DIAGONALISING AN ULTRAFILTER AND PRESERVING A P-POINT

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Abstract. With Ramsey-theoretic methods we show: It is consistent that there is a forcing that diagonalises one ultrafilter over ω and preserves another ultrafilter.

1. Introduction

Suppose M is a family of infinite sets such that M is large in some sense, e.g. unbounded or non-meagre, and that $\mathbb{P}$ is a notion of forcing. We can ask whether M or some closure of M or reinterpretation of M in any $\mathbb{P}$-generic extension $V^{\mathbb{P}}$ are still large in $V^{\mathbb{P}}$. If this is the case we say that $\mathbb{P}$ preserves that M is large. Various degrees of preservation can be distinguished: the forcing $\mathbb{P}$ preserves the largeness of one particular set M, e.g. M being the set of all ground model reals, or $\mathbb{P}$ preserves some large sets and makes others non-large, or $\mathbb{P}$ preserves any large set of a certain form.

We are concerned with partial preservation for the largeness notion of generating an ultrafilter over ω, the set of natural numbers. Suppose that M is a family of infinite subsets of of ω. We say M generates an ultrafilter if $\{Y : (\exists X \in M)(X \subseteq Y)\}$ is an ultrafilter. For the case of unboundedness and Mathias forcing with a filter, examples of preserving the ground model reals as an unbounded family are given by [6] and examples preserving all unbounded families are given in [8]. Examples for specific preservation of the non-meagerness and non-nullness are given in [11]. Forcings with Milliken–Taylor ultrafilters preserving a P-point while destroying all P-points that are superfilters of a certain filter are given in [9] and in [14]. It is not known whether these forcings diagonalise an ultrafilter from the ground model. Here we add a new kind of example: Preserving one P-point and diagonalising another.

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In Section 3 we introduce suitable families $\mathcal{H}$ (see Def. 3.3) in a combinatorial space $((\mathcal{P})^\omega, \leq)$ (see Def. 2.2(3)) and the notion "$\mathcal{H}$ avoids $\mathcal{V}$" for a P-point $\mathcal{V}$. In Theorem 3.17 we present an exact criterion for preservation of a given P-point and a suitable family. In the special case of a suitable family $\mathcal{H} \subseteq (\mathcal{P})^\omega$ that projects to an ultrafilter over ω the criterion yields the following result:

Theorem 1.1. *Suppose that $\mathcal{V}$ is a P-point and $\mathcal{U}$ is an ultrafilter over ω such that there is a suitable family $\mathcal{H}$ with projection $\Phi_2(\mathcal{H}) = \mathcal{U}$ and that $\Phi_2(\mathcal{H}) \not\leq_{RB} \mathcal{V}$. Then there is a proper notion of forcing that diagonalises $\mathcal{U}$ and preserves $\mathcal{V}$.*

The premises to the theorem are consistent with ZFC:

Theorem 1.2. *Under CH or Martin's Axiom, given any P-point $\mathcal{V}$ there is a suitable maximal centred family $\mathcal{C}$ with projection $\Phi_2(\mathcal{C}) = \mathcal{U}$ such that $\mathcal{U}$ is an ultrafilter and $\mathcal{U} \not\leq_{RB} \mathcal{V}$.*

Recall that a partial order is called centred if any finitely many elements have a common lower bound. Here $\mathcal{C}$ is equipped with the partial order $\leq$ of $(\mathcal{P})^\omega$, see Def. 2.2. We will prove the existence theorem in Section 4.

Remark 1.3. In the first theorem we just use that $\mathcal{H}$ is suitable and $(\Phi_2(\mathcal{H}), \subseteq)$ is centred. This might increase the versatility of suitable families. However, we do not know how to construct a family $(\mathcal{H}, \leq)$ as in the first theorem that is not centred. Moreover, maximality can be ensured in a natural way along the construction.

In Proposition 2.5 we show: If any two ultrafilters over ω are nearly coherent (see definitions below) then there are no examples $\mathcal{C}$, $\mathcal{V}$ with the stated properties.

We recall definitions and facts: For a set X, we denote its powerset by $\mathcal{P}(X)$. By a *filter over* ω we mean a non-empty subset of $\mathcal{P}(\omega)$ that is closed under supersets and under finite intersections and that does not contain the empty set. We call a filter *non-principal* if it contains all cofinite subsets of ω and we call it an ultrafilter if it is a maximal filter.

For $B \subseteq \omega$ and $f \colon \omega \to \omega$, we let $f[B] = \{f(b) : b \in B\}$ and $f^{-1}[B] = \{n : f(n) \in B\}$. For $\mathcal{B} \subseteq \mathscr{P}(\omega)$ we let $f(\mathcal{B}) = \{X : f^{-1}[X] \in \mathcal{B}\}$. This double lifting is an important function from $\mathscr{P}(\mathscr{P}(\omega))$ into itself. In analysis the special case of f being finite-on-one (that means that the preimage of each natural number is finite) is particularly useful, see e.g., [3].

Let $\mathcal{F}$ be a non-principal filter over ω and let $f \colon \omega \to \omega$ be finite-to-one. Then also $f(\mathcal{F})$ is a non-principal filter. It is the filter generated by $\{f[X] : X \in \mathcal{F}\}$. From now on we consider only non-principal filters and ultrafilters. Two filters $\mathcal{F}$ and $\mathcal{G}$ are *nearly coherent*, if there is some finite-to-one $f \colon \omega \to \omega$ such that $f(\mathcal{F}) \cup f(\mathcal{G})$ generates a filter. On the set of non-principal ultrafilters near coherence is an equivalence relation whose equivalence classes are called near-coherence classes.

The principle near coherence of filters (short NCF) says that any two non-principal ultrafilters over ω are nearly coherent. Blass and Shelah [5] showed that NCF is consistent relative to ZFC.

The set of infinite subsets of ω is denoted by $[\omega]^\omega$, the set of finite subsets of ω by $[\omega]^{<\omega}$, and the set of functions from ω to ω is written ${}^\omega\omega$. We say "A is almost a subset of B" and write $A \subseteq^* B$ iff $A \setminus B$ is finite. For $X \subseteq \omega$, we write X^c for $\omega \setminus X$.

Definition 1.4. Let κ be a regular uncountable cardinal. An ultrafilter $\mathcal{U}$ is called a *P_κ-point* if for every $\gamma < \kappa$, for every $A_i \in \mathcal{U}$, $i < \gamma$, there is some $A \in \mathcal{U}$ such that for all $i < \gamma$, $A \subseteq^* A_i$; such an A is called a *pseudo-intersection* of the A_i, $i < \gamma$. A $P_{\aleph_1}$-point is called a P-point.

Let $\mathbb{P}$ be a notion of forcing. We say that $\mathbb{P}$ *preserves* an ultrafilter $\mathcal{U}$ over I if

$$\Vdash_{\mathbb{P}} \text{“}(\forall X \subseteq I)(\exists Y \in \mathcal{U})(Y \subseteq X \vee Y \subseteq I \setminus X)\text{”}$$

and in the contrary case we say "$\mathbb{P}$ *destroys* $\mathcal{U}$". A particular way to destroy a non-principal ultrafilter is to diagonalise it, that means adding an infinite set X such that for any $Y \in \mathcal{U}$, $X \subseteq^* Y$.

If $\mathbb{P}$ preserves $\mathcal{U}$ then $\mathcal{U}$ generates an ultrafilter in $\mathbf{V}[G]$. If $\mathbb{P}$ is proper and preserves $\mathcal{U}$ as an ultrafilter and $\mathcal{U}$ is a P-point, then $\mathcal{U}$ generates a P-point in the extension, since any countable set of ground model sets in the extension has a countable superset in the ground model, see [5, Lemma 3.2].

By nowadays, techniques for preserving ultrafilters that do not involve preservation of P-points are much more difficult than the known proofs of P-point preservation, see e.g. [16]. This experience is, at least partially, based on mathematical reasons: Any forcing that adds a real destroys an ultrafilter [1, Theorem 3.5], whereas Miller forcing, Sacks forcing and a few other tree forcings preserve any P-point.

