ON THE GROUPWISE DENSITY NUMBER FOR FILTERS

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ABSTRACT. We consider the groupwise density number $\mathfrak{g}_f$ for groupwise dense ideals or for non-meagre filters. We answer a question by Taras Banakh on the value $\mathfrak{g}_f$ in the known models of $\mathfrak{g} < \mathfrak{mcf}$ and one by Boaz Tsaban on the value of $\mathfrak{g}_f$ in the Hechler model.

1. INTRODUCTION

In this note, we work with five cardinal characteristics

Definition 1.1. *(1)* $\mathfrak{b} = \min\{|F| \,:\, F \subseteq {}^\omega\omega \wedge (\forall g \in {}^\omega\omega)(\exists f \in F)(f \not\leq^* g)\}$ *is the bounding number.*

(2) $\mathfrak{u} = \min\{|B| \,:\, B$ *is a base for an ultrafilter*$\}$ *is the ultrafilter-base number.*

(3) $\mathfrak{g}$ *is the smallest number of groupwise dense sets whose intersection in empty (or not groupwise dense). A set* $\mathcal{G} \subseteq [\omega]^\omega$ *is groupwise dense iff it is closed under almost subsets and if for every* $\langle n_i \,:\, i < \omega\rangle$ *of strictly increasing natural numbers there is some infinite* A *such that* $\bigcup_{i \in A}[n_i, n_{i+1}) \in \mathcal{G}$.

(4) $\mathfrak{g}_f$ *is the smallest number of groupwise dense ideals whose intersection is empty.*

(5) $\mathfrak{mcf} = \min\{\mathrm{cf}(\omega^\omega/U, \leq_U) \,:\, U$ *is a free ultrafilter on* $\omega\}$*. Here* $[f]_U \leq_U [g]_U$ *iff* $\{n \,:\, f(n) \leq g(n)\} \in U$*, and* $\mathrm{cf}(L, \leq_L)$ *is the smallest size of a cofinal set in the linear order* $(L, \leq_L)$.

It is known that $\mathfrak{g} \leq \mathfrak{g}_f \leq \mathfrak{mcf}$ (see [4] or [1]) and that $Con(\mathfrak{b} = \mathfrak{g} < \mathfrak{mcf})$ [5]. The consistencies of the strict inequalities above $\mathfrak{b}$ are interesting because

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they are weak relatives of the items in the long-standing open problem on the reversibilities of the implications:

$$\mathfrak{u} < \mathfrak{g} \Leftrightarrow \text{semi filter trichotomy} \Rightarrow$$
$$\mathfrak{u} < \mathfrak{g}_f \Leftrightarrow \text{filter dichotomy} \Rightarrow$$
$$\mathfrak{u} < \mathfrak{mcf} \Leftrightarrow \text{near coherence of filters.}$$

For the principles, which will not be used in the current work, we refer the reader to [2]. In this note we prove:

Theorem 1.2. *It is consistent relative to* ZFC *that* $\mathfrak{b} = \mathfrak{g} = \mathfrak{g}_f = \aleph_1 < \mathfrak{mcf} = \mathfrak{c} = \aleph_2$.

For a filter F, $\{\omega \setminus X : X \in F\}$ is groupwise dense (and closed under finite unions) iff F is not meagre. Thus $\mathfrak{g}_f$ is also the smallest number of non-meagre filters whose intersection is meagre.

So far only $\mathfrak{g}$ and $\mathfrak{mcf}$ have been separated above $\mathfrak{b}$ in a a quite complex oracle c.c. iteration in [5]. We show that that forcing also separates $\mathfrak{g}_f$ from $\mathfrak{mcf}$. It is open whether $\mathfrak{g} < \mathfrak{g}_f$ is consistent relative to ZFC. In all our models, $\mathfrak{u}$ is $\aleph_2$ and there are $\aleph_2$ Cohen reals, though.

The same sufficient criterion for $\mathfrak{g}_f$ that we use in the proof of Theorem 1.2 will yield a short proof of the following

Theorem 1.3. *In the finite support iteration of Hechler forcing of uncountable length κ over a ground model of CH we have that $\mathfrak{g}_f = \aleph_1$.*

Yorioka proved that $\mathfrak{g} = \aleph_1$ in the Hechler model [8].

2. A sufficient criterion

The following sufficient criterion for $\mathfrak{g}_f$ being small is a modification Lemma 5.1 in [5] in which the second premise is now strengthened to finite unions.

To our knowledge neither for this criterion nor for the original criterion it is known whether they are also necessary.

Lemma 2.1. *Assume that $\{Y_\zeta : \zeta < \mathfrak{c}\} \subseteq [\omega]^\omega$, and κ is a cardinal such that:*

(1) For each meagre ideal $\mathbf{B} \subseteq [\omega]^\omega$, $|\{\zeta : Y_\zeta \notin \mathbf{B}\}| = \mathfrak{c}$.

(2) For each $A \in [\omega]^\omega$, every family of finite sequences $\bar{\zeta}$ with pairwise disjoint ranges such that for all members $\bar{\zeta}$ of the family, $A \subseteq^* Y_{\zeta_0} \cup \cdots \cup Y_{\zeta_{\lg(\bar{\zeta})-1}}$, has cardinality strictly less than κ.

Then $\mathfrak{g}_f \leq \kappa$.

