EXACTLY TWO AND EXACTLY THREE NEAR-COHERENCE CLASSES

HEIKE MILDENBERGER

Abstract. We prove that for $n = 2$ and $n = 3$ there is a forcing extension with exactly n near-coherence classes of non-principal ultrafilters. We introduce localised versions of Matet forcing and we develop Ramsey spaces of names. The evaluation of some of the new forcings is based on a relative of Hindman's theorem due to Blass 1987.

1. Introduction

We partially answer Banakh's and Blass' question [1, Question 31] on the finite part of near-coherence spectrum by providing forcing extensions with exactly two and with exactly three near-coherence classes of ultrafilters. Whereas the infinite part of the near-coherence spectrum is known, up to now only models with one near-coherence class have been known and the rest of the finite part of the spectrum has not been known. By Blass [4] the existence of only finitely many classes implies $\mathfrak{u} < \mathfrak{d}$ (since otherwise there are at least $2^{\mathfrak{u}}$ near-coherence classes). The cardinal characteristic $\mathfrak{u}$ is the smallest size of a base of a non-principal ultrafilter over ω, and $\mathfrak{d}$ is the dominating number. In 2006 Banakh and Blass [1] showed that the existence of infinitely many near-coherence classes implies that there are $2^{(2^\omega)}$ many classes. The latter is the over-all number of ultrafilters over ω.

We use a new type of proper forcing that preserves a P-point with a small base and destroys another ultrafilter. The witnesses for the second and the third near-coherence class in our models will be Ramsey ultrafilters that are not $P_{\aleph_2}$-points. The second coordinates of our forcing conditions, also called pure components, will be elements of selective coideals with the Ramsey property in the sense of Definitions 3.4 and 5.1. Such sets, together with suitable partial orders, form topological Ramsey spaces. The books [42] and [18] are good references to this rich field. Our proofs, however, will hardly cite from these sources and rather build on Blass's, Shelah's and Eisworth's work. The first coordinates of our forcings will not be finite initial segments of the second components, but rather carry less information.

We recall some results on the existence of special ultrafilters. The following is known about P-points or Ramsey ultrafilters in extensions that do not add unbounded reals: Kunen [27] proved that no Ramsey ultrafilter can be extended to

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a P-point after addition of any number of random reals at once, Shelah [39] constructed a model with no P-points, recently Chodounský and Guzmán [17] proved that there are no P-points in the Silver model and that no P-point from the ground model can be extended in a Silver extension. On the other hand [43] and [22] proved that there are Milliken–Taylor ultrafilters in the Sacks model, indeed, any Milliken–Taylor ultrafilter from the ground model is preserved. Blass [6, 8] proved that the minimum and the maximum projection of a Milliken–Taylor ultrafilter are non-nearly coherent Ramsey ultrafilters.

In the case of forcings adding an unbounded real, Ketonen proved that $\mathfrak{d} = \mathfrak{c}$ implies the existence of a P-point, Canjar [16] proved that any Ramsey filter of size $< \mathfrak{c}$ can be extended to a Ramsey ultrafilter under $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$, and Eisworth [19] proved that any Milliken–Taylor filter of size $< \mathfrak{c}$ can be extended to an Milliken–Taylor ultrafilter with the Galvin–Glazer technique (see, e.g. [26]) under the same condition. We refer the reader to [9] for the definitions of the cardinal characteristics $\mathrm{cov}(\mathcal{M})$, $\mathfrak{u}$, $\mathfrak{d}$, etc. We write $\mathfrak{c}$ for $2^{\aleph_0}$.

Here is a brief review on the number of near-coherence classes of (non-principal) ultrafilters over ω, see Definition 1.1. In 1987 Blass and Shelah [11] constructed a new type of forcing ("Blass–Shelah forcing") and a model with just one near-coherence class. Shortly after Blass and Shelah [12] found: Also a countable support iteration of Miller forcing of length ω_2 forces the existence of just one class. Around the same time Blass [7] showed that a Matet iteration does the same. Up to now, in ($\mathfrak{u} < \mathfrak{d}$)-models that are not variations of these three types, there are infinitely many near-coherence classes or the number of classes is not yet known or possibly not pinned down by the forcing. As already mentioned, in ($\mathfrak{u} \geq \mathfrak{d}$)-models there are infinitely many classes, and Banakh and Blass [1] showed: If the number of near-coherence classes is infinite then it is $2^{\mathfrak{c}}$.

In our construction we work with variants of Matet forcing [30] that come from various constraints on the reservoir of the pure components of the conditions. The full Matet forcing preserves any P-point from the ground model [19, Theorem 4] and destroys any Ramsey ultrafilter, since it adds an unbounded real. The (non-complete) subforcings with pure parts from a Milliken–Taylor ultrafilter have specific preservation properties; they destroy some P-points and preserve others, see [19, Cor. 2.5, Theorem 4]. We show that the reservoir Milliken–Taylor ultrafilter $\mathcal{U}$ can be extended to a new Milliken–Taylor ultrafilter after forcing. A new technical ingredient is the work with names for diagonal constructions.

One of our main results about one iterand is:

Theorem 3.3. *We assume* CH*, let $\mathcal{E}$ be a P-point and $\mathcal{U}$ be a Milliken–Taylor ultrafilter with $\Phi(\mathcal{U}) \not\leq_{\mathrm{RB}} \mathcal{E}$. Then in the forcing extension by $\mathbb{M}(\mathcal{U})$ the Milliken–Taylor ultrafilter $\mathcal{U}$ is destroyed and can be completed to a Milliken–Taylor ultrafilter $\mathcal{U}^{\mathrm{ext}} \supseteq \mathcal{U}$ with $\Phi(\mathcal{U}^{\mathrm{ext}}) \not\leq_{\mathrm{RB}} \mathcal{E}$.*

After this σ-centred forcing of size $\aleph_1$, CH still holds. The notion of a P-point will be explained in this section, the core $\Phi(\mathcal{U})$ is defined in Definition 2.12, the Rudin–Blass order $\leq_{\mathrm{RB}}$ is defined in Definition 2.13. Milliken–Taylor ultrafilters are defined in Definition 2.7(6), the forcing $\mathbb{M}(\mathcal{U})$ is defined in Definition 2.15. By [19, Theorem 2.5], $\mathcal{E}$ generates a P-point in the extension.

We prove a preservation theorem for countable support iterations and show that there is a model of $\aleph_1 = \mathfrak{u} < \mathfrak{d} = \mathfrak{c} = \aleph_2$ with at least three specifically named near-coherence classes of ultrafilters and a Milliken–Taylor ultrafilter of character

$\mathfrak{c}$. An iterable version of Theorem 3.3, namely Theorem 3.10, serves as a successor step in the forcing that is used in the following theorem:

Theorem 4.9. *Assume* CH. *Then there is a countable support iteration of proper iterands* $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{M}(\mathcal{U}_\beta) \,:\, \beta < \omega_2, \alpha \leq \omega_2 \rangle$ *such that in the extension there are at least three near-coherence classes of ultrafilters and there is a Milliken–Taylor ultrafilter of character* $\mathfrak{c}$.

We do not know the exact number of near-coherence classes in these forcing extensions. In these extensions $\mathfrak{b} = \mathfrak{u} = \aleph_1 < \mathfrak{d} = \aleph_2$.

We fix some $k \in \omega \setminus \{0\}$. We introduce Milliken–Taylor ultrafilters that are related to Hindman's theorem for colourings of Fin_k as Milliken–Taylor ultrafilters are related to Hindman's theorem, and investigate Matet forcing with Milliken–Taylor ultrafilters over Fin_k. We show that this generalisation gives nothing new in point of near-coherence classes.

In order to get exactly three near-coherence classes, we introduce a new kind of localisation of Matet forcing. We fix two non-nearly coherent Ramsey ultrafilters $\mathcal{R}_{\min}$, $\mathcal{R}_{\max}$. Under CH, we work with conditions $(s, \bar{a})$ such there is a Milliken–Taylor ultrafilter $\mathcal{U}$ such that $\min(\mathcal{U}) = \mathcal{R}_{\min}$ and $\max(\mathcal{U}) = \mathcal{R}_{\max}$ and $\bar{a} \in \mathcal{U}$. There is a better equivalent definition that does not need CH:

Definition 5.1. Let $\mathcal{R}_{\min}$, $\mathcal{R}_{\max}$ be two non-nearly coherent (nnc) Ramsey ultrafilters. If $\mathcal{R}_{\min}$, $\mathcal{R}_{\max}$ are clear form the context, we just write $\bar{\mathcal{R}}$ for $(\mathcal{R}_{\min}, \mathcal{R}_{\max})$. We write $\bar{a} \in (\mathrm{Fin})^\omega(\bar{\mathcal{R}})$ if $\bar{a} \in (\mathrm{Fin})^\omega$ and

$$(\forall X \in \mathcal{R}_{\min})(\forall Y \in \mathcal{R}_{\max})(\exists^{\min-\mathrm{unb}} s \in \mathrm{FU}(\bar{a}))(\min(s) \in X \wedge \max(s) \in Y).$$

The quantifier $(\exists^{\min-\mathrm{unb}} s)$ is defined in Definition 2.1(8). We let $\mathbb{M}(\bar{\mathcal{R}})$ be the Matet forcing with conditions $(s, \bar{a})$ such that $\bar{a} \in (\mathrm{Fin})^\omega(\bar{\mathcal{R}})$. Our main result about this notion of forcing is:

Theorem 5.22. *Assume* CH *and fix two pairwise nnc Ramsey ultrafilters* $\mathcal{R}_{\min,0}$, $\mathcal{R}_{\max,0}$. *Then there is a countable support iteration of proper iterands*

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{M}(\bar{\mathcal{R}}_\beta) \,:\, \beta < \omega_2, \alpha \leq \omega_2 \rangle$$

such that in the extension there are exactly three near-coherence classes of ultrafilters. Namely, one class is represented by a P*-point of character* ω_1 *and two other classes represented by Ramsey ultrafilters* $\mathcal{R}_{\min,\omega_2}$, $\mathcal{R}_{\max,\omega_2}$ *of character* $\aleph_2$ *that are not* $P_{\aleph_2}$*-points. The sequence of Ramsey ultrafilters* $\langle \mathcal{R}_{\min,\alpha} \,:\, \alpha \leq \omega_2 \rangle$ *is* $\subseteq$*-increasing in* α *and the same holds for the* $\mathcal{R}_{\max,\alpha}$.

In these extensions and also in the ones from Theorem 5.23 we again have $\mathfrak{b} = \mathfrak{u} = \aleph_1 < \mathfrak{d} = \aleph_2$. A slight variation of the model by dispensing with one of the Ramsey parameters gives exactly two near-coherence classes.

Theorem 5.23. *Under* CH *there is a countable support iteration* $\langle \mathbb{P}_\beta, \mathbb{Q}_\alpha \,:\, \beta \leq \omega_2, \alpha < \omega_2 \rangle$ *that forces that there are exactly two near-coherence classes and there is no Milliken–Taylor ultrafilter.*

By work of Mioduszewski our result has applications to analysis, namely the number of composants of $\beta(\mathbb{R}^+) - \mathbb{R}^+$ corresponds by [34, 35] to the number of near-coherence classes of ultrafilters. Blass [4] gives applications to cofinality classes of short non-standard models of arithmetic, and to the decomposition of the ideal of compact linear operators on a Hilbert space into proper subideals. His results on equivalent characterisations of indecomposability can be translated to: For $n \geq 1$, there is a decomposition of the ideal of compact operators into n proper subideals such that the union of any two of them is the whole ideal, if and only if there are exactly n near-coherence classes. This correspondence is defined by Blass and Weiss in [13].

In the remainder of the introduction we recall some definitions from the realm of near-coherence and special ultrafilters over ω.

Let S be a countable set. By a *filter over S* we mean a non-empty subset of $\mathcal{P}(S)$ that is closed under supersets and under finite intersections and that does not contain the empty set. We call a filter over S *non-principal* if it contains all cofinite subsets of S. A $\subseteq$-maximal filter is an ultrafilter.

For $B \subseteq \omega$ and $f \colon \omega \to \omega$, we let $f[B] = \{f(b) : b \in B\}$ and $f^{-1}[B] = \{n : f(n) \in B\}$. The set of all infinite subsets of ω is denoted by $[\omega]^\omega$. For $\mathcal{B} \subseteq \mathcal{P}(\omega)$ we let $f(\mathcal{B}) = \{X \subseteq \omega : f^{-1}[X] \in \mathcal{B}\}$. [1] This double lifting of f is an important function from $\mathcal{P}\mathcal{P}(\omega)$ into itself. In analysis the special case of f being finite-to-one (that means that the preimage of each natural number is finite) is particularly useful, see e.g., [5].

From now on all filters over ω will be non-principal filters over ω, though we write only "filter" over ω. If $f \colon \omega \to \omega$ is finite-to-one, then also $f(\mathcal{F})$ is a non-principal filter. It is the filter generated by $\{f[X] : X \in \mathcal{F}\}$.

Definition 1.1. (1) A non-empty family $\mathcal{G} \subseteq [\omega]^\omega$ is called a *filter subbase (over ω)* if any intersection of finitely many elements of $\mathcal{G}$ is infinite. We write

$$\operatorname{fil}(\mathcal{G}) = \{X \in [\omega]^\omega : (\exists n \in \omega)(\exists G_0 \dots \exists G_n \in \mathcal{G})(X \supseteq G_0 \cap \dots \cap G_n)\}$$

for the filter generated by $\mathcal{G}$. The *character* of a filter $\mathcal{F}$ is the smallest size of a generating subbase.

(2) Two filters $\mathcal{F}, \mathcal{G} \subseteq [\omega]^\omega$ are *nearly coherent*, if there is some finite-to-one $f \colon \omega \to \omega$ such that $f(\mathcal{F}) \cup f(\mathcal{G})$ generates a filter.

(3) On the set of non-principal ultrafilters near-coherence is an equivalence relation (for a proof see [4], e.g.) whose equivalence classes are called *near-coherence classes.*

(4) Two subsets $\mathcal{H}_1$, $\mathcal{H}_2$ of $[\omega]^\omega$ are called *nowhere nearly coherent (nnc)*, if for any $X_i \in \mathcal{H}_i$, $i = 1, 2$ and any finite-to-one h there are $Y_i \subseteq X_i$, $Y_i \in \mathcal{H}_i$, $i = 1, 2$ such that $h[Y_1] \cap h[Y_2] = \emptyset$.

(5) $\mathcal{H} \subseteq [\omega]^\omega$ is called *nowhere almost a filter* if for any $X \in \mathcal{H}$, and any finite-to-one h the set $\{h[Y] : Y \in \mathcal{H} \wedge Y \subseteq X\}$ does not have the finite intersection property.

(6) $\mathcal{H} \subseteq [\omega]^\omega$ is called *nowhere almost an ultrafilter* if for any $X \in \mathcal{H}$, and any finite-to-one h the set $\mathcal{S} = \{h[Y] : Y \in \mathcal{H} \wedge Y \subseteq X\}$ does not have the finite intersection property or there is some infinite Z that is not decided by $\mathcal{S}$, i.e., for any $W \in \mathcal{S}$, $W \not\subseteq^* Z$ and $W \not\subseteq^* (\omega \setminus Z)$.

For filters, nnc is the negation of near-coherence. Near-coherence is witnessed by a weakly increasing surjective finite-to-one function. A function f is weakly

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[1] If f is surjective, then $f(\mathcal{B})$ can be written as $\{f[X] : X \in \mathcal{B}\}$. In any case, $f(\mathcal{B})$ is contained in the set of supersets of members of the latter set.

increasing if $x < y \to f(x) \leq f(y)$. A coideal $\mathcal{H}$ is nowhere almost a filter if and only if it is nowhere almost an ultrafilter. We will use the property for coideals.

We say "A is almost a subset of B" and write $A \subseteq^* B$ if $A \setminus B$ is finite. Similarly, the symbol $=^*$ denotes equality up to finitely many exceptions between elements of $[S]^\omega$ for a set S.

Let κ be a regular uncountable cardinal. An ultrafilter $\mathcal{W}$ is called a *P_κ-point* if for every $\gamma < \kappa$, for every collection $A_i \in \mathcal{U}$, $i < \gamma$, there is some $A \in \mathcal{W}$ such that for any $i < \gamma$, $A \subseteq^* A_i$; such an A is called a *pseudo-intersection* of $\{A_i : i < \gamma\}$. A $P_{\aleph_1}$-point is called a P-point.

Let $\mathbb{P}$ be a notion of forcing. We say that $\mathbb{P}$ *preserves an ultrafilter $\mathcal{W}$ over I* if

$$\Vdash_{\mathbb{P}} \text{“}(\forall X \subseteq I)(\exists Y \in \mathcal{W})(Y \subseteq X \vee Y \subseteq I \setminus X)\text{”}$$

and in the contrary case we say "$\mathbb{P}$ destroys $\mathcal{W}$". In the first case $\{X \in [\omega]^\omega \cap \mathbf{V}[G] : (\exists Y \in \mathcal{W}) X \supseteq Y\}$ is an ultrafilter in $\mathbf{V}[G]$ and $\mathcal{W}$ generates an ultrafilter in $\mathbf{V}[G]$. We just say: $\mathcal{W}$ is an ultrafilter in $\mathbf{V}[G]$. If $\mathbb{P}$ is proper and preserves $\mathcal{W}$ and $\mathcal{W}$ is a P-point in the ground model, then $\mathcal{W}$ stays a P-point in the forcing extension by [11, Lemma 3.2].

An ultrafilter $\mathcal{W}$ over ω is called a *Q-point* if for every strictly increasing sequence $\langle n_i : i \in \omega \rangle$ of natural numbers there is $X \in \mathcal{W}$ such that for every i, $|X \cap [n_i, n_{i+1})| \leq 1$. Any Q-point from the ground model ceases to be a Q-point after adding an unbounded real.

An ultrafilter $\mathcal{R}$ is called *selective* (or *a Ramsey ultrafilter*) if it is a P-point and a Q-point. We use the von Neumann natural numbers $n = \{0, \dots, n-1\}$. We often use the following, equivalent (see [14, Theorem 4.9]) characterisation of selectivity:

(∗) For any $\subseteq$-descending sequence $\langle A_n : n \in \omega \rangle$ of sets $A_n \in \mathcal{R}$ there is $A \in \mathcal{R}$ such that $A \subseteq A_0$ and $(\forall n \in A)(A \setminus (n+1) \subseteq A_n)$.

Kunen [27] contains more information on Ramsey ultrafilters. Two Ramsey ultrafilters are nnc iff they are not isomorphic [3].

In forcing, we follow the Kunen style that the stronger condition is the *smaller* one. This corresponds to the close relationship between the $\leq$-relation in Matet forcing and the condensation relation $\sqsubseteq$ on the second components, the so-called pure components, of a condition in Matet forcing.

2. Matet forcing with Milliken–Taylor ultrafilters

In this section we review results of Blass, Eisworth, and Hindman and carry them a bit further. Our nomenclature follows Blass [6], Eisworth [19] and/or Todorčević [42]. Of course, we cannot reconcile all the traditions.

Definition 2.1.

(1) We let Fin denote the set of finite non-empty subsets of ω.

(2) An element $a \in \mathrm{Fin}$ is called a block.

(3) For $a, b \in \mathrm{Fin}$ we write $a < b$ if $(\forall m \in a)(\forall n \in b)(m < n)$.

(4) We define a well-order (of type ω) $\leq_{\mathrm{lex,Fin}}$ on the set Fin via $a <_{\mathrm{lex,Fin}} b$ if $\max(a) < \max(b)$ or $(\max(a) = \max(b)$ and $\min(a \triangle b) \in a)$.[2]

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[2] Note that this is not the usual lexicographic well-order; e.g., $\{0, 1\} <_{\mathrm{lex,Fin}} \{1\}$. The purpose of this well-order is to define the $\sqsubseteq$-largest common lower bound of two $\sqsubseteq$-compatible elements of $(\mathrm{Fin})^\omega$ by induction on the blocks. The relation $\sqsubseteq$ is defined in Definition 2.2.

(5) A sequence $\bar{a} = \langle a_n : n \in \omega \rangle$ of members of Fin is called *unmeshed* if for all n, $a_n < a_{n+1}$.

(6) By $(\mathrm{Fin})^\omega$ we denote the set of unmeshed infinite sequences of members in Fin. This is $\mathrm{FIN}^{[\infty]}$ in [42].

(7) Let a, b be blocks. We let $a \cup b$ be undefined unless $a < b$. Otherwise, $a \cup b$ is defined as the union.

(8) A set $X \subseteq \mathrm{Fin}$ is called *min-unbounded* if for any $n \in \omega$ there is some $x \in X$ with $\min(x) \geq n$. We write $(\exists^{\min-\mathrm{unb}} s)(\varphi(s))$ if the set of $s \in \mathrm{Fin}$ such that $\varphi(s)$ holds is a min-unbounded set.

(9) The structure $(\mathrm{Fin}, \cup)$ is a partial semigroup. The associative partial binary operation $\cup$ lifts to $\gamma(\mathrm{Fin})$, the space of min-unbounded ultrafilters over Fin, as follows (and we write $\cup$ for the lifted operation):

$$\mathcal{U}_1 \cup \mathcal{U}_2 = \{X \subseteq \mathrm{Fin} : \{s : \{t : s \cup t \in X\} \in \mathcal{U}_2\} \in \mathcal{U}_1\}.$$

For details and history see [26, Section 4.1].

(10) If X is a subset of Fin, we write $\mathrm{FU}(X)$ for the set of all unions of finitely many members of X. We write $\mathrm{FU}(\bar{a})$ instead of $\mathrm{FU}(\{a_n : n \in \omega\})$. The set $\mathrm{FU}(\bar{a})$ is denoted by $[\bar{a}]$ in [42]. We call X an FU-set if there is some unmeshed $\bar{a}$ such that $X = \mathrm{FU}(\bar{a})$.

(11) For $\emptyset \neq X \subseteq \mathrm{Fin}$ we let $\min_{<_{\mathrm{lex,Fin}}}(X)$ be the $\leq_{\mathrm{lex,Fin}}$-least element of X. For $\bar{a} \in (\mathrm{Fin})^\omega$ we let $\min_{\mathrm{Fin}}(\bar{a}) = \min_{<_{\mathrm{lex,Fin}}}\{a_n : n < \omega\}$, which is a_0.

(12) A *filter over* Fin is a non-empty subset of $\mathcal{P}(\mathrm{Fin})$ that is closed under finite intersections and supersets and does not contain the empty set.