The paper is organised as follows: In Section 2 we explain normed subsets of powersets, introduce $(\mathcal{P})^\omega$ and introduce suitable sets. We prove in Proposition 2.5 that the premises of Theorem 1.1 are independent of ZFC. In Section 3 we explain Blass–Shelah forcing and Blass–Shelah with a suitable set $\mathcal{H} \subseteq (\mathcal{P})^\omega$. We recall the definition of the Rudin–Blass order $\leq_{RB}$ and we recall Eisworth's work on the preservation of P-points for Matet forcing and prove Theorem 1.1. In Section 4 we prove Theorem 1.2. We close with a short discussion and some open questions.

In the forcing, the stronger condition is the *smaller* one. This direction fits to the $\leq$-relation on the sequences of possibilities (see Def. 2.2(4)) that form the second components of conditions. In addition we follow the alphabetical rule: Later letters are used for stronger conditions.

2. Sets of normed subsets of powersets

In this section we introduce a relative of Blass-Shelah forcing ([5]). Conditions in either version of Blass–Shelah forcing are of the form $p = (s, \bar{a})$. The component s is a finite subset of ω, which is usually called the trunk, and the component $\bar{a}$ is an ω-sequence of hereditarily finite sets and is usually called the pure part.

Now we define a space $\mathcal{P}$ from which the entries of the pure parts of our posets will be taken.

Definition 2.1. (1) A finite subset s of ω is called a *block*. A *set of possibilities* is a subset of the power set of a block that contains the empty set. We denote by $\mathcal{P}$ the set of all sets of possibilities. Typically we use variables s, t, ... for blocks and a, b, c, ... for sets of possibilities.

So sets of possibilities are one powerset operation higher than blocks.

(2) Let a be a set of possibilities and $Y \subseteq \omega$. We let $a \restriction Y = \{s : s \in a, s \subseteq Y\}$.

(3) We define $\mathrm{norm}\colon \mathcal{P} \to \omega$ as follows:
   (a) $\mathrm{norm}(a) \geq 0$, always,
   (b) $\mathrm{norm}(a) \geq 1$ iff $\bigcup a \neq \emptyset$,
   (c) $\mathrm{norm}(a) \geq k+1$ iff whenever $\bigcup a = Y_1 \cup Y_2$ then $\max(\mathrm{norm}(a \restriction Y_1), \mathrm{norm}(a \restriction Y_2)) \geq k$,
   (d) $\mathrm{norm}(a) = k$ iff $(\mathrm{norm}(a) \geq k$ and $\mathrm{norm}(a) \not\geq k+1)$.

(4) We define $\circ\colon \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ by $a \circ b = \{s \cup t : s \in a, t \in b\}$.

The structure $(\mathcal{P}, \circ)$ is a semigroup.
If $\mathrm{norm}(a) \geq 1$, then a contains a non-empty set.

Definition 2.2. (1) For $a, b \in \mathcal{P}$ we write $a < b$ iff $\bigcup a, \bigcup b \neq \emptyset$ and $(\forall n \in \bigcup a)(\forall m \in \bigcup b)(n < m)$.

(2) A sequence $\bar{a} = \langle a_n : n \in \omega \rangle$ of members of $\mathcal{P}$ is called *unmeshed* if for all n, $a_n < a_{n+1}$.

(3) By $(\mathcal{P})^\omega$ we denote the set of unmeshed sequences $\bar{a}$ such that $\lim_{n \to \omega} \mathrm{norm}(a_n) = \omega$.

(4) For sequences $\bar{a}$, $\bar{b} \in (\mathcal{P})^\omega$ we write $\bar{b} \leq \bar{a}$ or "$\bar{b}$ is stronger than $\bar{a}$" iff there is a strictly increasing function $g \in {}^\omega\omega$ such that for any n,

$$b_n \subseteq a_{g(n)} \circ a_{g(n)+1} \circ \cdots \circ a_{g(n+1)-1}.$$

So this means first $\langle a_n : n \in \omega \rangle$ is merged over finite intervals and then a subset of each set of possibilities is taken. We do not need to drop members from the sequence, since the empty set is contained in any a_n.

(5) For sequences $\bar{a}$, $\bar{b} \in (\mathcal{P})^\omega$ we write $\bar{a} \leq^* \bar{b}$ iff there is an n such that $\langle a_k : k \geq n\rangle \leq \bar{b}$.

(6) For sequences $\bar{a}$, $\bar{b} \in (\mathcal{P})^\omega$ we write $\bar{a} \perp^* \bar{b}$ if they are incompatible in $\leq^*$, i.e., if there is no $\bar{c} \in (\mathcal{P})^\omega$ such that $\bar{c} \leq^* \bar{a}, \bar{b}$.

So for any sequence $\bar{a} \in (\mathcal{P})^\omega$ we have for any n, $\mathrm{norm}(a_n) \geq 1$ and $\max(\bigcup a_n) \geq n$.

Lemma 2.3. *The relations $\leq$ and $\leq^*$ are transitive.*

The next two notions connect elements of $(\mathcal{P})^\omega$ with subsets of ω.

Definition 2.4. (1) For $\bar{a} \in (\mathcal{P})^\omega$ we let $\mathrm{set}(\bar{a}) = \bigcup\{\bigcup a_n : n \in \omega\}$.

(2) Let $\mathcal{H} \subseteq (\mathcal{P})^\omega$. The *projection of $\mathcal{H}$ into* $[\omega]^\omega$ is $\Phi_2(\mathcal{H}) = \{\mathrm{set}(\bar{a}) : \bar{a} \in \mathcal{H}\}$. [1]

We show that the premises to Theorem 1.1 are independent of ZFC:

Proposition 2.5. *Under NCF there are no $\mathcal{C}$ and $\mathcal{V}$ as in the conditions of Theorem 1.1.*

Proof. Let $\mathcal{V}$ be an ultrafilter over ω and let f be finite-to-one or at least so that $f(\mathcal{V})$ is non-principal. We show that diagonalising $f(\mathcal{V})$ means destroying $\mathcal{V}$. We fix a pseudo-intersection X, i.e., $(\forall A \in f(\mathcal{V}))(X \subseteq^* A)$. We claim that $f^{-1}[X] \notin \mathcal{V}$ and $(f^{-1}[X])^c \notin \mathcal{V}$. This is seen as follows. Let $\langle x_i : i \in \omega\rangle$ be an injective enumeration of X. We divide $f^{-1}[X]$ into two disjoint sets $B_1 = \bigcup\{f^{-1}[\{x_{2i}\}] : i \in \omega\}$, $B_2 = \bigcup\{f^{-1}[\{x_{2i+1}\}] : i \in \omega\}$. If $f^{-1}[X]$ were an element of $\mathcal{V}$, one of the B_j is in $\mathcal{V}$; and for this B_j we have $X \not\subseteq^* f[B_j]$. Hence X is not a diagonalisation of $f(\mathcal{V})$. If $(f^{-1}[X])^c = f^{-1}[X^c]$ were in $\mathcal{V}$, then $X^c \in f(\mathcal{V})$, so again X cannot be a pseudo-intersection of $f(\mathcal{V})$.

Now assume that $\mathcal{C}$ and $\mathcal{V}$ are as in the premises of Theorem 1.1 and NCF holds. Then there is a finite-to-one f such that $f(\Phi_2(\mathcal{C})) = f(\mathcal{V})$. We take a forcing as stated in Theorem 1.1, and in the extension there is a set X that diagonalises $\Phi_2(\mathcal{C})$. Then $f[X]$ diagonalises $f(\Phi_2(\mathcal{C})) = f(\mathcal{V})$. Hence $\mathcal{V}$ is destroyed. $\square$

Definition 2.6. Let $\bar{a} \in (\mathcal{P})^\omega$, $b \in \mathcal{P}$, $\bigcup b \neq \emptyset$. We write $(\bar{a} \ \mathrm{past}\, b)$ for $\langle a_i : i \in [k, \omega)\rangle$, where k is the minimal number such that $\max \bigcup b < \min \bigcup a_k$. For $\bar{a} \in (\mathcal{P})^\omega$, $n \in \omega$. We write $(\bar{a} \ \mathrm{pastn}\, n)$ for $(\bar{a} \ \mathrm{past}\, \{\{n\}\})$.

Now we introduce a version $\mathbb{Q}^{242}$ of Blass-Shelah forcing (see [5], which is number 242 in Shelah's list of articles).

[1] We write Φ_2 to distinguish it from the projection that is used for Matet forcing in [9].

