



Cardinal characteristics for Menger-bounded subgroups

Heike Mildenerger

Universität Wien, Kurt Gödel Research Center for Mathematical Logic, Währinger street 25, 1090 Vienna, Austria

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ABSTRACT

Machura, Shelah and Tsaban showed in [M. Machura, S. Shelah, B. Tsaban, Squares of Menger-bounded groups, *Trans. Amer. Math. Soc.*, in press, <http://arxiv.org/pdf/math.GN/0611353>, 2007] that under the condition, that a relative $\mathfrak{d}'(\mathcal{P})$ of the dominating number is at least \mathfrak{d} , there are subgroups of the Baer–Specker group whose k th power is Menger-bounded and whose $(k+1)$ st power is not. We show that the sufficient condition implies $\mathfrak{r} \geq \mathfrak{d}$ and indeed can be replaced by $\mathfrak{r} \geq \mathfrak{d}$. This result includes an affirmative answer to a question by Tsaban on a possibly weaker still sufficient condition. We show that it is consistent relative to ZFC that $\mathfrak{g} \leq \mathfrak{r} < \mathfrak{d}$ and there are subgroups of the Baer–Specker group whose k th power is Menger-bounded and whose $(k+1)$ st power is not.

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1. Introduction and some estimates

Machura, Shelah and Tsaban [12] showed that under the condition, that a relative $\mathfrak{d}'(\mathcal{P})$ of the dominating number is at least \mathfrak{d} , for every k there are groups $G \subseteq \mathbb{Z}^\omega$ whose k th power is Menger-bounded and whose $(k+1)$ st power is not Menger-bounded. The aim of this note is to give more information on the strength of this premise. We show that it implies $\mathfrak{r} \geq \mathfrak{d}$, that the possibly weaker $\mathfrak{r} \geq \mathfrak{d}$ is a sufficient condition as well, and that $\mathfrak{r} \geq \mathfrak{d}$ is not a necessary condition.

First we recall some definitions.

Definition 1.1. The *Baer–Specker group* is \mathbb{Z}^ω with pointwise addition. Let $G \subseteq \mathbb{Z}^\omega$ be a subgroup. For $g : \omega \rightarrow \mathbb{Z}$, we write $\hat{g}(n) = \max\{|g(m)| : m \leq n\}$. Let $k \in \omega \setminus \{0\}$. We say “ G^k is Menger-bounded” or “ G has Menger-bounded k th power” iff

$$(\exists f \in {}^\omega \omega) (\forall F \in [G]^k) (\exists^\infty n) (\forall g \in F) (\hat{g}(n) \leq f(n)).$$

This is syntactically the simplest of the equivalent characterisations given in [12, Theorem 5]. Menger-boundedness in a broader sense is defined for topological groups and also called o -boundedness. We refer the reader to [2] for more information.

Now we recall the definitions of the possibly new family of cardinal characteristics $\mathfrak{d}'(\mathcal{P})$ from [12] and of some relatives. A function from the natural numbers into the natural numbers is called weakly increasing if for all $n < m$, $f(n) \leq f(m)$. The set of all weakly increasing functions is denoted by $\omega^{\uparrow\omega}$. The set of all infinite subsets of ω is denoted by $[\omega]^\omega$. The quantifier \exists^∞ means “there are infinitely many” and the dual quantifier \forall^∞ means “for all but finitely many”.

E-mail address: heike@logic.univie.ac.at.

Definition 1.2.

(1) Let $\mathcal{P} = \{A_n : n < \omega\}$ be a partition of ω into infinite sets. We call a family $\mathcal{F} \subseteq \omega^{\uparrow\omega}$ good for $\mathfrak{d}_*(\mathcal{P})$ iff

$$(\forall h \in \omega^{\uparrow\omega}) (\exists A \in \mathcal{P}) (\exists f \in \mathcal{F}) (\forall^\infty n \in A) (f(h(n)) \geq h(n+1)).$$

We let

$$\mathfrak{d}_*(\mathcal{P}) = \min\{|\mathcal{F}| : \mathcal{F} \text{ is good for } \mathfrak{d}_*(\mathcal{P})\}.$$

(2) Let $A \in [\omega]^\omega$. We let

$$\mathfrak{d}_*(A) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^{\uparrow\omega} \wedge (\forall h \in \omega^{\uparrow\omega}) (\exists f \in \mathcal{F}) (\forall^\infty n \in A) (f(h(n)) \geq h(n+1))\}.$$

(3) Let $\mathcal{P} = \{A_n : n < \omega\}$ be a partition of ω into infinite sets such that for every n there are infinitely many i such that $i, i+1 \in A_n$. We call a family $\mathcal{F} \subseteq \omega^{\uparrow\omega}$ good for $\mathfrak{d}'(\mathcal{P})$ iff

$$(\forall h \in \omega^{\uparrow\omega}) (\exists A \in \mathcal{P}) (\exists f \in \mathcal{F}) (\forall^\infty n \in A) (f(h(n)) \geq h(n+1) \vee f(h(n+1)) \geq h(n+2) \vee n+1 \notin A).$$

We let

$$\mathfrak{d}'(\mathcal{P}) = \min\{|\mathcal{F}| : \mathcal{F} \text{ is good for } \mathfrak{d}'(\mathcal{P})\}.$$

(4) Let $A \in [\omega]^\omega$ be such that $(\exists^\infty i) (i, i+1 \in A)$.

$$\mathfrak{d}'(A) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^{\uparrow\omega} \wedge (\forall h \in \omega^{\uparrow\omega}) (\exists f \in \mathcal{F}) (\forall^\infty n \in A) (f(h(n)) \geq h(n+1) \vee f(h(n+1)) \geq h(n+2) \vee n+1 \notin A)\}.$$

Machura, Shelah and Tsaban's sufficient condition for the existence of subgroups of \mathbb{Z}^ω whose k th power is Menger-bounded but whose $(k+1)$ st power is not, is the following:

There is a partition $\mathcal{P} = \{A_n : n < \omega\}$ of ω into infinite sets such that

$$\text{for every } n \text{ there are infinitely many } i \text{ with } i, i+1 \in A_n \text{ and } \mathfrak{d}'(\mathcal{P}) \geq \mathfrak{d}. \tag{1.1}$$

There are numerous questions about modifications, e.g., we could also replace $\omega^{\uparrow\omega}$ by the set of all strictly increasing functions in the second appearance. We do not know whether the analogously defined cardinals might drop.