(13) For $X \subseteq \mathrm{Fin}$, the set $(\mathrm{FU}(X))^\omega$ denotes the collection of all infinite unmeshed sequences in $\mathrm{FU}(X)$. For $\bar{a} \in (\mathrm{Fin})^\omega$, the set $(\mathrm{FU}(\bar{a}))^\omega$ denotes the collection of all infinite unmeshed sequences in $\mathrm{FU}(\bar{a})$ (recall item (10)).

(14) For $Y \subseteq \mathrm{Fin}$ and $s \in \mathrm{Fin}$ we write $(Y\,; \text{past } s)$ for $\{u \in Y : \max(s) < \min(u)\}$. In [42] the latter is denoted by Y/s. For $\bar{a} \in (\mathrm{Fin})^\omega$ and $s \in \mathrm{Fin}$ we write $(\bar{a}\,; \text{past } s)$ for $\langle a_n : n \geq n_0 \rangle$, where $n_0 = \min\{n : \max(s) < \min(a_n)\}$.

Now the set $(\mathrm{Fin})^\omega$ is equipped with a partial order $\sqsubseteq$ that makes it into a topological Ramsey space in the sense of [42]. We will work with the (closed) subspace of sets of the form $\mathrm{FU}(\bar{a})$.

Definition 2.2. Given X and $Y \subseteq \mathrm{Fin}$, we say that Y *is a condensation of* X and we write $Y \sqsubseteq X$ if $Y \subseteq \mathrm{FU}(X)$. [3] We say Y is *almost a condensation of* X and we write $Y \sqsubseteq^* X$ if there is an $n \in \omega$ such that $(Y\,; \text{past } \{n\})$ is a condensation of X. In accordance with the convention $\mathrm{FU}(\bar{a}) = \mathrm{FU}(\{a_n : n < \omega\})$, we write $\bar{b} \sqsubseteq \bar{a}$ for $\mathrm{FU}(\bar{b}) \sqsubseteq \mathrm{FU}(\bar{a})$, i.e., for $\{b_n : n < \omega\} \subseteq \mathrm{FU}(\bar{a})$.

The "blurred" relation $\sqsubseteq^*$ is an $(< \omega_1)$-complete preorder.

Definition 2.3. Let $\bar{a}, \bar{b} \in (\mathrm{Fin})^\omega$. We say $\bar{a}$ and $\bar{b}$ are *compatible*, if there is a $\bar{c} \sqsubseteq \bar{a}, \bar{b}$. In the contrary case we write $\bar{a} \perp \bar{b}$.

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[3] In [42], the condensation order is denoted by $\leq$. We use $\leq$ for linear orders, in the sense of Definition 2.1(3) and for forcing orders.

Lemma 2.4. *If $\bar{a}$ and $\bar{b}$ are compatible, there is a weakest (i.e., $\sqsubseteq$-largest) $\bar{c} \in (\mathrm{Fin})^\omega$ such that $\bar{c} = \langle c_n : n < \omega\rangle \sqsubseteq \bar{b}, \bar{a}$. Moreover, $\bar{c}$ is given by the following procedure. By induction on n we define c_n as follows: $c_0 = \min_{<_{\mathrm{lex,Fin}}}(\mathrm{FU}(\bar{a}) \cap \mathrm{FU}(\bar{b}))$, $c_{n+1} = \min_{<_{\mathrm{lex,Fin}}}\{s \in \mathrm{FU}(\bar{a}) \cap \mathrm{FU}(\bar{b}) : c_n < s\}$.*

Proof. By definition $\bar{c} \sqsubseteq \bar{a}, \bar{b}$. To see that $\bar{c}$ is the largest witness, we suppose for a contradiction that there is $\bar{d} \sqsubseteq \bar{a}, \bar{b}$ such that $\bar{d} \not\sqsubseteq \bar{c}$. We take the first n such that $d_n \neq c_n$.

Since $\bar{a}$ and $\bar{b}$ are unmeshed sequences, there are $i_1 < \cdots < i_k$, $j_1 < \cdots < j_\ell$, such that

$$c_n = a_{i_1} \cup \cdots \cup a_{i_k} = b_{j_1} \cup \cdots \cup b_{j_\ell}, \tag{2.1}$$

and c_n is defined as in the lemma. There are $\varepsilon_1 < \cdots < \varepsilon_\mu$, $\zeta_1 < \cdots < \zeta_\nu$ such that $d_n = a_{\varepsilon_1} \cup \cdots \cup a_{\varepsilon_\mu} = b_{\zeta_1} \cup \cdots \cup b_{\zeta_\nu}$.

First case: $\max(d_n) = \max(c_n)$. First subcase $d_n <_{\mathrm{lex,Fin}} c_n$. This contradicts the choice of c_n.

Second subcase: $d_n >_{\mathrm{lex,Fin}} c_n$. Then d_n uses in its sum not all the summands $a_{i_1}, \ldots, a_{i_k}$ that combine to c_n and hence $d_n = \emptyset$, because leaving off summands on one side or on both sides of (2.1) is not possible by the choice of c_n. However, d_n is a block. Thus this case ends in a contradiction.

Second case: $\max(d_n) < \max(c_n)$. Since $\bar{a}$ and $\bar{b}$ are unmeshed sequences, the block c_n could be broken up in the following way. There is a block c'_n with $\max(c'_n) \leq \max(d_n)$ and there is $c''_n \in \mathrm{FU}(\bar{a}\ ; \text{past } c'_n) \cap \mathrm{FU}(\bar{b}\ ; \text{past } c'_n)$ such that $c_n = c'_n \cup c''_n$, and $\bar{c}$ could be replaced by $\bar{c}'$ with

$$\bar{a}, \bar{b} \sqsupseteq \bar{c}' = \langle c_0, \ldots, c_{n-1}, c'_n, c''_n, c_{n+1}, \ldots\rangle \sqsupset \bar{c}.$$

Now c'_n contradicts the $<_{\mathrm{lex,Fin}}$-minimality of c_n.

Third case: $\max(d_n) > \max(c_n)$. If $\min(d_n) \leq \max(c_n)$, there is a block d'_n with $\max(d'_n) \leq \max(c_n)$ and there is $d''_n \in \mathrm{FU}(\bar{a}\ ; \text{past } c_{n-1}) \cap \mathrm{FU}(\bar{b}\ ; \text{past } c_{n-1})$ such that $d_n = d'_n \cup d''_n$ and $\bar{d}$ could be replaced by $\bar{d}'$ with

$$\bar{a}, \bar{b} \sqsupseteq \bar{d}' = \langle d_0, \ldots, d_{n-1}, d'_n, d''_n, d_{n+1}, \ldots\rangle \sqsupset \bar{d}.$$

Now $\bar{d}' \not\sqsubseteq \bar{c}$ and n, c_n, d'_n are as in the first case or in the second case. If $\min(d_n) > \max(c_n)$, then we compare d_n with the first c_r, $r > n$, such that $d_n \cap c_r \neq \emptyset$ and continue similar reasoning. Again we get a contradiction.

□

Definition 2.5. If $\bar{a}$ and $\bar{b}$ are $\sqsubseteq$-compatible, we write $\bar{c} = \bar{a} \wedge \bar{b}$ for the largest common lower bound.

Definition 2.6. A non-empty subset $\mathcal{C} \subseteq (\mathrm{Fin})^\omega$ is called *centred*, if for any finite $C \subseteq \mathcal{C}$ there is $\bar{a} \in \mathcal{C}$ that is a condensation of any $\bar{c} \in C$ and if $\mathcal{C}$ is closed under finite alterations i.e., if $\bar{d} \in \mathcal{C}$ and $\bar{e} \in (\mathrm{Fin})^\omega$ then there are $n_0, m_0 \in \omega$ such that $\langle d_m : m \geq m_0\rangle = \langle e_n : n \geq n_0\rangle$ then $\bar{e} \in \mathcal{C}$.

We specialise $\mathcal{C}$ further.

Definition 2.7.

(1) Let $\mathcal{F}$ be a filter. A *basis* $\mathcal{B}$ of $\mathcal{F}$ is a subset of $\mathcal{F}$ such that $(\forall F \in \mathcal{F})(\exists B \in \mathcal{B})(B \subseteq F)$.

(2) A non-principal filter $\mathcal{F}$ over Fin is said to be a *union* filter if it has a basis of sets of the form $\mathrm{FU}(X)$ for $X \subseteq \mathrm{Fin}$ such that the elements of X are pairwise disjoint. Note $\mathrm{FU}(X)$ need not be an FU-set.

(3) A non-principal filter $\mathcal{F}$ over Fin is said to be a *min-unbounded* filter if $(\forall n \in \omega)(\exists X \in \mathcal{F})(\forall s \in X)(\min(s) > n)$.

(4) A non-principal filter $\mathcal{F}$ over Fin is said to be an *ordered-union* filter if it has a basis of sets of the form $\mathrm{FU}(\bar{d})$ for $\bar{d} \in (\mathrm{Fin})^\omega$.

(5) Let μ be an uncountable cardinal. A union filter is said to be $(< \mu)$*-stable* if, whenever it contains $\mathrm{FU}(X_\alpha)$ for $X_\alpha \subseteq \mathrm{Fin}$, $\alpha < \kappa$, for some $\kappa < \mu$, then it also contains some $\mathrm{FU}(Y)$ for some Y such that for each $\alpha < \kappa$ there is $n_\alpha \in \omega$ with $(Y \text{ ; past } \{n_\alpha\}) \subseteq \mathrm{FU}(X_\alpha)$. Such an Y is called a *lower bound* of $\{X_\alpha : \alpha < \kappa\}$. For "$< \omega_1$-stable" we say "stable".

(6) A stable ordered-union ultrafilter is also called a *Milliken–Taylor ultrafilter.*

(7) An ultrafilter is called *idempotent* if $\mathcal{U} \cup \mathcal{U} = \mathcal{U}$.

Ordered-union ultrafilters need not exist, as their existence implies the existence of Q-points [6] and there are models without Q-points [33]. Even union ultrafilters need not exist: Blass [8, Theorem 38] showed that the existence of a union ultrafilter implies the existence of at least two near-coherence classes of ultrafilters. In [11] Blass and Shelah show that it is consistent relative to ZFC to have exactly one near-coherence class of non-principal ultrafilters. Union ultrafilters are idempotent. Idempotent ultrafilters exist by the Ellis–Namakura Lemma [20, 36]. With the help of Hindman's theorem one shows that CH or Martin's Axiom for σ-centred posets and $< 2^\omega$ dense sets implies that (even $< 2^\omega$-) stable Milliken–Taylor ultrafilters exist [6]. We recall Hindman's theorem:

Theorem 2.8 (Hindman, [24, Corollary 3.3])**.** *If the set* Fin *is partitioned into finitely many pieces then there is a set* $\bar{d} \in (\mathrm{Fin})^\omega$ *such that* $\mathrm{FU}(\bar{d})$ *is included in one piece.*

The theorem also holds if instead of Fin we partition some $\mathrm{FU}(\bar{c})$ for a $\bar{c} \in (\mathrm{Fin})^\omega$ and search for a homogeneous sequence $\bar{d} \sqsubseteq \bar{c}$, see [6, p. 92].

Corollary 2.9 (See [6, p. 93])**.** *Under* CH *or* $\mathsf{MA}_{<2^\omega}(\sigma\text{-centred})$, *for every* $\bar{a} \in (\mathrm{Fin})^\omega$ *there is a Milliken–Taylor ultrafilter* $\mathcal{U}$ *such that* $\mathrm{FU}(\bar{a}) \in \mathcal{U}$.

For $X \subseteq \mathrm{Fin}$, we let $[X]^n_<$ be the set of increasing unmeshed n-sequences of members of X. For the evaluation of our forcings, Taylor's theorem [41] is utilised:

Theorem 2.10 (Taylor [41])**.** *Let* $\mathcal{U}$ *be a Milliken–Taylor ultrafilter, and let* $n \in \omega$. *Let* $[\mathrm{Fin}]^n_<$ *be partitioned into finitely many sets. Then there is* $A \in \mathcal{U}$ *such that* $[\mathrm{FU}(A)]^n_<$ *is monochromatic.*

Corollary 2.11 ([19, Cor. 1.3])**.** *Existence of diagonal lower bounds in* $((\mathrm{Fin})^\omega, \sqsubseteq)$. *Let* $\mathcal{U}$ *be a Milliken–Taylor ultrafilter, and let* $\langle A_n : n \in \omega\rangle$ *be a* $\sqsubseteq$*-descending sequence of members of* $\mathcal{U}$. *Then there is a* $B \in \mathcal{U}$ *such* $B \sqsubseteq A_0$ *and*

$$(\forall s \in B)((B \text{ ; past } s) \text{ is a condensation of } A_{\max(s)+1}). \tag{2.2}$$

Such a B *is called a* diagonal lower bound of $\langle A_n : n \in \omega\rangle$.

In Equation (2.2) we can equivalently let s range over $\mathrm{FU}(B)$. If B is a diagonal lower bound and $B' \sqsubseteq B$ then B' is a diagonal lower bound as well.

Definition 2.12. Let $\mathcal{H}$ be a subset of $\mathcal{P}(\mathrm{Fin})$.

(1) The *core of* $\mathcal{H}$ is the set $\Phi(\mathcal{H}) \subseteq [\omega]^\omega$ such that

$$X \in \Phi(\mathcal{H}) \text{ iff } (\exists Y \in \mathcal{H})(\bigcup Y \subseteq X).$$

(2) The *minimum projection of* $\mathcal{H}$ is the set

$$\min(\mathcal{H}) = \{\min[Y] \,:\, Y \in \mathcal{H}\},$$

where

$$\min[Y] = \{\min(y) \,:\, y \in Y\},$$

and analogously we define the *maximum projection* $\max(\mathcal{H})$.

(3) For $\bar{a} \in (\mathrm{Fin})^\omega$ we write $\min[\bar{a}] = \{\min(a_i) \,:\, i \in \omega\}$.

(4) For $\bar{a} \in (\mathrm{Fin})^\omega$ we write $\mathrm{set}(\bar{a}) = \bigcup\{a_i \,:\, i \in \omega\}$.

(5) For $\mathcal{H} \subseteq (\mathrm{Fin})^\omega$ and for $\bar{a} \in \mathcal{H}$, we let

$$\begin{array}{rcl} \mathrm{range}(\bar{a}) & = & \{a_n \,:\, n < \omega\} \\ \Phi(\mathcal{H}) & = & \Phi(\{\mathrm{range}(\bar{a}) \,:\, \bar{a} \in \mathcal{H}\}) \\ \min(\mathcal{H}) & = & \{\min[\bar{a}] \,:\, \bar{a} \in \mathcal{H}\}. \end{array}$$

Note that $\min[\bar{a}] = \min[\mathrm{FU}(\bar{a})]$ and $\bigcup\{a_n \,:\, n \in \omega\} = \bigcup \mathrm{FU}(\bar{a})$.

If $\mathcal{C} \subseteq (\mathrm{Fin})^\omega$ is centred, then the core of $\mathcal{C}$ is a filter over ω. For centred $\mathcal{C} \subseteq (\mathrm{Fin})^\omega$ and for filters $\mathcal{C}$ over Fin the each of the projections $\min(\mathcal{C})$, $\max(\mathcal{C})$ is a filter and $\min(\mathcal{C}), \max(\mathcal{C}) \supseteq \Phi(\mathcal{C})$. Blass ([6, 3.6–3.9] together with [8, Theorem 38]) showed that for a Milliken–Taylor ultrafilter $\mathcal{U}$, $\min(\mathcal{U})$ and $\max(\mathcal{U})$ are nnc Ramsey ultrafilters.

The cores of centred systems are just filters over ω. Even if $\mathcal{U}$ is an ultrafilter over Fin, it is possible that for any finite-to-one f, the filter $f(\Phi(\mathcal{U}))$ is not an ultrafilter over ω. Blass [8, Theorem 38]) showed that for union-ultrafilters $\mathcal{U}$, for any finite-to-one f, $f(\Phi(\mathcal{U}))$ is not an ultrafilter because among its supersets there are two nnc ultrafilters, namely the ultrafilters generated by the f-images of the minimum projection and the maximum projection.

If $\mathcal{U}$ is a Milliken–Taylor ultrafilter, then $\Phi(\mathcal{U})$ does not have a pseudointersection (see [19, Prop. 2.3]) and also any finite-to-one image of $\Phi(\mathcal{U})$ does not have a pseudointersection by the same proof. Hence, by Talagrand [40], the filter $\Phi(\mathcal{U})$ is not meagre. Thus the filter dichotomy principle, which says that every non-meagre filter is almost ultra (for more information see, e.g., [9]), precludes the existence of a Milliken–Taylor ultrafilter. The filter dichotomy principle is consistent relative to ZFC by [10].

Definition 2.13. The *Rudin–Blass ordering* for filters over ω is defined as follows: Let $\mathcal{F} \leq_{RB} \mathcal{G}$ if there is a finite-to-one f such that $f(\mathcal{F}) \subseteq f(\mathcal{G})$.[4]

For filters $\mathcal{F}$, $\mathcal{G}$, the relation $\mathcal{F} \leq_{\mathrm{RB}} \mathcal{G}$ implies that $\mathcal{F}$ is nearly coherent to $\mathcal{G}$. If $\mathcal{G}$ is an ultrafilter, also the converse holds.

Now we turn to forcing:

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[4] Also the definition $f(\mathcal{F}) \subseteq \mathcal{G}$ is used in the literature. If $\mathcal{G}$ is a P-point both definitions are closely related.

Definition 2.14. Conditions in *Matet forcing*, $\mathbb{M}$, are pairs $(s, \bar{c})$ such that $s \in \mathrm{Fin}$ and $\bar{c} \in (\mathrm{Fin})^\omega$ and $s < c_0$. The forcing order is $(t, \bar{d}) \leq (s, \bar{c})$ (recall the stronger condition is the smaller one) if $s \subseteq t$ and t is the union of s and finitely many of the c_n, and $\bar{d}$ is a condensation of $\bar{c}$.

Definition 2.15. For a family $\mathcal{H} \subseteq (\mathrm{Fin})^\omega$, the notion of forcing $\mathbb{M}(\mathcal{H})$ consists of all pairs $(s, \bar{a})$ such that $\bar{a} \in \mathcal{H}$. The forcing order is the same as in the Matet forcing.

For a centred system $\mathcal{C} \subseteq (\mathrm{Fin})^\omega$, the set $\Phi(\mathcal{C})$ is the filter $\mathrm{fil}(\{\mathrm{set}(\bar{a}) : \bar{a} \in \mathcal{C}\})$. The forcing $\mathbb{M}(\mathcal{C})$ diagonalises $\Phi(\mathcal{C})$. Let G be an $\mathbb{M}(\mathcal{C})$-generic filter over $\mathbf{V}$. Then the generic real

$$\mu := \bigcup \{s : \exists \bar{c} : (s, \bar{c}) \in G\}$$

is a pseudointersection of $\Phi(\mathcal{C})$.

The following property of Milliken–Taylor ultrafilters $\mathcal{U}$ will be important for our proof:

Theorem 2.16 (Eisworth [19, "→" Theorem 4, "←" Corollary 2.5, this direction works also with non-P ultrafilters])**.** *Let $\mathcal{U}$ be a Milliken–Taylor ultrafilter over* Fin *and let $\mathcal{W}$ be a P-point. Then $\mathcal{W} \not\geq_{RB} \Phi(\mathcal{U})$ if and only if $\mathcal{W}$ continues to generate an ultrafilter after we force with $\mathbb{M}(\mathcal{U})$.*

We remark:

Proposition 2.17. *Suppose that $\mathcal{U}$ is a Milliken–Taylor ultrafilter and $\mathcal{F}$ is a filter over ω and $\Phi(\mathcal{U}) \leq_{RB} \mathcal{F}$. Then $\mathbb{M}(\mathcal{U})$ forces that for any finite-to-one function f from the ground model, $f(\mathcal{F})$ does not generate an ultrafilter.*

Proof. Let $f \in \mathbf{V}$ be finite-to-one such that $f(\Phi(\mathcal{U})) \subseteq f(\mathcal{F})$. Let G be $\mathbb{M}(\mathcal{U})$-generic over $\mathbf{V}$, and let μ be a name for the generic real μ. We show:

$$(\forall (s, \bar{a}) \in \mathbb{M}(\mathcal{U}))(\forall Y \in \mathcal{F})(\exists (t, \bar{b}) \leq_{\mathbb{M}(\mathcal{U})} (s, \bar{a}))(t, \bar{b}) \Vdash_{\mathbb{M}(\mathcal{U})} f[Y] \cap f[\mu] \neq \emptyset$$

$$(\forall (s, \bar{a}) \in \mathbb{M}(\mathcal{U}))(\forall Y \in \mathcal{F})(\exists (t, \bar{b}) \leq_{\mathbb{M}(\mathcal{U})} (s, \bar{a}))(t, \bar{b}) \Vdash_{\mathbb{M}(\mathcal{U})} f[Y] \cap (\omega \smallsetminus f[\mu]) \neq \emptyset.$$

Let $(s, \bar{a})$ and Y be given. Since $f[\mathrm{set}(\bar{a})] \in f(\Phi(\mathcal{U})) \subseteq f(\mathcal{F})$, we have $f[\mathrm{set}(\bar{a})] \cap f[Y]$ is infinite. So there is $t \in \mathrm{FU}(\bar{a})$ such that $f[t] \cap f[Y] \neq \emptyset$. It follows that $(s \cup t, \bar{a} \text{ ; past } t) \Vdash_{\mathbb{M}(\mathcal{U})} f[Y] \cap f[\mu] \neq \emptyset$.

Now for the second property:

Next we define a colouring h of $[\mathrm{FU}(\bar{a} \text{ ; past } s)]^2_<$ by

$$h(u < v) = \begin{cases} 1 & \text{if } f[u] < f[v] \wedge \\ & \big(\max(f[u]), \min(f[v])\big) \cap f[Y] \neq \emptyset, \\ 0 & \text{else.} \end{cases}$$

Since $\mathcal{U}$ is a Milliken–Taylor ultrafilter by Theorem [41] there is a monochromatic $\bar{b} \sqsubseteq (\bar{a} \text{ ; past } s)$, $\bar{b} \in \mathcal{U}$. Since f is finite-to-one, the colour is 1. So we have $(s, \bar{b}) \Vdash_{\mathbb{M}(\mathcal{U})} f[Y] \cap (\omega \smallsetminus f[\mu]) \neq \emptyset$. □

Now let $\underset{\sim}{f}$ be a name for the function $n \mapsto |\mu \cap n|$. Then

$$\mathbb{M}(\mathcal{U}) \Vdash \underset{\sim}{f} \text{ is finite-to-one and } \underset{\sim}{f}[\mu] =^* \omega$$

and the above proof breaks down. Information on $\underset{\sim}{g}(\mathcal{F})$ for particular filters $\mathcal{F}$ with $\Phi(\mathcal{U}) \leq_{\mathrm{RB}} \mathcal{F}$ and any name $\underset{\sim}{g}$ for a finite-to-one function is contained in Theorem 3.26.

