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Meeting infinitely many cells of a partition once

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Abstract. We investigate several versions of a cardinal characteristic $\mathfrak{f}$ defined by Frankiewicz. Vojtáš showed $\mathfrak{b} \leq \mathfrak{f}$, and Blass showed $\mathfrak{f} \leq \min(\mathfrak{d}, \mathrm{unif}(\mathbf{K}))$. We show that all the versions coincide and that $\mathfrak{f}$ is greater than or equal to the splitting number. We prove the consistency of $\max(\mathfrak{b}, \mathfrak{s}) < \mathfrak{f}$ and of $\mathfrak{f} < \min(\mathfrak{d}, \mathrm{unif}(\mathbf{K}))$.

1. Introduction

We start with the definition of several cardinal characteristics. "There are infinitely many" is abbreviated by $\exists^\infty$, the dual quantifier "for all but finitely many" is $\forall^\infty$. In our context, a partition is a set of pairwise disjoint sets that combine to ω. The set of all functions from ω to ω is written as ω^ω; and the set of all infinite subsets of ω is written as $[\omega]^\omega$. For $f, g \in \omega^\omega$ the ordering of eventual dominance is defined by $f \leq^* g$ iff $\forall^\infty n\ f(n) \leq g(n)$. The set ω is equipped with the discrete topology. The Baire space ω^ω carries the product topology.

The well-known cardinal invariants we are dealing with are: the splitting number $\mathfrak{s} = \min\{|\mathscr{S}| : \mathscr{S} \subseteq [\omega]^\omega \ \wedge\ \forall X \in [\omega]^\omega\ \exists S \in \mathscr{S}\ |X \cap S| = |X \setminus S| = \omega\}$, the (un)bounding number $\mathfrak{b} = \min\{|\mathscr{B}| : \mathscr{B} \subseteq \omega^\omega \ \wedge\ \forall f \in \omega^\omega\ \exists b \in \mathscr{B}\ b \not\leq^* f\}$, the dominating number $\mathfrak{d} = \min\{|\mathscr{D}| : \mathscr{D} \subseteq \omega^\omega \ \wedge\ \forall f \in \omega^\omega\ \exists d \in \mathscr{D}\ f \leq^* d\}$, and the uniformity of the sets of first Baire category $\mathrm{unif}(\mathbf{K}) = \min\{|A| : A \subseteq \omega^\omega$ is not meager$\}$.

Definition 1. *For $r \in \omega$:*

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$$\mathfrak{f}_{1,r+1} := \min\{|\mathscr{A}| : \mathscr{A} \subseteq [\omega]^\omega \wedge \forall \textit{ partitions } \mathscr{P} \textit{ into finite intervals } \exists A \in \mathscr{A}\, \exists^\infty \textit{ pieces } P \in \mathscr{P}\ 1 \leq |P \cap A| \leq r+1\}.$$

$$\mathfrak{f}_{2} := \min\{|\mathscr{A}| : \mathscr{A} \subseteq [\omega]^\omega \wedge \forall \textit{ partitions } \mathscr{P} \textit{ into finite intervals } \exists A \in \mathscr{A}\, \exists r \in \omega\ \exists^\infty \textit{ pieces } P \in \mathscr{P}\ 1 \leq |P \cap A| \leq r+1\}.$$

$$\mathfrak{f}_{3} := \min\{|\mathscr{A}| : \mathscr{A} \subseteq [\omega]^\omega \wedge \exists r \in \omega\, \forall \textit{ partitions } \mathscr{P} \textit{ into finite intervals } \exists A \in \mathscr{A}\ \exists^\infty \textit{ pieces } P \in \mathscr{P}\ 1 \leq |P \cap A| \leq r+1\}.$$

If we replace in any of these definitions "finite intervals" by "finite sets", then we get an invariant that we denote with the same indexed letter but primed.

The families $\mathscr{A}$ in the different sets are called "good" for the cardinal in question, and the families $\mathscr{A}$ of minimal cardinality are called "witnesses" for the considered cardinal.