Some estimates for the cardinals are known: In [12] it is shown that for all \mathcal{P} that meet the conditions,

$$\max(\text{cov}(\mathcal{M}), \mathfrak{b}) \leq \mathfrak{d}'(\mathcal{P}) \leq \mathfrak{d}.$$

For the definitions of the cardinal characteristics \mathfrak{d} , $\text{cov}(\mathcal{M})$, \mathfrak{u} , \mathfrak{r} , \mathfrak{g} and of “groupwise dense” we refer the reader to Blass' handbook article [8].

In the International Conference on Set-Theoretic Topology in Kielce in August 2006 Tsaban asked whether the syntactically simpler family of cardinals $\mathfrak{d}_*(\mathcal{P})$ (see Definition 1.1(1)) enjoys similar properties. We do not know whether the cardinals do coincide, nor whether $\mathfrak{r} \geq \mathfrak{d}$ implies $(\exists \mathcal{P}) (\mathfrak{d}'(\mathcal{P}) = \mathfrak{d})$, however, we have the following main results.

Theorem 1.3. For every partition \mathcal{P} into infinitely many infinite sets we have $\mathfrak{d}_*(\mathcal{P}) = \min(\mathfrak{d}, \mathfrak{r})$.

Hence $\mathfrak{r} \geq \mathfrak{d}$ is equivalent to $(\exists \mathcal{P}) (\mathfrak{d}_*(\mathcal{P}) = \mathfrak{d})$ and to $(\forall \mathcal{P}) (\mathfrak{d}_*(\mathcal{P}) = \mathfrak{d})$. Nevertheless we still formulate the following theorem with the help of $\mathfrak{d}_*(\mathcal{P})$. The condition $\mathfrak{d}_*(\mathcal{P}) \geq \mathfrak{d}$ (combined with $\mathfrak{d}_*(\mathcal{P}) \leq \mathfrak{r}$) is handier for the construction than working with $\mathfrak{r} \geq \mathfrak{d}$.

Theorem 1.4. “There is a partition $\mathcal{P} = \{A_\ell : \ell \in \omega\}$ into infinite sets such that $\mathfrak{d}_*(\mathcal{P}) = \mathfrak{d}$ ” is a sufficient condition for the existence of subgroups of \mathbb{Z}^ω whose k th power is Menger-bounded but whose $(k+1)$ st power is not.

Corollary 1.5. $\mathfrak{r} \geq \mathfrak{d}$ is a sufficient condition for the existence of subgroups of \mathbb{Z}^ω whose k th power is Menger-bounded but whose $(k+1)$ st power is not.

In Section 2 we investigate the influence of \mathcal{P} , in Section 3 we show that $\mathfrak{d}_*(\mathcal{P}) \leq \mathfrak{r}$, in Section 4 we prove Theorem 1.3, in Section 5 we prove Theorem 1.4, and in the final section we show that $\mathfrak{r} \geq \mathfrak{d}$ is not a necessary condition, and we discuss some open questions.

2. The influence of the partition \mathcal{P}

It will be very convenient to know that $\mathfrak{d}_*(\mathcal{P})$ does not depend on \mathcal{P} , and even better, for every $\mathcal{F} \subseteq \omega^{\uparrow\omega}$ we have that \mathcal{F} is good for $\mathfrak{d}_*(\mathcal{P})$ iff it is good for any other $\mathfrak{d}_*(\mathcal{P}')$.

Proposition 2.1. *Let \mathcal{P} and \mathcal{P}' be partitions of ω into infinitely many infinite sets. For every $\mathcal{F} \subseteq \omega^{\uparrow\omega}$ we have that \mathcal{F} is good for $\mathfrak{d}_*(\mathcal{P})$ iff it is good for $\mathfrak{d}_*(\mathcal{P}')$. So $\mathfrak{d}_*(\mathcal{P})$ does not depend on the choice of \mathcal{P} .*

Proof. Let $\mathcal{P} = \{A_n : n \in \omega\}$ and $\mathcal{P}' = \{A'_n : n \in \omega\}$ be given. We show that $\mathfrak{d}_*(\mathcal{P}) \leq \mathfrak{d}_*(\mathcal{P}')$. We choose a strictly increasing function $e : \omega \rightarrow \omega$ such that for all n , $e[A_n] \subseteq A'_n$. In most cases e cannot be chosen as to be a bijection. We set $\tilde{e}(n) = \min\{k : e(k) \geq n\}$, then $\tilde{e}(e(n)) = n$.

Let \mathcal{F} be a family that is good for $\mathfrak{d}_*(\mathcal{P}')$. We claim that \mathcal{F} is also good for $\mathfrak{d}_*(\mathcal{P})$. Let $h \in \omega^{\uparrow\omega}$ be given. We take $h' = h \circ \tilde{e}$. This may be only weakly increasing. Then by the definition of \mathcal{F} being good for $\mathfrak{d}_*(\mathcal{P}')$, there are some $A' \in \mathcal{P}'$ and some $f \in \mathcal{F}$ such that $(\forall^\infty n \in A') ((f \circ h \circ \tilde{e})(n) \geq (h \circ \tilde{e})(n+1))$. For each $k \in A$ we have $\tilde{e}(e(k)+1) = \tilde{e}(e(k+1)) = k+1$. So $(\forall^\infty k \in A \subseteq e^{-1}[A']) (f(h(k)) \geq h(k+1))$. \square

So we have that $\mathfrak{d}_*(\mathcal{P})$ does not depend on \mathcal{P} . We point out that Aubrey [1] works with a cardinal \mathfrak{d}^* (the minimal cardinal of a finitely dominating family) and shows $\mathfrak{d}^* = \min(\mathfrak{r}, \mathfrak{d})$. $\mathfrak{d}_*(\mathcal{P}) \leq \mathfrak{r}$ will be shown in Section 3. In Section 4 we show $\mathfrak{d}_*(\mathcal{P}) \geq \min(\mathfrak{r}, \mathfrak{d})$. So $\mathfrak{d}_*(\mathcal{P}) = \mathfrak{d}^*$.