Definition 2.18. Let $\mathcal{F}$ be a filter. We let

$$\mathcal{F}^+ = \{X \in [\omega]^\omega \,:\, (\forall Y \in \mathcal{F})(Y \cap X \neq \emptyset)\}.$$

The elements of $\mathcal{F}^+$ are called positive w.r.t. $\mathcal{F}$ or $\mathcal{F}$-positive.

For a non-principal $\mathcal{F}$, $\mathcal{F}^+$ coincides with $\{X \in [\omega]^\omega \,:\, (\forall Y \in \mathcal{F})(Y \cap X \in [\omega]^\omega)\}$.

3. Ramsey-theoretic computations in $\mathbb{M}(\mathcal{U})$-extensions

In this section, we begin the mentioned work on Ramsey spaces of names. We consider the Ramsey space $((\mathrm{Fin})^\omega, \sqsubseteq)$ in an $\mathbb{M}(\mathcal{U})$-extension. We use the notation $\mathbf{V}^{\mathbb{P}}$ for any $\mathbf{V}[G]$, with a $\mathbb{P}$-generic filter G over $\mathbf{V}$. The first aim is to examine $\mathcal{U}$ in the forcing extension $\mathbf{V}^{\mathbb{M}(\mathcal{U})}$. Is there an extension of the destroyed Milliken–Taylor ultrafilter $\mathcal{U}$ to a new Milliken–Taylor ultrafilter? We show that under CH there are even 2^{ω_1} possibilities with pairwise nnc cores. We show that in $\mathbf{V}^{\mathbb{P}}$, the set of $(\mathcal{U} \restriction \mu)$-positive block sequences is Matet-adequate and nowhere almost a filter.

This will be the successor step of an iteration of iterands of type $\mathbb{M}(\mathcal{U}_\alpha)$. We introduce Ramsey-theoretic computations with names for elements of $(\mathrm{Fin})^\omega$ in order to establish the existence of names for Milliken–Taylor ultrafilters. Although in the end, like in Corollary 2.11, min-unbounded FU-subsets of Fin are the elements of Milliken–Taylor ultrafilters, intermediate work is better carried out with sequences $\bar{a} \in (\mathrm{Fin})^\omega$. We use letters $\bar{a}, \bar{b}, \dots, A, B, \dots$ for elements of $(\mathrm{Fin})^\omega$, where capital letters are in particular used in work with sequences of sequences. Capital letters are also used for subsets of Fin.

Definition 3.1. (1) Let $\bar{a} \in (\mathrm{Fin})^\omega$ and $X \in [\omega]^\omega$. We let $\bar{a} \restriction X = \langle a_n \,:\, n \in \omega,\ a_n \subseteq X\rangle$. Note, we do not take those a_n with $a_n \cap X \neq \emptyset$ that are not subsets of X.

(2) Let $\mathcal{U} \subseteq (\mathrm{Fin})^\omega$ and $X \in [\omega]^\omega$. We use the restriction symbol also for subsets of $(\mathrm{Fin})^\omega$ and let $\mathcal{U} \restriction X = \{\bar{a} \restriction X \,:\, \bar{a} \in \mathcal{U}\}$.

(3) Let $\mathcal{U}$ be a Milliken–Taylor ultrafilter and G be $\mathbb{M}(\mathcal{U})$-generic over V. Throughout the paper we let μ be a name for the Matet generic real

$$\bigcup\{s \,:\, \exists \bar{a}(s, \bar{a}) \in G\}.$$

Lemma 3.2. *Let $\mathcal{U}$ be a Milliken–Taylor ultrafilter.*

(1) $\mathbb{M}(\mathcal{U}) \Vdash (\forall \bar{a} \in \mathcal{U})(\bar{a} \restriction \mu \in (\mathrm{Fin})^\omega)$.

(2) $\mathbb{M}(\mathcal{U}) \Vdash \mathcal{U} \restriction \mu$ is an ordered-union filter.

Proof. (1) is an easy density argument. For (2), recall the meet operation $\wedge$ between compatible infinite block sequences in Definition 2.5. We use

$$\mathbb{M}(\mathcal{U}) \Vdash (\bar{a} \wedge \bar{b}) \restriction \mu = (\bar{a} \restriction \mu) \wedge (\bar{b} \restriction \mu) \text{ and } \mathrm{FU}(\bar{a} \restriction \mu) = \mathrm{FU}(\bar{a}) \restriction \mu.$$

□

Now we restate and prove

Theorem 3.3. *We assume* CH*, let* $\mathcal{E}$ *be a P-point and let* $\mathcal{U}$ *be a Milliken–Taylor ultrafilter such that* $\Phi(\mathcal{U}) \not\leq_{\mathrm{RK}} \mathcal{E}$*. In* $\mathbf{V}^{\mathbb{M}(\mathcal{U})}$ *there is a Milliken–Taylor ultrafilter* $\mathcal{U}^{\mathrm{ext}} \supseteq \mathcal{U} \restriction \mu$ *such that* $\Phi(\mathcal{U}^{\mathrm{ext}}) \not\leq_{RK} \mathcal{E}$*.*

In the remainder of this section we prove this theorem in an iterable form, Theorem 3.10.

Definition 3.4 (See [19, Definition 3.1])**.** A set $\mathcal{H} \subseteq (\mathrm{Fin})^\omega$ is called a *Matet-adequate family* if the following hold:

(i) $\mathcal{H}$ is closed $\sqsubseteq^*$-upwards.

(ii) $\mathcal{H}$ is countably closed, i.e., any $\sqsubseteq$-descending ω-sequence of members of $\mathcal{H}$ has a $\sqsubseteq^*$-lower bound in $\mathcal{H}$.

(iii) $\mathcal{H}$ has the *Hindman property*: If $\bar{a} \in \mathcal{H}$ and $\mathrm{FU}(\bar{a})$ is partitioned into two pieces then there is some $\bar{b} \sqsubseteq \bar{a}$, $\bar{b} \in \mathcal{H}$ such that $\mathrm{FU}(\bar{b})$ is a subset of a single piece of the partition.

After Lemma 3.6 we shall discuss the connection between a Matet-adequate $(\mathcal{H}, \sqsubseteq)$ and various Ramsey spaces from [42].

Definition 3.5. Let $\langle \bar{a}_n : n < \omega \rangle$ be $\sqsubseteq$-descending chain of elements $\bar{a}_n \in (\mathrm{Fin})^\omega$. A sequence $\bar{b} \in (\mathrm{Fin})^\omega$ is a *diagonal lower bound of* $\langle \bar{a}_n : n < \omega \rangle$ if

$$(\forall s \in \mathrm{FU}(\bar{b}))((\bar{b} \restriction \mathrm{past}\ s) \sqsubseteq \bar{a}_{\max(s)+1}). \tag{3.1}$$

It is equivalent to say $(\forall s \in \mathrm{range}(\bar{b})))((\bar{b} \restriction \mathrm{past}\ s) \sqsubseteq \bar{a}_{\max(s)+1})$ in (3.1).

The Hindman property of $\mathcal{H}$ together with the countable closure of $\mathcal{H}$ implies the existence of diagonal lower bounds of sequences in $\mathcal{H}$, see [19, Cor.1.3], which is based on the deep result [6, Theorem 4.2]. We give an alternative proof in Lemma 3.6 that shows that Matet-adequate families have equivalent properties that look prima facie stronger than the ones of Definition 3.4.

Lemma 3.6. *Let* $\mathcal{H}$ *be a Matet-adequate family.*

(a) Let $n \geq 1$*. If* $\bar{a} \in \mathcal{H}$ *and* $[\mathrm{FU}(\bar{a})]^n_<$ *is partitioned into two pieces then there is some* $\bar{b} \sqsubseteq \bar{a}$*,* $\bar{b} \in \mathcal{H}$ *such that* $[\mathrm{FU}(\bar{b})]^n_<$ *is a subset of a single piece of the partition.*

(b) $\mathcal{H}$ *contains for each descending sequence a diagonal lower bound.*

Proof. (a) is proved in [6, Theorem 4.2]. We write an alternative proof that also serves to show that ultraness is not used. The assertion is proved by induction on the exponent n. For simplicity we write the step from $n = 1$ to $n = 2$. We let $c \colon [\mathrm{FU}(\bar{a})]^2_< \to m$ for some finite $m \geq 1$. We enumerate $\mathrm{FU}(\bar{a})$ as $\langle s_\ell : \ell \in \omega \rangle$ such that for any ℓ, all the s_i with $\max(s_i) < \max(s_\ell)$ have $i < \ell$. Let $\bar{c}_{-1} = \bar{a}$. For each $s_\ell \in \mathrm{FU}(\bar{a})$ by induction hypothesis we may take a $\bar{c}_\ell \sqsubseteq (\bar{c}_{\ell-1} \restriction \mathrm{past}\ s_\ell)$ such that

$$\begin{aligned} f_\ell \colon\ & \mathrm{FU}(\bar{c}_{\ell-1} \restriction \mathrm{past}\ s_\ell) \to m \\ & t \mapsto c(s_\ell, t) \end{aligned}$$

is monochromatic on $\mathrm{FU}(\bar{c}_\ell)$. Let $\bar{b} \sqsubseteq^* \bar{c}_\ell$ for any $\ell < \omega$. Now we take $g \colon \omega \to \omega$ with $g(0) = 0$ and $g(\ell + 1) > g(\ell)$ so large that

$$(\forall r \leq g(\ell))\big((\bar{b} \restriction \mathrm{past}\ \{g(\ell+1)\}) \sqsubseteq \bar{c}_r\big). \tag{3.2}$$

Next we let for $s \in \mathrm{FU}(\bar{b})$, say $s = s_\ell$, $f(s) = c(s,t)$ for any $t \in \mathrm{FU}(\bar{c}_\ell) \cap \mathrm{FU}(\bar{b})$. By the induction hypothesis there is $\bar{c} \sqsubseteq \bar{b}$, $\bar{c} \in \mathcal{H}$, such that $\mathrm{FU}(\bar{c})$ is monochromatic under the colouring f. Then we colour $s \in \mathrm{FU}(\bar{c})$ with colour $j \in 2$ if $\max(s)$ is in $\bigcup\{[g(2r+j), g(2r+j+1)) \,:\, r \in \omega\}$, and take $\bar{d} \sqsubseteq \bar{c}$ such that $\mathrm{FU}(\bar{d})$ is monochromatic for the latter colouring. Finally we take $\bar{e} \sqsubseteq \bar{d}$ such that for each r there is at most one ℓ such that $e_\ell \cap [g(2r+j), g(2r+j+1)) \neq \emptyset$. The existence of such an $\bar{e} \in \mathcal{H}$ follows by [6, Theorem 3.9]. Then $[\mathrm{FU}(\bar{e})]^2_<$ is c-monochromatic.

(b) Let $\langle \bar{a}_n \,:\, n \in \omega \rangle$ be a $\sqsubseteq$-descending sequence of element of $\mathcal{H}$ and let $\bar{c} \in \mathcal{H}$ be a $\sqsubseteq^*$-lower bound such that $\bar{c} \sqsubseteq \bar{a}_0$. We colour $\mathrm{FU}(\bar{c})]^2_<$ via

$$c(u,v) = \begin{cases} 1 & \text{if } v \in \mathrm{FU}(\bar{a}_{\max(u)+1}); \\ 0 & \text{else.} \end{cases}$$

Since $\bar{c}$ is a $\sqsubseteq^*$-lower bound of $\langle \bar{a}_n \,:\, n < \omega \rangle$, any $\bar{d} \sqsubseteq \bar{c}$ such that $[\mathrm{FU}(\bar{d})]^2_<$ is c-monochromatic with colour 1 is a diagonal lower bound. By part (a), there is such a $\bar{d}$. The monochromatic colour can only be the colour 1. □

A Matet-adequate family $\mathcal{H}$ with the condensation order $\sqsubseteq$ is a selective coideal with the $\mathcal{H}$-Ramsey property (which is in our space the Hindman property) in the sense of [15]. The clause about the existence of diagonal lower bounds can be weakened to the condition of semiselectivity as follows:

Definition 3.7. A set $\mathcal{H} \subseteq (\mathrm{Fin})^\omega$ is called a *a semiselective Matet-adequate family* if the following hold:

(i) $\mathcal{H}$ is closed $\sqsubseteq^*$-upwards.

(ii) For any descending sequence $\langle \mathcal{D}_n \,:\, n < \omega \rangle$ of open (i.e., closed under stronger conditions) dense subsets $\mathcal{D}_n$ of $(\mathcal{H}, \sqsubseteq)$ there is a diagonal lower bound, i.e., some $\bar{a} = \langle a_i \,:\, i < \omega \rangle \in \mathcal{D}_0$ such that for any i, $(\bar{a}\ ;\mathrm{past}\ a_i) \in \mathcal{D}_{i+1}$. Recall, the notion $(\bar{a}\ ;\mathrm{past}\ s)$ was defined in Definition 2.1(14).

(iii) $\mathcal{H}$ has the *Hindman property*: If $\bar{a} \in \mathcal{H}$ and $\mathrm{FU}(\bar{a})$ is partitioned into two pieces then there is some $\bar{b} \sqsubseteq \bar{a}$, $\bar{b} \in \mathcal{H}$ such that $\mathrm{FU}(\bar{b})$ is a subset of a single piece of the partition.

In this context, Matet-adequate families could be called selective Matet-adequate families. Stepping up from the Hindman property to the Milliken–Taylor property as in (b) of Lemma 3.6 is possible also for semiselective Matet-adequate families. For centred families (see Definition 2.6) the notions of selectivity and semiselectivity coincide. The technical work with the stronger notion is somewhat simpler, since block sequences are simpler sets than open dense subsets of $\mathcal{H}$, and later we will see calculations with names.

Definition 3.8. Let $\mathcal{C} \subseteq (\mathrm{Fin})^\omega$ be centred. $\mathcal{C}^+ = \{\bar{a} \in (\mathrm{Fin})^\omega \,:\, \forall \bar{c} \in \mathcal{C}, \bar{c} \not\perp \bar{a}\}$. The elements of $\mathcal{C}^+$ are called the $\mathcal{C}$-positive sequences.

We introduce an abbreviation:

Definition 3.9. Let $\mathcal{H} \subseteq (\mathrm{Fin})^\omega$ and let $\mathcal{E}$ be a P-point. We say $\mathcal{H}$ *avoids* $\mathcal{E}$ if $\{\mathrm{set}(\bar{a}) \,:\, \bar{a} \in \mathcal{H}\}$ is nnc to $\mathcal{E}$. For nnc see Definition 1.1(4).

The technical core of the proof of Theorem 3.3 is:

Theorem 3.10. *Let $\mathcal{E}$ be a P-point and let $\mathcal{U}$ be a Milliken–Taylor ultrafilter such that $\Phi(\mathcal{U}) \not\leq_{\mathrm{RB}} \mathcal{E}$. After forcing with $\mathbb{M}(\mathcal{U})$, $(\mathcal{U} \restriction \mu)^+$ is a Matet-adequate family such that for any finite-to-one h, for any $\bar{a} \in (\mathcal{U} \restriction \mu)^+$,*

$$(\exists \bar{b}, \bar{c} \sqsubseteq \bar{a})(\bar{b} \in (\mathcal{U} \restriction \mu)^+ \wedge \bar{c} \in (\mathcal{U} \restriction \mu)^+ \wedge h[\mathrm{set}(\bar{b})] \cap h[(\mathrm{set}(\bar{c})] = \emptyset).$$

By the latter property, $(\mathcal{U} \upharpoonright \mu)^+$ avoids any ultrafilter from the ground model, in particular $\mathcal{E}$. Once Theorem 3.10 is proved, a routine downwards construction along ω_1 (see e.g., [6, Theorem 2.4]) completes the proof of Theorem 3.3: Under CH in $\mathbf{V}^{\mathbb{M}(\mathcal{U})}$ there is a Milliken–Taylor ultrafilter $\mathcal{U}^{\rm ext} \supseteq \mathcal{U}$ that avoids $\mathcal{E}$.

Now we start to prove Theorem 3.10. Showing adequacy requires some technical work to evaluate the forcing.

Definition 3.11. A condition $(s, \bar{b})$ with $(s, \bar{b}) \leq (s, \bar{a})$ is called *a pure extension of* $(s, \bar{a})$.

Lemma 3.12 ([19, Lemma 2.6])**.** *$\mathbb{M}(\mathcal{U})$ has the pure decision property, that is, for any φ in the forcing language for any $(s, \bar{a}) \in \mathbb{M}(\mathcal{U})$, there is $\bar{b} \in \mathcal{U}$, $\bar{b} \sqsubseteq \bar{a}$ such that $(s, \bar{b})$ decides φ.* ☐

Eisworth introduced the notion of a neat condition for a name of a subset of ω. We extend the notion of neatness for our purposes. [5]

Definition 3.13. (1) Let $\underset{\sim}{A}$ be a name for an infinite subset of Fin (that means, the weakest condition forces this). We say $(s, \bar{b})$ is *neat for* $\underset{\sim}{A}$ if

$$\begin{aligned}&(\forall t \in \mathrm{FU}(\bar{b}))(\forall r \in \mathrm{FU}(\bar{b}\ ;\mathrm{past}\ t))(\forall u \subseteq \max(r))\\ &(\forall r' \in \mathrm{FU}(\bar{b}\ ;\mathrm{past}\ r))\big((s \cup t \cup r', (\bar{b}\ ;\mathrm{past}\ r'))\\ &\text{ decides } u \in \underset{\sim}{A} \text{ and the decision does not depend on } r'\big).\end{aligned}$$

(2) Let $\underset{\sim}{h}$ be a name for a finite-to-one function such that $h(i) \leq i$. We say $(s, \bar{b})$ is *neat for* $\underset{\sim}{h}$ if

$$\begin{aligned}&(\forall t \in \mathrm{FU}(\bar{b}))(\forall r \in \mathrm{FU}(\bar{b}\ ;\mathrm{past}\ t))(\forall i \leq \max(r))\\ &(\forall r' \in \mathrm{FU}(\bar{b}\ ;\mathrm{past}\ r))\big((s \cup t \cup r', (\bar{b}\ ;\mathrm{past}\ r'))\\ &\text{ decides } \underset{\sim}{h}(i) \text{ and the decision does not depend on } r'\big).\end{aligned}$$

(3) Let $\underset{\sim}{c}$ be a name for a function $\underset{\sim}{c}\colon \mathrm{Fin} \to \{0, 1\}$. We say $(s, \bar{b})$ is *neat for* $\underset{\sim}{c}$ if

$$\begin{aligned}&(\forall t \in \mathrm{FU}(\bar{b}))(\forall r \in \mathrm{FU}(\bar{b}\ ;\mathrm{past}\ t))(\forall u \subseteq \max(r))\\ &(\forall r' \in \mathrm{FU}(\bar{b}\ ;\mathrm{past}\ r))\big((s \cup t \cup r', (\bar{b}\ ;\mathrm{past}\ r'))\\ &\text{ decides } \underset{\sim}{c}(u) \text{ and the decision does not depend on } r'\big).\end{aligned}$$

(4) Let $\underset{\sim}{\bar{A}} = \langle \underset{\sim}{A}_j \,:\, j < \omega \rangle$ be a name of a sequence of elements of $(\mathrm{Fin})^\omega$. We say $(s, \bar{b})$ is *neat for* $\langle \underset{\sim}{A}_j \,:\, j < \omega \rangle$ if

$$\begin{aligned}&(\forall t \in \mathrm{FU}(\bar{b}))(\forall r \in \mathrm{FU}(\bar{b}\ ;\mathrm{past}\ t))(\forall j \leq \max(r))(\forall u \subseteq \max(r))\\ &(\forall r' \in \mathrm{FU}(\bar{b}\ ;\mathrm{past}\ r))\big((s \cup t \cup r', (\bar{b}\ ;\mathrm{past}\ r'))\\ &\text{ decides } u \in \underset{\sim}{A}_j \text{ and the decision does not depend on } r'\big).\end{aligned}$$

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[5] It can be used for names of subsets of $H(\omega)$ and names for any ω-hierarchy of hereditary finite sets whose union is a subset $H(\omega)$. We define neatness by tailoring initial segments towards computations with diagonal lower bounds.

Lemma 3.14 ([19, Lemma 2.7, Lemma 2.8])**.** *Let $(s,\bar{a}) \in \mathbb{M}(\mathcal{U})$. Let $(\underset{\sim}{X}, \underset{\sim}{h}, \underset{\sim}{c}, \langle \underset{\sim}{A}_j : j < \omega\rangle)$ be given such that the condition $(s,\bar{a})$ forces: $\underset{\sim}{X}$ is a min-unbounded subset of* Fin*, $\underset{\sim}{h}$ is a surjective weakly increasing finite-to-one function, $\underset{\sim}{c}$ is a name for a colouring, $\langle \underset{\sim}{A}_j : j < \omega\rangle$ is a sequence of members of* $(\mathrm{Fin})^\omega$*. Then there is $\bar{b} \sqsubseteq \bar{a}$ such that $(s,\bar{b}) \in M(\mathcal{U})$ is neat for $\underset{\sim}{X}$, $\underset{\sim}{h}$, $\underset{\sim}{c}$, $\langle \underset{\sim}{A}_j : j < \omega\rangle$.* □

Remark 3.15. Since the proof does not use the fact that $\mathcal{U}$ is a filter, Lemma 3.14 also holds for $\mathbb{M}(\mathcal{H})$ for any Matet-adequate family $\mathcal{H}$.

Now we prove Theorem 3.10. It is obvious that the set of positive sets $(\mathcal{U} \restriction \mu)^+$ is upwards closed in the $\sqsubseteq^*$-order. Now we show that any $\sqsubseteq$-descending ω-sequence has a $\sqsubseteq^*$-lower bound. It is not harder to show directly that there is a diagonal lower bound.