2. Equalities

There are some obvious inequalities: $\mathfrak{f}_2 \leq \mathfrak{f}_3 \leq \mathfrak{f}_{1,r+1} \cdots \leq \mathfrak{f}_{1,1}$ for $r \in \omega$, and the same for the primed versions, as well as $\mathfrak{f}_x \leq \mathfrak{f}'_x$ for all meaningful subscripts. Now we show that each primed invariant is the same as the unprimed one. Thereafter, we will work only with the (unprimed) interval versions.

Theorem 1. $\mathfrak{f}_{1,r+1} = \mathfrak{f}'_{1,r+1}, \mathfrak{f}_j = \mathfrak{f}'_j$ *for* $r \in \omega, j = 2, 3$.

Proof. Let $\mathscr{A}$ be a witness for the definition of $\mathfrak{f}_{1,r+1}$. For $Y \in [\omega]^\omega$, we let e_Y denote the increasing bijection $\omega \to Y$. We set $\tilde{\mathscr{A}} = \{e_Y[A] : A, Y \in \mathscr{A}\} \cup \mathscr{A}$ and show that $\tilde{\mathscr{A}}$ meets any partition of ω into finite sets in infinitely many parts between 1 and $r+1$ times.

For any partition $\mathscr{P}$ of ω into finite sets, we define an increasing function $f_\mathscr{P} : \omega \to \omega$ in the following manner:

$$\begin{aligned} f_\mathscr{P}(0) &= 0, \\ f_\mathscr{P}(n+1) &= \max\{\bigcup P : P \in \mathscr{P}, P \cap [0, f_\mathscr{P}(n)] \neq \emptyset\} + 1. \end{aligned}$$

Given any increasing function $f \in \omega^\omega$, we interpret it as a partition $\mathscr{Q}(f)$ of ω into finite intervals:

$$\mathscr{Q}(f) = \{[0, f(0))\} \cup \{[f(i), f(i+1)) : i \in \omega\}.$$

We will write only f instead of $\mathscr{Q}(f)$. The choices of the open and the closed end matter only in the proof of theorem 3. We also have: $\forall P \in \mathscr{P}\ \exists n\ P \subseteq [f_\mathscr{P}(n), f_\mathscr{P}(n+2))$.

In the first step, we "treat" a partition gotten by combining pairs of consecutive blocks of $f_{\mathscr{P}}$. The properties of $\mathscr{A}$ yield:

$$\exists A \in \mathscr{A} \;\; \exists^{\infty} i \in \omega \;\; 1 \leq |A \cap [f(2i), f(2(i+1)))| \leq r+1.$$

We fix such an A.

First case:

$$\begin{aligned} \exists^{\infty} i \in \omega \;\; \exists P \in \mathscr{P} \qquad & (1 \leq |A \cap [f_{\mathscr{P}}(2i), f_{\mathscr{P}}(2(i+1)))| \leq r+1 \\ & \text{ and } A \cap [f_{\mathscr{P}}(2i), f_{\mathscr{P}}(2i+1)) \cap P \neq \emptyset \\ & \text{ and } A \cap [f_{\mathscr{P}}(2i+1), f_{\mathscr{P}}(2(i+1))) \cap P \neq \emptyset). \end{aligned}$$

For any $P \in \mathscr{P}$ such that $A \cap P \cap [f_{\mathscr{P}}(2i), f_{\mathscr{P}}(2i+1)) \neq \emptyset$ and $A \cap [f_{\mathscr{P}}(2i+1), f_{\mathscr{P}}(2(i+1))) \cap P \neq \emptyset$, by the definition of $f_{\mathscr{P}}$ we have $P \subseteq [f_{\mathscr{P}}(2i), f_{\mathscr{P}}(2(i+1)))$. So we take for each of those infinitely many i one or more $P \in \mathscr{P}$ with these two properties.