Proof. We now define κ sets and then show that they are groupwise dense ideals and that their intersection is empty.

Let $\langle \bar{n}^\zeta : \zeta < \mathfrak{c} \rangle$ list all strictly increasing sequences of natural numbers, each sequence appearing cofinally often. By induction on $\zeta < \mathfrak{c}$ we choose $\varepsilon_\zeta \leq \kappa$, $\gamma_\zeta < \mathfrak{c}$ and $C_\zeta \in [\omega]^\omega$ as follows.

If there is some $\varepsilon < \kappa$ such that for each $\xi < \zeta$ with $\varepsilon_\xi = \varepsilon$ we have $[n_i^\zeta, n_{i+1}^\zeta) \not\subseteq C_\xi$ for all but finitely many i, then we take as ε_ζ the minimal such ε. By the assumption (1), applied to the meagre ideal $\{A : \exists^{<\infty} i [n_i^\zeta, n_{i+1}^\zeta) \subseteq A\}$ we can choose γ_ζ to be the minimal $\gamma < \mathfrak{c}$ such that $\gamma \neq \gamma_\xi$ for all $\xi < \zeta$ and there are infinitely many i such that $[n_i^\zeta, n_{i+1}^\zeta) \subseteq Y_\gamma$. In this case we set $C_\zeta = \bigcup \{[n_i^\zeta, n_{i+1}^\zeta) : i \in \omega, [n_i^\zeta, n_{i+1}^\zeta) \subseteq Y_{\gamma_\zeta}\}$. Otherwise we set $\varepsilon_\zeta = \kappa$ and $C_\zeta = \omega$.

For each $\xi < \kappa$, define

$$\begin{aligned} \mathcal{G}_\xi = \{B \in [\omega]^\omega : &(\exists n < \omega)(\exists \zeta_1 \dots \zeta_n < \mathfrak{c})((\forall k \in [1, n])(\xi \leq \varepsilon_{\zeta_i} < \kappa) \text{ and} \\ &B \subseteq^* C_{\zeta_1} \cup \dots \cup C_{\zeta_n})\}. \end{aligned}$$

We show that each $\mathcal{G}_\xi$ is groupwise dense and the dual of a non-meagre filter $F_\xi = \{\omega \setminus X : X \in \mathcal{G}_\xi\}$. Clearly it is closed under almost subsets and under finite unions. Let an increasing sequence $\bar{n}$ be given. Then there are ζ_j, $j < \mathfrak{c}$, such that for all j, $\bar{n} = \bar{n}^{\zeta_j}$ and the ζ_j, $j < \mathfrak{c}$, are cofinal in $\mathfrak{c}$. Then, by our construction $\varepsilon_{\zeta_j} < \varepsilon_{\zeta_{j'}}$ for $j < j'$ if $\varepsilon_{\zeta_j} < \kappa$, or $\varepsilon_{\zeta_j} = \kappa$. So there is some j such that $\varepsilon_{\zeta_j} = \kappa$ or $\varepsilon_{\zeta_j} \in (\xi, \kappa)$. In both cases we have $(\exists \zeta)((\exists^\infty i)([n_i, n_{i+1}) \subseteq C_\zeta) \wedge \varepsilon_\zeta \geq \xi)$.

To see that $\bigcap \{\mathcal{G}_\xi : \xi < \kappa\} = \emptyset$, assume that B is infinite and for each ξ, $B \in \mathcal{G}_\xi$. Then for each $\xi < \kappa$, there is $(\beta_{1,\xi}, \dots, \beta_{n_\xi,\xi}) =: \bar{\beta}_\xi < \mathfrak{c}$ such that $\varepsilon_{\beta_{i,\xi}} \geq \xi$ and $B \subseteq^* \bigcup_{i<n_\xi} C_{\beta_{i,\xi}} \subseteq \bigcup_{i<n_\xi} Y_{\gamma_{\beta_{i,\xi}}}$. Since κ is regular, we can thin out and assume that if $\xi_1 < \xi_2$, then $\varepsilon_{\beta_{i,\xi_1}} \neq \varepsilon_{\beta_{j,\xi_2}}$ for all $i \leq n_{\xi_1}$ and all $j \leq n_{\xi_2}$. Thus we have that for $\xi_1 < \xi_2$, $\bar{\beta}_{\xi_1} =$ is disjoint from $\bar{\beta}_{\xi_2}$, and hence $\gamma_{\bar{\beta}_{\xi_1}} = (\gamma_{\beta_{1,\xi}}, \dots, \gamma_{\beta_{n_\xi,\xi}})$ is disjoint from $\gamma_{\bar{\beta}_{\xi_2}}$. Consequently, $\{\gamma_{\bar{\beta}_\xi} : \xi < \kappa\}$ is a family of pairwise disjont tuples $\gamma_{\bar{\beta}_\xi}$ of size κ. But $\{\gamma_{\bar{\beta}_\xi} : \xi < \kappa\} \subseteq \{(\zeta_1, \dots, \zeta_k) < \mathfrak{c} : B \subseteq^* \bigcup_{i<k} Y_{\zeta_i}$ and the $\bar{\zeta}$ are pairwise disjoint$\}$, contradicting the assumption (2). $\square$

3. The computation in the oracle c.c. iteration

Now we show that the oracle c.c. forcing from [5] yields that the $Y_\zeta^1[G_{\aleph_2}] = Y_\zeta$ fulfil the premises of Lemma 2.1. We cannot repeat the whole complicated construction form [5], and thus we give a sketch and point out the differences, where we claim that the Y_ζ has stronger properties than the ones used in the former work. For an introduction ot oracle-c.c.-forcing and for the explanation of the expression "S_δ guesses $\langle \mathcal{U}_\alpha, g_\alpha) : \alpha < \aleph_1\rangle$" we refer to the third and second section of the mentioned work.