Now let \mathcal{P} be as in the definition of $\mathfrak{d}'(\mathcal{P})$. Obviously $\mathfrak{d}'(\mathcal{P}) \leq \mathfrak{d}_*(\mathcal{P})$, because the disjunction in the definition of $\mathfrak{d}'(\mathcal{P})$ is weaker than the requirement in $\mathfrak{d}_*(\mathcal{P})$.

For the $\mathfrak{d}'(\mathcal{P})$ the transition from one partition $\{A_\ell : \ell \in \omega\}$ to another $\{A'_\ell : \ell \in \omega\}$ is more difficult, since now we require from the reduction e that it preserves for all k $((n \in A_k \text{ and } n+1 \in A_k) \rightarrow (e(n), e(n+1) \in A'_k \text{ and } e(n+1) = e(n)+1))$.

Definition 2.2. Let $A \subseteq \omega$ be infinite and coinfinite.

$$\|A\| = \sup\{n : (\exists^\infty k) (k, k+1, \dots, k+n-1 \in A)\}$$

is between 1 and ω , inclusively. Now let $\mathcal{P} = \{A_n : n < \omega\}$ and $\mathcal{P}' = \{A'_n : n < \omega\}$ be two partitions of ω into infinite sets. We let $\mathcal{P} \leq \mathcal{P}'$ if $\exists \sigma : \omega \rightarrow \omega$ bijective such that for all i $\|A_i\| \leq \|A'_{\sigma(i)}\|$.

Proposition 2.3. *If $\mathcal{P} \leq \mathcal{P}'$ then every family $\mathcal{F} \subseteq \omega^{\uparrow\omega}$ that is good for $\mathfrak{d}'(\mathcal{P}')$ is also good for $\mathfrak{d}'(\mathcal{P})$, and hence $\mathfrak{d}'(\mathcal{P}) \leq \mathfrak{d}'(\mathcal{P}')$.*

Proof. Let $\|A_i\| \leq \|A'_{\sigma(i)}\|$. Let \mathcal{F} be a family that is good for $\mathfrak{d}'(\mathcal{P}')$. We choose a strictly increasing function $e : \omega \rightarrow \omega$ such that for all n , $e[A_n] \subseteq A'_{\sigma(n)}$ and such that $k, k+1 \in A_n \rightarrow e(k+1) = e(k)+1 \in A'_{\sigma(n)}$. We set $\tilde{e}(n) = \min\{k : e(k) \geq n\}$, then $\tilde{e}(e(n)) = n$.

We claim that \mathcal{F} is also good for $\mathfrak{d}_*(\mathcal{P})$. Let $h \in \omega^{\uparrow\omega}$ be given. We take $h' = h \circ \tilde{e}$. Then by the definition of \mathcal{F} being good for $\mathfrak{d}'(\mathcal{P}')$, there are some $A' \in \mathcal{P}'$ and some $f \in \mathcal{F}$ such that $(\forall^\infty n \in A') ((f \circ h \circ \tilde{e})(n) \geq (h \circ \tilde{e})(n+1) \vee (f \circ h \circ \tilde{e})(n+1) \geq (h \circ \tilde{e})(n+2) \vee n+1 \notin A')$. For each $k \in A$ we have $\tilde{e}(e(k)+1) = \tilde{e}(e(k+1)) = k+1$ and the if $k+1 \in A$, then $e(k+1) = e(k)+1 \in A'$. So $(\forall^\infty k \in A \subseteq e^{-1}[A']) (f(h(k)) \geq h(k+1) \vee (f(h(k+1)) \geq h(k+2) \vee k+1 \notin A)$. \square

3. All $\mathfrak{d}'(\mathcal{P})$, $\mathfrak{d}_*(\mathcal{P})$ are bounded by the reaping number

In [12] it is shown that if there is a \mathcal{P} such that $\mathfrak{d}'(\mathcal{P}) \geq \mathfrak{d}$ then for every $k \geq 1$ there is a subgroup of \mathbb{Z}^ω such that G^k is Menger-bounded but G^{k+1} is not. In [7, Theorem 3.1] is shown that $\mathfrak{u} < \mathfrak{g}$ implies that for all subgroups of \mathbb{Z}^ω whose square is Menger-bounded all their finite powers are Menger-bounded (also simultaneously). So $\mathfrak{u} < \mathfrak{g}$ implies $\mathfrak{d}'(\mathcal{P}) < \mathfrak{d}$. Now we give a direct proof of a stronger statement. Let $A_0 \dot{\cup} A_1 = \omega$. We read the definitions of $\mathfrak{d}_*(\mathcal{P})$ and of $\mathfrak{d}'(\mathcal{P})$ in a natural way also for partitions of ω into finitely many infinite parts. Then of course we get larger or equal cardinals.

Theorem 3.1. $\mathfrak{d}_*({A_0, A_1}) \leq \mathfrak{r}$ and $\mathfrak{d}'(\{\omega\}) \leq \mathfrak{r}$.

Proof. Let \mathcal{B} be a refining family of size \mathfrak{r} . Refining means: $(\forall A \in [\omega]^\omega) (\exists B \in \mathcal{B}) (B \subseteq^* A \vee B \subseteq^* \omega \setminus A)$. For each $B \in \mathcal{B}$ we let $f_B : \omega \rightarrow \omega$ be defined by letting $f_B(n)$ be the n th element of B . We shall show that $\{f_B : B \in \mathcal{B}\}$ is a family \mathcal{F} as in the computation of $\mathfrak{d}_*(\mathcal{P})$. We assume that the contrary is the case. So

$$(\exists h \in \omega^{\uparrow\omega}) (\forall B \in \mathcal{B}) (\forall \ell \in \{0, 1\}) (\exists^\infty n \in A_\ell) (f_B(h(n)) < h(n+1)).$$

We enumerate the infinitely many $n \in A_\ell$ with $f_B(h(n)) < h(n+1)$ as $n_{\ell,k}^B$, $k \in \omega$. Now since $f_B(h(n_{\ell,k}^B)) \geq h(n_{\ell,k}^B)$, we have that

$$(\forall B \in \mathcal{B}) (B \cap [h(n_{\ell,k}^B), h(n_{\ell,k}^B + 1)]) \neq \emptyset. \tag{3.1}$$

We set $C_\ell = \bigcup_{k \in \omega, B \in \mathcal{B}} [h(n_{k,\ell}^B), h(n_{k,\ell}^B + 1))$. Since $n_{k,\ell}^B \in A_\ell$ and since the $A_0 \cap A_1 = \emptyset$, we have and $C_0 \cap C_1 = \emptyset$. So (3.1) shows that the set $A = C_0$ is a counterexample to \mathcal{B} 's being refining.