Second case:

$$\begin{aligned} \exists^{\infty} i \in \omega \qquad\qquad & (1 \leq |A \cap [f_{\mathscr{P}}(2i), f_{\mathscr{P}}(2(i+1)))| \leq r+1 \\ \text{and } \forall P \in \mathscr{P} \quad & (A \cap P \cap [f_{\mathscr{P}}(2i), f_{\mathscr{P}}(2i+1)) = \emptyset \\ & \text{ or } A \cap P \cap [f_{\mathscr{P}}(2i+1), f_{\mathscr{P}}(2(i+1))) = \emptyset)). \end{aligned}$$

Now we define a new partition, that is coarser and shifted to the odd arguments: We enumerate those infinitely many i's in the case hypothesis increasingly as $\langle i_n : n \in \omega \rangle$. We take the partition defined by $g(n) = f_{\mathscr{P}}(2i_n + 1)$. We think of this partition shrunk to the domain A, explicitly: $g_{0,A}(0) = |[0, g(0)) \cap A|$, $g_{0,A}(n+1) = g_{0,A}(n) + |[g(n), g(n+1)) \cap A|$.

This shrinkage procedure yields: If A' is good for $g_{0,A}$, then $e_A[A']$ is good for g. Then we have $A' \in \mathscr{A}$ such that $e_A[A']$ is good for the partition g. Since $e_A[A'] \subseteq A$, for infinitely many n it meets the interval $[f_{\mathscr{P}}(2i_n+1), f_{\mathscr{P}}(2i_{n+1}+1))$ between 1 and $r+1$ times in a piece P of $\mathscr{P}$ such that P is not met by A (and hence neither by $e_A[A']$) again in the part of P possibly sticking out into $[f_{\mathscr{P}}(2i_n), f_{\mathscr{P}}(2i_n+1))$ or into $[f_{\mathscr{P}}(2i_{n+1}+1), f_{\mathscr{P}}(2i_{n+1}+2))$.

For the other versions, we can use almost the same proof: If in the second use of $\mathscr{A}$ a larger r appears, we just take this as a final r. □

Remark: Indeed, our proof gives a morphism from the primed relation into the sequential composition of two copies of the corresponding unprimed relation; for details about morphism constructions see [1].

Now we show that all the versions coincide; and we shall call the invariant $\mathfrak{f}$.

Proposition 1. $\mathfrak{f}_{1,1} \leq \mathfrak{f}_2$.

Proof. For any $A \in [\omega]^\omega$, $r \in \omega$, we thin out A as follows: Let $\langle a(n) : n \in \omega \rangle$ be the strictly increasing enumeration of A. We set

$$s(A, r) = \{a(n \cdot (r + 1)) : n \in \omega\}.$$

Let $\mathscr{A}$ be a witness for $\mathfrak{f}_2$. We show that $\tilde{\mathscr{A}} = \{s(A, r) : A \in \mathscr{A}, r \in \omega\}$ is a set good in the sense of $\mathfrak{f}_{1,1}$.

Let $\mathscr{P} = \langle p(n) : n \in \omega \rangle$ be a partition of ω into intervals. As $\mathscr{A}$ is good for $\mathfrak{f}_2$ we have

$$\exists r \ \ \exists^\infty n \ \ |[p(n), p(n+1)) \cap A| = r + 1.$$

For those infinitely many n, $[p(n), p(n+1)) \cap A$ consists of $r + 1$ consecutive elements of A. Hence we have $|[p(n), p(n+1)) \cap s(A, r)| = 1$. □

3. Inequalities

In this section we show in *ZFC* that $\max(\mathfrak{b}, \mathfrak{s}) \leq \mathfrak{f} \leq \min(\mathfrak{d}, \mathrm{unif}(\mathbf{K}))$.

If we work with the strictly increasing enumeration $\langle a_n : n \in \omega \rangle$ of $A \in \mathscr{A}$ and the increasing function p for a partition $\mathscr{P}$, "A meets infinitely many parts of $\mathscr{P}$ in one element" translates to

$$\exists^\infty n \ \exists k \ a(k-1) < p(n) \leq a(k) < p(n+1) \leq a(k+1) =: R(p, a).$$

For each $p \in \omega^{\omega\uparrow}$, the set of all strictly increasing functions from ω to ω, the set

$$R_p := \{a \in \omega^{\omega\uparrow} : R(p, a)\}$$

is a comeager subset of the Baire space $\omega^{\omega\uparrow}$. Any non-meager set $\mathscr{A} \subseteq [\omega]^\omega$ will intersect all the R_p's and hence $\mathfrak{f} \leq \mathrm{unif}(\mathbf{K})$.