Definition 3.1. *We use a finite support iteration $\langle \mathbb{P}_\delta, \underset{\sim}{\mathbb{Q}}_\delta, : \delta < \aleph_2\rangle$ of c.c.c. forcing notions, and choose constant or increasing oracles $\bar{M}^\delta$, such that $\mathbb{P}_\delta$ has the $\bar{M}^\delta$-c.c. for each δ. We start with a ground model satisfying $\diamondsuit^*_{\aleph_1}$ and $\diamondsuit_{\aleph_2}(S^2_1)$. Let $\langle S_\delta : \delta \in S^2_1\rangle$ be a $\diamondsuit_{\aleph_2}(S^2_1)$-sequence.*

There are three possibilities for $\mathbb{Q}_\delta$. If $\mathrm{cf}(\delta) = \aleph_0$ or if δ is a successor, then $\mathbb{Q}_\delta$ is the Cohen forcing.

If $\mathrm{cf}(\delta) = \aleph_1$ and $\Vdash_{\mathbb{P}_\delta}$ "S_δ guesses a sequence of ultrafilters $\mathcal{U}_\alpha$ and of functions g_α, $\alpha < \aleph_1$", then we choose A_α, $\alpha < \aleph_1$, as in Lemma [5, 4.1] *but with additional provisos as in the next definition and force with $\mathbb{Q}_\delta = \mathbb{Q}(\langle A_\alpha, g_\alpha : \alpha < \aleph_1\rangle)$. Here,*

$$\mathbb{Q} = \mathbb{Q}(A_\alpha, g_\alpha : \alpha < \gamma) = \{(n, h, F) : n \in \omega, h \in {}^n\omega, F \in [\gamma]^{<\aleph_0}\},$$

with $(n_1, h_1, F_1) \le (n_2, h_2, F_2)$ if $n_1 \le n_2$, $h_2 \restriction n_1 = h_1$, $F_1 \subseteq F_2$, and

$$(\forall \alpha \in F_1)(\forall n \in [n_1, n_2) \cap A_\alpha)(g_\alpha(n) \le h_2(n)).$$

Otherwise, we set $\mathbb{Q}_\delta = \{0\}$.

Definition 3.2. *For $\gamma \le \aleph_2$ we consider the class $\mathcal{K}_\gamma$ of γ-*approximations

$$\langle (\mathbb{P}_\delta, \underset{\sim}{\mathbb{Q}}_\delta, \bar{M}^\delta, W_1, W_2) : \delta < \gamma\rangle$$

with the following properties:

- (a) *$\langle \mathbb{P}_\delta, \underset{\sim}{\mathbb{Q}}_\delta : \delta < \gamma\rangle$ is a finite support iteration of partial orders such that for each $\delta < \gamma$, $|\mathbb{P}_\delta| \le \aleph_1$.*
- (b) *$\langle \bar{M}^\delta : \delta < \gamma\rangle$ is a constant sequence of oracles such that for all δ, $\mathbb{P}_\delta$ satisfies the $\bar{M}^\delta$-c.c. and for $\delta + 1 < \gamma$, $\Vdash_{\mathbb{P}_\delta}$ "$\underset{\sim}{\mathbb{Q}}_\delta$ satisfies the $(\bar{M}^{\delta+1})^*$-c.c." (as in Lemma* [6, IV.3.1]*). The constant value of the oracle sequence is some oracle $\bar{M}$ as in Lemma* [5, 3.9], *keeping* $\mathrm{cov}(\mathcal{M}) = \aleph_1$.
- (c) *$W_1, W_2 \subseteq \aleph_2 \setminus S^2_1$, W_1 and W_2 are disjoint and if γ is a limit of cofinality $\aleph_1$, then $W_1 \cap \gamma$, $W_2 \cap \gamma$ are both cofinal in γ.*
- (d) *If $\beta \in (W_1 \cup W_2) \cap \gamma$ then $\underset{\sim}{\mathbb{Q}}_\beta$ is the Cohen forcing adding the real $\underset{\sim}{r}_\beta \in {}^\omega 2$.*

(e) *If $\delta \in S^2_1 \cap \gamma$ and S_δ guesses $\langle (\mathcal{U}_\alpha(\delta), g_\alpha(\delta)) : \alpha < \aleph_1 \rangle$, then there is some strictly increasing enumeration $\langle \zeta_\alpha(\delta) : \alpha < \aleph_1 \rangle$ of a cofinal part of $W_2 \cap \delta$, and for every $\alpha < \aleph_1$ there is $\ell_{\zeta_\alpha(\delta)} \in \{0,1\}$ such that $Y^{\ell_{\zeta_\alpha(\delta)}}_{\zeta_\alpha(\delta)} := r^{-1}_{\zeta_\alpha(\delta)}(\{\ell_{\zeta_\alpha(\delta)}\}) \in \mathcal{U}_\alpha$, and $\mathbb{Q}_\delta = \mathbb{Q}(Y^{\ell_{\zeta_\alpha(\delta)}}_{\zeta_\alpha(\delta)}, g_\alpha(\delta) : \alpha < \aleph_1)$.*