Definition 2.7. In the forcing order $\mathbb{Q}^{242}$, conditions are pairs $(s, \bar{a})$ such that $s \in [\omega]^{<\omega}$ and $\bar{a} \in (\mathcal{P})^\omega$ and $(\forall n \in s)(\forall m \in \bigcup a_0)(n < m)$. We let $(t, \bar{b}) \le (s, \bar{a})$ (recall the stronger condition is the smaller one) iff $s \subseteq t$ and there are $k \in \omega$ such that $t \setminus s \in a_0 \circ \cdots \circ a_{k-1}$ and $\bar{b} \le (\bar{a} \text{ past } a_{k-1})$, where $\le$ is from Def. 2.2(4).

For readers of [5] we give a brief description of the differences: Instead of sequences of relations $\langle r_k : k < \omega\rangle$, $r_k \in K_{n_k,m_k} \subseteq \mathcal{P}(\mathcal{P}(n_k) \times \mathcal{P}(m_k))$, $n_k < m_k \le n_{k+1}$, we work with $\bar{a} \in (\mathcal{P})^\omega$. In our version $(s,t) \in r \in K_{n,m}$ from [5] is replaced by $s \in a_i \wedge (t \setminus s \in a_j)$ for some $i < j$ with $\max(\bigcup a_i) < n$ and $n \le \min(\bigcup a_j) \le \max(\bigcup a_j) < m$. Hence the set

$$\{(t, \bar{b}) : (s \cup t, \bar{b}) \ge (s, \bar{a})\}$$

depends only on $\bar{a}$ and not on s. A forcing with this property is called "forgetful" in the classification of creature forcings in [15].

Note: The trunks contain no information about the block structure. Moreover, there are gaps: if $(t, \bar{b}) \le (s, \bar{a})$, and $(t \setminus s) \cap \bigcup a_n \ne \emptyset$, then $\bar{b}$ must begin after the maximum of $\bigcup a_n$ and not just after $\max(t)$ as would be the case in Mathias forcing.

Definition 2.8. (1) Let $(s, \bar{a}) \in \mathbb{Q}^{242}$. We define a tree $T(s, \bar{a})$ as follows: Elements of the tree are $\{s \cup t : (\exists n \in \omega)(\exists i_0 < i_1 < \cdots < i_{n-1})(t \in a_{i_0} \circ \cdots \circ a_{i_{n-1}})\}$. The tree is ordered by end extension.

(2) We let $\mathrm{Lev}_{<0}(s, \bar{a}) = \{s\}$ and for $k \ge 1$ we let

$$\mathrm{Lev}_{<k}(s, \bar{a}) = \{s \cup t : (\exists n \le k)(\exists i_0 < i_1 < \cdots < i_{n-1} < k)(t \in a_{i_0} \circ \cdots \circ a_{i_{n-1}})\}.$$

In other words, $t \in T(s, \bar{a})$ iff $(t, \bar{a} \text{ past } t) \le (s, \bar{a})$. Here and henceforth we write $(t, \bar{a} \text{ past } r)$ for $(t, (\bar{a} \text{ past } r))$.

Definition 2.9. Let $n \in \omega$, $(s, \bar{a})$, $(t, \bar{b}) \in \mathbb{Q}^{242}$ and assume $(t, \bar{b}) \le (s, \bar{a})$. We say $(t, \bar{b})$ is a 0-extension of $(s, \bar{a})$ and write $(t, \bar{b}) \le_0 (s, \bar{a})$ if $t = s$.

3. Blass–Shelah forcing with suitable sets

Now we thin out the reservoir $(\mathcal{P})^\omega$ to a suitable subfamily. We choose $\mathcal{H}$ in allusion to a happy family.

Definition 3.1. We assume that $\langle \bar{a}_n : n \in \omega\rangle$ is a $\le$-descending sequence of members $\bar{a}_n \in (\mathcal{P})^\omega$. We say $\bar{b}$ is a *diagonal lower bound of* $\langle \bar{a}_n : n \in \omega\rangle$ if $\bar{b} \le \bar{a}_0$ and for any $n \in \omega$

$$(\bar{b} \text{ past } b_n) \le \bar{a}_{\max(\bigcup b_n)+1}.$$

Remark 3.2. If $\bar{c} \le \bar{b}$ and $\bar{b}$ is a diagonal lower bound of $\langle \bar{a}_n : n \in \omega\rangle$ then also $\bar{c}$ is a diagonal lower bound of $\langle \bar{a}_n : n \in \omega\rangle$.

Definition 3.3. Compare to Mathias [13]. A set $\mathcal{H} \subseteq (\mathcal{P})^\omega$ is called a *suitable set* if the following hold:

(1) (Non-emptyness, freeness, upwards closure) $\mathcal{H} \subseteq (\mathcal{P})^\omega$, $\mathcal{H} \neq \emptyset$. If $\bar{a} \in \mathcal{H}$ and $\bar{b} \geq^* \bar{a}$ then $\bar{b} \in \mathcal{H}$.

(2) (Existence of diagonal lower bounds) Any $\leq$-descending ω-sequence in $\mathcal{H}$ has a diagonal lower bound in $\mathcal{H}$.

(3) (Fullness) For any $Y \subseteq \omega$ and $\bar{a} \in \mathcal{H}$ there is $\bar{b} \leq \bar{a}$, $\bar{b} \in \mathcal{H}$ such that $\mathrm{set}(\bar{b}) \subseteq Y$ or $\mathrm{set}(\bar{b}) \subseteq Y^c$.

(4) (Ramsey property) For any $C \colon [\omega]^{<\omega} \to 2$ and any $(s, \bar{a})$ with $\bar{a} \in \mathcal{H}$ and $\max(s) < \min(\bigcup a_0)$ there is $(t, \bar{b}) \leq (s, \bar{a})$, $\bar{b} \in \mathcal{H}$ such that either $C \restriction T(t, \bar{b})$ is constantly 1 or $C \restriction (T(s, \bar{b}) \setminus \{s\})$ is constantly 0.

Here is a family of subforcings of $\mathbb{Q}^{242}$:

Definition 3.4. Given a suitable set $\mathcal{H}$ in $(\mathcal{P})^\omega$, the notion of forcing $\mathbb{Q}^{242}(\mathcal{H})$ consists of all pairs $(s, \bar{a}) \in \mathbb{Q}^{242}$ such that $\bar{a} \in \mathcal{H}$. The order relation is as in $\mathbb{Q}^{242}$ (see Def. 2.7).

In Definition 2.2(3), we take the limit requirement following [5]. All the proofs in this paper could be carried out equally with the requirement $\limsup_n \mathrm{norm}(a_n) = \infty$ instead. We do not know any difference in the effect of the two variants. For the full forcing with $(\mathcal{P})^\omega$, the two variants are equivalent, since any sequence with $\limsup_n(a_n) = \infty$ has a subsequence that satisfies the lim-requirement.

Now we repeat some propositions from [5] for $\mathbb{Q}^{242}(\mathcal{H})$. Our proofs are slightly different from the original proofs since the second coordinates of conditions need to be elements of the given family $\mathcal{H}$ and $\mathcal{H}$ is less closed than $(\mathcal{P})^\omega$. We let $\chi \geq (2^\omega)^+$ be a regular cardinal and we fix a well-ordering $\lhd$ of $H(\chi)$. For a sequence $\langle \bar{a}_n : n \in \omega \rangle$ with $\bar{a}_n \in (\mathcal{P})^\omega$ we let $\bar{a}_n = \langle a_{n,\ell} : \ell \in \omega \rangle$ be the indexing of the components.

Lemma 3.5. *(See* [5, Proposition 2.4]*) Let τ_i, $i \in \omega$, be $\mathbb{Q}^{242}(\mathcal{H})$ names for ordinals. Then every condition $(s, \bar{a})$ has a 0-extension $(s, \bar{b})$ with the following property: If*

(1) $\ell \geq 0$, $t \in \mathrm{Lev}_{<\ell+1}(s, \bar{b})$, and $i \leq \max(\bigcup b_\ell)$, and

(2) $(t, \bar{b} \text{ past } b_\ell)$ has a 0-extension forcing a particular value for τ_i,

then $(t, \bar{b} \text{ past } b_\ell)$ forces a particular value for τ_i.