We next give a proof of Vojtáš' and Blass' observations. Then we show $\mathfrak{f} \geq \mathfrak{s}$.

Theorem 2 (Vojtáš, Blass). $\mathfrak{b} \leq \mathfrak{f} \leq \mathfrak{d}$.

Proof. First inequality, which is proved in [5]: Assuming that $\mathscr{A} \subseteq [\omega]^\omega$ has cardinality strictly less than $\mathfrak{b}$ we give a partition $\mathscr{P}$ of ω into finite intervals that $\forall r \in \omega \ \forall A \in \mathscr{A}$ for all but finitely many pieces P of $\mathscr{P}$, the piece P is met by A in more than r points. This shows that even if we leave out the $1 \leq |A \cap P|$ in the requirement for $\mathfrak{f}_2$, we will get an invariant greater or equal than $\mathfrak{b}$. (Indeed, then we get exactly $\mathfrak{b}$, which is proved in [5].) We enumerate $\mathscr{A}$ as $\langle A_\alpha : \alpha < \gamma < \mathfrak{b} \rangle$, and define $g_\alpha \colon \omega \to \omega$, increasing, $g_\alpha(0) = 0$, $g_\alpha(n+1)$ =the $(n+1)$-st element in A_α after $g_\alpha(n)$.

There is some $g \in \omega^\omega$ that dominates all the g_α. We define $h(0) = g(0)$, $h(n+1) = g(h(n) + 1)$, and consider the partition defined by h. We show:

$$\forall^\infty n \ \ |[h(n), h(n+1)) \cap A_\alpha| \geq h(n).$$

We take n_0 such that $\forall n \geq n_0$, $g(h(n)+1) \geq g_\alpha(h(n)+1)$. Then we have for $n \geq n_0$: $h(n+1) = g(h(n)+1) \geq g_\alpha(h(n)+1)$ = the $(h(n)+1)$st element of A_α after $h(n)+1$.

The proof of the second inequality is based upon the same ideas and shows $\mathfrak{f}_{1,1} \leq \mathfrak{d}$. We take a dominating family $\{g_\alpha : \alpha \in \mathfrak{d}\}$. Again, we define $h_\alpha(0) = g_\alpha(0)$, $h_\alpha(n+1) = g_\alpha(h_\alpha(n)+1)$, and we take $A_\alpha = \text{range}(h_\alpha)$. Suppose we are given a partition $\mathscr{P} = \langle f(n) : n \in \omega \rangle$. We choose an α such that $f \leq^* g_\alpha$, and show that A_α is good for $\mathscr{P}$ in the sense of $\mathfrak{f}_{1,1}$, that is $\exists^\infty n \ \ |[f(n), f(n+1)) \cap A_\alpha| = 1$. As A_α is an infinite set, $\exists^\infty n \ \ [f(n), f(n+1)) \cap A_\alpha \neq \emptyset$. We show that for all but finitely many of those n there is exactly one element in the intersection.

Suppose that $\forall n \geq n_0 \ \ g_\alpha(n) \geq f(n)$ and that $n \geq n_0$ and that k is minimal such that $f(n) \leq h_\alpha(k) < f(n+1)$. Then $h_\alpha(k+1) = g_\alpha(h_\alpha(k)+1) \geq f(h_\alpha(k)+1) \geq f(f(n)+1) \geq f(n+1)$; and hence $h_\alpha(k)$ is the only element in the intersection. □

Theorem 3. $\mathfrak{f} \geq \mathfrak{s}$.