(f) *For all $\delta \leq \gamma$, $\Vdash_{\mathbb{P}_\delta}$ "$(\forall A \in [\omega]^\omega)$ every set of the form $\{\bar\beta \in W_1 \cap \delta : A \subseteq^* \bigcup_{i < \lg(\bar\beta)} \underset{\sim}{Y}^1_{\beta_i}$ and the $\bar\beta$ are pairwise disjoint$\}$ is at most countable." Here, for $\delta = \gamma$ limit, $\mathbb{P}_\gamma$ is the direct limit of $\langle \mathbb{P}_\beta : \beta < \gamma \rangle$, and for $\delta = \gamma = \beta + 1$, $\mathbb{P}_\gamma = P_\beta * \underset{\sim}{\mathbb{Q}}_\beta$.*

Now the technical core is to prove the following.

Theorem 3.3. *If $V \models \diamondsuit^*_{\aleph_1}$ and $\diamondsuit_{\aleph_2}(S^2_1)$, then for each $\gamma \leq \aleph_2$, $\mathcal{K}_\gamma$ is not empty.*

Let V fulfil the premises and let $\mathbb{P}_{\aleph_2}$ be the direct limit of the first components of an $\aleph_2$-approximation. If G is a $\mathbb{P}_{\aleph_2}$-generic filter and $\underset{\sim}{Y}^1_\zeta[G_{\aleph_2}] = Y_\zeta$ for $\zeta \in W_1$, then we have in the final model a sequence $\langle Y_\zeta : \zeta < \mathfrak{c} \rangle$ as in Lemma 2.1 with $\kappa = \aleph_1$. For the $2^\omega = \aleph_2 \geq \mathfrak{mcf} \geq \mathrm{cov}(\mathcal{D}_{\mathrm{fin}}) \geq \aleph_2$-part, which is not affected by the difference between the current Definition 3.2(f) and the former version, we refer the reader to [5]. Thus Theorem 3.3 yields:

Corollary 3.4. $V^{\mathbb{P}_{\aleph_2}} \models \mathrm{cov}(\mathcal{M}) = \mathfrak{g}_f = \aleph_1 < \mathrm{cov}(\mathcal{D}_{\mathrm{fin}}) = \aleph_2$.

Theorem 3.3 is proved by induction on γ. The witnesses are end-extensions of former witnesses. For some γ's, one has to work to show item (e). For this the work in [5] suffices. For all γ's but maybe the successor steps of points not in S^2_1, one has to carefully revise the work from [5] in order to show that item 3.2(f) can be preserved in the induction. For completeness sake, we carry this out in Lemma 3.7 to Lemma 3.10.

Lemma 3.5. *Consider a successor $\gamma = \delta + 1$, $\delta \in S^2_1$. Given any $\aleph_1$-oracle $(\bar M^{\delta+1})^*$, the sequence $\langle \zeta_\alpha(\delta) : \alpha < \aleph_1 \rangle$ can be chosen as in (e) so that the forcings given in item (e) have the $(\bar M^{\delta+1})^*$-c.c.*

Proof. This is literally as in [5, Lemma 5.4].

Choice 3.6. *We start with $\bar M$ as described. By Lemma* [6, IV,3.1], *all the $\mathbb{P}_\delta$, $\delta \leq \aleph_2$, have the $\bar M$-c.c. as soon as we can arrange that all the $\mathbb{Q}_\delta$ have the $(\bar M)^*$-c.c. in $V^{\mathbb{P}_\delta}$. The Cohen forcing has the $\bar M$-c.c. for any $\bar M$. The $\mathbb{Q}_\delta$ in the steps $\delta \in S^2_1$ can be chosen by the previous lemma so that they have the $(\bar M)^*$-c.c.*

Lemma 3.7. *If $\delta \in S^2_1$, $\mathbb{Q}_\delta$ is chosen as in Lemma 3.5, and $\mathbb{P}_\delta$ satisfies (f) of Definition 3.2, then $\mathbb{P}_{\delta+1}$ has the property stated in item (f).*

Proof. Suppose that $p \Vdash_{\mathbb{P}_{\delta+1}}$ "$\underset{\sim}{A} \in [\omega]^\omega$ and $|\{\bar\zeta \in W_1 \cap \delta : \underset{\sim}{A} \subseteq^* \bigcup_{i<\lg(\bar\zeta)} \underset{\sim}{Y}^{\ell_{\zeta_i}}_{\zeta_i}$ and the $\bar\zeta$ are pairwise disjoint$\}| = \aleph_1$", and w.l.o.g. $p \Vdash_{\mathbb{P}_{\delta+1}}$ "$\underset{\sim}{A} \in [\omega]^\omega$ and $\{\bar\xi \in W_1 \cap \delta : \underset{\sim}{A} \subseteq^* \bigcup_{i<\lg(\bar\zeta)} \underset{\sim}{Y}^{\ell_{\zeta,i}}_{\zeta,i}\}$ is increasingly enumerated by $\{\bar\xi_\alpha : \alpha < \aleph_1\} = W_1(A)$".