We recall that $\min_{<,\mathrm{lex},\mathrm{Fin}}$ was defined in Definition 2.1(11). A $\mathbb{Q}$-name is a set of the form $\tau = \{\langle \sigma, q\rangle : \langle \sigma, q\rangle \in \tau\}$ with names σ of lower rank. For $x \in \mathbf{V}$ we have the $\mathbb{Q}$-name $\check{x} = \{\langle \check{y}, q\rangle : y \in x, q \in \mathbb{Q}\}$. We drop the $\check{x}$-sign.

The following technique is one of the cornerstones of our forcing constructions and interesting for itself:

Lemma 3.16 (Existence of diagonal lower bounds)**.** *Let $\mathcal{U}$ be a Milliken–Taylor ultrafilter, $\mathcal{E}$ be a P-point, $\Phi(\mathcal{U}) \not\leq_{\mathrm{RB}} \mathcal{E}$. Let $\mathbb{Q} = \mathbb{M}(\mathcal{U})$ and let μ be the name for the generic real. Let $\bar{\underset{\sim}{X}} = \langle \underset{\sim}{X}_n : n \in \omega\rangle$ be a $\mathbb{Q}$-name of a sequence of elements of* $(\mathrm{Fin})^\omega$ *such that*

$$\mathbb{Q} \Vdash (\forall n \in \omega)(\underset{\sim}{X}_n \in (\mathcal{U} \restriction \mu)^+ \wedge \underset{\sim}{X}_{n+1} \sqsubseteq \underset{\sim}{X}_n).$$

Then

(3.3)

$$\begin{aligned}\underset{\sim}{D} = \bigl\{\langle t,(s,\bar{a})\rangle \,:\, &(s,\bar{a}) \in \mathbb{Q} \wedge (\exists k \in \omega)(\exists t_0 < t_1 < \dots < t_{k-1} \in [\mathrm{Fin}]^k_<)\\ &\bigl(t_{k-1} < t_k = t \wedge (s,\bar{a}) \Vdash \text{“}t_0 = \min_{<_{\mathrm{lex},\mathrm{Fin}}}((\underset{\sim}{X}_0 \restriction \mu) \cap \mathrm{FU}(\bar{a})) \wedge \\ &\qquad \bigwedge_{i<k} (t_{i+1} = \min_{<_{\mathrm{lex},\mathrm{Fin}}}(((\underset{\sim}{X}_{\max(t_i)+1} \restriction \mu) \;;\mathrm{past}\ t_i)) \cap \mathrm{FU}(\bar{a}))\text{”}\bigr)\bigr\}\end{aligned}$$

fulfils

$$\mathbb{Q} \Vdash \underset{\sim}{D} \in (\mathcal{U} \restriction \mu)^+ \wedge \underset{\sim}{D} \sqsubseteq \underset{\sim}{X}_0 \wedge (\forall t \in \underset{\sim}{D})((\underset{\sim}{D} \;;\mathrm{past}\ t) \sqsubseteq \underset{\sim}{X}_{\max(t)+1}). \tag{3.4}$$

Proof. Let $(s,\bar{a})$ be given such that $(s,\bar{a})$ is neat for $\underset{\sim}{D}$ and for $\langle \underset{\sim}{X}_i \restriction \mu : i < \omega\rangle$. We show that there is $(s,\bar{b}) \leq_{\mathbb{Q}} (s,\bar{a})$ that forces that $\underset{\sim}{D} \restriction \mu \in (\mathcal{U} \restriction \mu)^+$.

Let $k \in \omega$ and $u \in \mathrm{FU}(\bar{a})$. We say that u *is good for* $(s,\bar{a})$, $\bar{t} = (t_0,\dots,t_{k-1}) \in [\mathrm{Fin}]^k_<$, $\bar{\underset{\sim}{X}}$, if

$$\begin{aligned}(s \cup u, \bar{a} \;;\mathrm{past}\ u) \Vdash t_0 = &\min_{<_{\mathrm{lex},\mathrm{Fin}}}((\underset{\sim}{X}_0 \restriction \mu) \cap \mathrm{FU}(\bar{a})) \wedge \\ &\bigwedge_{i<k-1} (t_{i+1} = \min_{<_{\mathrm{lex},\mathrm{Fin}}}(((\underset{\sim}{X}_{\max(t_i)+1} \restriction \mu) \;;\mathrm{past}\ t_i) \cap \mathrm{FU}(\bar{a})).\end{aligned}$$

Note that goodness requires $t_i \in \mathrm{Fin}$ and not just names for elements of Fin. We define a colouring of $[\mathrm{FU}(\bar{a})]^2_<$ as follows

$$F(u<v)=\begin{cases}1 & \text{if for any } \bar t \text{ such that}\\ & u \text{ is good for } (s,\bar a), \bar t, \underset{\sim}{\bar X},\\ & \text{there is a proper end extension } \bar{t'} \text{ of } \bar t\\ & \ \text{such that } u\cup v \text{ is good for } (s,\bar a), \bar{t'}, \underset{\sim}{\bar X};\\ 0 & \ \text{else.}\end{cases} \tag{3.5}$$

By Taylor's Theorem 2.10 there is $(s,\bar b)\leq_{\mathbb{Q}}(s,\bar a)$ such that F is monochromatic on $[\mathrm{FU}(\bar b)]^2_<$. We argue that the monochromatic colour can only be 1: It suffices to find $u<v\in \mathrm{FU}(\bar b)$ such that $F(u,v)=1$. Suppose that $u\in\mathrm{FU}(\bar b)$ is good for $(s,\bar a)$, $\bar t=(t_0,\dots,t_{k-1})$, $\underset{\sim}{\bar X}$. If $k=0$, we let $X_{\max(t_{k-1})+1}=X_0$. Then $(s\cup u,\bar b)\Vdash ((\underset{\sim}{X}_{\max(t_{k-1})+1}\restriction\mu)\ ;\text{past } t_{k-1})\cap\mathrm{FU}(\bar b))\in(\mathcal{U}\restriction\mu)^+$. So $(s\cup u,\bar b)\Vdash \exists t\in((\underset{\sim}{X}_{\max(t_{k-1})+1}\restriction\mu)\ ;\text{past } t_{k-1})\cap\mathrm{FU}(\bar b)$.

Now we show

$$(s,\bar b)\Vdash \underset{\sim}{D}\in(\mathcal{U}\restriction\mu)^+\wedge \underset{\sim}{D}\sqsubseteq\underset{\sim}{X}_0\wedge(\forall t\in\mathrm{FU}(\underset{\sim}{D}))((\underset{\sim}{D}\ ;\text{past } t)\sqsubseteq\underset{\sim}{X}_{\max(t)+1}). \tag{3.6}$$

The proof comes in four parts. First we show that $\underset{\sim}{D}$ is a min-unbounded subset of Fin. Given any $k\in\omega$, the procedure above for finding v, t' is iterated k times, starting at stage $k=0$ with $u=s$. Thus we find v_i and t_i such that

$$\begin{aligned}(s\cup v_0,\bar b\ ;\text{past } v_0)&\Vdash t_0=\min_{<_{\mathrm{lex},\mathrm{Fin}}}((\underset{\sim}{X}_0\restriction\mu)\cap\mathrm{FU}(\bar b)),\\ (s\cup v_0\cup v_1,\bar b\ ;\text{past } v_1)&\Vdash t_1=\min_{<_{\mathrm{lex},\mathrm{Fin}}}(((\underset{\sim}{X}_{\max(t_0)+1}\restriction\mu)\ ;\text{past } t_0)\cap\mathrm{FU}(\bar b)),\\ &\ \,\vdots\\ (s\cup v_0\cup\dots\cup v_k,\bar b\ ;\text{past } v_k)&\Vdash t_k=\min_{<_{\mathrm{lex},\mathrm{Fin}}}(((\underset{\sim}{X}_{\max(t_{k-1})+1}\restriction\mu)\ ;\text{past } t_{k-1})\cap\mathrm{FU}(\bar b)),\\ (s\cup v_1\cup\dots\cup v_k,\bar b\ ;\text{past } v_k)&\Vdash t_0<\dots<t_k\in\underset{\sim}{D}.\end{aligned}$$

By neatness, the later part $(\bar b\ ;\text{past } v_i)$ does not have an influence on the truth of $t_i=\min_{<_{\mathrm{lex},\mathrm{Fin}}}(((\underset{\sim}{X}_{\max(t_{i-1})+1}\restriction\mu)\ ;\text{past } t_{i-1})\cap\mathrm{FU}(\bar b))$ and the forcing statements say that t_i is the only block in $\mathrm{FU}(s,v_0,\dots,v_i)$ that is minimal in $(\underset{\sim}{X}_{\max(t_{k-i})+1}\restriction\mu)\ ;\text{past } t_{i-1})$. Next we show $(s,\bar b)\Vdash\underset{\sim}{D}\in(\mathcal{U}\restriction\mu)^+$. Suppose for a contradiction, that $\bar c\in\mathcal{U}$ and $(s',\bar b')\leq(s,\bar b)$ and $(s',\bar b')\Vdash\mathrm{FU}(\bar c\restriction\mu)\cap\mathrm{FU}(\underset{\sim}{D})=\emptyset$. Then we take $\bar d\sqsubseteq\bar b',(\bar c\ ;\text{past } s')$, such that $\bar d\in\mathcal{U}$ and see that $(s',\bar d)\Vdash(\underset{\sim}{D}\ ;\text{past } s')\subseteq\mathrm{FU}(\bar c\restriction\mu)$. Contradiction.

Since $(s,\bar a)\Vdash\underset{\sim}{X}_{n+1}\sqsubseteq\underset{\sim}{X}_n$, by definition of $\underset{\sim}{D}$, $(s,\bar b)\Vdash\underset{\sim}{D}\subseteq\underset{\sim}{X}_0$.

For the last conjunctive clause in Equation (3.6), we work with the characterisation of diagonal lower bound that is given immediately after Definition 3.5. We suppose that $(s\cup v,\bar b\ ;\text{past } v)\Vdash t<t'\in\underset{\sim}{D}$. Then by the definition of $\underset{\sim}{D}$ $(s\cup v,\bar b\ ;\text{past } v)\Vdash t'\in\mathrm{FU}(\underset{\sim}{X}_{\max(t)+1}\restriction\mu\ ;\text{past } t)$. $\square$

Recall $\mathrm{set}(\langle a_n : n<\omega\rangle)=\bigcup\{a_n : n<\omega\}$.

Lemma 3.17. *Let* $(s,\bar a) \Vdash_{\mathbb{M}(\mathcal{U})} \underset{\sim}{\bar c} \in (\mathcal{U}\restriction\mu)^+ \wedge \underset{\sim}{h}$ *is finite-to-one, onto and monotone. Then there are* $E \in \mathcal{E}$ *and* $(s,\bar b) \le (s,\bar a)$ *and* $\underset{\sim}{\bar d^j}$, $j = 0,1$, *such that*

$$\begin{aligned}(s,\bar b) \Vdash &\bigwedge_{j=0,1} (\underset{\sim}{\bar d^j} \in (\mathcal{U}\restriction\mu)^+ \cap \mathrm{FU}(\bar b) \wedge \underset{\sim}{\bar d^j} \sqsubseteq \underset{\sim}{\bar c}) \wedge \\ &\underset{\sim}{h}[\mathrm{set}(\underset{\sim}{\bar d^0})] \cap \underset{\sim}{h}[\mathrm{set}(\underset{\sim}{\bar d^1})] = \emptyset \wedge \bigvee_{j=0,1} \underset{\sim}{h}[E] \cap \underset{\sim}{h}[\mathrm{set}(\underset{\sim}{\bar d^j})] = \emptyset.\end{aligned} \tag{3.7}$$

Proof. We assume w.l.o.g. that $(s,\bar a)$ is neat for $\bar c$ and h. We define names $\underset{\sim}{\bar d^j}$, $j = 0,1$, such that densely many q below p force $\underset{\sim}{h}[\mathrm{set}(\underset{\sim}{\bar d^0})] \cap \underset{\sim}{h}[\mathrm{set}(\underset{\sim}{\bar d^1})] = \emptyset \wedge \bigwedge_{j=0,1}(\underset{\sim}{\bar d^j} \sqsubseteq \bar c)$. Again this is proved with a colouring.

Let $k < \omega$ and $u \in \mathrm{FU}(\bar a)$. We say that u *is good for* $(s,\bar a)$, $\bar d = (d_0,\dots,d_{k-1})$, $\underset{\sim}{\bar c}$, *and* $\underset{\sim}{h}$ if

$$\begin{aligned}(s \cup u, \bar a \text{ ; past } u) \Vdash &\underset{\sim}{d_0} = \min_{<_{\mathrm{lex,Fin}}} ((\underset{\sim}{\bar c}\restriction\mu) \cap \mathrm{FU}(\bar a)) \wedge \\ &\bigwedge_{i<k-1} (\underset{\sim}{d_{i+1}} = \min_{<_{\mathrm{lex,Fin}}} \{d \in \mathrm{FU}(\underset{\sim}{\bar c}\restriction\mu) : \underset{\sim}{h}[\underset{\sim}{d_i}] \cap \underset{\sim}{h}[d] = \emptyset\}).\end{aligned}$$

Again the d_i are in the ground model, not names. This, though, is not important, since we do not use them as indices. We define a colouring of $[\mathrm{FU}(\bar a)]^2_<$ as follows

$$F(u<v) = \begin{cases} 1 & \text{if for any } \bar d \text{ such that} \\ & u \text{ is good for } (s,\bar a), \bar d, \underset{\sim}{\bar c}, \underset{\sim}{h} \\ & \text{there is a proper end extension } \bar d' \text{ of } \bar d \\ & \text{such that } u \cup v \text{ is good for } (s,\bar a), \bar d', \underset{\sim}{\bar c}, \underset{\sim}{h}; \\ 0 & \text{else.}\end{cases} \tag{3.8}$$

By Taylor's Theorem 2.10 there is $(s,\bar b) \le_{\mathbb{M}(\mathcal{U})} (s,\bar a)$ such that F is monochromatic on $[\mathrm{FU}(\bar b)]^2_<$. Since for any k, $((\underset{\sim}{\bar c}\restriction\mu) \text{ ; past } k)$ is forced to be in $(\mathcal{U}\restriction\mu)^+$, the monochromatic colour can only be 1 and there is again the diagonal property. By the maximal principle (see [28, Ch. VII, Theorem 8.2] or [39, Ch. I, Lemma 3.1] "existential completeness") there are names $\underset{\sim}{d_i}$ such that

$$\begin{aligned}(s,\bar b) \Vdash &\langle \underset{\sim}{d_{2k+1}} : k \in \omega\rangle \in (\mathcal{U}\restriction\mu)^+ \wedge \\ &\langle \underset{\sim}{d_{2k}} : k \in \omega\rangle \in (\mathcal{U}\restriction\mu)^+ \wedge \\ &\langle \underset{\sim}{d_k} : k \in \omega\rangle \sqsubseteq \underset{\sim}{\bar c} \wedge \\ &\underset{\sim}{h}[\bigcup\{\underset{\sim}{d_{2k}} : k \in \omega\}] \cap \underset{\sim}{h}[\bigcup\{\underset{\sim}{d_{2k+1}} : k \in \omega\}] = \emptyset.\end{aligned} \tag{3.9}$$

The first two conjunctive clauses are shown as in the proof of Equation (3.6). The last conjunctive clause follows from the new definition of goodness. We let $(s,\bar b) \Vdash \underset{\sim}{\bar d^j} = \langle \underset{\sim}{d_{2k+j}} : k < \omega\rangle$ for $j = 0,1$. Since $\mathbb{M}(\mathcal{U}) \Vdash \underset{\sim}{h}(\mathcal{E})$ is an ultrafilter, $(s,\bar b)$ forces there are a $j = 0,1$ and an $E \in \mathcal{E}$ such that $\underset{\sim}{h}[E] \cap \underset{\sim}{h}[\mathrm{set}(\underset{\sim}{\bar d^j})] = \emptyset$. □

The next lemma is the most important step in the proof of Theorem 3.10. Indeed, it includes again a proof that positive diagonal lower bounds exist.

Lemma 3.18. *In* $\mathbf{V}^{\mathbb{M}(\mathcal{U})}$, $(\mathcal{U}\restriction\mu)^+$ *has the Hindman property.*

For the proof of this lemma, we adapt a theorem of Eisworth.

Theorem 3.19 ([19, Theorem 5])**.** *Let $\mathcal{F}$ be an ordered-union filter generated by $< \mathrm{cov}(\mathcal{B})$ sets and let c be a partition of* Fin *into finitely many sets. Then there is an $\bar{a} \in \mathcal{F}^+$ such that* $\mathrm{FU}(\bar{a})$ *is included in one piece of the partition.*

At a crucial point in Eisworth's proof a Cohen real over an elementary submodel provides a name in a Galvin–Glazer framework. We show that also a Matet-real and even an $\mathbb{M}(\mathcal{U})$-generic real can be used. We recall the Galvin–Glazer [25] technique.

Definition 3.20. We denote by $\gamma(\mathrm{Fin})$ the set of min-unbounded (see Definition 2.7(3)) ultrafilters over Fin. This set is endowed with the topology generated by

$$\{\{\mathcal{U} \in \gamma(\mathrm{Fin}) \,:\, A \in \mathcal{U}\} \,:\, A \subseteq \mathrm{Fin}\}.$$

The space $\gamma(\mathrm{Fin})$ is a compact zero-dimensional Hausdorff space. With the named topology and the semigroup operation $\cup$ from Definition 2.1(9), the semigroup $(\gamma(\mathrm{Fin}), \cup)$ is a right-topological semigroup. Details can be found in [26].

Lemma 3.21 (Ellis [20])**.** *Each compact subsemigroup of* $(\gamma(\mathrm{Fin}), \cup)$ *contains an idempotent ultrafilter.*

Now we apply Ellis' lemma to $\{\mathcal{U} \in \gamma(\mathrm{Fin}) \,:\, \mathcal{U} \supseteq \mathcal{F}\}$ for a min-unbounded filter $\mathcal{F}$.

Lemma 3.22 ([19, Prop. 4.2])**.** *Let $\mathcal{F}$ be a min-unbounded filter on* Fin*. There is an idempotent ultrafilter $\mathcal{U} \in \gamma(\mathrm{Fin})$ that extends $\mathcal{F}$.*

Now we prove Lemma 3.18. Let p force that $\underset{\sim}{c}$ is a partition of $\mathrm{FU}(\bar{\underset{\sim}{b}}_0) \in (\mathcal{U} \restriction \mu)^+$ into finitely many pieces and $\langle \bar{\underset{\sim}{b}}_n \,:\, n < \omega\rangle$ is a $\sqsubseteq$-descending sequence of elements $\bar{\underset{\sim}{b}}_n \in (\mathcal{U} \restriction \mu)^+$. By Lemma 3.22 there is an $\mathbb{M}(\mathcal{U})$-name $\underset{\sim}{\mathcal{U}}^i$ — the superscript i refers to idempotency — such that

$$\mathbb{M}(\mathcal{U}) \Vdash \underset{\sim}{\mathcal{U}}^i \supseteq \mathrm{fil}((\mathcal{U} \restriction \mu) \cup \{\{\underset{\sim}{b}_{n,m} \,:\, m \in \omega\} \,:\, n \in \omega\}) \wedge \underset{\sim}{\mathcal{U}}^i \cup \underset{\sim}{\mathcal{U}}^i = \underset{\sim}{\mathcal{U}}^i.$$

For $X \subseteq \mathrm{Fin}$ and $t \in \mathrm{Fin}$ we set

$$X \ominus t = \{s \,:\, s \cup t \in X\}.$$

Since $\underset{\sim}{\mathcal{U}}^i$ is forced to be idempotent,

$$\mathbb{M}(\mathcal{U}) \Vdash (\forall X \in \underset{\sim}{\mathcal{U}}^i)(\{t \,:\, X \ominus t \in \underset{\sim}{\mathcal{U}}^i\} \in \underset{\sim}{\mathcal{U}}^i).$$

Now we use again the Milliken–Taylor trick. We assume that $(s, \bar{a})$ is neat for $\underset{\sim}{c}$, $\langle \underset{\sim}{b}_n \,:\, n < \omega\rangle$.

Let $n \geq 1$. We call $u \in \mathrm{FU}(\bar{a})$ *good for* $(s, \bar{a})$, $(\underset{\sim}{X}_m, \underset{\sim}{d}_m \,:\, m < n)$ if $(s \cup u, \bar{a}\, ; \text{past } u)$ forces the following statements:

(1) $\underset{\sim}{X}_0$ is the piece of the partition $\underset{\sim}{c}$ of $\mathrm{FU}(\bar{b})$ that is in $\underset{\sim}{\mathcal{U}}^i$.

(2) We let $d_{-1} = \{-1\}$. For any $0 \leq m < n$ $\underset{\sim}{d}_m$ is the $\leq_{\mathrm{lex},\mathrm{Fin}}$-least element of

$$\begin{aligned}\{d \in \underset{\sim}{X}_m \cap \mathrm{FU}(\{a_k \,:\, k \in \omega\} \restriction \mu) \cap \mathrm{FU}(\bar{b}_{\max(\underset{\sim}{d}_{m-1})+1}) \,:\,\\ \underset{\sim}{X}_m \ominus d \in \underset{\sim}{\mathcal{U}}^i \text{ and } \min(d) > \max(\underset{\sim}{d}_{m-1})\}\end{aligned} \tag{3.10}$$

(3) For any $0 \leq m < n-1$, $\underset{\sim}{X}_{m+1} = \underset{\sim}{X}_m \cap (\underset{\sim}{X}_m \ominus \underset{\sim}{d}_m)$.

Here we allow names. Only the natural numbers are meant to be pinned down. We colour $[\mathrm{FU}(\bar{a}\ ;\mathrm{past}\ s)]^2_<$ as follows:

$$F(u<v)=\begin{cases}1 & \text{if for any } (\underset{\sim}{X}_m, \underset{\sim}{d}_m\ :\ m<n) \text{ such that}\\ & u \text{ is good for } (s,\bar{a}), (\underset{\sim}{X}_m, \underset{\sim}{d}_m\ :\ m<n),\\ & \text{there is a proper end extension } (\underset{\sim}{X}_m, \underset{\sim}{d}_m\ :\ m<n')\\ & \text{such that } u\cup v \text{ is good for } (s,\bar{a}), (\underset{\sim}{X}_m, \underset{\sim}{d}_m\ :\ m<n');\\ 0 & \text{else.}\end{cases} \tag{3.11}$$

Then we find a monochromatic $\bar{b}\in\mathcal{U}$ with $(s,\bar{b})\le(s,\bar{a})$. Since $\mathcal{U}^i$ is forced to be idempotent and $(s,\bar{b})\Vdash \underset{\sim}{\bar{b}}_n\in\mathcal{U}\subseteq\mathcal{U}^i$, the set in (3.10) is in forced to be in $\mathcal{U}^i$. Hence the monochromatic colour can only be 1 and again there is the diagonal self-strengthening with $\mathrm{FU}(\bar{b})$ instead of $\mathrm{FU}(\bar{a})$. We let $\underset{\sim}{\bar{e}}$ be a name such that $(s,\bar{b})\Vdash(\forall n)\langle \underset{\sim}{e}_0,\dots,\underset{\sim}{e}_{n-1}\rangle=\langle \underset{\sim}{d}_0,\dots,\underset{\sim}{d}_{n-1}\rangle$.