Proof. The main part is the following

Observation: Let $\langle a(n) : n \in \omega \rangle$ be an increasing enumeration of a set A, and let $r \in \omega$. For convenience, we set $a(-1) = -1$. We partition ω into $r+1$ pieces $Y(A, i, r)$, $i \leq r$:

$$Y(A, i, r) = \bigcup \{[a((r+1)n + i - 1) + 1, a((r+1)n + i) + 1) : n \in \omega\}.$$

Assume we have a partition $\mathscr{P} = \{[0, p(0))\} \cup \{[p(k), p(k+1)) : k \in \omega\}$ such that $\exists i \leq r \ \forall k \in \omega \ p(k) \in Y(A, i, r)$. Then we have:

$$\forall k \in \omega \ \ \exists \ell \in \omega \ \ |[p(k), p(k+1)) \cap A| = \ell(r+1).$$

The best way to see this is drawing a picture with a line, some points and looking at it. □ (observation)

Now suppose we have $\mathscr{A} \subset [\omega]^\omega$ of cardinality less than $\mathfrak{s}$. Then also

$$\mathscr{A}' = \{Y(A, i, r) : A \in \mathscr{A}, r \in \omega, i \leq r\}$$

has cardinality less than $\mathfrak{s}$. Hence there is a $p \in \omega^{\omega\uparrow}$ such that range(p) is not split by any element of $\mathscr{A}'$, i.e.

$$\forall A \in \mathscr{A} \ \forall r \in \omega \ \exists i \leq r \ \text{range}(p) \subseteq^* Y(A, i, r).$$

Above some $p(n)$, the observation is applicable and yields

$$\forall r \in \omega \ \forall^\infty n \in \omega \ |[p(n), p(n+1)) \cap A| \notin \{1, 2, \dots r\},$$

so $\mathscr{A}$ is not a family as in the definition of $\mathfrak{f}_2$. □

4. Consistency results

In this section, we show: In *ZFC*, $\mathfrak{f}$ cannot be pinned down as $\max(\mathfrak{b}, \mathfrak{s})$ nor as $\min(\mathfrak{d}, \mathrm{unif}(\mathbf{K}))$.

A forcing notion $\mathbf{P}$ is called ω^ω-bounding iff for every $\mathbf{P}$-generic filter G over V:

$$\forall f \in \omega^\omega \cap V[G] \quad \exists g \in \omega^\omega \cap V \quad f \leq^* g,$$

or even without an *; that does not make any difference here.

We are now thinking in terms of the $\mathfrak{f}_{1,1}$ version and use the following two abbreviations: For $A \subseteq \omega$ and a partition p we say "A is good for p" iff $\exists^\infty n \ |A \cap [p(n), p(n+1))| = 1$. For $\mathscr{A} \subseteq [\omega]^\omega$, we say "$\mathscr{A}$ is good for p" iff $\exists A \in \mathscr{A}$ such that A is good for p.

Proposition 2. *ω^ω-bounding forcing does not increase $\mathfrak{f}$.*

We prove a lemma that immediately yields the above proposition.

For $g \in \omega^\omega$, let $\tilde{g}$ be defined by

$$\begin{aligned}\tilde{g}(0) &= g(0),\\ \tilde{g}(n+1) &= g(\tilde{g}(n)).\end{aligned}$$

As in Theorem 1, for $A \in [\omega]^\omega$ and a partition $h \in \omega^{\omega\uparrow}$ let $h_{0,A}$ be the partition of ω that is given by h shrunk to A, explicitly: $h_{0,A}(0) = |[0, h(0)) \cap A|$, $h_{0,A}(n+1) = h_{0,A}(n) + |[h(n), h(n+1)) \cap A|$.

Let e_A be the increasing enumeration of A, $e_A \colon \omega \overset{bijective}{\rightarrow} A$. As in Theorem 1 we will use: If A' is good for $h_{0,A}$, then $e_A[A']$ is good for h.

If A is good for $\langle h(2n) : n \in \omega \rangle$, we define h_A: We take an increasing enumeration $\langle i_n : n \in \omega \rangle$ of the infinitely many i's such that $|[h(2i), h(2i+2)) \cap A| = 1$ and set $h_A(n) = h(2i_n + 1)_{0,A}$.