We take for $n \in \omega$ a maximal antichain $\{p_{n,i} : i \in \omega\}$ above p deciding the statements "$\check n \in \underset{\sim}{A}$" with truth value $t_{n,i}$. Let $C_{n,i} = \{\varepsilon \le \delta : p_{n,i}(\varepsilon) \ne 1\}$. For $\varepsilon \in C_{n,i} \cap S^2_1$ with $\mathbb{Q}_\varepsilon \ne \{0\}$, let $p_{n,i}(\varepsilon) = (m_{n,i}(\varepsilon), h_{n,i}(\varepsilon), F_{n,i}(\varepsilon))$. Let $F'_{n,i}(\varepsilon) = \{\zeta_\alpha(\varepsilon) : \alpha \in F_{n,i}(\varepsilon)\}$. We assume that all these are objects not just names. For $\varepsilon \in C_{n,i} \setminus S^2_1$ let $p_{n,i}(\varepsilon) = h_{n,i}(\varepsilon)$, $m_{n,i}(\varepsilon) = |h_{n,i}(\varepsilon)|$ and set the other two components for simplicity zero. Set $m_{n,i} = \max\{m_{n,i}(\varepsilon) : \varepsilon \in C_{n,i}\}$. Set

$$\bar C = \langle\langle(m_{n,i}(\varepsilon), h_{n,i}(\varepsilon), F_{n,i}(\varepsilon), F'_{n,i}(\varepsilon), \langle g_\alpha(\varepsilon) \restriction m_{n,i} : \alpha \in F_{n,i}(\varepsilon)\rangle) : \varepsilon \in C_{n,i}\rangle : n, i \in \omega\rangle.$$

For each $\beta \in \aleph_1$, let $p_\beta \ge p$, $p_\beta \Vdash_{\mathbb{P}_{\delta+1}}$ "$\underset{\sim}{A} \cap [s_\beta, \infty) \subseteq \bigcup_{i<\lg(\bar\xi)} \underset{\sim}{Y}^{\ell_{\xi_{i,\beta}}}_{\xi_{i,\beta}}$" and p_β shall decide the value of $\ell_{\bar\xi_\beta} \in 2$ and $s_\beta \in \omega$. For $\beta < \aleph_1$ we set $C_\beta = \{\varepsilon \le \delta : p_\beta(\varepsilon) \ne 1\}$. If $\varepsilon \in C_\beta \cap S^2_1$, then $p_\beta(\varepsilon) = (m_\beta(\varepsilon), h_\beta(\varepsilon), F_\beta(\varepsilon))$. If $\varepsilon \in C_\beta \setminus S^2_1$, then $p_\beta(\varepsilon) = h_\beta(\varepsilon)$, ${}_\beta(\varepsilon) = |h_\beta(\varepsilon)|$ and $F_\beta(\varepsilon) = \emptyset$. For all β, $\varepsilon \in C_\beta$, let $F'_\beta(\varepsilon) = \{\zeta_\alpha(\varepsilon) : \alpha \in F_\beta(\varepsilon)\} \subseteq W_2$.

Set

$$R_\beta(m) = \langle(m_\beta(\varepsilon), h_\beta(\varepsilon), F_\beta(\varepsilon), F'_\beta(\varepsilon), \langle g_\alpha(\varepsilon) \restriction m : \alpha \in F_\beta(\varepsilon)\rangle) : \varepsilon \in C_\beta\rangle.$$

These are finite arrays of finite sets.

Now we thin out: First we assume that for some $k \in \omega$ for all $\beta < \aleph_1$, $|C_\beta| = k$, $s_\beta \le k$. We apply the delta system lemma to C_β, $\beta \in \aleph_1$, get a root C. We assume that $\delta \in C$, as this is the difficult case. We apply the delta lemma for each $\varepsilon \in C$ to the $F_\beta(\varepsilon)$, $\beta \in \aleph_1$, and get a root $F(\varepsilon)$, and to $F'_\beta(\varepsilon)$, $\beta \in \aleph_1$, and get a root $F'(\varepsilon)$. We further assume that for each β in the delta system and for all $\varepsilon \in C$, all $F_\beta(\varepsilon) \setminus F(\varepsilon)$ are above $\max(\bigcup_{\varepsilon' \in C}(F(\varepsilon')) \cup (C \setminus \{\delta\}))$ and same for the primed ones. All $F'_{\alpha_i}(\varepsilon) \setminus F$ are above $\max F'(\varepsilon)$. This goes only ε-wise, because in the definition of $\mathcal{K}_\gamma$ in item (e) we did not require coherence in the enumerations $\langle \zeta_\alpha(\varepsilon) : \alpha \in \aleph_1\rangle$. We thin out further and assume that there are $(m(\varepsilon), h(\varepsilon), F(\varepsilon))$ such that for all $\beta < \aleph_1$, for all $\varepsilon \in C$, $m_\beta(\varepsilon) = m(\varepsilon)$, $h_\beta(\varepsilon) = h(\varepsilon) \in {}^{m(\varepsilon)}\omega$, and for the $\varepsilon \in C_\beta \setminus C$, the increasingly enumerated ε's in $C_\beta = \{\varepsilon^\beta_i : i < k\}$, are isomorphic to the lexicographically first $\langle \varepsilon_i : i < k\rangle$, i.e., $m_\beta(\varepsilon_i^\beta) = m(\varepsilon_i)$, $h_\beta(\varepsilon_i^\beta) = h(\varepsilon_i) \in {}^{m(\varepsilon_i)}\omega$, and we use a delta system argument on the $F_\beta(\varepsilon_i^\beta)$ giving a root $F(\varepsilon_i)$ and again impose on the parts $F_\beta(\varepsilon_i^\beta) \setminus F(\varepsilon_i)$, that they have to lie above $\bigcup_{i<k} F(\varepsilon_i)$ and are all of the same size. The analogous thinning out is done for the primed parts, that have to lie above $\max(\bigcup_{i<k}(F'(\varepsilon_i)) \cup (C \setminus \{\delta\}))$, be for all i of the same size $|F'_\beta(\varepsilon_i^\beta)|$ independently of β (but depending on i), and all of the $\langle F'_\beta(\varepsilon_i^\beta) : i < k\rangle$ shall have the same $\leq$ or $\geq$-relations with the members of $C_\beta(\varepsilon_i)$. Moreover, if ε is a Cohen coordinate in C_β, then $p_\beta(\varepsilon)$ does not depend on β.