Now we turn to $\mathfrak{d}'(\{\omega\}) \leq \tau$. We assume that $f_B, B \in \mathcal{B}$, is not a family as in the computation of $\mathfrak{d}'(A)$. Then

$$(\exists h \in \omega^{\uparrow\omega}) (\forall B \in \mathcal{B}) (\exists^\infty n) (f_B(h(n)) < h(n+1) \wedge f_B(h(n+1)) < h(n+2)).$$

We enumerate the infinitely many n such that

$$f_B(h(n)) < h(n+1) \wedge f_B(h(n+1)) < h(n+2)$$

as $n_k^B, k \in \omega$. Now we let $C_0 = \bigcup_{k \in \omega, B \in \mathcal{B}, n_k^B \text{ is even}} [h(n_k^B), h(n_k^B + 1))$ and $C_1 = \bigcup_{k \in \omega, B \in \mathcal{B}, n_k^B \text{ is odd}} [h(n_k^B), h(n_k^B + 1))$. Then $C_0 \cap C_1 = \emptyset$ and

$$(\forall B \in \mathcal{B}) (B \cap C_0 \neq \emptyset \wedge B \cap C_1 \neq \emptyset). \tag{3.2}$$

So (3.2) contradicts \mathcal{B} 's being refining. \square

Only for the case of having only one part in the partition and only one inequality there is the opposite result, that $\mathfrak{d}_*(\omega) > \tau$ is consistent. This is because $\tau < \mathfrak{d}$ is consistent (see [9–11,5]) and the following result, obtained by Boaz Tsaban and Petr Simon independently:

Theorem 3.2. $\mathfrak{d}_*(\omega) \geq \mathfrak{d}$.

4. $\mathfrak{d}_*(\mathcal{P}) = \min(\tau, \mathfrak{d})$

For the proof we use the following partition order. Let $\Pi = \langle \pi_i : i \in \omega \rangle$ for a strictly increasing sequence $\pi_i, i < \omega$, be a partition of ω into the cells $[\pi_i, \pi_{i+1})$. We say Π dominates Π' if each interval in Π , with finitely many exceptions, includes an interval in Π' . It is easy to see and shown in [8] that there is a family of \mathfrak{d} interval partitions that every interval partition is dominated by a member of the family and that fewer than \mathfrak{d} interval partitions do not suffice. Our first lemma is actually Simon's and Tsaban's theorem (with a different proof). For $X \in [\omega]^\omega$, we define the next-function $\text{next}(X, \cdot) : \omega \rightarrow \omega$ by $\text{next}(X, n) = \min\{k \in X : k \geq n\}$.

Lemma 4.1. For every $\mathcal{F} \subseteq \omega^{\uparrow\omega}$, if $|\mathcal{F}| < \mathfrak{d}$ then

$$(\exists h \in \omega^{\uparrow\omega}) (\forall f \in \mathcal{F}) (\exists^\infty n) (f(h(n)) < h(n+1)).$$

Proof. Since \mathcal{F} is not dominating, there is some $g \in \omega^{\uparrow\omega}$ such that for every $f \in \mathcal{F}$ there are infinitely many n with $f(n) < g(n)$. Let for $f \in \mathcal{F}$, X_f be an infinite subset of $\{n : f(n) < g(n)\}$ such that for

$$(\forall n \in X_f) (g(n) \leq \text{next}(X_f, n)).$$

Identify the increasing enumeration of X_f with a partition $\Pi_f = \langle \pi_{f,n} : n \in \omega \rangle$ of ω . Then, by Blass' results, there is a partition Π such that for all f , $\Pi_f := \langle \pi_{f,n} : n \in \omega \rangle$ does not dominate Π in the partition order, that means

$$\text{for all } f \text{ there are infinitely } n \text{ such that there is no point } \pi_j \text{ is in } [\pi_{f,n}, \pi_{f,n+1}).$$

Now take $h \in \omega^{\uparrow\omega}$ being the increasing enumeration of Π . Given $f \in \mathcal{F}$, take n , such that there is no point π_j is in $[\pi_{f,n}, \pi_{f,n+1})$. and then take k such that k is the maximal k with $h(k) \leq \pi_{f,n}$. Now

$$f(h(k)) \leq f(\pi_{f,n}) < g(\pi_{f,n}) \leq \text{next}(X_f, \pi_{f,n}) = \pi_{f,n+1} \leq h(k+1).$$

Since there are infinitely many n to start from, there are infinitely many such k . \square

Lemma 4.2. Let $|\mathcal{F}| < \min(\tau, \mathfrak{d})$. Then there is a partition \mathcal{P} such that \mathcal{F} is not good for $\mathfrak{d}_*(\mathcal{P})$.

Proof. Since $|\mathcal{F}| < \mathfrak{d}$, be the previous lemma there is $h \in \omega^{\uparrow\omega} (\forall f \in \mathcal{F}) (\exists^\infty n) (f(h(n)) < h(n+1))$. Enumerate these n 's as $X_f = \{n_{f,i} : i < \omega\}$. The family $X_f, f \in \mathcal{F}$, is not reaping, and hence there are an infinite set, call it A_0 , and its complement, call it A'_1 , such that for all $f \in \mathcal{F}$, both sets $X_f \cap A_0$ and $X_f \cap A'_1$ are infinite. Now we continue along these lines and partition A'_1 into A_1 and A'_2 . After ω steps, the partition $\mathcal{P} = \{A_\ell : \ell \in \omega\}$ is as required and the function $h \in \omega^{\uparrow\omega}$ witnesses that \mathcal{F} is not good for $\mathfrak{d}_*(\mathcal{P})$. \square

So we have proved Theorem 1.3.

Remark 4.3. The partition \mathcal{P} in the proof of Theorem 1.3 depends on \mathcal{F} and this does not necessarily prove that $\tau \geq \mathfrak{d}$ implies that there is a single \mathcal{P} with $\mathfrak{d}'(\mathcal{P}) = \mathfrak{d}$.