Proof. By induction on ℓ we choose a $\leq$-decreasing sequence $\langle \bar{a}_\ell : \ell \in \omega \rangle$ of elements of $\mathcal{H}$ with the following properties:

(a) $\bar{a}_0 = \bar{a}$,

(b) $\bar{a}_{\ell+1} = (\bar{a}_{\ell+1} \text{ past } a_{\ell,\ell})$,

(c) For any $\ell \in \omega$, $t \in T(s, \bar{a}_0)$ such that $\max(t) \le \max(\bigcup a_{\ell,\ell})$, $i \le \max(\bigcup a_{\ell,\ell})$: If $(t, (a_{\ell,\ell+1}, a_{\ell,\ell+2} \dots,))$ has a 0-extension forcing a value to τ_i then $(t, \bar{a}_{\ell+1})$ forces a value to τ_i. The condition $t \in T(s, \bar{a}_0)$ is not a misprint, we want to include stems coming from weaker conditions.

Note that item (c) implies: For any $\ell \in \omega$, $t \in T(s, \bar{a}_0)$ with $\max(t) \le \max(\bigcup a_{\ell,\ell})$, $i \le \max(\bigcup a_{\ell,\ell})$: If $(t, a_{\ell+1})$ has a 0-extension forcing a value to τ_i then $(t, \bar{a}_{\ell+1})$ itself forces a value to τ_i. Moreover, the implication also holds for any $\bar{b} \le \bar{a}_{\ell+1}$.

The induction starts with $\bar{a}_0 = \bar{a}$. Suppose $\ell \ge 0$ and $\bar{a}_\ell$ is chosen. Let $\{(t_j, i_j) : 0 \le j < k\}$ be the $\lhd$-least enumeration of $\mathrm{Lev}_{<\ell+1}(s, \bar{a}) \times (\max(\bigcup a_{\ell,\ell}) + 1)$.

Now by a subinduction on $j \le k$ we choose $\bar{a}^j$, $j = 0, \dots, k$. We start with $\bar{a}^0 = (a_{\ell,\ell+1}, a_{\ell,\ell+1} \dots) = (\bar{a}_\ell \text{ past}\, a_{\ell,\ell})$. Given $\bar{a}^j$ we do the following: If there is $(t_j, \bar{b}) \le (t_j, \bar{a}^j)$ forcing a value to τ_{i_j} then we let $\bar{a}^{j+1}$ be the $\lhd$-least such $\bar{b}$. Otherwise we let $\bar{a}^{j+1} = \bar{a}^j$. In the end we let $\bar{a}_{\ell+1} = (\bar{a}_{\ell+1} \text{ past}\, a_{\ell,\ell}) = \bar{a}^k$.

Having defined $\langle \bar{a}_\ell : \ell < \omega \rangle$, we let $\bar{b} \in \mathcal{H}$ be a the $\lhd$-least diagonal lower bound of $\langle \bar{a}_\ell : \ell \in \omega \rangle$ in $\mathcal{H}$. [2] The condition $(s, \bar{b})$ has the desired properties: Assume $\ell \ge 0$ and $t \in \mathrm{Lev}_{<\ell+1}(s, \bar{b})$ and $i \le \max(\bigcup b_\ell) \le \max(a_{\max(\bigcup b_\ell), \max(\bigcup b_\ell)})$ and suppose that there is a 0-extension of $(t, \bar{b} \text{ past}\, b_\ell)$ forcing a value to τ_i. Then $t \in T(s, \bar{a}_0)$ and $\max(t) \le \max(\bigcup b_\ell)$. Since $\bar{b}$ is a diagonal lower bound we have $(\bar{b} \text{ past}\, b_\ell) \le \bar{a}_{\max(\bigcup b_\ell)+1} = (\bar{a}_{\max(\bigcup b_\ell)+1} \text{ past}\, a_{\max(\bigcup b_\ell), \max(\bigcup b_\ell)})$. So there is a 0-extension of $(t, \bar{a}_{\max(\bigcup b_\ell)+1})$ forcing a value to τ_i. Hence by construction $(t, \bar{a}_{\max(\bigcup b_\ell)+1})$ itself forces a value to τ_i. Again since $(\bar{b} \text{ past}\, b_\ell) \le \bar{a}_{\max(\bigcup b_\ell)+1}$, also the condition $(t, \bar{b} \text{ past}\, b_\ell)$ forces a value to τ_i. $\square$

Lemma 3.6. $\mathbb{Q}^{242}(\mathcal{H})$ *is proper.*

Proof. This is derived from Lemma 3.5 as in [5, Proposition 2.5]. Indeed, given $M \prec (H(\chi), \in, \lhd)$ and an enumeration τ_i, $i \in \omega$, of all names in M of ordinals and $(s, \bar{a}) \in M$, the condition $(s, \bar{b})$ constructed in Lemma 3.5 is $(M, \mathbb{Q}^{242}(\mathcal{H}))$-generic. $\square$

Can we work with weaker properties than suitability? We do not know. At least the requirement of fullness seems to be natural. In order to explain this we name the generic reals:

Definition 3.7. Let G be $\mathbb{Q}^{242}(\mathcal{H})$-generic over $\mathbf{V}$. We call

$$W_G = \bigcup \{s : \exists \bar{a} (s, \bar{a}) \in G\}$$

the $\mathbb{Q}^{242}(\mathcal{H})$-generic real and let W be a name for it.

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[2] Note that $\bar{b}$ can be incompatible with the sequence $\langle a_{n,n} : n \in \omega \rangle$, which, if a member of $\mathcal{H}$, could serve as a $\bar{b}$. Moreover, $\langle a_{n,n} : n \in \omega \rangle$ is a diagonal lower bound of $\langle \bar{a}_n : n \in \omega \rangle$ iff for $n \in \omega$, $\bar{a}_{n+1} \le \bar{a}_{\max(\bigcup a_{n,n})+1}$, so only in very special cases.

The generic real of the full Blass–Shelah forcing $\mathbb{Q}^{242}$ is not split by any real in the ground model:

Lemma 3.8. *If $\bar{a} \in (\mathcal{P})^\omega$ and $X \subseteq \omega$ then there is $\bar{b} \leq \bar{a}$ such that* $\mathrm{set}(\bar{b}) \subseteq X$ *or* $\mathrm{set}(\bar{b}) \subseteq (\omega \setminus X)$.

Proof. By definition, for $a \in \mathcal{P}$, $\mathrm{norm}(a \restriction X) \geq \mathrm{norm}(a) - 1$ or $\mathrm{norm}(a \restriction X^c) \geq \mathrm{norm}(a) - 1$. Now we choose $Y = X$ or $Y = X^c$ such that $C = \{n : \mathrm{norm}(a_n \restriction Y)) \geq \mathrm{norm}(a_n) - 1\}$ is infinite. Let $(n_k)_{k \in \omega}$ enumerate C and let $\bar{b} = \langle a_{n_k} \restriction Y : k \in \omega \rangle$. $\square$

By the fullness requirement we have the same result for the subforcing $\mathbb{Q}^{242}(\mathcal{H})$:

Lemma 3.9. *Let $\mathcal{H}$ be a suitable set. In $\mathbf{V}^{\mathbb{Q}^{242}(\mathcal{H})}$, for any $X \subseteq \omega$, $X \in \mathbf{V}$, $W_G \subseteq^* X$ or $W_G \subseteq^* X^c$.*

Proof. Given $p = (s, \bar{a}) \in \mathbb{Q}^{242}(\mathcal{H})$ and $X \subseteq \omega$, by fullness there is $\bar{b} \leq \bar{a}$, $\bar{b} \in \mathcal{H}$ with $\mathrm{set}(\bar{b}) \subseteq X$ or $\mathrm{set}(\bar{b}) \subseteq X^c$. Hence $D = \{p : p \Vdash W \subseteq^* X \text{ or } p \Vdash W \subseteq^* X^c\}$ is dense. $\square$

So in the generic extension we have the set $\mathcal{U}' := \{X \in \mathbf{V} \cap [\omega]^\omega : W_G \subseteq^* X\}$ that decides each old subset of ω. If $\mathcal{U}' \in \mathbf{V}$ then $\mathcal{U}'$ can serve as the ultrafilter $\mathcal{U}$ that is diagonalised, as required in Theorem 1.1. This ends the discussion of fullness. Further below we work with centred suitable families, for which $\mathcal{U}' \in \mathbf{V}$.

Now we work towards preserving a given P-point.