We let m_{max} be the the maximum of the $m(\varepsilon)$ and of the lengths of all the finitely many Cohen coordinates for all β in the delta system. Let $\triangleleft$ denote the initial segment relation for finite sequences. We thin out further and assume that all the $R_\beta(m_{max})$ have the same quantifier free $(<_{\aleph_1}, \triangleleft)$-type over $\mathrm{Ran}(\bar{C}) \cup \mathrm{Ran}(\mathrm{Ran}(\bar{C}))$. Speaking about components of five tuples $(m, h, F, F', \bar{g})$ separately is allowed as well as evaluating $\bar{g}$ and the members of all involved finite sets. There are only countably many quantifier types in this language that can be fulfilled by a (finite) sequence $R_\beta(m_{max})$ in our delta system.

Let G_δ be a subset of $\mathbb{P}_\delta$ that is generic over V such $W^* = \{\gamma \in W_1(A) \cap \delta : p_\gamma \restriction \delta \in G_\delta\}$ is uncountable.

For $\gamma \in W^*$, let in $V[G_\delta]$,

$$B_\gamma = \{n \in \omega : \exists p' \in \mathbb{P}_{\delta+1}, p' \geq p_\gamma, p' \restriction \delta \in G_\delta, \text{ and } p' \Vdash_{\mathbb{P}_{\delta+1}} n \in \underset{\sim}{A}\}.$$

$B_\gamma \subseteq^* \bigcup_{i<\lg(\bar{\xi})} Y_{\xi_{i,\alpha}}^{\ell_{\xi_{i,\alpha}}}[G]$, and the latter is fully evaluated by G, because $\bar{\xi}_\alpha \in W_1 \subseteq \delta + 1$ for $\alpha < \aleph_1$, and $\delta \notin W_1$.

We shall show that for β, $\gamma \in W^*$, $B_\beta \cap [k, \infty) = B_\gamma \cap [k, \infty) = B \in V[G]$. Then B is a counterexample to $\langle(\mathbb{P}_\varepsilon, \mathbb{Q}_\beta, M^\varepsilon, W_1, W_2) : \varepsilon \leq \delta, \beta < \delta\rangle \in \mathcal{K}_\delta$.

Let $||_{\mathbb{P}_{\delta+1}}$ denote the compatibility relation in $\mathbb{P}_{\delta+1}$. If $n \in B_\beta$, then $p_\beta ||_{\mathbb{P}_{\delta+1}} p_{n,i}$ for the one i such that $p_{n,i} \in G$, and for this i we have $t_{n,i} = true$. The same holds for $n \notin B_\beta$ with $false$. So our claim that $B_\beta \cap [k, \infty) = B_\gamma \cap [k, \infty)$ for all $\beta, \gamma \in W^*$ now follows from

Claim 3.8. *For all β, γ in W^*:*

$$p_\beta ||_{\mathbb{P}_{\delta+1}} p_{n,i} \textit{ iff } p_\gamma ||_{\mathbb{P}_{\delta+1}} p_{n,i}.$$

Proof. The point is the coordinate δ, since the restrictions to δ are in G_δ, and hence compatible. Assume $p_{n,i}(\delta) = (m_{n,i}, h_{n,i}, F_{n,i})$, $p_\beta(\delta) = (m_\beta, h_\beta, F_\beta)$, $p_\gamma(\delta) =$

$(m_\gamma, h_\gamma, F_\gamma)$. We do not write the δ at these points, but will not suppress it completely. We assume that $p_\beta(\delta)$ is compatible with $p_{n,i}(\delta)$. Since $\zeta_\alpha(\delta) \in W_2$, we can now literally use the proof of [5, Claim 5.8].