Lemma 1. *If $f \leq^* g$ and A is good for $\langle \tilde{g}(2n) : n \in \omega \rangle$ and A' is good for $\tilde{g}_A$, then $e_A[A']$ is good for f.*

Proof. We show that all but finitely many of those infinitely many n such that $|A' \cap [\tilde{g}_A(n), \tilde{g}_A(n+1))| = 1$ there exists some $k(n)$ such that the function k is injective and such that $|e_A[A'] \cap [f(k(n)), f(k(n)+1))| = 1$. We take n such that $|A' \cap [\tilde{g}_A(n), \tilde{g}_A(n+1))| = 1$ and such that for all $k \geq n$, $f(k) \leq g(k)$. For such an n, we define $k(n)$ as the unique k such that the singleton $e_A[A'] \cap [\tilde{g}(2i_n + 1), \tilde{g}(2i_{n+1}+1)) \subseteq [f(k), f(k+1))$. We show that $e_A[A']$ does not hit $[f(k), f(k+1))$ in $[f(k), f(k+1)) \setminus [\tilde{g}(2i_n + 1), \tilde{g}(2i_{n+1} + 1))$. So we suppose that the latter is not empty and consider the two cases:

First case: $f(k) \leq \tilde{g}(2i_n + 1) < f(k+1) \leq \tilde{g}(2i_n + 2)$. Then $\tilde{g}(2i_n) < f(k)$, and since $A \cap [\tilde{g}(2i_n + 1), \tilde{g}(2i_n + 2)) = A \cap [\tilde{g}(2i_n + 1), f(k+1)) \neq \emptyset$, we have $e_A[A'] \cap [f(k), \tilde{g}(2i_n + 1)) \subseteq A \cap [\tilde{g}(2i_n), \tilde{g}(2i_n + 1)) = \emptyset$.

Second case: $\tilde{g}(2i_{n+1}) < f(k) \leq \tilde{g}(2i_{n+1}+1) < f(k+1)$. Then $f(k+1) \leq \tilde{g}(2i_{n+1}+2)$ and we have $e_A[A'] \cap [\tilde{g}(2i_{n+1}+1), f(k+1)) \subseteq A \cap [\tilde{g}(2i_{n+1}+1), \tilde{g}(2i_{n+1}+2)) = \emptyset$.

This also shows that k is injective. □

The lemma gives us: If $f \leq^* g$ and $\mathscr{A}$ is good for $\langle \tilde{g}(2n) : n \in \omega \rangle$ and good for $\tilde{g}_A$ for $A \in \mathscr{A}$, then $\{e_A[A'] : A, A' \in \mathscr{A}\}$ is good for f, which is just a more constructive form of the proposition. □ (proposition)

Now we get

Theorem 4. $\mathfrak{b} = \mathfrak{s} = \mathfrak{f} = \aleph_1 \ \wedge \ \mathfrak{d} = \mathrm{unif}(\mathbf{K}) = \aleph_2$ *is consistent.*

Proof. We start with a model of CH and first add $\aleph_2$ Cohen reals with finite support and then we force with the measure algebra on $2^{\aleph_2}$, called $B_{\aleph_2}$. The Cohen reals increase $\mathfrak{d}$ and keep the rest as $\aleph_1$ (for reference to proofs see [2]). The random reals increase unif($\mathbf{K}$) while not decreasing $\mathfrak{d}$ and not increasing $\mathfrak{f}$, because $B_{\aleph_2}$ is ω^ω bounding (Lemma 3.1.2 in [2]). □

Now we begin working towards the complementary result.

Definition 2. *Define a forcing* $(Q, \leq)$ *as follows: Conditions are pairs* (σ, F), *where* $\sigma \in \omega^{<\omega}$ *is strictly increasing and* $F \subseteq [\omega]^\omega$ *is finite. The order is defined by letting* $(\sigma, F) \leq (\tau, H)$ *iff* $\tau \subseteq \sigma$, $H \subseteq F$ *and*

$$\forall i \in |\sigma| \setminus (|\tau| \cup \{0\}) \forall a \in H \ \ |[\sigma(i-1), \sigma(i)) \cap a| \neq 1.$$

Lemma 2. *Let* $\sigma \in \omega^{<\omega}$ *be strictly increasing and let* $n, k \in \omega$. *Suppose* μ *is a* Q*-name such that* $\Vdash_Q \mu \in \omega$. *There exists* $i^* < \omega$ *such that whenever* $F \subseteq [\omega]^\omega$ *has size* n *and* $|[\sigma(|\sigma|-1), k) \cap a| \geq 2$ *for all* $a \in F$, *then it is not the case that* $(\sigma, F) \Vdash_Q \mu \geq i^*$.