The monochromaticity statement

$$\mathbb{M}(\mathcal{U})\Vdash\mathrm{FU}(\underset{\sim}{\bar{e}})\subseteq\underset{\sim}{X}_0$$

is proved literally as in Eisworth [19, page 460].

By item (2) in the current definition of "good", the sequence $\underset{\sim}{\bar{e}}$ is a diagonal lower bound of $\langle\underset{\sim}{\bar{b}_n}\ :\ n<\omega\rangle$. Now we show that $\underset{\sim}{\bar{e}}$ is positive. For this we use the conditions on $\mathrm{FU}(\bar{a})$ in the goodness clause (2) and the fact that the monochromatic good $\bar{b}$ contains only pairs $u<v$ that are good for itself $\bar{b}$. Suppose for a contradiction that $\underset{\sim}{\bar{e}}$ is not forced to be $(\mathcal{U}\restriction\mu)$-positive. Hence there is $q\in\mathbb{M}(\mathcal{U})$, $\bar{c}\in\mathcal{U}$ such that q is neat for $\underset{\sim}{\bar{e}}$ and μ, $q\le(s,\bar{b})$, and

$$q\Vdash_{\mathbb{M}(\mathcal{U})}\mathrm{FU}(\underset{\sim}{\bar{e}})\cap\mathrm{FU}(\bar{c}\restriction\mu)=\emptyset.$$

Since $\mathcal{U}$ is a filter, we can assume $q=(t,\bar{c})$. We produce an extension r of q that forces the contrary.

There is a minimal m such that q does not determine e_m. So q determines $e_0=d_0$, …, $e_{m-1}=d_{m-1}$.

Since $[\mathrm{FU}(\bar{c}\ ;\mathrm{past}\ t)]^2_<$ has colour 1,

$$\begin{aligned}(t,\bar{c})\Vdash Y=&\{d\in\underset{\sim}{X}_m\cap\mathrm{FU}(\{c_k\ :\ k\in\omega\}\restriction\mu)\cap\mathrm{FU}(\bar{b}_{\max(\underset{\sim}{d}_{m-1})+1})\ :\\ &\underset{\sim}{X}_m\ominus d\in\underset{\sim}{\mathcal{U}}^i \text{ and } \min(d)>\max(\underset{\sim}{d}_{m-1})\}\in\underset{\sim}{\mathcal{U}}^i.\end{aligned}$$

Since $[\mathrm{FU}(\bar{c})]^2_<$ has colour 1 and since q is neat for $\underset{\sim}{\bar{e}}$ and μ, there is $r\le q$ of the form $(t\cup u,\bar{c}\ ;\mathrm{past}\ u)$ and there is $d\in\mathrm{Fin}$

$$r\Vdash d\in\mathrm{FU}(\underset{\sim}{\bar{e}})\cap\mathrm{FU}(\bar{c}\restriction\mu),$$

in contradiction to the assumptions on q. $\square_{3.18,3.10,3.3}$

Henceforth we drop the tildes underneath the names.

Now we return to filters over ω and answer some instances of the question left open in the previous section: What happens to filters with $\Phi(\mathcal{U})\le_{\mathrm{RB}}\mathcal{F}$?

Mathias introduced the following notion under the name "happy family" [31, Definition 0.1]. Louveau studied it in the special case of ultrafilters [29]. Todorcevic [42, Chapter 7] uses the name "selective coideal" for a happy family.

Definition 3.23 (See [31, Definition 0.1.], [42, Definition 7.3])**.** A set $\mathcal{H} \subseteq [\omega]^\omega$ is called a *selective coideal/happy family* if the following hold:

(i) $\mathcal{I}_\mathcal{H} := \mathcal{P}(\omega) \smallsetminus \mathcal{H}$ is an ideal that contains all singletons.

(ii) If $\langle A_i \,:\, i \in \omega\rangle$ is a $\subseteq$-descending sequence of elements $A_i \in \mathcal{H}$, then there is $B \in \mathcal{H}$ such that $B \subseteq A_0$ and $(\forall i \in B) B \smallsetminus (i+1) \subseteq A_i$. We call such a B a *diagonal lower bound of* $\langle A_i \,:\, i \in \omega\rangle$.

We write $\mathcal{F}_\mathcal{H} = \{\omega \smallsetminus X \,:\, X \in \mathcal{I}_\mathcal{H}\}$ for the filter that is dual to $\mathcal{I}_\mathcal{H}$. Then $\mathcal{H}$ coincides with the $\mathcal{F}_\mathcal{H}$-positive sets, i.e.,

$$\mathcal{H} = \mathcal{F}_\mathcal{H}^+ := \{X \in [\omega]^\omega \,:\, (\forall Y \in \mathcal{F}_\mathcal{H})(X \cap Y \neq \emptyset)\}.$$

Lemma 3.24. *Let $\mathcal{F}$ be a filter over ω and let $\mathcal{R}$ be an ultrafilter over ω. $\mathcal{R} \subseteq \mathcal{F}^+$ iff $\mathcal{R} \supseteq \mathcal{F}$.*

The forward implication, which will be invoked many times, uses that $\mathcal{R}$ is ultra.

Remark 3.25. Let $\mathcal{H} \subseteq [\omega]^\omega$ and let $\mathcal{E}$ be a filter. $\mathcal{H}$ and $\mathcal{E}$ are nnc iff for any finite-to-one function f and $X \in \mathcal{H}$ there are a $E \in \mathcal{E}$ and a $Y \subseteq X$, such that $Y \in \mathcal{H}$ and such that $f[E] \cap f[H] = \emptyset$.

The following theorem provides information on $\min(\mathcal{U})$ and $\max(\mathcal{U})$.

Theorem 3.26. *We assume* CH *and we fix a P-point $\mathcal{E}$ and we let $\mathcal{U}$ be a Milliken–Taylor ultrafilter with $\Phi(\mathcal{U}) \not\leq_{\rm RK} \mathcal{E}$. We let $\mathcal{R} \in \{\min(\mathcal{U}), \max(\mathcal{U})\}$ and we let μ denote the generic real. After forcing with $\mathbb{M}(\mathcal{U})$, the set of positive sets*

$$(\mathrm{fil}(\mathcal{R} \cup \{\mu\}))^+ = \{X \in ([\omega]^\omega)^{\mathbf{V}^{\mathbb{M}(\mathcal{U})}} \,:\, (\forall Y \in \mathcal{R})|X \cap Y \cap \mu| = \omega\}$$

is a happy family that is nowhere almost a filter in $\mathbf{V}^{\mathbb{M}(\mathcal{U})}$ (and hence it is nnc to $\mathcal{E}$) and by [31, Prop. 0.11] *there is a Ramsey ultrafilter $\mathcal{R}^{\rm ext} \supseteq \mathcal{R} \cup \{\mu\}$ that is nnc to $\mathcal{E}$.*

Proof. The theorem is proved like Theorem 3.10, however, it is much easier. Lemma 3.16, giving diagonal lower bounds, and Lemma 3.17, showing the nnc-part, are adapted to names for $(\mathcal{R}\cup\{\mu\})$-positive subsets of ω. There are no new ideas. $\square$

Lemma 3.27. *Let $\mathcal{R} \in \{\min(\mathcal{U}), \max(\mathcal{U})\}$ and $\mathcal{E}$ be a P-point such that $\Phi(\mathcal{U}) \not\leq_{\rm RK} \mathcal{E}$. Let $\underset{\sim}{Y}$ be a $\mathbb{Q}$-name for a $\mathrm{fil}(\mathcal{R} \cup \{\mu\})$-positive set and let $\underset{\sim}{h}$ be a $\mathbb{Q}$-name for a finite-to-one function and let $(s, \bar{a}) \in \mathbb{Q}$. Then there are $E \in \mathcal{E}$ and $(s, \bar{b}) \leq (s, \bar{a})$ and $(s, \bar{b}) \Vdash \underset{\sim}{Z} \in \mathcal{R}^+$ such that*

$$(s, \bar{b}) \Vdash \underset{\sim}{h}[E] \cap \underset{\sim}{h}[\underset{\sim}{Z} \cap \underset{\sim}{Y} \cap \mu] = \emptyset. \tag{3.12}$$

Proof. W.l.o.g we can assume that $(s, \bar{a})$ is neat for $\underset{\sim}{Y}$ and $\underset{\sim}{h}$. An application of a colouring similar to above shows: For any $p = (s, \bar{a})$, $\underset{\sim}{h}$ for a finite-to-one function for any name $\underset{\sim}{X}$ for a set in $(\mathrm{fil}(\mathcal{R} \cup \{\mu\}))^+$ there are $q \leq p$, $\underset{\sim}{X}^0$, $\underset{\sim}{X}^1$,

$$q \Vdash \underset{\sim}{X}^0, \underset{\sim}{X}^1 \subseteq \underset{\sim}{X} \wedge \underset{\sim}{X}^0, \underset{\sim}{X}^1 \in (\mathrm{fil}(\mathcal{R} \cup \{\mu\}))^+ \wedge h[\underset{\sim}{X}^0] \cap h[\underset{\sim}{X}^1] = \emptyset.$$

Since $\mathcal{E}$ is a filter, there is a $j \in 2$ and an $E \in \mathcal{E}$ such that $q \Vdash \underset{\sim}{h}[E] \cap \underset{\sim}{h}[\underset{\sim}{X}^j \cap \mu] = \emptyset$. Again we refer the reader to the Fin-version in 3.17 for a detailed proof. $\square$

$\square_{3.26}$

Remark 3.28. We remark that by an analogous proof to Mathias' [31, Prop. 0.11], under CH any happy family that is nnc to $\mathcal{E}$ contains a Ramsey ultrafilter as a subset that is nnc to $\mathcal{E}$. We see that instead of heading for a Milliken–Taylor ultrafilter $\mathcal{U}^{\text{ext}} \supseteq \mathcal{U} \restriction \mu$ that yields of course $\min(\mathcal{U}^{\text{ext}}) \supseteq \min(\mathcal{U}) \cup \{\mu\}$ and the same for the maximum projection, we could proceed into a different direction: By Theorem 3.26, we can extend the minimum and maximum projections $\min(\mathcal{U})$, $\max(\mathcal{U})$ to new nnc Ramsey ultrafilters in $\mathbf{V}^{\mathbb{M}(\mathcal{U})}$ and not care whether these extensions are the minimum and the maximum of a Milliken–Taylor ultrafilter extending $\mathcal{U}$. This direction will be important in Section 5.

4. A name for a Matet-adequate family at limit stages

In this section we establish names of Ramsey spaces in limit steps of countable support iterations. To this end, we prove a new preservation theorem. At the end of the section, we discuss generalisations to block sequences over Fin_k.

We define by induction on $\alpha \le \omega_2$ a countable support iteration $\mathbb{P}_\alpha = \langle \mathbb{P}_\beta, \mathbb{M}(\mathcal{U}_\gamma) : \beta \le \alpha, \gamma < \alpha\rangle$ and a sequence $\langle \mathcal{U}_\beta, \mu_\gamma : \gamma < \alpha, \beta \le \alpha\rangle$ of names such that for any $\gamma < \alpha$,

$$\begin{aligned}
&\mathbb{P}_\gamma \Vdash \mathcal{E} \text{ is a } P\text{-point},\\
&\mathbb{P}_\gamma \Vdash \mathcal{U}_\gamma \supseteq \bigcup\{(\mathcal{U}_\delta \restriction \mu_\delta) : \delta < \gamma\},\\
&\qquad\qquad\quad \text{is a Milliken–Taylor ultrafilter that avoids } \mathcal{E},\\
&\mathbb{P}_{\gamma+1} = \mathbb{P}_\gamma * \mathbb{M}(\mathcal{U}_\gamma), \text{ and}\\
&\mathbb{P}_{\gamma+1} \Vdash \mu_\gamma = \bigcup\{s : (s, \bar{a}) \in G_{\mathbb{M}(\mathcal{U}_\gamma)}\}.
\end{aligned} \tag{4.1}$$

In Theorem 3.3 we proved that there are extensions of Milliken–Taylor ultrafilters in the successor steps

$$\begin{aligned}
\mathbb{P}_\gamma \Vdash \Big(&\mathbb{M}(\mathcal{U}_\gamma) \Vdash \mu_\gamma = \bigcup\{s : (s, \bar{a}) \in G_{\mathbb{M}(\mathcal{U}_\gamma)}\}\\
&\wedge (\exists \mathcal{U}_{\gamma+1} \supseteq \mathcal{U}_\gamma \restriction \mu_\gamma)\\
&(\mathcal{U}_{\gamma+1} \text{ is a Milliken–Taylor ultrafilter and } \mathcal{U}_{\gamma+1} \text{ avoids } \mathcal{E})\Big).
\end{aligned}$$

This guarantees the continuation of our construction in the successor steps, via $\mathcal{U}_{\gamma+1} \supseteq \mathcal{U}_\gamma \restriction \mu_\gamma$.

Now we consider limit steps α. If $\mathrm{cf}(\alpha) > \omega$, we can just take $\mathbb{P}_\alpha \Vdash \mathcal{U}_\alpha = \bigcup_{\gamma<\alpha} \mathcal{U}_\gamma$ and the inductive hypotheses will be carried on, since in proper forcing every real appears at a step of at most countable cofinality, with the only exception that for $\alpha = \aleph_2$ the CH gets lost. So we concentrate on the hard case, $\mathrm{cf}(\alpha) = \omega$.

Theorem 4.1. *Suppose* CH *in the ground model and that* $\mathcal{E}$ *and* $\mathbb{P}_\beta$, $\mathcal{U}_\beta$, $\beta < \alpha$, *are as in Equation* (4.1), $\alpha < \omega_2$, $\mathrm{cf}(\alpha) = \omega$, $\mathbb{P}_\alpha$ *is the countable support limit of* $\langle \mathbb{P}_\beta, \mathbb{M}(\mathcal{U}_\beta) : \beta < \alpha\rangle$. *In* $\mathbf{V}^{\mathbb{P}_\alpha}$, *the filter* $\mathcal{E}$ *still generates a* P*-point and the set of positive sets*

$$\Big(\bigcup_{\gamma<\alpha} (\mathcal{U}_\gamma \restriction \mu_\gamma)\Big)^+$$

forms a Matet-adequate family such that for any $\bar{a} \in (\bigcup_{\gamma<\alpha}(\mathcal{U}_\gamma \restriction \mu_\gamma))^+$ *and finite-to-one* h,

$$(\exists \bar{b}, \bar{c} \sqsubseteq \bar{a})(\bar{b}, \bar{c} \in \Big(\bigcup_{\gamma<\alpha} (\mathcal{U}_\gamma \restriction \mu_\gamma)\Big)^+ \wedge h[\mathrm{set}(\bar{b})] \cap h[(\mathrm{set}(\bar{c})] = \emptyset).$$

As in Theorem 3.10, the latter implies avoidance of $\mathcal{E}$. Again CH and a routine enumeration along ω_1 gives the following.

Corollary 4.2. *Suppose* CH *in the ground model and that* $\mathcal{E}$ *and* $\mathbb{P}_\beta, \mathcal{U}_\beta$, $\beta < \alpha$, *are as in Equation* (4.1) *and* $\mathbb{P}_\alpha$ *is the countable support limit of* $\langle \mathbb{P}_\beta, \mathbb{M}(\mathcal{U}_\beta) : \beta < \alpha \rangle$ *and that* CH *holds in* $\mathbf{V}^{\mathbb{P}_\alpha}$. *Then*

$$\mathbb{P}_\alpha \Vdash \exists \mathcal{U}_\alpha \big(\mathcal{U}_\alpha \textit{ is a Milliken–Taylor ultrafilter that avoids } \mathcal{E}, \textit{ and} \\ \mathcal{U}_\alpha \supseteq \bigcup_{\gamma < \alpha} (\mathcal{U}_\gamma \restriction \mu_\gamma) \big).$$

Now we prove Theorem 4.1. Blass and Shelah [11, Theorem 4.1] showed that in $\mathbf{V}^{\mathbb{P}_\alpha}$ the closure of $\mathcal{E}$ under almost supersets is a P-point. For $\alpha < \omega_2$, the CH holds in $\mathbf{V}^{\mathbb{P}_\alpha}$ by [38, Theorem III.4.1]. For the new part, by induction we define an increasing sequence $\bar{R} = \langle R_\gamma : \gamma < \alpha \rangle$ of relations R_γ in $\mathbf{V}^{\mathbb{P}_\gamma}$ such that a property called

$$\text{“}\mathbb{P}_\gamma \text{ is } R_\gamma\text{-preserving”} \tag{4.2}$$

is a notion we want to carry from $\gamma < \alpha$ to α in addition to the property (4.1) and properness in the inductive choice of the iteration.

In $\mathbf{V}^{\mathbb{P}_\alpha}$ we define the relation R_α for which we want to preserve statements of the form $(\forall f)(\exists \bar{g})(f R_\alpha \bar{g})$. The relation R_α will be a Borel relation on the Baire space in $\mathbf{V}^{\mathbb{P}_\alpha}$. The requirement of α-positivity on the domain and on the range of the relation contains the sequence of names $\langle \mathcal{U}_\beta : \beta < \alpha \rangle$ from the ground model as a parameter.

Definition 4.3. By induction on $\alpha \leq \omega_2$ we define the following relations.

1. We say that a $\mathbb{P}_\alpha$-name $\bar{a}$ for an element of $(\mathrm{Fin})^\omega$ is *α-positive* if $\mathbb{P}_\alpha \Vdash \bar{a} \in (\bigcup \{\mathcal{U}_\gamma \restriction \mu_\gamma : \gamma < \alpha\})^+$.
2. Assume that $\langle \mathcal{U}_\gamma : \gamma < \alpha \rangle$ is an ascending sequence of Milliken–Taylor ultrafilters $\mathcal{U}_\gamma \in \mathbf{V}^{\mathbb{P}_\gamma}$, such that $\mathbb{P}_\gamma \Vdash \Phi(\mathcal{U}_\gamma) \not\leq_{\mathrm{RB}} \mathcal{E}$ and $\forall \gamma < \delta < \alpha$, $\mathbb{P}_\delta \Vdash \mathcal{U}_\gamma \restriction \mu_\gamma \subseteq \mathcal{U}_\delta$. We say $f R_\alpha \bar{g}$ if the following holds in $\mathbf{V}^{\mathbb{P}_\alpha}$:
    - (a) $f = (\bar{A}, h, c)$,
    - (b) $\bar{A} = \langle A_\ell : \ell \in \omega \rangle$ is a $\sqsubseteq$-descending sequence of α-positive sequences $A_\ell \in (\mathrm{Fin})^\omega$,
    - (c) $h \colon \omega \to \omega$ is finite-to-one,
    - (d) c is a partition of $\mathrm{FU}(A_0)$ into finitely many parts.
    - (e) For $j = 0, 1$ we let $\bar{g}^j := \langle g_{2n+j} : n \in \omega \rangle$. Then
        - (i) For $j = 0, 1$, $\bar{g}^j$ is an α-positive diagonal lower bound of $\bar{A}$.
        - (ii) For $j = 0, 1$, $\mathrm{FU}(\bar{g}^j)$ is in one piece of the partition c.
        - (iii) $h[\mathrm{set}(\bar{g}^0)] \cap h[\mathrm{set}(\bar{g}^1)] = \emptyset$.

The object R_α is a $\mathbb{P}_\alpha$-name for a relation.

Definition 4.4. We say $\mathbb{P}_\alpha$ is *R_α-preserving* if $\mathbb{P}_\alpha$ is proper and

$$\mathbb{P}_\alpha \Vdash \forall f \in \mathrm{dom}(R_\alpha) \exists \bar{g} (f R_\alpha \bar{g}).$$

There are two main differences to the known “Case A” of iteration theorem [39, Ch. XVIII], [23]: For our R_α, it is not the case that for countably many tasks f_n, $n \in \omega$, there is one answer $\bar{g}$, such that $\forall n f_n R_\alpha \bar{g}$. One reason for this is that there are $\sqsubseteq^*$-incompatible first components A_0 of tasks f. A second aspect is that not only the quests f but also the answers $\bar{g}$ are now from the forcing extension. This differs from the traditional applications in the preservation of cardinal invariants, see e.g. [2]. We do not write tildes below the R_α's.

Lemma 4.5. *Assume $\alpha < \omega_2$, $\mathrm{cf}(\alpha) < \omega_1$. If $\mathbb{P}_\alpha$ is R_α-preserving then*

$$\begin{aligned}
\mathbb{P}_\alpha \Vdash & (\bigcup\{\mathcal{U}_\beta \restriction \mu_\beta \,:\, \beta < \alpha\})^+ \text{ is a Matet-adequate family such that} \\
& (\forall \text{ finite-to-one } h)(\forall \bar{a} \in (\bigcup\{\mathcal{U}_\beta \restriction \mu_\beta \,:\, \beta < \alpha\})^+)(\exists \bar{b}, \bar{c} \sqsubseteq \bar{a}) \\
& (\bar{b}, \bar{c} \in (\bigcup\{\mathcal{U}_\beta \restriction \mu_\beta \,:\, \beta < \alpha\})^+ \wedge h[\mathrm{set}(\bar{b})] \cap h[(\mathrm{set}(\bar{c})] = \emptyset).
\end{aligned}$$

Proof. This follows from Definition 4.4. □

Now we carry the preservation property upwards by induction.