So the claim is proved and with it also Lemma 3.7. $\square$

Lemma 3.9. *(1) If* $\mathrm{cf}(\gamma) = \aleph_1$ *and* $\underset{\sim}{\mathbb{Q}}$ *and* $\bar{M}^\gamma$ *are as in the previous lemma and if* $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \bar{M}^\beta, W_1, W_2) : \beta < \gamma\rangle \in \mathcal{K}_\gamma$*, then*

$$\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \bar{M}^\beta, W_1, W_2) : \beta < \gamma\rangle {}^\frown \langle \mathbb{P}_\gamma, \underset{\sim}{\mathbb{Q}}, \bar{M}^\gamma\rangle \in \mathcal{K}_{\gamma+1}.$$

(2) If $\mathrm{cf}(\gamma) = \aleph_0$ *and if* $\langle \mathbb{P}_\delta, \underset{\sim}{\mathbb{Q}}_\beta, \bar{M}^\beta, W_1, W_2) : \beta < \gamma\rangle \in \mathcal{K}_\gamma$*, then*

$$\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \bar{M}^\beta, W_1, W_2) : \beta < \gamma\rangle {}^\frown \langle \mathbb{P}_\gamma, \mathbb{C}, \bar{M}^\gamma\rangle \in \mathcal{K}_{\gamma+1}.$$

(3) If $\mathrm{cf}(\gamma) = \aleph_0$ *and if* $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \bar{M}^\beta, W_1, W_2) : \beta < \gamma\rangle \restriction \beta \in \mathcal{K}_\beta$ *for each* $\beta < \gamma$*, then* $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \bar{M}^\beta, W_1, W_2) : \beta < \gamma\rangle \in \mathcal{K}_\gamma$*.*

(4) If $\mathrm{cf}(\gamma) = \aleph_1$ *or* $\gamma = \aleph_2$*, and if* $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \bar{M}^\beta, W_1, W_2) : \beta < \gamma\rangle \restriction \beta \in \mathcal{K}_\beta$ *for each* $\beta < \gamma$*, then* $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \bar{M}^\beta, W_1, W_2) : \beta < \gamma\rangle \in \mathcal{K}_\gamma$*.*

Proof. (1) This was proved in Lemma 3.7.

(2) If A is an almost subset of uncountably many $\bigcup_{i<\lg(\bar{\zeta})} Y_{\zeta_i}$'s, then there is some $\gamma_0 < \gamma$ that there are uncountably many such $\bar{\zeta}$ below γ_0. A is possibly a name using the last, new forcing. But this is just Cohen forcing. So there is some finite part of a Cohen condition forcing that $\underset{\sim}{A}$ is in uncountably many Y_ζ's. But then also the forcing $\mathbb{P}_\gamma$ already contains a name for some infinite $B \subseteq \omega$ almost contained in the intersection of uncountably many $\bigcup_{i<\lg(\bar{\zeta})} Y_{\zeta_i}$'s with $\zeta < \gamma_0$. So P_γ does not fulfil property (f) and hence the induction hypothesis is not fulfilled.

(3) First we use the pigeonhole principle for the $Y_{\zeta,i}$'s as in the previous item. Then we use the following

Lemma 3.10. *Assume*

(a) $\langle \mathbb{P}_n : n \in \omega\rangle$ *is a* $\lessdot$*-increasing sequence of c.c.c. forcing notions with union* $\mathbb{P}$*,*

(b) $\mathcal{Y}$ *is a set of* $\mathbb{P}_0$*-names of infinite subsets of* ω*,*

(c) for $n \in \omega$ *we have* $\Vdash_{\mathbb{P}_n}$ *"*$\kappa = \mathrm{cf}(\kappa) > |\{\bar{\underset{\sim}{Y}} \in \mathcal{Y}^{<\omega} : \underset{\sim}{B} \subseteq^* \bigcup_{i<\lg(\bar{Y})} \underset{\sim}{Y}_i\}|$*", whenever* $\underset{\sim}{B}$ *is a* $\mathbb{P}_n$*-name of an infinite subset of* ω*.*

Then condition (c) holds for $\mathbb{P}$ *too.*

Proof. Since $\mathbb{P}$ is a c.c.c. forcing notion, also in $V^{\mathbb{P}}$ we have κ is a regular cardinal.

If the desired conclusion fails, then we can find a $\mathbb{P}$-name $\underset{\sim}{B}$ of an infinite subset of ω and a sequence $\langle (p_\alpha, \underset{\sim}{Y}_\alpha, m_\alpha) : \alpha < \kappa \rangle$ such that

(α) $m_\alpha \in \omega$,

(β) $\underset{\sim}{\bar{Y}}_\alpha \in \mathcal{Y}$ without repetitions,

(γ) $p_\alpha \in \mathbb{P}$, $p_\alpha \Vdash_{\mathbb{P}} \underset{\sim}{B} \setminus m_\alpha \subseteq \bigcup_{i<\lg(\bar{Y}_\alpha)} \underset{\sim}{Y}_{i,\alpha}$.

Since $\mathrm{cf}(\kappa) > \aleph_0$, for some $n(*), m(*) \in \omega$ the set $S =^{\mathrm{df}} \{\alpha < \kappa : p_\alpha \in \mathbb{P}_{n(*)}, m_\alpha = m(*)\}$ has cardinality κ. We identify it with κ.

Now for every large enough $\alpha \in S$ we have

$$p_\alpha \Vdash_{\mathbb{P}} \kappa = |\{\beta \in S : p_\beta \in \underset{\sim}{G}_{\mathbb{P}_{n(*)}}\}|.$$

Why? Else for an end segment of $\alpha < \kappa$ there is $q_\alpha \geq p_\alpha$ such that for all but $< \kappa$ many $\beta \in S$, $q_\alpha \Vdash p_\beta \notin \underset{\sim}{G}_{\mathbb{P}_{n(*)}}$. That means that for an end segments of $\alpha < \kappa$, w.l.o.g., for all $\alpha \in \kappa$, $\mathrm{Perp}_\alpha := \{\beta \in S : q_\beta \perp q_\alpha\}$ contains an end segment of S. Then we take the diagonal intersection D of all these end segments of S. Since κ is regular, D contains a club in κ. But then $\{q_\beta : \beta \in D\}$ is an antichain in $\mathbb{P}_{n(*)}$ of size κ. Contradiction.