5. The proof of Theorem 1.4

Lemma 5.1. *Let $\delta < \min(\mathfrak{d}, \mathfrak{r})$ and let $\Pi_\gamma, \gamma < \delta$, be partitions of ω into finite intervals and let $\Pi_\gamma = \langle \pi_{\gamma,i}; i \in \omega \rangle$. Then there are a partition $\mathcal{P} = \{A_\ell; \ell < \omega\}$ of ω into infinite sets and a partition $\Pi = \langle \pi_i; i < \omega \rangle$, such that for every infinite $\ell \in \omega$ for every $\gamma < \delta$ there are infinitely many $i \in A_\ell$ such that $[\pi_i, \pi_{i+1})$ contains at least two points $\pi_{\gamma,j}, \pi_{\gamma,j+1}$.*

Proof. Since $\gamma < \mathfrak{d}$ there is a partition $\Pi = \langle \pi_i; i < \omega \rangle$ such that for every $\gamma < \delta$ there are infinitely many $i \in \omega$ such that $[\pi_i, \pi_{i+1})$ contains at least two points $\pi_{\gamma,j}, \pi_{\gamma,j+1}$. Enumerate these i 's as $\{i_{\gamma,n}; n \in \omega\} =: X_\gamma$. Since $\delta < \mathfrak{r}$, the family $X_\gamma, \gamma < \delta$, is not reaping, and hence there are an infinite set, call it A_0 , and its complement, call it A'_1 , such that for all $\gamma < \delta$, both sets $X_\gamma \cap A_0$ and $X_\gamma \cap A'_1$ are infinite. Now we continue along these lines and partition A'_1 into A_1 and A'_2 . After ω steps, the partition $\Pi = \langle \pi_i; i < \omega \rangle$ and the partition $\mathcal{P} = \{A_\ell; \ell \in \omega\}$ are as required in the lemma. \square

Proof of Theorem 1.4. Suppose the $\mathcal{F} \subseteq \omega^{\uparrow\omega}$ and $|\mathcal{F}| < \mathfrak{d}_*(\mathcal{P})$. Then $(\exists h \in \omega^{\uparrow\omega}) (\forall f \in \mathcal{F}) (\forall \ell \in \omega) (\exists^\infty m \in A_\ell) (f(h(m)) < h(m+1))$. Fix such an h . Let $\langle m'_{f,\ell,k}; k \in \omega \rangle$ enumerate these m 's. Then each $\langle m'_{f,\ell,k}; k \in \omega \rangle$ out in order to get a sequence $\langle m'_{f,\ell,k}; k \in \omega \rangle$ such that for all $f, \ell, k, h(m'_{f,\ell,k}) < m'_{m,\ell,k+1}$.

Since $|\mathcal{F}| < \mathfrak{d}'(\mathcal{P}) \leq \mathfrak{d}, \mathfrak{r}$, there are a partition $\langle \pi_i; i \in \omega \rangle$ and a partition $\mathcal{P}' = \{A'_r; r \in \omega\}$ such that for all $r \in \omega, \langle [\pi_i, \pi_{i+1}); i \in A'_r \rangle$ is not dominated by all the partitions $\langle m'_{f,\ell,2k}; k \in \omega \rangle, f \in \mathcal{F}, \ell \in \omega$, in the partition order. Set $j(i) = \pi_i$ and set $e(i) = r$ if $(i \in A'_r \text{ and } i \geq r)$ otherwise set $e(i) = 0$ (that is, to react onto the matrix which is just used to build the vector). For technical reasons (i.e., for Eq. (5.13)) we need that $e(i) \leq i$. Then

$$(\forall f \in \mathcal{F}) (\forall \ell, r) (\exists^\infty n) \\ (e(n-1) = r \wedge \text{the next } m'_{f,\ell,k} \text{ after } j(n) \text{ is called } m'_{f,\ell,k} \text{ the last } m'_{f,\ell,k'} \text{ strictly before } j(n+1) \text{ is called } m'_{f,\ell,k'} \wedge k < k' \wedge f(h(m'_{f,\ell,k})) < h(m'_{f,\ell,k} + 1) \leq h(m'_{f,\ell,k'}) \wedge f(h(m'_{f,\ell,k'})) < h(m'_{f,\ell,k'} + 1) \leq j(n+1)). \tag{5.1}$$

We say j is chosen well for \mathcal{F}, h .

Now one can continue the proof of [12, Theorem 9] with h_α from there being j for a j chosen as above for $\mathcal{F} = M_\alpha$ (for the construction at stage $\alpha < \mathfrak{d}$) and h as chosen above and the functions g_i^α (from Eq. (5.7)) are constant on intervals $[j(n), j(n+1))$ with the matrix C_r if $e(n) = r$ (that is, iff $n \in A'_r$ and we never use the original partition $\{A_\ell; \ell \in \omega\}$) and hence everything is carried out similarly with $p < j(n)$ in the role of the case description $p < h_\alpha(n)$ from there.

This is a vague description and it is dangerous to claim that all goes through despite these weakenings in the preparation. Hence we carry out the proposed modification of Machura, Shelah and Tsaban's construction from [12]. The gist is: the matrix coding can be chosen anew at each step $\alpha < \mathfrak{d}$.

We enumerate $\mathbb{Z}^{k \times (k+1)}$ as $\{C_m; m \in \omega\}$ such that the sequence $C_m, m \in A'_\ell$, is constant for each ℓ . We also fix a dominating family of increasing functions $d_\alpha, \alpha < \mathfrak{d}$. We write $\|v\|$ or $\|v_0, \dots, v_k\|$ for $\max\{|v_0|, \dots, |v_k|\}$.

We carry out a modified construction by induction on $\alpha < \mathfrak{d}$. In step α we define $\varphi_{\alpha,m} \in \omega^\omega$ for $m \in \omega$ by

$$\varphi_{\alpha,m}(n) = \min\{\|v\|; v \in \mathbb{Z}^{k+1} \wedge \|v\| \geq d_\alpha(n) \wedge C_m v = \vec{0}\} \tag{5.2}$$

and

$$\varphi_\alpha(n) = \max\{\varphi_{\alpha,m}(n); m \leq n\}. \tag{5.3}$$

Note that the functions $\varphi_{\alpha,m}, \varphi_\alpha \in \omega^{\uparrow\omega}$, this will be important in the definition of I in Eq. (5.14). We let $M_\alpha < (H_\chi, \epsilon)$ be of cardinality $|\alpha| \cdot \omega < \mathfrak{d}$ and contain $\varphi_{\alpha,m}, \varphi_\alpha$ and all functions defined in the stages $< \alpha$. By the definition of $\mathfrak{d}_*(\mathcal{P})$ there is an increasing h_α such that

$$(\forall f \in M_\alpha \cap \omega^\omega) (\forall \ell) (\exists^\infty n) (n \in A_\ell \wedge f(h_\alpha(n)) < h_\alpha(n+1)). \tag{5.4}$$

Now choose j_α , and a matrix chooser e_α for M_α, h_α as above.