Definition 3.10. Let $\mathcal{S}, \mathcal{S}' \subseteq [\omega]^\omega$ be closed under almost supersets. We write $\mathcal{S} \leq_{\mathrm{RB}} \mathcal{S}'$ and say $\mathcal{S}$ *is Rudin-Blass below* $\mathcal{S}'$ iff there is a finite-to-one f such that $f(\mathcal{S}) \subseteq f(\mathcal{S}')$.

This definition with f on both sides follows [9]. The frequently used variant of the definition in which $f(\mathcal{S}) \subseteq f(\mathcal{S}')$ is replaced by $f(\mathcal{S}) \subseteq \mathcal{S}'$ has very similar preservation properties when used for ultrafilters $\mathcal{S}'$.

Definition 3.11. We assume that $\mathcal{V}$ is a P-point and $\mathcal{H} \subseteq (\mathcal{P})^\omega$. We let for $\bar{a} \in \mathcal{H}$, $\mathcal{H} \restriction \bar{a} = \{\bar{b} \in \mathcal{H} : \bar{b} \leq \bar{a}\}$. We say $\mathcal{H}$ *avoids* $\mathcal{V}$ iff $(\forall \bar{a} \in \mathcal{H})(\Phi_2(\mathcal{H} \restriction \bar{a}) \not\leq_{\mathrm{RB}} \mathcal{V})$.

Since $\mathcal{V}$ is an ultrafilter, $\mathcal{H}$ avoids $\mathcal{V}$ holds iff for any finite-to-one function h and any $\bar{a} \in \mathcal{H}$ there is $V \in \mathcal{V}$ and $\bar{b} \in \mathcal{H} \restriction \bar{a}$ such that $h[\mathrm{set}(\bar{b})] \cap h[V] = \emptyset$.

Definition 3.12. (1) Let $\langle \bar{a}_n : n \in \omega \rangle$ be a $\leq$-descending sequence of elements on $(\mathcal{P})^\omega$. A sequence $\bar{b} \in (\mathcal{P})^\omega$ is called a lower bound iff $(\forall n \in \omega)(\bar{b} \leq^* \bar{a}_n)$.

(2) $\mathcal{H} \subseteq (\mathcal{P})^\omega$ is called stable if any $\leq$-descending sequence of elements of $\mathcal{H}$ has a lower bound in $\mathcal{H}$.

Since diagonal lower bounds are lower bounds the double projection $\Phi_2(\mathcal{H})$ of any suitable $\mathcal{H}$ contains lower bounds for $\subseteq^*$-descending sequences. So if the double projection of a suitable family $\mathcal{H}$ is an ultrafilter, it is a P-point.

The following deep theorem is crucial for the construction and the evaluation of the forcing.

Theorem 3.13. *(*[5, Theorem 2.6]*, see also* [2, Theorem 7.4.20]*)* $(\mathcal{P})^\omega$ *has the Ramsey property.*

The full set $(\mathcal{P})^\omega$ is suitable.

Remark 3.14. If we let for $\bar{a} \in (\mathcal{P})^\omega$, $n \in \omega$, $r_n(\bar{a}) = \langle a_0, \dots, a_{n-1}\rangle$ then the triple $((\mathcal{P})^\omega, <, (r_n)_{n\in\omega})$ reminds of a topological Ramsey space in the sense of [17, Section 5.1]. The Ramsey property is (at least formally) incomparable with the pigeonhole principle which could be translated into: For any $C \colon \mathcal{P} \to 2$ and any $(s, \bar{a})$ with $\bar{a} \in \mathcal{H}$ and $\max(s) < \min(\bigcup a_0)$ there is $(s, \bar{b}) \leq (s, \bar{a})$, $\bar{b} \in \mathcal{H}$ such that for any $\bar{c} \leq \bar{b}$, $C(c_0) = 0$ or for any $\bar{c} \leq \bar{b}$, $C(c_0) = 1$. In our forcing the trunk s is just a set, not a member of $(\mathcal{P})^{<\omega}$, which would be the case in the setting of a topological Ramsey space. Forcing with trunks $\langle a_0, \dots, a_{n-1}\rangle \in (\mathcal{P})^{<\omega}$ and pure parts from $(\mathcal{P})^\omega$ would better fit to the space $((\mathcal{P})^\omega, \leq, (r_n)_{n\in\omega})$. This forcing does not help for our purpose since it adds a dominating real and thus destroys any ultrafilter. This is true also if we restrict the reservoir of the pure parts of conditions to a subreservoir that has the pigeonhole property. To see this we take for $X \subseteq \omega$ the colouring $C_X(a) = 1$ iff $[\min(\bigcup a), \max(\bigcup a)) \cap X \neq \emptyset$ and $C_X(a) = 0$ otherwise. For this colouring C_X, any monochromatic set of sets of possibilities that is closed under mergings has colour 1. By letting X range over a dominating set we see that forcing with trunks $\langle a_0, \dots, a_{n-1}\rangle \in (\mathcal{P})^{<\omega}$ and pure parts from $(\mathcal{P})^\omega$ adds a dominating real. Since $(\mathcal{P}, \circ)$ is a semigroup one can use the Galvin–Glazer technique (see e.g. [10]) to show that $((\mathcal{P})^\omega, \leq_{\text{sub}}, (r_n)_n)$ is a topological Ramsey space, where $\leq_{\text{sub}}$ is a subrelation of $\leq$ allowing only dropping and merging a_n's and not going over to subsets when strengthening $\bar{a}$. Note that here the sets of possibilities are coloured, not the blocks, as in Theorem 3.13. We do not know whether $((\mathcal{P})^\omega, \leq, (r_n)_{n\in\omega})$ is a Ramsey space.

In preparation for the next theorem we need a consequence of the Ramsey property:

Proposition 3.15. *(See* [5, Prop. 2.9]*) Let $\underset{\sim}{A}$ be a $\mathbb{Q}^{242}(\mathcal{H})$-name for a subset of ω. Then every condition $(s, \bar{a})$ has an extension $(t, \bar{b})$ with the following property: If $\ell \geq 0$, $t' \in \mathrm{Lev}_{<\ell+1}(t, \bar{b})$, and $i \leq \max(b_\ell)$, then $(t', \bar{b} \operatorname{past} b_\ell)$ decides whether $i \in \underset{\sim}{A}$.*

Proof. We let $\tau_i = 0$ iff $i \not\in \underset{\sim}{A}$ and $\tau_i = 1$ else. We assume that $(s, \bar{a})$ already has the property stated for $(s, \bar{b})$ in Lemma 3.5 for the sequence τ_i, $i \in \omega$. We define

$C\colon T(s,\bar{a}) \to 2$ by

$$C(t) = 1 \text{ iff } (\forall \ell \geq 0)\big(t \in \mathrm{Lev}_{<\ell+1}(s,\bar{a}) \to \\ (\forall i \leq \max(\bigcup a_\ell))((t, \bar{a} \text{ past } a_\ell) \text{ decides } i \in \underset{\sim}{A})\big).$$

By the Ramsey property there is $(t,\bar{b}) \leq (s,\bar{a})$ such that for each $t' \in T(t,\bar{b})$, $C(t') = 0$ or there for each $t' \in T(s,\bar{b}) \smallsetminus \{s\}$, $C(t') = 1$. We fix such a $(t,\bar{b})$.

The first possibility is ruled out: Let $\ell \geq 0$ and $(t',\bar{c}) \leq (t,\bar{b})$, $t' \in \mathrm{Lev}_{<\ell+1}(s,\bar{a})$ be such that it decides $i \in A$ and $i \leq \max(\bigcup a_\ell)$. Then $(t',\bar{c}) \leq_0 (t', \bar{b} \text{ past } a_\ell) \leq_0 (t', \bar{a} \text{ past } a_\ell)$ and by the choice of $(s,\bar{a})$ as in Lemma 3.5, the condition $(t', \bar{a} \text{ past } a_\ell)$ decides $i \in \underset{\sim}{A}$. So the uniform colour cannot be 0.