Let $G_{n(*)}$ be a subset of $\mathbb{P}_{n(*)}$ generic over V, and let $S_* := \{\beta \in S : p_\beta \in G_{n(*)}\}$. We choose $G_{n(*)}$, such that $|S_*| = \kappa$. We let $B' = \cap\{\underset{\sim}{Y}_\beta \setminus m(*) : \beta \in S_*\}$. Then in $V[G_{n(*)}]$, B' is an infinite subset of ω included in $\bigcup_{i<\lg(\bar{Y}_\alpha)} \underset{\sim}{Y}_{i,\alpha}$ for κ pairwise disjoint members $\bar{Y}_\alpha$ of $\mathcal{Y}^{<\omega}$, contradicting the assumption. So Lemma 3.10 is proved. □

(4) If $\mathbb{P}_\delta$ adds some A, then this already comes earlier, say in $V^{\mathbb{P}_\varepsilon}$, $\varepsilon < \delta$, because $A \subseteq \omega$ and because of the c.c.c. If $A \subseteq^* Y_\zeta$ is forced, then $\zeta < \varepsilon$. This contradicts the induction hypothesis for $\mathbb{P}_\varepsilon$. This completes the proof of Lemma 3.9. □

The lemmas together give that there is an $\aleph_2$-approximation, and the proofs of Theorem 3.3 and of Theorem 1.2 are completed. □

As in [5, 5.11], with some extra care our proof can be modified to yield the following (cf. [7, 3]).

Theorem 3.11. *It is consistent (relative to ZFC) that all of the following assertions hold:*

(1) *Each unbounded set of* ω^ω *contains an unbounded subset of size* $\aleph_1$*,*

(2) *Each nonmeagre subset of* ω^ω *contains a nonmeagre subset of size* $\aleph_1$*,*

(3) $\mathfrak{g}_f = \aleph_1$*; and*

(4) $\mathrm{cov}(\mathcal{D}_{\mathrm{fin}}) = \mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \aleph_2$.

4. The situation in the Hechler model

The proof of Theorem 1.3 consists of Lemma 2.1 and the following lemma:

Lemma 4.1. *Let* $\mathrm{cf}(\kappa) \geq \aleph_1$*. Let* $\mathbb{P}$ *be the finite support iteration adding* κ *Hechler reals over a ground model satisfying the* CH*. We call the generic reals* $h_\zeta \in {}^\omega\omega$*,* $\zeta < \kappa$*. We set* $Y_\zeta = \{h_\zeta(n) : n < \omega\}$*. Then the family* $\{Y_\zeta : \zeta < \kappa\}$ *satisfies the two premises of Lemma 2.1*

Proof. For every meagre set $\mathbf{B}$ there are $r \in {}^\omega 2$ and a strictly increasing sequence $\bar{k}$ such that

$$\mathbf{B} \subseteq B_{r,\bar{k}} := \{s \in {}^\omega 2 : (\forall^\infty n) r \restriction [k_n, k_{n+1}) \neq s \restriction [k_n, k_{n+1})\}.$$

Now r and $\bar{k}$ appear in some step of the iteration, say that they are in $V[G_{<\zeta_0}]$. We show that all later Y_ζ, $\zeta \geq \zeta_0$, are not in $B_{r,\bar{k}}$. Let $p = (s, f) \in \mathbb{Q}_\zeta$. Then for all $n \in \omega$ there are some $q \geq p$, $m \geq n$, $q \in \mathbb{Q}_\zeta$, such that $q \Vdash \underset{\sim}{Y}_\zeta \restriction [k_m, k_{m+1}) = r \restriction [k_n, k_{n+1})$, because $h_\zeta \geq f$ on all arguments above $|s|$ is compatible with $(\exists m \geq n)(Y_\zeta \restriction [k_m, k_{m+1}) = \chi(\{h_\zeta(a) : a \in \omega\} \cap [k_m, k_{m+1})))$. To see this, we just take m sufficiently large and put no points $h(a)$ into $\min(Y_\zeta \restriction [k_m, k_{m+1}))$. Then we take $q = (s\hat{\ }h \restriction (h^{-1}[k_m, k_{m+1})), f)$.

Also premise (2) is fulfilled: $B \subseteq Y_{\zeta_1} \cup \cdots \cup Y_{\zeta_n}$ means that the next function of B eventually dominates the minimum of the next functions of the Y_{ζ_k}, $1 \leq k \leq n$. Again, B is in some intermediate model, say in $V[G_{<\zeta_0}]$. Then if all Y_{ζ_k} come later, by a density argument, the next function of B does not dominate the minimum of their next functions. So $B \subseteq^* Y_{\zeta_1} \cup \cdots \cup Y_{\zeta_n}$ means $\zeta_0 \cap \mathrm{range}(\bar{\zeta}) \neq \emptyset$, and there are strictly less than κ pairwise disjoint tuples $\bar{\zeta}$ of this kind. □

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