Define $k+1$ elements $g_0^\alpha, \dots, g_k^\alpha \in \mathbb{Z}^\omega$. For each n let $v \in \mathbb{Z}^{k+1}$ be a witness for the definition of $\varphi_{\alpha,e_\alpha(n)}(j_\alpha(n+1))$, namely

$$\varphi_{\alpha,e_\alpha(n)}(j_\alpha(n+1)) = \|v\| \geq d_\alpha(j_\alpha(n+1)), \tag{5.5}$$

and

$$C_{e_\alpha(n)} v = \vec{0}, \tag{5.6}$$

and define

$$\begin{pmatrix} g_0^\alpha(j_\alpha(n)) \\ \vdots \\ g_k^\alpha(j_\alpha(n)) \end{pmatrix} = v, \tag{5.7}$$

so that

$$C_{e_\alpha(n)} \cdot \begin{pmatrix} g_0^\alpha(j_\alpha(n)) \\ \vdots \\ g_k^\alpha(j_\alpha(n)) \end{pmatrix} = \vec{0}. \tag{5.8}$$

The remaining values of the functions g_i^α are defined by declaring these functions constant on each interval $[j_\alpha(n), j_\alpha(n+1))$. By Eqs. (5.5) and (5.7)

$$\|g_0^\alpha(j_\alpha(n)), \dots, g_k^\alpha(j_\alpha(n))\| = \varphi_{\alpha, e(n)}(j_\alpha(n+1)) \tag{5.9}$$

for all n . We take G as the subgroup of \mathbb{Z}^ω that is generated by $\{g_i^\alpha : i \leq k, \alpha < \mathfrak{d}\}$. We show that G is as required in the theorem.

G^{k+1} is not Menger-bounded. Let $f \in \omega^\omega$. We take $\alpha < \mathfrak{d}$ such that $f \leq^* d_\alpha$. We fix m_0 such that for all $m \geq m_0$, $f(m) \leq d_\alpha(m)$. Let n be such that $m-1 \in [j_\alpha(n), j_\alpha(n+1))$. Then

$$\|g_0^\alpha(m-1), \dots, g_k^\alpha(m-1)\| = \|g_0^\alpha(j_\alpha(n)), \dots, g_k^\alpha(j_\alpha(n))\| = \varphi_{\alpha, e_\alpha(n)}(j_\alpha(n+1)) \geq d_\alpha(j_\alpha(n+1)) \geq d_\alpha(m) \geq f(m).$$

G^k is Menger-bounded. We take $f(n) = n^2$. We prove that $(\forall F \in [G]^k) (\exists^\infty n) (\forall g \in F) (\hat{g}(n) \leq f(n))$.

Fix $F = \{g_0, \dots, g_{k-1}\}$. Then there is $M \in \omega$ and there are $\alpha_1 < \dots < \alpha_M < \mathfrak{d}$ and matrices $B_1, \dots, B_M \in \mathbb{Z}^{k \times (k+1)}$ such that

$$\begin{pmatrix} g_0 \\ \vdots \\ g_{k-1} \end{pmatrix} = B_1 \begin{pmatrix} g_0^{\alpha_1} \\ \vdots \\ g_k^{\alpha_1} \end{pmatrix} + \dots + B_M \begin{pmatrix} g_0^{\alpha_M} \\ \vdots \\ g_k^{\alpha_M} \end{pmatrix}.$$

Let $g_{0,0} = \dots = g_{k-1,0} = 0$ and for each $m = 1, \dots, M$ let

$$\begin{pmatrix} g_{0,m} \\ \vdots \\ g_{k-1,m} \end{pmatrix} = B_1 \begin{pmatrix} g_0^{\alpha_1} \\ \vdots \\ g_k^{\alpha_1} \end{pmatrix} + \dots + B_m \begin{pmatrix} g_0^{\alpha_m} \\ \vdots \\ g_k^{\alpha_m} \end{pmatrix}. \tag{5.10}$$

We prove by induction on $m = 0, \dots, M$, that there is a constant c_m and there are infinitely many j such that

$$\|\hat{g}_{0,m}(j), \dots, \hat{g}_{k-1,m}(j)\| \leq c_m \cdot (j+1).$$

By the definition of our increasing chain of elementary submodels, then there is an infinite set of such j 's, call it J_m , that is an element of $M_{\alpha_{m+1}}$. By the definition of f this is sufficient. The case $m = 0$ is vacuous. We show how to step up from $m-1$ to m . Assume that

$$J_{m-1} = \{j : \|\hat{g}_{0,m-1}(j), \dots, \hat{g}_{k-1,m-1}(j)\| \leq c_{m-1} \cdot (j+1)\} \in M_{\alpha_{m-1}+1} \subseteq M_{\alpha_m}$$

is infinite. Hence also the function

$$g_{<c_{m-1}}(n) := \min\{j : n \leq j \in J_{m-1}\} \tag{5.11}$$

is well defined and in M_{α_m} . For each $i \leq k$ and each n such that

$$e_{\alpha_m}(n-1) = m' \wedge B_m = C_{m'}, \tag{5.12}$$

we get by Eq. (5.5)

$$|g_i^{\alpha_m}(j_{\alpha_m}(n-1))| \leq \varphi_{\alpha_m, e_{\alpha_m}(n-1)}(j_{\alpha_m}(n)) \leq \varphi_{\alpha_m}(j_{\alpha_m}(n)).$$

As φ_{α_m} and j_{α_m} are non-decreasing, and by Eq. (5.3) we can take also the $n' < n$ into (as hidden in the \hat{g} 's) the latter inequality

$$|g_i^{\alpha_m}(j_{\alpha_m}(n'-1))| \leq \varphi_{\alpha_m, e_{\alpha_m}(n'-1)}(j_{\alpha_m}(n')) \leq \varphi_{\alpha_m}(j_{\alpha_m}(n))$$

and get

$$\|\hat{g}_0^{\alpha_m}(j_{\alpha_m}(n-1)), \dots, \hat{g}_k^{\alpha_m}(j_{\alpha_m}(n-1))\| \leq \varphi_{\alpha_m}(j_{\alpha_m}(n)). \tag{5.13}$$