So we have the second possibility. We fix some $t' \in T(s,\bar{b}) \smallsetminus \{s\}$, $t' \in \mathrm{Lev}_{<\ell_0+1}(s,\bar{a})$ for some $\ell_0 \geq 0$. Then $(t', \bar{b} \text{ past } b_{\ell_0})$ has the following property: If $\ell \geq 0$ and if $t' \in (\mathrm{Lev}_{<\ell+1}(s,\bar{a})) \cap (T(s,\bar{b}) \smallsetminus \{s\})$, and if $i \leq \max(\bigcup a_\ell)$, then $(t', \bar{b} \text{ past } a_\ell)$ decides whether $i \in \underset{\sim}{A}$. □

We recall a very useful theorem:

Theorem 3.16. *(Eisworth* [9, "→" Theorem 4, "←" Cor. 2.5, this direction works also with non-P ultrafilters]*) Let $\mathcal{U}$ be a stable ordered-union ultrafilter over $[\omega]^{<\omega} \smallsetminus \{\emptyset\}$ and let $\mathcal{V}$ be a P-point. Then we have: $\Phi(\mathcal{U}) \not\leq_{RB} \mathcal{V}$ iff $\mathcal{V}$ continues to generate an ultrafilter after we force with $\mathbb{M}(\mathcal{U})$.*

Here $\mathbb{M}(\mathcal{U})$ stands for the Matet forcing [12] with a stable ordered-union ultrafilter $\mathcal{U}$, see [9]. Stable ordered-union ultrafilters are also called Milliken–Taylor ultrafilters. They are ultrafilters over the space $\mathbb{F}$ of non-empty finite subsets of ω. The projection to a filter over ω is $\Phi(\mathcal{U}) = \{X : \exists R \in \mathcal{U}, X \supseteq \bigcup R\}$. For more details we refer to [9] and [4]. Stable ordered-union ultrafilters and the projection function Φ will not be used in this work.

Here is an analogous theorem for $\mathbb{Q}^{242}(\mathcal{H})$:

Theorem 3.17. *Let $\mathcal{V}$ be a P-point and let $\mathcal{H} \subseteq (\mathcal{P})^\omega$ be a suitable set. Then the following holds:*

(1) If $\mathcal{H}$ avoids $\mathcal{V}$, then $\mathcal{V}$ continues to generate an ultrafilter after we force with $\mathbb{Q}^{242}(\mathcal{H})$.

(2) If $\mathcal{H}$ does not avoid $\mathcal{V}$ then there is a condition in $\mathbb{Q}^{242}(\mathcal{H})$ that forces that $\mathcal{V}$ is destroyed.

Proof. (1) We adapt the proof of [5, Theorem 3.3]. Let A be a name for a subset of ω. By genericity, it suffices to show that, if $(s,\bar{a})$ forces $A \subseteq \omega$, then some extension forces either $B \subseteq A$ or $B \subseteq A^c$, for some $B \in \mathcal{V}$. According to Proposition 3.15, we may assume that, for $\ell \geq 0$, $i \leq \max(\bigcup a_\ell)$, $t \in \mathrm{Lev}_{<\ell+1}(s, \bar{a})$ the condition $(t, \bar{a} \operatorname{past} a_\ell)$ decides whether $i \in A$. Consider any $t \in T(s, \bar{a})$.

Then $t \in \mathrm{Lev}_{<\ell+1}(s, \bar{a})$ for all sufficiently large ℓ. Thus, for any fixed $i \in \omega$, $(t, \bar{a} \operatorname{pastn} z)$ will decide whether $i \in A$ once there is ℓ such that $i \leq \max(\bigcup a_\ell) \leq z$; the decisions agree as z varies, since $(t, \bar{a} \operatorname{pastn} z')$ extends $(t, \bar{a} \operatorname{pastn} z)$ if $z' \geq z$. Let $A(t)$ be the set of those $i \in \omega$ for which the decision is positive.

Partition $T(s, \bar{a})$ by putting into one class all those $t \in T(s, \bar{a})$ for which $A(t) \in \mathcal{V}$. By the Ramsey property, we can extend $(s, \bar{a})$ to some $(s', \bar{b}) \in \mathbb{Q}^{242}(\mathcal{H})$ such that all of $T(s', \bar{b})$ is in a single class. When we form this extension, we do not destroy the fact that, for $t \in T(s', \bar{b})$ for $i \in A(t)$ (resp. $i \notin A(t)$), $(t, \bar{b} \operatorname{pastn} z)) \Vdash i \in A$ $(i \notin A)$ for all z such that there is ℓ such that $i \leq \max(\bigcup b_\ell) \leq z$. We assume henceforth that $A(t) \in \mathcal{V}$ for all $t \in T(s', \bar{b})$; in the other case A is replaced by its complement. As $\mathcal{V}$ is a P-point, let $B \in \mathcal{V}$ be almost included in each $A(t)$.

For $\ell \geq 0$, we let $n(\ell) = \max(\bigcup b_\ell) + 1 \geq \ell$. Inductively we define a sequence $\langle \zeta(k) : k \in \omega \rangle$ of natural numbers, starting with $\zeta(0) = 1$, and increasing so rapidly that, if $t \in \mathrm{Lev}_{<\zeta(k)+1}(s', \bar{b})$, then

(i) $B \setminus A(t) \subseteq \zeta(k+1)$, and

(ii) $\zeta(k+1) > n(\zeta(k))$.

We think of $n \circ \zeta$ as partitioning ω into blocks $[n(\zeta(k)), n(\zeta(k+1)))$, $k \in \omega$, and consider the four sets X_m, $m = 0, 1, 2, 3$, obtainable by taking the union of every fourth block:

$$X_m = \bigcup \{[n(\zeta(j)), n(\zeta(j+1))) : j = m \mod 4\}.$$

As $\mathcal{V}$ is an ultrafilter, it contains exactly one of these sets. By omitting a few terms (at most 3) from the sequence ζ, we may assume $m = 2$. Replacing B with $X_2 \cap B$, which is also in $\mathcal{V}$, we may assume $B \subseteq X_2$.

Let h_1 map $i \in [n(\zeta_{2k}), n(\zeta_{2k+2}))$ to $h_1(i) = k$, let h_2 map $i \in [n(\zeta_{2k+1}), n(\zeta_{2k+3}))$ to $h_2(i) = k$. Since $h_1(\Phi_2(\mathcal{H} \restriction \bar{b})) \not\subseteq h_1(\mathcal{V})$ there are $\bar{c}_1 \in \mathcal{H}$, $\bar{c}_1 \leq \bar{b}$ and $B' \in \mathcal{V}$, $B' \subseteq B \cap X_2$, such that

$$h_1[\mathrm{set}(\bar{c}_1)] \cap h_1[B'] = \emptyset.$$

Again since $h_2(\Phi_2(\mathcal{H} \restriction \bar{c}_1)) \not\subseteq h_2(\mathcal{V})$ there are $\bar{c} \in \mathcal{H}$, $\bar{c} \leq \bar{c}_1$ and $B'' \in \mathcal{V}$, $B'' \subseteq B' \cap X_2$, such that

(iii) $$h_j[\mathrm{set}(\bar{c})] \cap h_j[B''] = \emptyset \text{ for } j = 1, 2.$$

To complete the proof of the theorem, we show that $(s', \bar{c})$ forces $B'' \subseteq A$.

We fix an element $i \in B''$ and an extension $(t, \bar{d})$ of $(s', \bar{c})$ deciding whether $i \in A$. Since $B'' \subseteq B \subseteq X_2$, there is k such that $i \in [n(\zeta(4k+2)), n(\zeta(4k+3)))$. We fix k. By the choice of $\bar{b} \leq \bar{a}$ we can assume that $\bar{d} = \bar{c} \operatorname{pastn} n(\zeta(4k+3))$ and that $t \in \mathrm{Lev}_{<\zeta(4k+3)+1}(s', \bar{b})$ and $t \in T(s', \bar{c})$. We show that the decision is positive. Now by (iii), the set $\mathrm{set}(\bar{c})$ avoids the interval $[n(\zeta(4k+1)), n(\zeta(4k+4)))$, and hence $t \in \mathrm{Lev}_{<\zeta(4k+1)+1}(s', \bar{b})$.

Since $t \in \mathrm{Lev}_{<\zeta(4k+1)+1}(s', \bar{b})$ and $i \in B''$ and $i \geq n(\zeta(4k+2)) \geq \zeta(4k+2)$, clause (i) in the definition of ζ implies that $i \in A(t)$. So $(t, \bar{b} \operatorname{pastn} z) \Vdash i \in A$ if there is ℓ such that $i \leq \max(\bigcup b_\ell) < z$. Since $i < n(\zeta(4k+3))$, the number $\ell = \zeta(4k+3)$ is a witness. Hence $(t, \bar{b} \operatorname{pastn} n(\zeta(4k+3))) \Vdash i \in A$ and $(t, \bar{d}) \Vdash i \in A$.

