 \Vdash \neg\varphi(c, v)$ or there is $s_1 \in \mathbb{C}$ such that $s_1 \Vdash \varphi(c, v)$. In either case we can find $p \in \mathbb{FSP}$ as in Lemma 7.3 such that $[p] \subseteq [s_0]$ or $[p] \subseteq [s_1]$ and every $x \in [p]$ is Cohen over V. We claim that p satisfies the required property.

- Case $[p] \subseteq [s_1]$: note every $x \in [p]$ is Cohen over V, and so $V[x] \models \varphi(x, v)$. Hence $V[G] \models \forall x \in [p](\varphi(x, v))$, which means $V[G] \models [p] \subseteq X$.
- Case $[p] \subseteq [s_0]$: we argue analogously and get $V[G] \models \forall x \in [p](\neg\varphi(x, v))$, which means $V[G] \models [p] \cap X = \emptyset$.

□

Theorem 7.4. *Assume there exists an inaccessible cardinal. There is a model for* ZF+DC *where all sets are $\mathbb{V}$-measurable (and so $\mathbb{S}$-measurable as well, by Remark 3.3) but there is a set which is not $\mathbb{FSP}$-measurable.*

Proof. The key idea is to get a complete Boolean algebra B and a B-name Y for a set of elements in 2^ω such that in the corresponding extension $V[G]$, for a B-generic filter G, the following hold:

(1) every subset of 2^ω in $L(\omega^\omega, Y)$ is $\mathbb{V}$-measurable
(2) Y is not $\mathbb{FSP}$-measurable.

Hence, we obtain that in $L(\omega^\omega, Y)^{V[G]}$ every subset of 2^ω is $\mathbb{V}$-measurable, but there is a set which is not $\mathbb{FSP}$-measurable.

We start with a definition.

Definition 7.5. A complete Boolean algebra B is $(\mathbb{V}, Y)$-*homogeneous*, if for every Silver algebras $B_0, B_1 \lessdot B$ and any isomorphism $\varphi\colon B_0 \to B_1$, there exists an automorphism $\varphi^+ \supseteq \varphi$ of B such that

$$\Vdash_B \varphi^+[Y] = Y.$$

To construct a $(\mathbb{V}, Y)$-homogenous Boolean algebra, we use Shelah's amalgamation. We start by sketching out such Shelah's procedure.

One *basic amalgamation step* consists of ω substeps and looks as follows. Given a Boolean algebra $B = \mathsf{Am}^0(B, \varphi)$, two complete subalgebras $B_0, B_1 \lessdot B$ and an isomorphism $\varphi\colon B_0 \to B_1$, the amalgamation process provides us with the pair $(\mathsf{Am}(B, \varphi), \varphi^1)$ such that $B \lessdot \mathsf{Am}(B, \varphi)$, there are two isomorphic copies $e_0[B]$, $e_1[B] \lessdot \mathsf{Am}(B, \varphi)$ of B and $\varphi^1 \supseteq \varphi$ such that $\varphi^1\colon e_0[B] \to e_1[B]$ is an isomorphism. Such a procedure can be repeated, now with $e_i[B]$, φ^1 and $\mathsf{Am}(B, \varphi) = \mathsf{Am}^1(B, \varphi)$, and thus we get $\mathsf{Am}^2(B, \varphi)$ and φ^2. At the limit stage ω we take the smallest complete superalgebra $\mathsf{Am}^\omega(B, \varphi)$ of $\mathsf{Am}^n(B, \varphi)$, $n < \omega$, as described in details in [7, p.10], or more briefly in [8, p.736]. We let $\varphi^\omega = \bigcup \varphi^n$. The corresponding automorphism $\varphi^\omega : \mathsf{Am}^\omega(B, \varphi) \to \mathsf{Am}^\omega(B, \varphi)$ fulfills $\varphi^\omega \supseteq \varphi$.

We construct B by a recursive construction of length κ for a strongly inaccessible cardinal κ. We partition κ into four cofinal sets S_i, $i = 0, 1, 2, 3$. By induction on $\alpha \leq \kappa$ we choose B_α and Y_α. We start with $B_0 = \{0\}$, $Y_0 = \emptyset$.

(a) For $\alpha \in S_0$, we let

$$B_{\alpha+1} = B_\alpha * \mathbb{A}(\mathbb{V}),$$
$$B_{\alpha+1} \Vdash \dot{Y}_{\alpha+1} = \dot{Y}_\alpha.$$

The amoeba for Silver forcing is denoted by $\mathbb{A}(\mathbb{V})$. It is a forcing for adding a Silver tree consisting of $\mathbb{V}$-generic branches, and it is used in the variant of Solovay's Lemma in order to obtain $\mathbb{V}$-measurability.

(b) For $\alpha \in S_1$, we use a standard book-keeping argument to hand us down all situations of the following kind: $B_\alpha \lessdot B' \lessdot B_\kappa$ and $B_\alpha \lessdot B'' \lessdot B_\kappa$ are such that B_α forces $(B' : B_\alpha) \approx (B'' : B_\alpha) \approx \mathbb{V}$ and $\varphi_\alpha \colon B' \to B''$ an isomorphism s.t. $\varphi_\alpha \restriction B_\alpha = \mathrm{Id}_{B_\alpha}$. So suppose that $\varphi_\alpha \colon B' \to B''$, where B' and B'' are two Silver algebras of B_α, is handed down by the book-keeping. Then we let

$$B_{\alpha+1} = \mathrm{Am}^\omega(B_\alpha, \varphi_\alpha),$$
$$B_{\alpha+1} \Vdash \dot{Y}_{\alpha+1} = \dot{Y}_\alpha \cup \{\varphi_\alpha^j(\dot{y}), \varphi_\alpha^{-j}(\dot{y}) : \dot{y} \in Y_\alpha, j \in \omega\}.$$