By Eq. (5.1) and by our assumptions on $M_{\alpha_m}, h_{\alpha_m}, e_{\alpha_m}, j_{\alpha_m}$,

$$I = \{n : e_{\alpha_m}(n-1) = m' \wedge (\exists n' < n'' \in [j_{\alpha_m}(n), j_{\alpha_m}(n+1))) (\varphi_{\alpha_m}(j_{\alpha_m}(n')) < h_{\alpha_m}(n'+1) \leq j_{\alpha_m}(n+1) \wedge g_{<c_{m-1}}(h_{\alpha_m}(n'')) = j < h_{\alpha_m}(n''+1) \leq j_{\alpha_m}(n+1))\} \tag{5.14}$$

is infinite.

Let $n \in I$. Then $e_\alpha(n) = m'$ and $C_{m'} = B_m$ and thus by Eqs. (5.6) and (5.8)

$$B_m \cdot \begin{pmatrix} g_0^{\alpha_m}(j_{\alpha_m}(n)) \\ \vdots \\ g_k^{\alpha_m}(j_{\alpha_m}(n)) \end{pmatrix} = B_m \cdot \begin{pmatrix} g_0^{\alpha_m}(j_{\alpha_m}(n+1) - 1) \\ \vdots \\ g_k^{\alpha_m}(j_{\alpha_m}(n+1) - 1) \end{pmatrix} = \vec{0}.$$

By Eq. (5.10) for each $i < k$,

$$g_{i,m} \upharpoonright [j_{\alpha_m}(n), j_{\alpha_m}(n+1)] = g_{i,m-1} \upharpoonright [j_{\alpha_m}(n), j_{\alpha_m}(n+1)]. \tag{5.15}$$

As $n \in I$, there is $j \in J_{m-1}$ and there are $n' < n'' \in [j_{\alpha_m}(n), j_{\alpha_m}(n+1))$ such that

$$h_{\alpha_m}(n'') \leq j < h_{\alpha_m}(n'' + 1). \tag{5.16}$$

From Eq. (5.13) we get

$$\begin{aligned} \|\hat{g}_0^{\alpha_m}(j_{\alpha_m}(n) - 1), \dots, \hat{g}_k^{\alpha_m}(j_{\alpha_m}(n) - 1)\| &= \|\hat{g}_0^{\alpha_m}(j_{\alpha_m}(n-1)), \dots, \hat{g}_k^{\alpha_m}(j_{\alpha_m}(n-1))\| = \varphi_{\alpha_m}(j_{\alpha_m}(n)) \\ &\leq \varphi_{\alpha_m}(j_{\alpha_m}(n')) < h_{\alpha_m}(n' + 1). \end{aligned} \tag{5.17}$$

We want to show that $j \in J_m$ for a suitable choice of c_m (not depending on j). Let $p \in [0, j]$.

Case 1: $p \geq j_{\alpha_m}(n)$. As $j < h_{\alpha_m}(n'' + 1)$,

$$[j_{\alpha_m}(n), j + 1] \subseteq [j_{\alpha_m}(n), j_{\alpha_m}(n + 1)],$$

and by Eq. (5.15) and the membership $j \in J_{m-1}$

$$|g_{i,m}(p)| = |g_{i,m-1}(p)| \leq \hat{g}_{i,m-1}(j) \leq c_{m-1}(j + 1) \tag{5.18}$$

for all $i < k$.

Case 2: $p < j_{\alpha_m}(n)$. Let C be the maximal absolute value of a coordinate of B_m . For all $i < k$, by the definition of $g_{i,m}$,

$$|g_{i,m}(p)| \leq |g_{i,m-1}(p)| + (k + 1)C \max\{|g_i^{\alpha_m}(p)| : i < k\}. \tag{5.19}$$

As $p < j_{\alpha_m}(n) \leq j \in J_{m-1}$, $|g_{i,m-1}(p)| \leq \hat{g}_{i,m-1}(j) \leq c_{m-1}(j + 1)$. Using $p < j_{\alpha_m}(n)$ and Eq. (5.17) and $g_i^{\alpha_m}$ being constant on $[j_{\alpha_m}(n-1), j_{\alpha_m}(n))$ and $h_{\alpha_m}(n' + 1) \leq h_{\alpha_m}(n'') \leq j$, we get from Eq. (5.17)

$$|g_i^{\alpha_m}(p)| \leq \hat{g}_i^{\alpha_m}(j_{\alpha_m}(n) - 1) \leq \|\hat{g}_1^{\alpha_m}(j_{\alpha_m}(n) - 1), \dots, \hat{g}_k^{\alpha_m}(j_{\alpha_m}(n) - 1)\| = \varphi_{\alpha_m}(j_{\alpha_m}(n)) < h_{\alpha_m}(n' + 1) \leq j$$

for each $i < k$. Together with Eq. (5.18) we have now

$$|g_{i,m}(p)| \leq |g_{i,m-1}(p)| + (k + 1)C \cdot \max\{|g_i^{\alpha_m}(p)| : i < k\} \leq c_{m-1}(j + 1) + (k + 1)Cj \leq (c_{m-1} + (k + 1)C) \cdot (j + 1).$$

So we take $c_m = c_{m-1} + (k + 1)C$. Since I is infinite, also J_m is infinite and this completes the inductive proof.

6. $\tau \geq \mathfrak{d}$ is not necessary

We collect the lower bounds and the upper bounds on $\mathfrak{d}'(\mathcal{P})$:

$$\text{cov}(\mathcal{M}), \mathfrak{b} \leq \mathfrak{d}'(\mathcal{P}) \leq \tau, \mathfrak{d}.$$

There are models in which every lower bound is \aleph_1 and every upper bound is \aleph_2 : On p. 384 in [4] a model of $\mathfrak{b} = \text{cov}(\mathcal{M}) = \aleph_1$ and $\text{cov}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c} = \aleph_2$ is given: Start with a ground model $\mathbf{V} \models \mathfrak{b}, \text{cov}(\mathcal{M}) < \mathfrak{d} = \aleph_2$ and then force with $B(\aleph_2)$, adding \aleph_2 random reals. Also since $\tau \geq \text{cov}(\mathcal{N})$, we have $\tau = \aleph_2$ and by Theorem 1.3, $\mathfrak{d}_*(\mathcal{P}) = \mathfrak{d}$. There are two possibilities for refining this choice by refining and modifying the choice the ground model:

First, there is model gotten by c.c.c. forcing, namely we start with two regular cardinals $\nu < \delta$ and we get a model $\nu = \tau = \mathfrak{u} = \mathfrak{d} = \delta$ as given in [11]. The notation in the following theorem is taken from the paper [11], and we also draw on [13]. For more details, the reader is referred to these two references.