(2) Suppose that $\bar{a} \in \mathcal{H}$ and $\mathcal{H} \restriction \bar{a} \leq_{\mathrm{RB}} \mathcal{V}$. We fix a finite-to-one function f that witnesses the latter, i.e.,

$$(\forall \bar{b} \leq \bar{a})\big(\bar{b} \in \mathcal{H} \rightarrow (\forall V \in \mathcal{V})(|f[V] \cap f[\mathrm{set}(\bar{b})]| = \omega)\big). \tag{3.1}$$

We show that this implies

$$(\emptyset, \bar{a}) \Vdash (\forall V \in \mathcal{V})(f[V] \not\subseteq^* f[W] \wedge f[V] \not\subseteq^* \omega \setminus f[W]).$$

Let $(s, \bar{b}) \leq (\emptyset, \bar{a})$, $\bar{b} \in \mathcal{H}$ and $V \in \mathcal{V}$. We partition V into two infinite parts V_1, V_2 such that each of the V_j avoids infinitely many sets $f^{-1}[\{i\}]$ with $i \in f[\mathrm{set}(\bar{b})] \cap f[V]$. By (3.1) there is such a partition. By fullness there is $\bar{c} \in \mathcal{H}$, $j \in 2$ with $\mathrm{set}(\bar{c}) \subseteq V_j$. Then $(s, \bar{c}) \Vdash f[V] \not\subseteq^* f[W]$. Again by (3.1), $(s, \bar{c}) \Vdash f[V] \not\subseteq^* \omega \setminus f[W]$. So $(\emptyset, \bar{a})$ forces that $f(\mathcal{V})$ is not an ultrafilter. Then also $\mathcal{V}$ is not an ultrafilter in any generic extension with $(\emptyset, \bar{a}) \in G$. □

Definition 3.18. (1) A subset $\mathcal{C} \subseteq (\mathcal{P})^\omega$ is called centred if any finitely many members of $\mathcal{C}$ have a common lower bound in $\mathcal{C}$.

(2) A centred subset $\mathcal{C} \subseteq (\mathcal{P})^\omega$ is called maximal if for any $\bar{a} \notin \mathcal{C}$ there is $\bar{b} \in \mathcal{C}$, $\bar{a} \perp^* \bar{b}$.

In the case of a suitable centred family $\mathcal{C}$, we have: $\mathcal{C}$ avoids $\mathcal{V}$ iff $\Phi_2(\mathcal{C}) \not\leq_{RB} \mathcal{V}$. Now we read Theorem 3.17 for the special case of $\mathcal{H} = \mathcal{C}$ and thus finish the proof of Theorem 1.1.

4. Existence of centred suitable sets under CH or MA

In this section we show that under CH, given a P-point $\mathcal{V}$ there is a suitable maximal centred family $\mathcal{C}$ such that $\mathcal{C}$ avoids the $\mathcal{V}$. A natural $\leq^*$-descending construction will give just a stable family. We add diagonal lower bounds in the family by explicit construction steps.

Remark 4.1. If we replace the lim-requirement in Def.2.2(3) by the weaker lim sup-requirement then maximal centred sets (in the altered space) are full.

Proof. Let $\mathcal{C}$ be a maximal centred suitable set. Let $X \subseteq \omega$ be given. By the definition of the norm and the modified definition of $(\mathcal{P})^\omega$, for any $\bar{c} \in \mathcal{C}$ we have $\bar{c} \restriction X \in (\mathcal{P})^\omega$ or $\bar{c} \restriction X^c \in (\mathcal{P})^\omega$. Since $\mathcal{C}$ is centred, we have for $Y = X$ or for $Y = X^c$, for any

$\bar{c} \in \mathcal{C}$, $\bar{c} \restriction Y \in (\mathcal{P})^\omega$. So $\mathcal{C}' = \{\bar{b} \,:\, (\exists \bar{c} \in \mathcal{C})(\bar{b} \geq^* \bar{c} \restriction Y)\}$ is a centred suitable set. Since $\mathcal{C}$ was maximal and $\mathcal{C}' \supseteq \mathcal{C}$, we have $\mathcal{C}' = \mathcal{C}$. So there is $\bar{d} \in \mathcal{C}$ with $\bar{d} = \bar{c} \restriction Y$ and hence $\mathrm{set}(\bar{d}) \subseteq Y$. □

However, in the original definition of the space $(\mathcal{P})^\omega$ we do not know whether maximal centred families are full. For $\mathcal{H} = (\mathcal{P})^\omega$, the lim and the lim sup requirement on the norms $\mathrm{norm}(a_n)$, $n \in \omega$, give equivalent forcings. We do not know whether this is still true for arbitrary (suitable) families.

If $\mathcal{C}$ is a suitable maximal centred set, then forcing with $\mathbb{Q}^{242}(\mathcal{C})$ diagonalises the ultrafilter $\Phi_2(\mathcal{C})$ by adding W_G. Hence the fullness and the maximality of $\mathcal{C}$ are destroyed.

Lemma 4.2. *Under CH or under Martin's Axiom for σ-centred forcings and for $< 2^\omega$ dense sets, given an P-point $\mathcal{V}$ there is a suitable maximal centred set $\mathcal{C}$ such that $\mathcal{C}$ avoids $\mathcal{V}$.*

Proof. Let $\langle (C_\alpha, n_\alpha, D_\alpha, X_\alpha, h_\alpha, \bar{a}_\alpha) \,:\, \alpha < 2^\omega \rangle$ enumerate all tuples $(C, n, D, X, h, \bar{a})$ such that $C \colon [\omega]^{<\omega} \to 2$ is a colouring, $n \in \omega$, $D = \langle \bar{d}_n \,:\, n \in \omega \rangle$ is a $\leq$-descending sequence, $X \subseteq \omega$, h is a finite-to-one function, and $\bar{a} \in (\mathcal{P})^\omega$, such that each tuple appears cofinally often in the enumeration. By induction on $\alpha < 2^\omega$ we choose $\bar{c}_\alpha \in (\mathcal{P})^\omega$ such that $(\forall \beta < \alpha) \bar{c}_\alpha \leq^* \bar{c}_\beta$. We let $\bar{c}_0$ be any element of $(\mathcal{P})^\omega$.

In the successor steps, given $\bar{c}_\alpha$ and C_α we first take care of the Ramsey property: By Theorem 3.15 we can choose $\bar{c}_{\alpha+0.3} \leq (\bar{c}_\alpha \,\mathrm{pastn}\, n_\alpha)$ such that for some s, we have $(s, \bar{c}_{\alpha+0.3}) \leq (\emptyset, \bar{c}_\alpha \,\mathrm{pastn}\, n_\alpha)$ and the colouring C_α is monochromatic with colour 0 on $T(\emptyset, \bar{c}_{\alpha+0.3}) \setminus \{\emptyset\}$ or C_α is monochromatic with colour 1 on $T(s, \bar{c}_{\alpha+0.3})$.

Now we take care of the diagonal lower bounds: Given a $\leq$-decreasing sequence $D_\alpha = \langle \bar{d}_n \,:\, n \in \omega \rangle$, we distinguish two cases: First case: Each $\bar{d}_n \not\perp^* \bar{c}_{\alpha+0.3}$. We choose a sufficiently fast growing, increasing function $j \colon \omega \to \omega$ such that for $n \in \omega$, there is some $\bar{e}_n \in (\mathcal{P})^\omega$ such that for any $k \leq n$, $\bar{e}_n \leq (\bar{c}_{\alpha+0.3} \,\mathrm{past}\, j(n)), (\bar{d}_k \,\mathrm{past}\, j(n))$. The sequence $\bar{e}_n$ need not be decreasing. We choose $\bar{c}_{\alpha+0.5} = \langle c_k \,:\, k \in \omega \rangle$ by induction on k. We let $c_0 = e_{0,i}$ with an i such that $e_{0,i}$ has norm at least 1. For $k \geq 0$, given c_k, we take c_{k+1} such that $c_{k+1} = e_{\max(\bigcup c_k), i}$ with i so large that c_{k+1} has norm at least $k+1$. By construction $\bar{c}_{\alpha+0.5} = \bar{c}$ is a diagonal lower bound of D and $\bar{c}_{\alpha+0.5} \leq \bar{c}_{\alpha+0.3}$. Second case: There is n such that $\bar{d}_n \perp^* \bar{c}_{\alpha+0.3}$. Then we let $\bar{c}_{\alpha+0.5} = \bar{c}_{\alpha+0.3}$.