Lemma 4.6. *Assume $\alpha < \omega_2$, and $\mathrm{cf}(\alpha) = \omega$. Let $\mathbb{P}_\alpha$ be the countable support limit of $\mathbb{P}_\beta$, $\beta < \alpha$. If for $\beta < \alpha$ such that $\mathrm{cf}(\beta) < \omega_1$, $\mathbb{P}_\beta$ is R_β-preserving and for any $\beta < \alpha$ Equation* (4.1) *holds then $\mathbb{P}_\alpha$ is R_α-preserving.*

Lemma 4.6 will be proved with Lemma 4.8. For definiteness, we can take $\chi = (2^{|\mathbb{P}_\alpha|})^+$. Under CH, for $\alpha < \omega_2$, $|\mathbb{P}_\alpha| \leq \aleph_1$ by [38, page 96]. So $\chi = (2^{\aleph_1})^+$ is sufficiently large. The following lemma on the translation to countable elementary submodels is well-known, see [39, Theorem 2.11 and Ch. XVIII].

Lemma 4.7. *The following are equivalent.*

(1) $\mathbb{P}_\alpha$ is R_α-preserving.

(2) For all countable $M \prec H(\chi)$ such that $\mathbb{P}_\alpha, \langle \mathcal{U}_\beta \,:\, \beta < \alpha\rangle \in M$, for all $\mathbb{P}_\alpha$-names $f \in M$, for all $p \in \mathbb{P}_\alpha \cap M$, if $p \Vdash f \in \mathrm{dom}(R_\alpha)$, then there is an $(M, \mathbb{P}_\alpha)$-generic $q \leq p$ and there is a $\mathbb{P}_\alpha$-name $\bar{g} \in M$ such that $q \Vdash f R_\alpha \bar{g}$.

The property of Lemma 4.7(2) is carried on by induction in the following slightly stronger technical form that is suitable for induction.

Lemma 4.8. *The induction lemma. Suppose $\xi < \zeta \leq \aleph_2$, and $\mathrm{cf}(\zeta) < \omega_1$. Let $M \prec H(\chi)$, M countable, $\zeta \in M$, $\mathbb{P}_\zeta \in M$, $p \in \mathbb{P}_\zeta \cap M$ and $q_0 \leq p \restriction \xi$ be $(M, \mathbb{P}_\xi)$-generic. Let $f \in M$ be a $\mathbb{P}_\zeta$-name such that*

$$p \Vdash_{\mathbb{P}_\zeta} f = (\langle A_n \,:\, n \in \omega\rangle, c, h) \in \mathrm{dom}(R_\zeta).$$

Then there are some $(M, \mathbb{P}_\zeta)$-generic condition $q \leq p \restriction [\xi, \eta) \cup q_0$ and a name $\bar{g} \in M$ such that $q \Vdash (\exists \bar{g})(f R_\zeta \bar{g})$.

Proof. Moreover we get $\mathrm{dom}(q) \setminus \xi \subseteq \zeta \cap M$. We go by induction on ζ. For $\zeta = 0$ there is nothing to prove, for ζ successor a proof is included in the proof of Theorem 3.10, namely in Lemmata 3.16, 3.17, 3.18. So let ζ be a limit. We fix a strictly increasing sequence $\langle \zeta_\ell \,:\, \ell < \omega\rangle$ with $\zeta_0 = \xi$, $\zeta_\ell \in M$, and $\sup \zeta_\ell = \zeta$. We also fix an enumeration $\langle D_n \,:\, n < \omega\rangle$ of the dense subsets of $\mathbb{P}_\zeta$ that are elements of M. We let $\mathcal{U}^i$ be a $\mathbb{P}_\zeta$-name in M such that

$$\begin{aligned}
&\mathbb{P}_\zeta \Vdash \mathcal{U}^i \text{ is idempotent and} \\
&\qquad \mathcal{U}^i \supseteq \mathrm{fil}(\bigcup\{(\mathcal{U}_\gamma \restriction \mu_\gamma) \,:\, \gamma < \zeta\} \cup \{A_n \,:\, n \in \omega\}).
\end{aligned} \tag{4.3}$$

By Lemma 3.22 such a name $\mathcal{U}^i$ exists. We choose by induction on n, $q_n \in \mathbb{P}_{\zeta_n}$, a $\mathbb{P}_{\zeta_n}$-name for a $\mathbb{P}_{\zeta_n,\zeta}$-name d_n for an element of Fin, a $\mathbb{P}_{\zeta_n}$-name for a $\mathbb{P}_{\zeta_n,\zeta}$-name X_n for an element of $\mathcal{U}^i$, and a $\mathbb{P}_{\zeta_n}$-name of a $p_n \in \mathbb{P}_{\zeta_n,\zeta}$ with the following properties:

(a) $q_n \in \mathbb{P}_{\zeta_n}$, $\mathrm{dom}(q_n) \smallsetminus \xi \subseteq M \cap \zeta_n$, $q_{n+1} \restriction \zeta_n = q_n$,

(b) q_n is $(M, \mathbb{P}_{\zeta_n})$-generic,

(c) $q_{n+1} \Vdash_{\mathbb{P}_{\zeta_n}} p_{n+1} \in D_n \cap M \cap G$,

(d) $p_0 \in M$, $p_0 \restriction \zeta_0 \geq q_0$ in $\mathbb{P}_{\zeta_0}$,

(e) $q_{n+1} \Vdash_{\mathbb{P}_{\zeta_{n+1}}} (p_{n+1} \restriction [\zeta_n, \zeta_{n+1}) \leq p_n \restriction [\zeta_n, \zeta_{n+1}))$ (in $\mathbb{P}_{\zeta_n,\zeta_{n+1}}$),

(f) q_1, p_1, X_0, d_0, X_1 are such that

$$\begin{aligned} q_1 \Vdash_{\mathbb{P}_{\zeta_1}} & p_1 \in M \cap D_0 \cap G \wedge X_0 \sqsubseteq \mathrm{FU}(\pi_1(q_1(\zeta_0))) \wedge \\ & \Big(p_1 \Vdash_{\mathbb{P}_{\zeta_1,\zeta}} \big(X_0 \text{ is the piece of the partition } c \restriction \mathrm{FU}(A_0) \text{ that lies in } \mathcal{U}^i \\ & \wedge d_0 = \min_{\mathrm{lex,Fin}} \{d \in X_0 \cap \mathrm{FU}(A_0) \,:\, X_0 \ominus d \in \mathcal{U}^i\} \\ & \wedge\ X_1 \subseteq X_0 \cap (X_0 \ominus d_0) \wedge X_1 \in \mathcal{U}^i\big)\Big). \end{aligned}$$

Moreover $q_1 \Vdash (p_1 \Vdash d_0 \in M$, $q_1 \Vdash p_1 \in M \cap G \cap D_0)$.

(g) q_{n+1}, p_{n+1}, d_n, X_{n+1} are such that

$$\begin{aligned} q_{n+1} \Vdash_{\mathbb{P}_{\zeta_{n+1}}} & \Big(p_{n+1} \in M \cap G \cap D_n \wedge X_{n+1} \sqsubseteq \mathrm{FU}(\pi_1(q_{n+1}(\zeta_n))) \wedge \\ & p_{n+1} \Vdash_{\mathbb{P}_{\zeta_{n+1},\zeta}} \big(d_n = \min_{\mathrm{lex,Fin}} \{d \in X_n \cap \mathrm{FU}(A_n) \\ & \quad : X_n \ominus d \in \mathcal{U}^i \text{ and } \min(d) > \max(d_{n-1}) \ \wedge\ h[d_{n-1}] \cap h[d] = \emptyset\} \\ & \wedge\ X_{n+1} \subseteq X_n \cap (X_n \ominus d_n) \wedge X_{n+1} \in \mathcal{U}^i\big)\Big). \end{aligned}$$

Since the name d_n is defined from elements in M, $q_{n+1} \Vdash_{\mathbb{P}_{\zeta_{n+1}}} (p_{n+1} \Vdash_{\mathbb{P}_{\zeta_{n+1},\zeta}} d_n \in M)$.

Since $M \prec H(\chi)$, such a sequence exists by the induction hypothesis and the maximum principle.

In the end we let $\bar{g}$ such that

$$q_n \Vdash_{\mathbb{P}_{\zeta_n}} (p_n \Vdash_{\mathbb{P}_{\zeta_n,\zeta}} \bar{g} \restriction n = \langle d_i \,:\, i < n\rangle)$$

and $q = \bigcup_{n<\omega} q_n$ and we let $\bar{g}^j$ be a name such that $q \Vdash \bar{g}^j = \langle g_{2n+j} \,:\, n < \omega\rangle$.

Now it is easy to see that q, $\bar{g}^j$ are as desired, i.e., $q \leq p$ is $(M, \mathbb{P}_\zeta)$ generic and

$$q \Vdash f R_\alpha \bar{g}, \tag{4.4}$$

but for the verification that $\bar{g}^j$, $j = 0, 1$, is an α-positive element.

We prove that q forces that such a $\bar{g}^j$ is α-positive, independently of the choice of the p_n and X_n. For simplicity we just write $\bar{g}$. Let $q' \leq q$ be any stronger condition. Let q' force that $\mathrm{FU}(\bar{c}) \in \bigcup\{\mathcal{U}_\gamma \restriction \mu_\gamma : \gamma < \zeta\}$ be a name for $\bar{c}[G] \in \mathcal{F}_\alpha = \bigcup\{\mathcal{U}_\gamma \restriction \mu_\gamma : \gamma < \alpha\}[G]$. Then $\bar{c}[G] \in V[G_{\zeta_{n_0}}]$ for some n_0 by the nature of union. By the definition of the successor extension $\mathcal{U}_{\gamma+1}$ we have $\bigcup\{\mathcal{U}_\gamma \restriction \mu_\gamma : \gamma < \alpha\}[G] = \bigcup\{\mathcal{U}_\gamma : \gamma < \alpha\}[G]$. Then means that $\mathrm{FU}(\bar{c}[G]) \in \mathcal{U}_{\zeta_n}$ for any $n \geq n_0$. Moreoever $\mathrm{FU}(\bar{g}) \in \mathcal{U}^i$ since $\mathcal{U}^i$ extends $\bigcup\{\mathcal{U}_\gamma \restriction \mu_\gamma : \gamma < \alpha\}[G]$. The condition r with support

$\{\zeta_n : n \geq n_0\}$ and $r(\zeta_n) = (\emptyset, \bar{c})$ for $n \geq n_0$ is compatible with any $q' \leq q$, let r' be stronger than q' and r. Then

$$r' \Vdash \mathrm{FU}(\bar{g}) \cap \mathrm{FU}(\bar{c}) \text{ is infinite.}$$

This is seen as follows: Suppose for a contradiction that there is some $r'' \leq r'$ and there is some $k \in \omega$ such that $r'' \Vdash \bar{g} \cap \mathrm{FU}(\bar{c})$ has at most k blocks, let k be minimal, so that this holds for r''. We assume that r'' determines $g_0, \dots, g_{z-1}$ for some natural number z, without loss of generality we assume that $z \geq \max(n_0, \mathrm{dom}(\pi_0(q(\zeta_{n_0})))$. We show that there is $r^+ \leq r''$ such that r^+ forces at least $k+1$ blocks of $\bar{g}$ that are in $\mathrm{FU}(\bar{c})$ and thus will have a contradiction.

For a condition $(s, \bar{a})$ we $\pi_0((s, \bar{a})) = s$ and $\pi_1((s, \bar{a})) = \bar{a}$. We work in $V[G_{\zeta_{n_0}}]$. Let $\ell = \max(\pi_0(r''(\zeta_{n_0}))$. Now A_z is forced by q, hence also by r'' to be an element of $\mathcal{U}^i$ and all elements of $\mathcal{U}^i$ are ζ-positive. Hence there is a strengthening r^+ of r'' and there is $(b \in \mathrm{FU}(A_z) \cap X_z \cap \mathrm{FU}(\bar{c}))[G_{\zeta_{n_0}}]$ which lies past g_{z-1} such that $X_z \ominus b \in \mathcal{U}^i$. We assume that $b = c_{j_0} \cup \dots \cup c_\ell$. By the rules of $\mathbb{P}_\zeta$, all the blocks of b are blocks or unions of blocks of the given $\bar{c}[G_{\zeta_{n_0}}] \in \mathcal{U}_{\zeta_{n_0}}$. Now b is possibly not minimal, so not the block fulfilling the definition for d_z, which is like the one for b but with the minimality requirement and no explicit requirement on being in $\mathrm{FU}(A_z) \cap X_z \cap \mathrm{FU}(\bar{c})$, just a requirement on being in $\mathrm{FU}(A_z) \cap X_z$. Now by (g) for n_0, $n_0 + 1, \dots, z-1$, we have $q \Vdash X_z \subseteq \dots \subseteq X_{n_0} \subseteq \mathrm{FU}(\pi_1(q_{n_0+1}(\zeta_{n_0}))) \subseteq \mathrm{FU}(\bar{c})$. So taking the minimal b with this seemingly lesser requirement of being in $\mathrm{FU}(A_z) \cap X_z$ will result in a block that is an element of $\mathrm{FU}(\bar{c})$. Hence there is $r^+ \leq r''$ that forces

$$r^+[G_{\zeta_{n_0}}] \Vdash_{\mathbb{P}_{\zeta_{n_0}, \zeta}} g_z[G_{\zeta_{n_0}}] \in \mathrm{FU}(A_z)[G_{\zeta_{n_0}}] \cap X_z[G_{\zeta_{n_0}}] \cap V.$$

In addition by (g) and since $r' \leq r, q$, we have

$$r'[G_{\zeta_{n_0}}] \Vdash_{\mathbb{P}_{\zeta_{n_0}, \zeta}} X_z[G_{\zeta_{n_0}}] \mathbin{;} \mathrm{past}\ (\mathrm{dom}(\pi_0(q(\zeta_{n_0})))) \subseteq \mathrm{FU}(\bar{c})[G_{\zeta_{n_0}}].$$

Hence $r^+[G_{\zeta_{n_0}}] \Vdash g_z[G_{\zeta_{n_0}}] \in \mathrm{FU}(\bar{c})[G_{\zeta_{n_0}}]$. So we reached a contradiction to r'' forcing that there are at most k blocks of $\bar{g}$ that are elements of $\mathrm{FU}(\bar{c})$. This argumentation can be carried out for even and for odd z, so that $\bar{g}^j$ is forced to be positive for $j = 0, 1$. $\Box_{4.8,4.1}$

Putting Theorem 3.10, Theorem 4.1 and Blass' ω_1 long downwards induction [6, Theorem 2.4] together yields the following.

Theorem 4.9. *Let $\mathcal{E}$ be a P-point and assume* CH*. Then there is a countable support iteration of proper iterands $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{M}(\mathcal{U}_\beta) : \beta < \omega_2, \alpha \leq \omega_2 \rangle$ such that in the extension $\mathcal{E}$ is a P-point, there at least three near-coherence classes of ultrafilters and there is a Milliken–Taylor ultrafilter of character $\mathfrak{d} = \mathfrak{c} = \aleph_2$.*

We note that recently Fernández-Bréton [21] built a model in which, like in ours, $\mathrm{cov}(\mathcal{M}) = \aleph_1 < \mathfrak{d} = \aleph_2$ with a Milliken–Taylor ultrafilter.

Generalisation to Fin_k. We comment on the parallel of the results of Sections 2, 3, and the beginning of Section 4 for the space Fin_k for $k \geq 2$ instead of Fin. In this setting, $(\mathrm{Fin}, \cup)$ corresponds to $(\mathrm{Fin}_1, +)$ via taking characteristic functions. This subsection shows that the spaces Fin_k, $k \geq 2$, are barren ground with respect to near-coherence classes. It is not used for our main result.

Definition 4.10. Let $k \in \omega \setminus \{0\}$ unless stated otherwise. For $p\colon \omega \to k+1$ we let $\operatorname{supp}(p) = \{n \in \omega \,:\, p(n) \neq 0\}$.

$$\mathrm{Fin}_k = \{p\colon \omega \to k+1 \,:\, \operatorname{supp}(p) \text{ is finite } \wedge k \in \operatorname{range}(p)\}.$$

$(\mathrm{Fin}_k)^\omega$ is the set of unmeshed infinite sequences $\langle p_n \,:\, n < \omega\rangle$. Unmeshed means now that for $n \in \omega$, $\operatorname{supp}(p_n) < \operatorname{supp}(p_{n+1})$ in the sense of $<$ from Definition 2.1(3). For $p < q \in \mathrm{Fin}_k$ and $i \in \omega$ we let $(p+q)(i) = p(i) + q(i)$. For $(\bar{a} \in \mathrm{Fin}_k)^\omega$, the $\mathrm{FU}_k(\bar{a})$ is the set of all unmeshed infinite sequences $\bar{b}$ such that each b_n is a sum of finitely many of the a_r. Some a_r may be dropped. We write $\bar{b} \sqsubseteq_k \bar{a}$ if $\{b_n \,:\, n < \omega\} \subseteq \mathrm{FU}_k(\bar{a})$. We let $\operatorname{set}(\bar{a}) = \bigcup\{\operatorname{supp}(a_n) \,:\, n < \omega\}$ and for a subset $\mathcal{U}$ of $(\mathrm{Fin}_k)^\omega$ we let $\Phi(\mathcal{U}) = \{\operatorname{set}(\bar{a}) \,:\, \bar{a} \in \mathcal{U}\}$.

For $1 \leq j \leq k$ we let

$$\begin{aligned}
&\operatorname{set}_j[\bar{a}] = \bigcup\{a_n^{-1}[\{j\}] \,:\, n < \omega\},\\
&\Phi_j(\mathcal{U}) = \{\operatorname{set}_j[\bar{a}] \,:\, \bar{a} \in \mathcal{U}\},\\
&\min_j[\bar{a}] = \{\min(a_n^{-1}[\{j\}]) \,:\, n < \omega\},\\
&\min_j(\mathcal{U}) = \{\min_j[\bar{a}] \,:\, \bar{a} \in \mathcal{U}\}, \text{and analogously for max.}
\end{aligned} \tag{4.5}$$

We do not include the tetris operation (e.g., defined in [42, Page 34]). Hindman's theorem is generalised to $(\mathrm{Fin}_k)^\omega$:

Theorem 4.11 (Hindman [24, Cor. 3.3])**.** *Let $k \geq 1$ and $\bar{a} \in (\mathrm{Fin}_k)^\omega$. For every finite colouring of $\mathrm{FU}_k(\bar{a})$ there is a infinite block sequence $\bar{b} \sqsubseteq_k \bar{a}$ such that the elements of $\mathrm{FU}_k(\bar{b})$ are monochromatic.*

There is no problem in generalising from Fin to Fin_k and we get:

Theorem 4.12. *Assume CH and let $k \geq 1$. Let $\mathcal{E}$ be a P-point and let $\mathcal{U}_0$ be a Milliken–Taylor ultrafilter on Fin_k such that $\Phi(\mathcal{U}) \not\leq_{\mathrm{RB}} \mathcal{E}$. Then there is a countable support iteration of proper iterands $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{M}_k(\mathcal{U}_\beta) \,:\, \beta < \omega_2, \alpha \leq \omega_2\rangle$ such that in the extension $\mathcal{E}$ is a P-point and there is a Milliken–Taylor ultrafilter $\mathcal{U}$ over Fin_k of character $\mathfrak{c} = \aleph_2 = \mathfrak{d}$. Again the sequence $\langle \mathcal{U}_\alpha \,:\, \alpha \leq \omega_2\rangle$ is increasing in α.*

Now we are concerned with the number of near-coherence classes among the ultrafilters that contain $\Phi(\mathcal{U})$ as a subset.

The following generalises [8, Theorem 38] and in addition concerns the ultrafilters on each side.

Theorem 4.13. *For any union-ultrafilter $\mathcal{U}$ over Fin_k, all the ultrafilters among the projections*

$$\min_i(\mathcal{U}), i = 1, 2 \ldots, k$$

are nearly coherent to $\min_k(\mathcal{U})$ and that class is different from the class of $\max_k(\mathcal{U})$, which is the class of any of the ultrafilters among

$$\max_j(\mathcal{U}), j = 1, \ldots, k.$$

Proof. We let $M \subseteq \{1, \ldots k-1\}$ be maximal such that for any $\bar{a} \in \mathcal{U}$ there is $\bar{b} \sqsubseteq \bar{a}$, $\bar{b} \in \mathcal{U}$ such that for any n, $\operatorname{range}(b_n) = M \cup \{k\}$. Then exactly for $m \in M \cup \{k\}$, $\max_m(\mathcal{U})$ and $\min_m(\mathcal{U})$ are ultrafilters by [6, Proposition 3.9]. Recall, ultrafilter means non-principal ultrafilter. Let $A = \mathrm{FU}_k(\bar{a}) \in \mathcal{U}$ be such that for any n $\mathrm{range}(a_n) = M \cup \{k\}$. We define $I_0 = [0, \max(\mathrm{supp}(a_0)))$, $I_{n+1} = [\max(\mathrm{supp}(a_n)), \max(\mathrm{supp}(a_{n+1})))$. We take a finite-to-one function $h_{\bar{a}}$ that is constant on I_n for $n \in \omega$. Then for any $i, j \in M$,

$$(\forall n \in \omega)(h_{\bar{a}}[\mathrm{set}_i(a_n)] \cap h_{\bar{a}}[\mathrm{set}_j(a_n)] \neq \emptyset).$$

Since $\mathcal{U}$ is centred, for any $\mathrm{FU}_k(\bar{b}), \mathrm{FU}_k(\bar{c}) \in \mathcal{U}$ there is $\bar{d} \sqsubseteq_k \bar{a}, \bar{b}, \bar{c}$, $\mathrm{FU}_k(\bar{d}) \in \mathcal{U}$. Hence we have for $\bar{d} = \langle d_\ell : \ell < \omega \rangle$ for any $\ell \in \omega$ that there are natural numbers $n, m, r \geq 1$ and natural numbers $i_1 < \cdots < i_n$, $j_1 < \cdots < j_m$, $k_1 < \cdots < k_r$, such that

$$d_\ell = a_{i_1} + \cdots + a_{i_n} = b_{j_1} + \cdots + b_{j_m} = c_{k_1} + \cdots + c_{k_r}.$$

Then

$$h_{\bar{a}}[\min_i(d_\ell)] \cap h_{\bar{a}}[\min_j(d_\ell)] \neq \emptyset.$$

Hence

$$h_{\bar{a}}[\min_i[\bar{d}]] \cap h_{\bar{a}}[\min_j[\bar{d}]] \text{ is infinite.}$$

In [8, Section 6] Blass proves that the minimum and the maximum class are different.

□

Theorem 4.14. *Let $\mathcal{U}$ be a Milliken–Taylor ultrafilter over* Fin_k. *Let M be as above. The cores $\Phi_j(\mathcal{U})$, $j \in M \cup \{k\}$, are all nearly coherent.*

Proof. Since $\min_i(\mathcal{U})$ is nearly coherent to $\min_j(\mathcal{U})$, also there subfilters $\Phi_i(\mathcal{U})$ and $\Phi_j(\mathcal{U})$ are nearly coherent. □

Remark 4.15. Theorem 4.14 can also be proved with the help of a Fin_k-parallel to Theorem 2.16 and the following folklore result.