Theorem 6.1. *In the models of [11] there are groups with Menger-bounded k th power but non-Menger-bounded $(k + 1)$ st power.*

Proof. Let $r_\eta, \eta < \delta$, be the Cohen reals and let $s_\alpha, \alpha < \nu$, be the Mathias reals as there. Let $\mathcal{U}_\mathcal{P}$ be the ultrafilter generated by the latter. Since by [3, Proposition 19] and a modification of [13, Theorem 3.6], $r_\eta, \eta < \delta$, is $\leq_{\mathcal{U}_\mathcal{P}}$ dominating, we know by [14] that $r_\eta \circ \text{next}(s_\alpha, \cdot), \eta < \delta, \alpha < \nu$, is \leq^* dominating.

Now we imitate a construction à la [12] along a layering $M_\alpha, \alpha < \nu$, such that $M_\alpha \subseteq V(\delta, h(\alpha))$ for some increasing continuous function $h : \nu \rightarrow \nu$, and $M_\alpha \prec (H(\chi), \in)$ is neither dominating nor refining. In the step from α to $\alpha + 1$, $\varphi_{\eta,\alpha}$ has to dominate $r_\eta \circ \text{next}(s_\alpha, \cdot)$ for all $\eta < \delta$, so that G^{k+1} will be dominating in the end.

For this aim we set $\varphi_{\eta,\alpha} = r_\eta \circ \text{next}(s_{\alpha+1}, \cdot)$. Now the analog to the functions h_α , j_α , and e_α for the model $V(\eta, h(\alpha) + 1)$ (which contains the functions $\varphi_{\eta,\alpha}$, $\eta < \delta$) can be found in $V(\eta, h(\alpha + 1))$ for some $h(\alpha + 1) < \delta$, since $V(\eta, h(\alpha) + 1)$ is neither refining nor dominating in $V(\delta, \nu)$. Then we define $g^{\eta,\alpha}$ for $\varphi_{\eta,\alpha}$, h_α , j_α , and e_α for each η separately, as in the original construction. The estimation, the G^k is Menger bounded, is conducted by induction on α . Now in one induction step finitely many $g^{\eta,r,\alpha}$, $r < R$, for some $R \in \omega$, have to be considered in the sums like (5.10). We take the maxima over the respective R functions before forming I as in (5.14). So in the end, $M_{\alpha+1}$ contains δ elements more than M_α , but is still neither dominating nor refining. \square

Since $\mathfrak{b} \leq \mathfrak{d}'(\mathcal{P}) \leq \mathfrak{d}_*(\mathcal{P}) \leq \mathfrak{r}$, in these models the new cardinal characteristics are pinned down as $\mathfrak{d}'(\mathcal{P}) = \mathfrak{d}_*(\mathcal{P}) = \nu$ and thus show that the sufficient condition is not necessary. We still can add random reals and get that the groups in the ground model are still k -Menger-bounded and not $k + 1$ -Menger-bounded. There are new examples of subgroups of \mathbb{Z}^ω with bounded k th power and unbounded $(k + 1)$ st power in the extension by the random reals, because the random reals increase \mathfrak{r} and hence make $\mathfrak{d}_*(\mathcal{P}) = \mathfrak{d}$.

Now we look at a second model of $\mathfrak{N}_1 = \text{cov}(\mathcal{M}) = \mathfrak{b} < \mathfrak{r} = \mathfrak{d} = \mathfrak{c} = \mathfrak{N}_2$: We start with a ground model V of $\mathfrak{u} < \mathfrak{g}$ gotten, e.g., by adding \mathfrak{N}_2 Miller reals [10] or Blass–Shelah reals [9] with countable support to a model of CH. In this model there are no groups with Menger-bounded k th power and not Menger-bounded $(k + 1)$ st power. Thereafter we add \mathfrak{N}_2 random reals. Then $\mathfrak{g} = \mathfrak{N}_1$ (by [5]) and $\mathfrak{r} = \mathfrak{N}_2$ and $\mathfrak{d} = \mathfrak{N}_2$. So in this model there groups with Menger-bounded k th power and non-Menger-bounded $(k + 1)$ st power added by forcing with random reals. We are interested whether $\mathfrak{r} \geq \mathfrak{d}$ implies $(\exists \mathcal{P}) (\mathfrak{d}'(\mathcal{P}) = \mathfrak{d})$ and hence we ask:

Question 6.2. *What is the value of $\mathfrak{d}'(\mathcal{P})$ in this type of forcing extensions?*

Separating the cardinal characteristics seems to be a challenge, because there is not much elbow room. However, since the non-existence result for $\mathfrak{u} < \mathfrak{g}$ mentioned in the beginning of Section 3 works only from $k = 2$ onwards, the following is most interesting:

Question 6.3. *Does $\mathfrak{u} < \mathfrak{g}$ imply that there is no Menger-bounded subgroup of \mathbb{Z}^ω whose square is not Menger-bounded?*

It is well possible that $\mathfrak{u} < \mathfrak{g}$ is not enough for non-existence and that a deeper analysis of one of the forcings given in [9,11,5] (i.e., the three main forcings for $\mathfrak{u} < \mathfrak{g}$) or an entirely new forcing order could answer affirmatively:

Question 6.4. *Is it consistent relative to ZFC that there are no Menger-bounded subgroup of \mathbb{Z}^ω whose square is not Menger-bounded?*

Similar questions on k -domination for various k , without groups, lead also into realm of $\mathfrak{u} < \mathfrak{g}$ versus “there are at least $k + 1$ near-coherence classes”, or $\mathfrak{r} \geq \mathfrak{d}$, or even $\mathfrak{r} \geq \mathfrak{c}$, and are considered in [6,7].

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