Next we ensure an instance of fullness: We choose $\bar{c}_{\alpha+0.7} \leq \bar{c}_{\alpha+0.5}$ that that $\mathrm{set}(\bar{c}_{\alpha+0.7})$ is a subset of X_α or of X_α^c.

Next we take care of the task $h_\alpha(\Phi_2(\mathcal{C})) \not\subseteq h_\alpha(\mathcal{V})$. The set $\{h_\alpha^{-1}[\mathrm{set}(\bar{b})] \,:\, \bar{b} \leq \bar{c}_{\alpha+0.7}\}$ is not a filter base. Hence the $\{h_\alpha^{-1}[\mathrm{set}(\bar{b})] \,:\, \bar{b} \leq \bar{c}_{\alpha+0.7}\}$ is not a subset of $\{h_\alpha^{-1}[X] \,:\, X \in \mathcal{V}\}$. Thus there is an $\bar{c}_{\alpha+0.9} \leq \bar{c}_{\alpha+0.7}$ such that $\omega \setminus h_\alpha^{-1}[\mathrm{set}(\bar{c}_{\alpha+0.9})] \in \mathcal{V}$.

Now we ensure an instance of maximality: If $\bar{c}_{\alpha+0.9} \not\perp^* \bar{a}_\alpha$ we take $\bar{c}_{\alpha+1} \leq \bar{c}_{\alpha+0.9}, \bar{a}_\alpha$, otherwise we let $\bar{c}_{\alpha+1} = \bar{c}_{\alpha+0.9}$.

This finishes the successor step.

In the limit steps $\alpha < 2^\omega$ of countable cofinality, we choose a cofinal sequence α_n, $n \in \omega$, converging to α, and then we take as $\bar{c}_\alpha$ a $\le^*$-lower bound of $\bar{c}_{\alpha_n}$, $n \in \omega$. If the continuum is larger and Martin's Axiom holds, in the limit steps α of uncountable cofinality in the construction we force a lower bound with the following σ-centred approximation forcing $\mathbb{Q}$: Conditions are pairs $(\bar{a}, F)$ such that $\bar{a} \in (\mathcal{P})^{<\omega}$, $\mathrm{norm}(a_i) \ge i+1$ for any $i < \mathrm{lg}(\bar{a})$ and F is a finite subset of $\{\bar{c}_\beta \,:\, \beta < \alpha\}$. We let $(\langle b_0, \dots, b_{n-1}\rangle, F') \le_{\mathbb{Q}} (\langle a_0, \dots, a_{m-1}\rangle, F)$ iff $F' \supseteq F$, $n \ge m$, $b_i = a_i$ for $i < m$ and for $i \in [m, n)$ the set of possibilities b_i is for every $\bar{c} \in F$ an element of a condensation of $\bar{c}$ and has norm at least $i+1$.

Having chosen $\bar{c}_\alpha$, $\alpha < 2^\omega$, we let

$$\mathcal{C} = \{\bar{a} \,:\, (\exists \alpha < 2^\omega)(\bar{a} \ge^* \bar{c}_\alpha)\}.$$

It is clear that $\mathcal{C}$ is centred, maximal, full and closed under diagonal lower bounds. We show the Ramsey property. Let $C \colon [\omega]^{<\omega} \to 2$ and $(s, \bar{a})$ be given such that $\bar{a} \in \mathcal{C}$. We take $\alpha < 2^\omega$ such that $\bar{a} \ge (\bar{c}_\alpha \text{ past}\, n_\alpha)$, $\max(s) < n_\alpha$, and $C_\alpha(t) = C(s \cup t)$ for $t \in T(\emptyset, \bar{c}_\alpha \text{ past}\, n_\alpha)$. By construction there is some t, $(t, \bar{c}_{\alpha+0.3}) \le (\emptyset, \bar{c}_\alpha \text{ past}\, n_\alpha)$ such that the colouring C_α is monochromatic with colour 0 on $T(\emptyset, \bar{c}_{\alpha+0.3}) \smallsetminus \{\emptyset\}$ or that C_α is monochromatic with colour 1 on $T(t, \bar{c}_{\alpha+0.3})$. Then C is monochromatic with colour 0 on $T(s, \bar{c}_{\alpha+0.3}) \smallsetminus \{s\}$ or that C is monochromatic with colour 1 on $T(s \cup t, \bar{c}_{\alpha+0.3})$. $\square$

Thus we finish the proof of Theorem 1.2.

5. Discussion and questions

Recall in Mathias forcing with an ultrafilter $\mathcal{U}$ conditions are of the form (s, A), $s \in [\omega]^{<\omega}$, $\max(s) < \min(A)$, $A \in \mathcal{U}$ and $(t, B) \le (s, A)$ if $t \smallsetminus s \subset A$ and $t \supseteq s$ and $B \subseteq A$.

Our result can be seen as an answer to the following question.

Question 5.1. *Is there a relative of Mathias forcing with an ultrafilter $\mathcal{U}$ that preserves another ultrafilter?*

For an affirmative answer the ultrafilter $\mathcal{U}$ cannot be rapid and hence cannot be a Ramsey ultrafilter, as Mathias forcing would add a dominating real and thus destroy any ultrafilter. On the other hand, for preserving a P-point Ramsey-theoretic properties of the forcing are often very useful. Thus we provided for Ramsey-theoretic properties by superposing more structure and norms and letting $\Phi_2(\mathcal{H}) = \mathcal{U}$ be an ultrafilter. There is a projection mapping $(s, \bar{a}) \in \mathbb{Q}^{242}(\mathcal{H})$ to $(a, \mathrm{set}(\bar{a}))$. However, we do not know whether this projection can be inverted to a complete embedding.

Ramsey ultrafilters over ω exist (even generically exist) under $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ [7], where $\mathcal{M}$ is the ideal of meagre subets of the reals and $\mathrm{cov}(\mathcal{M})$ is the covering number of this ideal, i.e., smallest size of a set of meagre sets whose union is the real line. Eisworth [9] showed that also stable ordered-union ultrafilters generically exist under $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$. This motivates the next question.

Question 5.2. *Can in Theorem 1.2 Martin's Axiom for σ-centred sets be replaced by $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$?*

Another topic is the possibility of iterating forcings of the type $\mathbb{Q}^{242}(\mathcal{H})$. We are interested in a tower of suitable maximal centred $\mathcal{C}$ in successive forcing extensions. For building such a tower it would be some ease of work if our list of requirements were redundant.

Here are some questions about redundancy. Let $\mathcal{H} \subseteq (\mathcal{P})^\omega$.

Question 5.3. *Does "$\mathcal{H}$ is maximal centred" imply "$\mathcal{H}$ is full"?*

Question 5.4. *Does the Ramsey property together with the existence of (diagonal) lower bounds for $\mathcal{H}$ imply fullness? If we add maximal centredness to the premises?*

Question 5.5. *Does the Ramsey property together with the existence of (diagonal) lower bounds and centredness for $\mathcal{H}$ imply maximal centredness? If we add fullness to the premises?*

The following question is about iteration.

Question 5.6. *Let $\mathcal{C}$ be maximal centred and suitable and avoiding $\mathcal{V}$. Is there a forcing $\mathbb{Q}$ of size at most 2^ω that preserves $\mathcal{V}$ such that in $\mathbf{V}^{\mathbb{Q}^{242}(\mathcal{C})*\mathbb{Q}}$, is there $\mathcal{C}' \supseteq \mathcal{C}$ that is centred and suitable?*

If so, in the iteration step of countable cofinality of iterands of the type $\mathbb{Q}^{242}(\mathcal{C}_\alpha) * \mathbb{Q}_\alpha$ is there a suitable maximal centred $\mathcal{C}$ extending the families used in the preceding iterands?

Question 5.6 has the following interesting sub-question.

Question 5.7. *Is there a forcing $\mathbb{Q}$ as in the previous question such that in the extension by $\mathbb{Q}^{242}(\mathcal{C}) * \mathbb{Q}$, the Ramsey property for a given coloring is witnessed by a sequence $\bar{b} \in (\mathcal{P})^\omega$ that is positive in the sense of the family $\mathcal{C}$?*

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