Definition 4.16. Let $X \in [\omega]^\omega$. The function $\mathrm{next}(\cdot, X)\colon \omega \to \omega$ is defined by $\mathrm{next}(n, X) = \min(X \smallsetminus (n+1))$.

Proposition 4.17. *Any forcing that diagonalises two nnc filters adds a dominating real.*

Proof. Let $\mathcal{F}$ and $\mathcal{F}' \in \mathbf{V}$ be nnc filters and let $x, y \in \mathbf{V}[G]$ be such that $(\forall F \in \mathcal{F})(x \subseteq^* F)$, $(\forall F' \in \mathcal{F}')(y \subseteq^* F')$. Then by the next lemma the function $\max(\mathrm{next}(\cdot, x), \mathrm{next}(\cdot, y))$ is a dominating function.

Lemma 4.18 (Proof of [5, Theorem 3.2])**.** *Let $\mathcal{V}$, $\mathcal{W}$ be non-principal filters over ω. $\mathcal{V}$ is nnc to $\mathcal{W}$ iff*

$$\{\max(\mathrm{next}(\cdot, X), \mathrm{next}(\cdot, Y)) : X \in \mathcal{V}, Y \in \mathcal{W}\}$$

is a $\leq^*$-dominating family.

We give a proof for the direction we use: Assume that $\mathcal{V}$ and $\mathcal{W}$ are not nearly coherent. Let $h \in {}^\omega\omega$ be given, w.l.o.g. we assume that h is strictly increasing and $h(0) > 0$. We let $\tilde{h}$ be the iterate of h: $\tilde{h}(0) = 0$, $\tilde{h}(n+1) = h(\tilde{h}(n))$. We let $f_e(n) = i$ for $n \in [\tilde{h}(2i), \tilde{h}(2i+2))$ and we let $\tilde{h}(-1) = 0$ and $f_o(n) = i$ for $n \in [\tilde{h}(2i-1), \tilde{h}(2i+1))$. Then there are $V_e, V_o \in \mathcal{V}$ and $W_e, W_0 \in \mathcal{W}$ such that $f_e[V_e] \cap f_e[W_e] = \emptyset$ and $f_o[V_o] \cap f_o[W_o] = \emptyset$. Since $\mathcal{V}$ and $\mathcal{W}$ are filters, we can assume $V_e = V_o$ and $W_e = W_o$. Then for any $n \in \omega$ we have $\max(\mathrm{next}(n, V_e), \mathrm{next}(n, W_e)) \geq h(n)$. □

This concludes the proof of Proposition 4.17.

Now we finish the alternative proof of Theorem 4.14: $\mathbb{M}_k(\mathcal{U})$ preserves $\mathcal{E}$ and hence does not add a dominating real. However $\mathbb{M}_k(\mathcal{U})$ diagonalises $\Phi_j(\mathcal{U})$ for $j \in M \cup \{k\}$. By Proposition 4.17 all the $\Phi_j(\mathcal{U})$, $j \in M \cup \{k\}$, are nearly coherent.

5. Localised Matet Forcing

In this section we take up the Ramsey space mentioned in the introduction and investigate Matet forcing when localised to this type of space.

Definition 5.1. Let $\mathcal{R}_{\min}$, $\mathcal{R}_{\max}$ be two nnc Ramsey ultrafilters. We write $\bar{a} \in (\mathrm{Fin})^\omega(\bar{\mathcal{R}})$ if $\bar{a} \in (\mathrm{Fin})^\omega$ and

$$(\forall X \in \mathcal{R}_{\min})(\forall Y \in \mathcal{R}_{\max})(\exists^{\min-\mathrm{unb}} s \in \mathrm{FU}(\bar{a}))(\min(s) \in X \wedge \max(s) \in Y). \tag{5.1}$$

The following theorem is due to Blass [6, Theorem 2.2].

Theorem 5.2. *Let $\mathcal{R}_{\min}$, $\mathcal{R}_{\max}$ be two nnc Ramsey ultrafilters.*

(1) Let $\bar{a} \in (\mathrm{Fin})^\omega(\bar{\mathcal{R}})$ and c be a colouring of $\mathrm{FU}(\bar{a})$ into finitely many colours. Then there is a $\bar{b} \sqsubseteq \bar{a}$, such that $\bar{b} \in (\mathrm{Fin})^\omega(\bar{\mathcal{R}})$ and $\mathrm{FU}(\bar{b})$ is c-monochromatic.

(2) Let $\langle \bar{a}_n : n \in \omega\rangle$, be a $\sqsubseteq$-descending sequence of elements of $(\mathrm{Fin})^\omega(\bar{\mathcal{R}})$. Then there is some $\bar{b} \in (\mathrm{Fin})^\omega(\bar{\mathcal{R}})$ such that for any n, $\bar{b} \sqsubseteq^* \bar{a}_n$.

(3) Under CH there is a Milliken–Taylor ultrafilter $\mathcal{U}$ such that $\min(\mathcal{U}) = \mathcal{R}_{\min}$ and $\max(\mathcal{U}) = \mathcal{R}_{\max}$.

Definition 5.3. Given a Matet-adequate family $\mathcal{H} \subseteq (\mathrm{Fin})^\omega$, the notion of forcing $\mathbb{M}(\mathcal{H})$ consists of all pairs $(s, \bar{a})$ such that $\bar{a} \in \mathcal{H}$. The forcing order is the same as in Matet forcing.

By the theorem, the family $(\mathrm{Fin})^\omega(\bar{\mathcal{R}})$ is Matet-adequate. Note that also the properties from Lemma 3.6 hold for $(\mathrm{Fin})^\omega(\bar{\mathcal{R}})$.

Definition 5.4. For $\mathcal{H} = (\mathrm{Fin})^\omega(\bar{\mathcal{R}})$ we just write $\mathbb{M}(\bar{\mathcal{R}})$ for the Matet forcing $\mathbb{M}(\mathcal{H})$.

Definition 5.5. For $n \in \omega$, $p, q \in \mathbb{M}(\bar{\mathcal{R}})$, we let $q = (t, \bar{b}) \leq_n p = (s, \bar{a})$ if $s = t$ and for $i < n$, $a_i = b_i$.

By Theorem 5.2 we have the pure decision property.

Lemma 5.6. $\mathbb{M}(\bar{\mathcal{R}})$ *has the* pure decision property, *i.e., for any $\varphi \in \mathcal{L}(\in)$, $(s, \bar{a}) \in \mathbb{M}(\bar{\mathcal{R}})$ there is $(s, \bar{b}) \leq_0 (s, \bar{a})$ such that $(s, \bar{b}) \Vdash \varphi$ or $(s, \bar{b}) \Vdash \neg\varphi$.*

Lemma 5.7. *The forcing poset $(\mathbb{M}(\bar{\mathcal{R}}), \leq, (\leq_n)_{n<\omega}))$ fulfils Axiom A and hence is proper.*

Proof. A derivation of Axiom A for the relations $(\leq_n)_n$ from the pure decision property can be found, e.g., in [2, Section 7.1]. Properness alone can also be proved as in [11, 2.3, 2.4, 2.5]. The latter proof uses the existence of lower bounds, i.e., statement (2) of Theorem 5.2, but not the pure decision property. □

Definition 5.8. A filter $\mathcal{U}$ over Fin *avoids $\mathcal{E}$* if $\Phi(\mathcal{U}) \not\leq_{\mathrm{RB}} \mathcal{E}$.

Now we investigate the number of near-coherence classes in models of our new forcing.

Definition 5.9. Let $X \in [\omega]^\omega$. We let $f_X(n) = |X \cap n|$.

We start with a density argument for evaluating our forcings.

Lemma 5.10. *Let* $\mathcal{W}$ *be an ultrafilter over* ω *such that* $\mathcal{W} \not\geq_{RB} \mathcal{R}_x$ *for* $x = \min, \max$*. Then for any* $\bar{a} \in (\mathrm{Fin})^\omega(\bar{\mathcal{R}})$ *and finite-to-one* h *there are some* $W \in \mathcal{W}$ *and some* $\bar{b} \sqsubseteq \bar{a}$*,* $\bar{b} \in (\mathrm{Fin})^\omega(\bar{\mathcal{R}})$*, such that* $h[\mathrm{set}(\bar{b})] \cap h[W] = \emptyset$*.*

Proof. Using the property from part (a) of Lemma 3.6, we take a preliminary strengthening $\bar{c} \sqsubseteq \bar{a}$ such that for $n \neq m$,

$$h^{-1}[[\min(c_{n-1}), \min(c_n))] \cap h^{-1}[[\min(c_{m-1}), \min(c_m))] = \emptyset.$$

Here we let $\min(c_{-1}) = 0$.

We let $I_n = [\max(c_{n-1}), \max(c_n))$ for $n \in \omega$. For $j = 0, 1$ we define $h_j \colon \omega \to \omega$ by letting $h_j(k) = n$ if $k \in I_{2n+j} \cup I_{2n+j+1}$ and $h_1(k) = 0$ for $k \in I_0$.

We take $W \in \mathcal{W}$, $X_{\min} \in \mathcal{R}_{\min}$, $X_{\min} \subseteq \{\min(c_n) : n < \omega\}$, $X_{\max} \in \mathcal{R}_{\max}$, $X_{\max} \subseteq \{\max(c_n) : n < \omega\}$ such that

$$(\forall j)(h_j[W] \cap (h_j[X_{\min}] \cup h_j[X_{\max}]) = \emptyset).$$

We let $\bar{b}$ be the increasing enumeration of $\{c_n : \min(c_n) \in X_{\min} \vee \max(c_n) \in X_{\max}\}$. It is clear that $\bar{b} \in (\mathrm{Fin})^{<\omega}(\bar{\mathcal{R}})$.

We show the disjointness. To this end we suppose that I_n contains an element of $\mathrm{set}(\bar{b})$. Then I_n contains an element of the form $\min(c_r)$ or $\max(c_r)$ for some r such that c_r belongs to $\bar{b}$. Since between any I_n that is met by any element of $X_{\min}$ or of $X_{\max}$ and any I_k that is met by W there is one I_s such that I_s does not contain any element of $X_{\min}$, $X_{\max}$, W, we have

$$c_r \cap W = \emptyset.$$

By the choice of $\bar{c}$ with respect to h, also $h[c_r] \cap h[W] = \emptyset$. □

The following proposition gives more information on the decompositions of the iterands.

Proposition 5.11. *We let* $\mathbb{Q}_{\mathrm{pure}} = (\mathrm{Fin}^\omega(\bar{\mathcal{R}}), \sqsubseteq^*)$ *and we let* $\underset{\sim}{\mathcal{U}} = \{\langle \bar{a}, \check{\bar{a}} \rangle : \bar{a} \in \mathbb{Q}_{\mathrm{pure}}\}$*. Then the following hold:*

(1) $\mathbb{Q}_{\mathrm{pure}}$ *is* ω*-closed.*

(2) $\mathbb{Q}_{\mathrm{pure}}$ *forces that* $\underset{\sim}{\mathcal{U}}$ *is a Milliken–Taylor ultrafilter with* $\min(\underset{\sim}{\mathcal{U}}) = \mathcal{R}_{\min}$ *and* $\max(\underset{\sim}{\mathcal{U}}) = \mathcal{R}_{\max}$*.*

(3) The map $(s, \bar{a}) \mapsto (\bar{a}, (s, \bar{a}))$ *is a dense embedding of the forcing* $\mathbb{M}(\bar{\mathcal{R}})$ *into the forcing* $\mathbb{Q}_{\mathrm{pure}} * \mathbb{M}(\underset{\sim}{\mathcal{U}})$*.*

(4) $\mathbb{Q}_{\mathrm{pure}}$ *forces that* $\Phi(\underset{\sim}{\mathcal{U}})$ *is nnc to any filter from the ground model that is nnc* $\mathcal{R}_x$*,* $x = \min, \max$*.*

Proof. (1) The forcing order $\mathbb{Q}_{\mathrm{pure}}$ is ω-closed by (2) of Theorem 5.2.

(2) Details can be found in [19, Proposition 3.2].

(3) By Theorem 5.2 and density arguments, the first component $\mathbb{Q}_{\mathrm{pure}}$ forces that the generic filter $\underset{\sim}{\mathcal{U}}$ is a Milliken–Taylor ultrafilter. Since $\mathbb{Q}_{\mathrm{pure}}$ does not add reals, only colourings and descending ω-sequences from the ground model have to be considered.

Statement (4) follows from Lemma 5.10, applied to $\mathbb{Q}_{\mathrm{pure}}$. □

Theorem 5.12 (Adaptation of [19, Theorem 4])**.** *Let $\mathcal{R}_{\min}$, $\mathcal{R}_{\max}$ be as above and assume that $\mathcal{E}$ is a P-point with $\mathcal{E} \not\geq_{RB} \mathcal{R}_{\min}, \mathcal{R}_{\max}$. Then $\mathcal{E}$ continues to generate an ultrafilter after we force with $\mathbb{M}(\bar{\mathcal{R}})$.*

Proof. Lemma 5.10 shows that $\mathbb{Q}_{\text{pure}}$ forces that $\Phi(\underset{\sim}{\mathcal{U}})$ avoids $\mathcal{E}$. Now the rest of the proof is just Eisworth's theorem Theorem 2.16. □

The following result is based on [11, Proposition 3.4]. Recall, f_μ is the name for the generic finite-to-one function $n \mapsto |n \cap \mu|$.

Theorem 5.13. *Assume* CH. *Let $\mathcal{E}$ be a P-point and let $\mathcal{W}$ be an ultrafilter over ω such that $\mathcal{W}$, $\mathcal{E} \not\geq_{RB} \mathcal{R}_x$ for $x = \min, \max$. Then*

$$\mathbb{M}(\bar{\mathcal{R}}) \Vdash_{\mathbb{M}(\bar{\mathcal{R}})} f_\mu(\mathcal{W}) = f_\mu(\mathcal{E}).$$

Proof. Since $f_\mu(\mathcal{E})$ is an ultrafilter in $\mathbf{V}^{\mathbb{M}(\bar{\mathcal{R}})}$, the equality will follow from

$$\mathbb{M}(\bar{\mathcal{R}}) \Vdash (\forall E \in \mathcal{E})(\exists W \in \mathcal{W}) f_\mu[W] \subseteq^* f_\mu[E].$$

Given $(s, \bar{a}) \in \mathbb{M}(\bar{\mathcal{R}})$ and $E \in \mathcal{E}$, we colour $[\mathrm{FU}(\bar{a})]^2_<$ with two colours, $c(t < t') = 0$ if $(\max(t), \min(t')) \cap E \neq \emptyset$ and $c(t < t') = 1$ otherwise. There is a monochromatic $\bar{b} \sqsubseteq \bar{a}$ in our space $\mathrm{Fin}(\bar{\mathcal{R}})$ and its colour is 0. By Lemma 5.10, there are $\bar{c} \sqsubseteq \bar{b}$, $\bar{c} \in (\mathrm{Fin})^\omega(\bar{\mathcal{R}}_\alpha)$, and $W \in \mathcal{W}$ such that $\mathrm{set}(\bar{c}) \cap W = \emptyset$. Now

$$(s, \bar{c}) \Vdash_{\mathbb{M}(\bar{\mathcal{R}}_\alpha)} f_\mu[W] \subseteq^* f_\mu[E].$$

□

Now we are concerned with the second iterand. The following follows from an easy density argument. We drop the tildes again.

Lemma 5.14. *For any $x = \min, \max$, we have*

$$\mathbb{M}(\bar{\mathcal{R}}) \Vdash \mathcal{R}_x \cup \{\mu\} \text{ is a filter subbase.}$$

Now we prove the parallel of Theorem 3.26.

Theorem 5.15. *We assume* CH.

$$\begin{aligned} \mathbb{M}(\bar{\mathcal{R}}) \Vdash & (\forall x \in \{\min, \max\}) \\ & (\mathrm{fil}(\mathcal{R}_x \cup \{\mu\}))^+ \text{ is a happy family} \\ & \text{that is nowhere almost a filter}). \end{aligned} \tag{5.2}$$

and hence by Mathias' [31, Prop. 0.11] *combined with* [4, proof of Theorem 14][6]

$$\begin{aligned} \mathbb{M}(\bar{\mathcal{R}}) \Vdash & (\exists \langle \mathcal{R}_x^{\text{ext}} : x = \min, \max \rangle) \Big(\bigwedge_{x = \min, \max} \big(\mathcal{R}_x^{\text{ext}} \supseteq (\mathcal{R}_x \cup \{\mu\}) \wedge \\ & \mathcal{R}_x^{\text{ext}} \text{ is a Ramsey ultrafilter that is nnc to } \mathcal{E} \big) \\ & \text{and the extensions are pairwise nnc} \Big). \end{aligned}$$

Proof. Theorem 5.2 allows us to transfer the proof of Theorem 3.26 to $\mathbb{M}(\bar{\mathcal{R}})$. $\square_{5.15}$

We rework Section 4, the iteration theory for limit steps, for iterands of the form $\mathbb{M}(\bar{\mathcal{R}})$. We choose under CH two nnc Ramsey ultrafilters $\mathcal{R}_{\min,0}$, $\mathcal{R}_{\max,0}$

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[6] Alternatively, one can force with $(\mathrm{fil}(\mathcal{R}_{\min} \cup \{\mu\}))^+, \subseteq^*) \times (\mathrm{fil}(\mathcal{R}_{\max} \cup \{\mu\}))^+, \subseteq^*)$ and get a generic pair of nnc Ramsey ultrafilters $\underset{\sim}{\bar{\mathcal{R}}}^{\text{ext,gen}} \supset \bar{\mathcal{R}}$.

in the ground model and let $\bar{\mathcal{R}}_0 = (\mathcal{R}_{\min,0}, \mathcal{R}_{\max,0})$. We define by induction on $\alpha \leq \omega_2$ a countable support iteration (in the sense of [39, Definition III, 3.1]) $\mathbb{P}_\alpha = \langle \mathbb{P}_\beta, \mathbb{M}(\bar{\mathcal{R}}_\gamma) : \beta \leq \alpha, \gamma < \alpha\rangle$ and a sequence $\langle \bar{\mathcal{R}}_\beta, \mu_\gamma : \beta \leq \alpha, \gamma < \alpha\rangle$ of names such that for any $\beta < \alpha$,

$$\begin{aligned} \mathbb{P}_{\beta+1} =& \mathbb{P}_\beta * \mathbb{M}(\bar{\mathcal{R}}_\beta) \\ \mathbb{P}_\beta \Vdash & \bigwedge_{x=\min,\max} \big(\mathcal{R}_{x,\beta} \supseteq \bigcup \{\mathcal{R}_{x,\gamma} \cup \{\mu_\gamma\} : \gamma < \beta\} \text{ and} \\ & \qquad \mathcal{R}_{x,\beta} \text{ is a Ramsey ultrafilter that is nnc to } \mathcal{E}\big) \\ & \text{and } \mathcal{R}_{\min,\beta} \text{ is nnc to } \mathcal{R}_{\max,\beta}. \end{aligned} \tag{5.3}$$

We use for names the same letters as for the corresponding evaluated names.

Here $\mathbb{P}_{\beta+1}$ forces that μ_β is the $\mathbb{M}(\bar{\mathcal{R}}_\beta)$-generic real. In Theorem 5.15 we proved that there are extensions in the successor steps. This guarantees the existence of $\mathcal{R}_{\min,\beta+1}$, $\mathcal{R}_{\max,\beta+1}$ with the desired properties.

Now we consider limit steps α. If $\mathrm{cf}(\alpha) > \omega$, we can just take $\mathbb{P}_\alpha \Vdash \mathcal{R}_{x,\alpha} = \bigcup_{\gamma<\alpha} \mathcal{R}_{x,\gamma}$ and the inductive hypotheses will be carried on, since in proper forcing every real appears at a step of at most countable cofinality, with the only exception that for $\alpha = \aleph_2$ the CH gets lost. So again we concentrate on the hard case, $\mathrm{cf}(\alpha) = \omega$. The statement

$$\begin{aligned} \mathbb{P}_\alpha \Vdash (\exists \bar{\mathcal{R}}_\alpha)\Big(& \bigwedge_x \big(\mathcal{R}_{x,\alpha} \supseteq \bigcup \{\mathcal{R}_{x,\beta} \cup \{\mu_\beta\}) : \beta < \alpha\} \text{ and} \\ & \qquad \mathcal{R}_{x,\alpha} \text{ is nnc to } \mathcal{E}\big) \\ & \text{and } \mathcal{R}_{\min,\alpha} \text{ is nnc to } \mathcal{R}_{\max,\alpha}\Big) \end{aligned} \tag{5.4}$$

will follow from the CH, a routine enumeration according to [31, Prop. 0.11] and [4, proof of Theorem 14] the following theorem:

Theorem 5.16. *We start with a ground model with* CH. *Suppose that* $\alpha < \omega_2$, $\mathrm{cf}(\alpha) = \omega$, *and that* $\mathbb{P}_\beta, \bar{\mathcal{R}}_\beta$, μ_β, $\beta < \alpha$, *are as in Equation* (5.3) *and* $\mathbb{P}_\alpha$ *is the countable support limit of* $\langle \mathbb{P}_\beta, \mathbb{M}(\bar{\mathcal{R}}_\beta) : \beta < \alpha\rangle$. *In* $\mathbf{V}^{\mathbb{P}_\alpha}$, *for any* $x = \min, \max$, *the set of positive sets*

$$\big(\bigcup_{\gamma<\alpha} (\mathcal{R}_{x,\gamma} \cup \{\mu_\gamma\})\big)^+$$

forms a happy family that is nowhere almost a filter.

As in the proof of Theorem 4.1 we introduce an increasing sequence $\langle R_\gamma : \gamma < \alpha\rangle$ of relations R_α in $\mathbf{V}^{\mathbb{P}_\alpha}$ such that a property called

$$\text{“}\mathbb{P}_\alpha \text{ is } R_\alpha\text{-preserving”} \tag{5.5}$$

is carried in addition to the property (5.3) and properness in the inductive choice of the iteration.