(c) For $\alpha \in S_2$, we let

$$B_{\alpha+1} := \mathrm{B}_\alpha * \prod_{p \in \mathbb{FSP}^{V_{B_\alpha}}} \dot{\mathbb{FSP}}_p,$$

where $\prod_{p \in \mathbb{FSP}^{V_{B_\alpha}}} \dot{\mathbb{FSP}}_p$ is the full support product and $\mathbb{FSP}_p := \{q \in \mathbb{FSP} : q \leq p\}$ and

$$B_{\alpha+1} \Vdash \dot{Y}_{\alpha+1} := \dot{Y}_\alpha \cup \{\dot{y}_p : p \in \mathbb{FSP}^{V_{B_\alpha}}\},$$

where $\dot{y}_p$ is the standard name for the $\mathbb{FSP}_p$-generic real over V^{B_α}.

(d) For $\alpha \in S_3$, we let

$$\mathrm{B}_{\alpha+1} := \mathrm{B}_\alpha * \mathrm{Coll}(\omega, \alpha),$$

and $B_{\alpha+1} \Vdash \dot{Y}_{\alpha+1} := \dot{Y}_\alpha$. Here $\mathrm{Coll}(\omega, \alpha)$ is the Lévy collapse of α to ω, i.e., the set of $p \colon n \to \alpha$, $n \in \omega$, ordered by end-extension.

(e) Finally, for any limit ordinal $\lambda \leq \kappa$, we take the direct limit $B_\lambda = \lim_{\alpha<\lambda} B_\alpha$ and $B_\lambda \Vdash \dot{Y}_\lambda = \bigcup_{\alpha<\lambda} \dot{Y}_\alpha$.

We let $B = B_\kappa$ and $Y = Y_\kappa$ and show that they are as in (1) and (2).

When amalgamating over Silver forcing (as in the construction [8, pp. 740-741]) in order to get $\mathbb{V}$-measurability, we need to isolate a particular property shared by the $\mathbb{FSP}$-generic reals (namely *unreachability*, i.e., reals which are not captured by any ground model slalom (introduced in [8, Def. 12])), which is both preserved under amalgamation ([8, Lemma 15]) and under iteration with Silver forcing ([8, Lemma 16]).

We recall here the definition of *unreachability* and some main remarks for the reader convenience.

Definition 7.6.

- $\Gamma_k = \{\sigma \in \mathrm{HF}^\omega : \forall n \in (|\sigma(n)| \leq 2^{kn})\}$ and $\Gamma = \bigcup_{k \in \omega} \Gamma_k$, where HF denotes the hereditary finite sets;
- $g(n) = 2^{n \cdot n}$;

- $\{J_n : n \in \omega\}$ is defined via $J_0 = \{0\}$ and $J_{n+1} = \Big[\sum_{j\leq n} g(j), \sum_{j\leq n+1} g(j)\Big)$, for every $n \in \omega$;
- Given $x \in 2^\omega$, define $h_x(n) = x{\restriction}J_n$.
- One says that $z \in 2^\omega$ is *unreachable over* V if

$$\forall \sigma \in \Gamma \cap \mathrm{V} \exists n \in \omega (h_z(n) \notin \sigma(n)).$$

By Lemma 6.2, applied for each $k \geq 1$ to $f_k(n) = 2^{kn}$, with our modification J_n instead of $I_{k,n}$ we have for the generic $\dot{y}_p$ of $\mathbb{FSP}_p$: For each $p \in \mathbb{FSP}$ the real $\dot{y}_p$ is unreachable over V.

The proof of Theorem 7.4 is concluded as follows:

Let G be a B_κ-generic filter over V. Then

$$\mathrm{V}[G] \models \text{“ } Y \text{ is not } \mathbb{FSP}\text{-measurable”}.$$

One has to prove that for every $p \in \mathbb{FSP}$, both $Y \cap [p] \neq \emptyset$ and $[p] \not\subseteq Y$.

The proof follows the line of [8, Lemma 28]. More specifically, to prove the part $Y \cap [p] \neq \emptyset$ it is enough to use item (c) of the construction, by choosing $\alpha < \kappa$ sufficiently large so that $p \in V[G{\restriction}\alpha]$ (possible by κ-cc) and then picking a $\mathbb{FSP}_p$-generic real over $V[G{\restriction}\alpha]$, call it $y \in Y_{\alpha+1} \cap [p]$. The part $[p] \not\subseteq Y$ follows from the fact that if $p \in V[G{\restriction}\alpha]$ then any new real added at stages $\beta > \alpha$ in $[p]$ cannot be in Y; the elements that enter Y under clause (b) come from former stages and hence are not identical to the new real unless there were identical elements already in a former stratum of Y, and the elements entering Y under clause (c) are not in $V[G{\restriction}\alpha]$. For a more detailed proof we refer to [8, Lemma 29]. The argument is similar to the proof of [7, Theorem 6.2], specifically in the part to show that the set Γ cannot have the Baire property, where the property of “being unreachable” replaces the property of “being unbounded”.

To see that every subset of the reals in $L(\omega^\omega, Y)$ is $\mathbb{V}$-measurable, we use the fact that any isomorphism between copies of smaller Silver algebras can be extended to an automorphism of B that fixes Y. This is a slightly more complex variant of the usual homogeneity of Levy Collapse, providing us with a way to apply a variant of Solovay's Lemma (e.g., see [7, Theorem 6.2.b] for Lebesgue measurability or [8, Lemma 24] for Silver measurability) in order to show that $(\mathbb{V}, Y)$-homogeneity implies that all sets in $L(\omega^\omega, Y)$ are Silver measurable. □

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