Again we define a $\mathbb{P}_\alpha$-name of a relation R_α and are concerned with statements of the form $(\forall f)(\exists g)(f R_\alpha g)$ for a Borel relation R_α on the Baire space in $\mathbf{V}^{\mathbb{P}_\alpha}$. This time the requirement on positivity on the domain and on the range of the relation contains the sequence of names $\langle \bar{\mathcal{R}}_\beta : \beta < \alpha\rangle$ from the ground model as a parameter.

Definition 5.17. By induction on $\alpha \leq \omega_2$ we define the following relations.

(1) Let $\mathbb{P}_\alpha = \langle \mathbb{P}_\beta, \mathbb{M}(\bar{\mathcal{R}}_\gamma) : \beta \leq \alpha, \gamma < \alpha \rangle$ be defined with (5.3). Let $x \in \{\min, \max\}$. We say that a $\mathbb{P}_\alpha$-name for a pair $g = (g_{\min}, g_{\max})$ of elements $g_x \in [\omega]^\omega$ is *α-min-max-positive* if for $x = \min, \max$, $1 \Vdash_{\mathbb{P}_\alpha} g_x \in (\bigcup\{\mathrm{fil}(\mathcal{R}_{x,\gamma} \cup \{\mu_\gamma\}) : \gamma < \alpha\})^+$.

(2) Let $\mathbb{P}_\alpha = \langle \mathbb{P}_\beta, \mathbb{M}(\bar{\mathcal{R}}_\gamma) : \beta \leq \alpha, \gamma < \alpha \rangle$ be defined with (5.3). We say $f R_\alpha g$ if the following holds in $\mathbf{V}^{\mathbb{P}_\alpha}$:
   (a) $f = (\bar{A}, h, x)$,
   (b) $\bar{A} = \langle (A_{\min,\ell}, A_{\max,\ell}) : \ell \in \omega \rangle$ is a $\subseteq^2$-descending sequence of α-min-max-positive members of $[\omega]^\omega$,
   (c) h is finite-to-one,
   (d) for $j = 0, 1$ there are $(g^j_{\min}, g^j_{\max}) \in ([\omega]^\omega)^2$, with the following properties:
      (i) g^j is an α-min-max-positive,
      (ii) for $x = \min, \max$, g^j_x is a diagonal lower bound of $\langle A_{x,\ell} : \ell < \omega \rangle$ and
      (iii) for $x = \min, \max$, $h[g^0_x] \cap h[g^1_x] = \emptyset$.

Definition 5.18. We say $\mathbb{P}_\alpha$ is R_α-preserving if $\mathbb{P}_\alpha$ is proper and

$$\mathbb{P}_\alpha \Vdash (\forall f \in \mathrm{dom}(R_\alpha))(\exists g)(f R_\alpha g).$$

Lemma 5.19. *We assume $\alpha < \omega_2$ and $\mathrm{cf}(\alpha) = \omega$. If $\mathbb{P}_\alpha = \langle \mathbb{P}_\beta, \mathbb{M}(\bar{\mathcal{R}}_\gamma) : \beta \leq \alpha, \gamma < \alpha \rangle$ is R_α-preserving then*

$$\mathbb{P}_\alpha \Vdash (\bigcup\{\mathrm{fil}(\mathcal{R}_{x,\beta} \cup \{\mu_\beta\}) : \beta < \alpha\})^+$$
$$\textit{is a happy family that is nowhere almost a filter}).$$

Proof. This follows from Definition 5.18. □

As mentioned a routine enumeration gives:

Proposition 5.20. *If $\mathbb{P}_\alpha \Vdash$ CH and $\mathbb{P}_\alpha$ is R_α-preserving then statement* (5.4) *holds.*

We carry the preservation property upwards by induction.

Lemma 5.21. *Assume* CH, *$\alpha < \omega_2$, $\mathrm{cf}(\alpha) = \omega$. Let $\mathbb{P}_\alpha$ be the countable support limit of $\mathbb{P}_\beta$, $\beta < \alpha$. If for $\beta < \alpha$ such that $\mathrm{cf}(\beta) < \omega_1$, $\mathbb{P}_\beta$ is R_β-preserving and for any $\beta < \alpha$,* (5.3) *holds then $\mathbb{P}_\alpha$ is R_α-preserving.*

This is proved similarly to Lemma 4.6.
This finishes the proof of Theorem 5.16. $\square_{5.16}$

Now there are two new forcing extensions.

Theorem 5.22. *Assume* CH *and fix two pairwise nnc Ramsey ultrafilters $\mathcal{R}_{\min,0}$, $\mathcal{R}_{\max,0}$. Then there is a countable support iteration of proper iterands*

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{M}(\bar{\mathcal{R}}_\beta) : \beta < \omega_2, \alpha \leq \omega_2 \rangle$$

such that in the extension there are exactly three near-coherence classes of ultrafilters. Namely, one class is represented by a P-point of character ω_1 and two other classes represented by Ramsey ultrafilters $\mathcal{R}_{\min,\omega_2}$, $\mathcal{R}_{\max,\omega_2}$ of character $\aleph_2$ that are *not* $P_{\aleph_2}$*-points. The sequence of Ramsey ultrafilters* $\langle \mathcal{R}_{\min,\alpha} : \alpha \leq \omega_2 \rangle$ *is increasing in* α *and the same holds for the* $\mathcal{R}_{\max,\alpha}$.

Proof. We start with a ground model $\mathbf{V}$ that fulfils CH. By CH there are a P-point $\mathcal{E}$ and two Ramsey ultrafilters $\mathcal{R}_{\min,0}$, $\mathcal{R}_{\max,0}$ such that any two of $\mathcal{E}$, $\mathcal{R}_{\min,0}$, $\mathcal{R}_{\max,0}$ are nnc. A countable support iteration $\langle \mathbb{P}_\beta, \mathbb{M}(\bar{\mathcal{R}}_\gamma) : \beta \leq \omega_2, \gamma < \omega_2 \rangle$ of proper forcings with (5.3) at any stage β forces: There are at least the near-coherence classes of $\mathcal{E}$, $\mathcal{R}_{\min,\omega_2}$, $\mathcal{R}_{\max,\omega_2}$.

Why are there no other near-coherence classes in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$? For a contradiction, we suppose that $\mathcal{W}$ is a non-principal ultrafilter in $\mathbf{V}[G]$ for some $\mathbb{P}_{\omega_2}$-generic filter G and $\mathcal{W}$ is nnc to any $\mathcal{R}_{x,\omega_2}$, $x = \min, \max$ and nnc to $\mathcal{E}$. Then by [11, Lemma 5.10] there are a $p \in \mathbb{P}_{\omega_2} \cap G$, say $p \in \mathbb{P}_{\alpha_0}$ for an $\alpha_0 \in \omega_2$ (by [11, Lemma 5.6]), and an ω_1-club $C \subseteq [\alpha_0, \omega_2)$ such that for each $\alpha \in C \cup \{\omega_2\}$, $\mathcal{W} \cap \mathbf{V}[G \cap \mathbb{P}_\alpha]$ has a $\mathbb{P}_\alpha$-name (for which we again write $\mathcal{W} \cap \mathbf{V}[G \cap \mathbb{P}_\alpha]$) and

$$p \Vdash_{\mathbb{P}_\alpha} \mathcal{W} \cap \mathbf{V}[G \cap \mathbb{P}_\alpha] \text{ is an ultrafilter that is nnc to } \mathcal{R}_{x,\alpha} \text{ and nnc to } \mathcal{E}.$$

We fix $\alpha = \min(C)$.

By Theorem 5.13 and Theorem 5.12 we have for any $\alpha \in C$,

$$p \Vdash_{\mathbb{P}_{\alpha+1}} f_{\mathrm{supp}(\mu_\alpha)}(\mathcal{W} \cap \mathbf{V}[G \cap \mathbb{P}_\alpha]) = f_{\mathrm{supp}(\mu_\alpha)}(\mathcal{E}) \wedge \mathcal{E} \text{ generates an ultrafilter.}$$

Since

$$p \Vdash_{\mathbb{P}_{\omega_2}} \mathcal{W} \text{ is an ultrafilter and } \mathcal{W} \supseteq \mathcal{W} \cap \mathbf{V}[G \cap \mathbb{P}_\alpha],$$

we have $p \Vdash_{\mathbb{P}_{\omega_2}} f_{\mathrm{supp}(\mu_\alpha)}(\mathcal{W}) = f_{\mathrm{supp}(\mu_\alpha)}(\mathcal{E})$. □

Theorem 5.23. *Under* CH *there is an iteration* $\langle \mathbb{P}_\beta, \mathbb{Q}_\alpha : \beta \leq \omega_2, \alpha < \omega_2 \rangle$ *that forces that there are exactly two near-coherence classes and there is no Milliken–Taylor ultrafilter.*

Proof. We fix a $\diamondsuit(S^{\aleph_2}_{\aleph_1})$-sequence $\langle D_\alpha : \alpha \in S^{\aleph_2}_{\aleph_1} \rangle$ in the ground model or add one by a preliminary forcing that preserves CH.

For $\alpha < \aleph_2$ of countable cofinality, we choose $\mathbb{Q}_\alpha = \mathbb{M}(\bar{\mathcal{R}}_\alpha)$, such that top ultrafilters $\bar{\mathcal{R}}_\alpha$ are chosen as in Theorem 5.15 and Theorem 5.16.

Now for $\alpha < \aleph_2$ with $\mathrm{cf}(\alpha) = \aleph_1$, we choose $\mathbb{Q}_\alpha = \mathbb{M}(\bar{\mathcal{R}}_\alpha)$ as follows. The Ramsey ultrafilter $\mathcal{R}_{\min,\alpha}$ extends the ultrafilters $\mathcal{R}_{\min,\beta}$, $\beta < \alpha$ according to Theorem 5.15 and Theorem 5.16.

The other top ultrafilter $\mathcal{R}_{\max,\alpha}$ is varied according to a $\diamondsuit(S^{\aleph_2}_{\aleph_1})$ sequence: Suppose the diamond hands down a name D_α that is a name for an ultrafilter that is nnc to $\mathcal{E}$ and nnc to $\mathcal{R}_{\min,\alpha}$. Then $\mathcal{R}_{\max,\alpha}$ is forced to be some Ramsey ultrafilter that is nnc to the ultrafilter D_α, to $\mathcal{R}_{\min,\alpha}$ and to $\mathcal{E}$. [7] According to Theorem 5.13 in stage $\mathbf{V}^{\mathbb{P}_{\alpha+1}}$ we have that D_α is nearly coherent to $\mathcal{E}$. Thus the existence of a third near-coherence class is prevented. If $\mathcal{W} \in \mathbf{V}^{\mathbb{P}_{\omega_2}}$ extends D_α, then in the final model $\mathcal{W}$ is nearly coherent to $\mathcal{E}$ via f_{μ_α}. At stages α of uncountable cofinality and below conditions where the diamond does not hand down an ultrafilter that is nnc to $\mathcal{E}$ and nnc to $\mathcal{R}_{\min,\alpha}$, or at stages of countable cofinality or successor stages we choose for $\mathcal{R}_{\max,\alpha}$ any Ramsey ultrafilter that is nnc to any of the $\mathcal{R}_{\min,\alpha}$ and nnc to $\mathcal{E}$. Alternatively we can define $\mathbb{Q}_\alpha$ for these α's by working in $\mathbf{V}^{\mathbb{P}_\alpha}$ with

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[7] Details about book-keeping functions that are defined from diamond sequences can for example be found in [32, Section 2].

the space $(\mathrm{Fin})^\omega(\mathcal{R}_{\min,\alpha})$ which is defined by dropping the condition on $\max(s)$ in Equation (5.1).

Since for any $\mathcal{P}_{\omega_2}$-name τ for an ultrafilter there are stationarily many $\alpha \in S^{\aleph_2}_{\aleph_1}$ such that $\tau \cap \mathbf{V}^{\mathbb{P}_\alpha} = D_\alpha$ is an ultrafilter at stage α, any ultrafilter $\mathcal{W}$ in the $\mathbb{P}_{\omega_2}$-extension is nearly coherent to $\mathcal{E}$ or nearly coherent to some of the $\mathcal{R}_{\min,\omega_2}$.

If there were a Milliken–Taylor ultrafilter in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$, say $\mathcal{U}$, its minimum or its maximum projection would be nearly coherent to $\mathcal{E}$, since there are only two classes. However, for any finite-to-one h, the projections $h(\min(\mathcal{U}))$, $h(\max(\mathcal{U}))$ are both rapid ultrafilters (that means the enumeration functions of the ultrafilter members form a $\leq^*$-dominating family), and $h(\mathcal{E})$ is not. This contradiction shows that there is no Milliken–Taylor ultrafilter. $\square$

As mentioned in the introduction, for exactly one class there is more knowledge: For $n = 1$ we can take the Blass–Shelah model [11], the Miller model [12], or the Matet model [7]. Each can be modified by Sacks iterands or by mixing Blass–Shelah, Miller and Matet iterands. The expression "(forcingname)-model" is jargon, it does not mean a fixed model, it rather means the notion of forcing defined as a csi of (forcingname) of length $\aleph_2$.

6. A GENERIC MODEL AND OPEN QUESTIONS

Now we return to the forcings of Theorem 4.9 and modify at stages $\alpha < \aleph_2$ with $\mathrm{cf}(\alpha) \leq \omega$ the choice of $\mathcal{U}_\alpha \supseteq \bigcup\{\mathcal{U}_\beta \restriction \mu_\beta \, : \, \beta < \alpha\}$. We replace the downwards $\aleph_1$-long inductive choice that was used in Theorem 3.3 and in the limits of countable cofinality and is modelled after [6, Theorem 2.4] by a generic choice. At stage α the set of α-positive sequences is Matet-adequate, we order it by $\sqsubseteq^*$ and get a countably closed forcing notion.

Definition 6.1. We let $\mathbb{P}^{\mathrm{gen}}_\alpha$ be defined by induction. $\mathbb{P}^{\mathrm{gen}}_0$ is the trivial forcing $\{1\}$. We define in $\mathbf{V}^{\mathbb{P}^{\mathrm{gen}}_\alpha}$,

$$\mathcal{H}_\alpha = \{\bar{a} \in (\mathrm{Fin})^\omega \, : \, \bar{a} \text{ is } \alpha\text{-positive}\}.$$

(α-positive is defined in Definition 4.3(1), we replace $\mathbb{P}_\alpha$ by $\mathbb{P}^{\mathrm{gen}}_\alpha$ there).

We let $\mathbb{Q}_{\mathrm{pure},\alpha} = (\mathcal{H}_\alpha, \sqsubseteq^*)$ and let $\mathcal{U}^{\mathrm{gen}}_\alpha$ be a name for the $\mathbb{Q}_{\mathrm{pure},\alpha}$-generic filter. So we have

$$\mathbb{Q}^{\mathrm{gen}}_\alpha = \mathbb{Q}_{\mathrm{pure},\alpha} * \mathbb{M}(\mathcal{U}^{\mathrm{gen}}_\alpha),$$

and by known decomposition results, the forcings $\mathbb{M}(\mathcal{H}_\alpha)$ and $\mathbb{Q}^{\mathrm{gen}}_\alpha$ are equivalent. The successor step is $\mathbb{P}^{\mathrm{gen}}_{\alpha+1} = \mathbb{P}^{\mathrm{gen}}_\alpha * \mathbb{Q}^{\mathrm{gen}}_\alpha$. In the limit steps α, we let $\mathbb{P}^{\mathrm{gen}}_\alpha$ the countable support limit of $\langle \mathbb{P}^{\mathrm{gen}}_\beta, \mathbb{Q}^{\mathrm{gen}}_\beta \, : \, \beta < \alpha\rangle$.

Like in Sections 3 and 4 one verifies that

$$\mathbb{P}^{\mathrm{gen}}_\alpha \Vdash \mathcal{H}_\alpha \text{ is Matet adequate.}$$

Hence $\mathbb{Q}^{\mathrm{gen}}_\alpha = \mathbb{M}(\mathcal{H}_\alpha)$ and $\mathbb{P}^{\mathrm{gen}}_\alpha * \mathbb{Q}_{\mathrm{pure},\alpha}$ forces that $\mathcal{U}^{\mathrm{gen}}_\alpha$ is a Milliken–Taylor ultrafilter. In limit steps $\alpha \leq \omega_2$ of uncountable cofinality, the family $\mathcal{H}_\alpha = \mathcal{U}^{\mathrm{gen}}_\alpha$ is a Milliken–Taylor ultrafilter. Therefore $\mathbb{P}^{\mathrm{gen}}_{\omega_2}$ also forces, like the forcing used in Theorem 4.9, that there are at least three near-coherence classes and a Milliken–Taylor ultrafilter. There is the following parallel to Theorem 5.13.

Lemma 6.2. *Assume* CH*. We fix a P-point* $\mathcal{E}$ *in the ground model. Let* $\alpha < \omega_2$*. Then*

$$\mathbb{P}^{\rm gen}_{\alpha+1} \Vdash (\forall \textit{ ultrafilters } \mathcal{W} \in \mathbf{V}^{\mathbb{P}^{\rm gen}_\alpha})(\Phi(\mathcal{U}^{\rm gen}_\alpha) \leq_{\rm RB} \mathcal{W} \vee f_\mu(\mathcal{W}) = f_\mu(\mathcal{E})).$$

Proof. We choose a name $\mathcal{W}$ of an ultrafilter in $\mathbf{V}^{\mathbb{P}^{\rm gen}_\alpha}$.

We argue now in $\mathbf{V}^{\mathbb{P}^{\rm gen}_\alpha * \mathbb{Q}_{{\rm pure},\alpha}}$. We assume that $\Phi(\mathcal{U}^{\rm gen}_\alpha) \not\leq_{\rm RB} \mathcal{W}$. Since $f_{\mu_\alpha}(\mathcal{E})$ is still an ultrafilter, the equality will follow from

$$\mathbb{M}(\mathcal{U}^{\rm gen}_\alpha) \Vdash (\forall E \in \mathcal{E})(\exists W \in \mathcal{W}) f_{\mu_\alpha}[W] \subseteq^* f_{\mu_\alpha}[E].$$

Given $(s, \bar{a}) \in \mathbb{M}(\mathcal{U}^{\rm gen}_\alpha)$ and $E \in \mathcal{E}$, we colour $[{\rm FU}(\bar{a})]^2_<$ with two colours, $c(t < t') = 0$ if $(\max(t), \min(t')) \cap E \neq \emptyset$ and $c(t < t') = 1$ otherwise. There is a monochromatic $\bar{b} \sqsubseteq \bar{a}$ in our space $\mathcal{U}^{\rm gen}_\alpha$ and its colour is 0. Since $\Phi(\mathcal{U}^{\rm gen}_\alpha) \not\leq_{\rm RB} \mathcal{W}$, there are a sequence $\bar{c} \sqsubseteq \bar{b}$, $\bar{c} \in \mathcal{U}^{\rm gen}_\alpha$, and a set $W \in \mathcal{W}$ such that ${\rm set}(\bar{c}) \cap W = \emptyset$. Now

$$(s, \bar{c}) \Vdash_{\mathbb{M}(\mathcal{U}^{\rm gen}_\alpha)} f_{\mu_\alpha}[W] \subseteq^* f_{\mu_\alpha}[E].$$

□

The forcing $\mathbb{P}^{\rm gen}_{\omega_2}$ also forces, like Theorem 4.9, that there are at least three near-coherence classes and an Milliken–Taylor ultrafilter. We do not know the exact number of near coherence classes in $\mathbf{V}^{\mathbb{P}^{\rm gen}_{\omega_2}}$. The choice of $\mathcal{U}_{\alpha+1}$ from the proof of Theorem 3.3 can be performed so that the parallel of Lemma 6.2 holds as well for the forcings and the names from models of Theorem 4.9. A negative answer to one of the following questions would give exactly three classes and a Milliken–Taylor ultrafilter.

Question 6.3. *Suppose that* $\alpha < \omega_2$ *and* ${\rm cf}(\alpha) = \omega_1$*.*

(1) *In* $\mathbf{V}^{\mathbb{P}^{\rm gen}_\alpha}$ *the union* $\mathcal{U}^{\rm gen}_{<\alpha} = \mathcal{U}^{\rm gen}_\alpha$ *of* $\mathcal{U}^{\rm gen}_\beta$*,* $\beta < \alpha$ *is a Milliken–Taylor ultrafilter. Let* $\mathcal{E}$ *be a fixed P-point from the ground model. Is there an ultrafilter in* $\mathcal{W} \in \mathbf{V}^{\mathbb{P}^{\rm gen}_\alpha}$ *that is not nearly coherent to* $\mathcal{E}$*, not nearly coherent to* $\min(\mathcal{U}^{\rm gen}_\alpha)$ *and not nearly coherent to* $\max(\mathcal{U}^{\rm gen}_\alpha)$ *but* $\mathcal{W} \supseteq \Phi(\mathcal{U}^{\rm gen}_\alpha)$*?*

(2) *Same question for* $\mathbb{P}_\alpha$*,* $\mathcal{U}_\alpha$ *from Theorem 4.9.*

Note that the parallels of Lemma 5.10, Theorem 5.13, Theorem 5.12, and Theorem 5.22, which would give a negative answer to the questions, would break down. The search for a $\bar{b}$ in Lemma 5.10 does not work for the iterations of $\mathbb{M}(\mathcal{U}_\alpha)$ in the case of $\Phi(\mathcal{U}_\alpha) \leq_{\rm RB} \mathcal{W}$, and the same holds for iterations with $\mathcal{U}^{\rm gen}_\alpha$.

More generally, we ask:

Question 6.4. *How many near-coherence classes are in the model* $\mathbf{V}^{\mathbb{P}^{\rm gen}_{\omega_2}}$ *among the ultrafilters* $\mathcal{W} \supseteq \Phi(\mathcal{U}^{\rm gen}_{\omega_2})$*?*

There is a complementary question about the model from Theorem 5.22.

Question 6.5. *Is there a Milliken–Taylor ultrafilter in any of the models of Theorem 5.22?*

Necessarily, such a Milliken–Taylor ultrafilter $\mathcal{U}$ would have to satisfy $\min(\mathcal{U}) = \mathcal{R}_{\min,\omega_2}$ and $\max(\mathcal{U}) = \mathcal{R}_{\max,\omega_2}$.

Question 6.6. *For* $n \geq 4$*, is it consistent relative to* ZFC *to have exactly* n *near-coherence classes?*

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Heike Mildenberger, Albert-Ludwigs-Universität Freiburg, Mathematisches Institut, Abteilung für math. Logik, Ernst–Zermelo–Strasse 1, 79104 Freiburg im Breisgau, Germany

Email address: `heike.mildenberger@math.uni-freiburg.de`