Proof. Otherwise there exist $F_i \subseteq [\omega]^\omega$ of size n such that $|[\sigma(|\sigma|-1), k) \cap a| \geq 2$ for all $a \in F_i$ and $(\sigma, F_i) \Vdash_Q \mu \geq i$, for all $i < \omega$. Let $F_i = \{a^i_j : j < n\}$. By compactness, we may find $B \in [\omega]^\omega$ and $a_j \subseteq \omega$, $j < n$, such that $\lim_{i \in B} a^i_j = a_j$ for all $j < n$, i.e.

$$\forall m \ \exists i \ \forall i' \in B \setminus i \ (a^{i'}_j \cap m = a_j \cap m).$$

Let $K_0 = \{j < n : |a_j| < \omega\}$ and $K_1 = n \setminus K_0$. Note that $|[\sigma(|\sigma|-1), k) \cap a_j| \geq 2$ for all $j < n$. Let

$$m^* = \max\{\max(a_j) + 1 : j \in K_0\}.$$

Find $(\tau, H) \leq (\sigma, \{a_j : j \in K_1\})$ such that (τ, H) decides μ, say as i_0, and $\tau(|\sigma|) > m^*$. Choose $i > i_0$ such that for all $j < n$

$$a^i_j \cap \tau(|\tau|-1) = a_j \cap \tau(|\tau|-1). \qquad (*)$$

We claim that $(\tau, H \cup F_i) \leq (\tau, H)$ and $(\tau, H \cup F_i) \leq (\sigma, F_i)$, which is a contradiction as (τ, H) and (σ, F_i) force contradictory statements about μ. The first inequality is clear. For the second we have to show that if $l \in |\tau| \setminus (|\sigma| \cup \{0\})$ and $j < n$, then $|[\tau(l-1), \tau(l)) \cap a^i_j| \neq 1$. Suppose first $j \in K_0$. If $l = |\sigma|$ then this is true since $\tau(|\sigma|) > m^*$. If $l > |\sigma|$, then $[\tau(l-1), \tau(l)) \cap a^i_j = \emptyset$ for the same reason and by $(*)$. Now suppose $j \in K_1$. Then $|[\tau(l-1), \tau(l)) \cap a_j| \neq 1$ since $(\tau, H) \leq (\sigma, \{a_j : j \in K_1\})$, and hence by $(*)$ we are done. □

Corollary 1. *Suppose that $U \subseteq \omega^\omega$ is unbounded (with respect to $\leq^*$). Then U is unbounded after forcing with Q.*

Proof. Suppose that ρ is a Q-name for a function in ω^ω. By Lemma 3.5, for every triple $(\sigma, n, k) \in \omega^{<\omega} \times \omega \times \omega$ with σ strictly increasing we have a function $h_{\sigma,n,k} \in \omega^\omega$ such that whenever $F \subseteq [\omega]^\omega$ has size n and $|\sigma(|\sigma|-1), k) \cap a| \geq 2$ for all $a \in F$, then it is not the case that for some $l < \omega$, $(\sigma, F) \Vdash_Q \rho(l) \geq h_{\sigma,n,k}(l)$. Choose $h \in \omega^\omega$ such that $h >^* h_{\sigma,n,k}$ for all (σ, n, k). Find $g \in U$ such that $h \not\geq^* g$. Suppose there were $(\sigma, F) \in Q$ and $l^* < \omega$ such that

$$(\sigma, F) \Vdash_Q \forall l \geq l^* \; \rho(l) > g(l).$$

Without loss of generality we may assume $|\sigma| > 0$. Let $n = |F|$ and let k be large enough such that $|[\sigma(|\sigma|-1), a) \cap k| \geq 2$ for all $a \in F$. Find $l > l^*$ such that $h(l) > h_{\sigma,n,k}(l)$ and $g(l) > h(l)$. By the definition of $h_{\sigma,n,k}$ we may find $(\tau, H) \leq (\sigma, F)$ such that $(\tau, H) \Vdash_Q \rho(l) < h_{\sigma,n,k}(l)$ and hence $(\tau, H) \Vdash_Q \rho(l) < g(l)$. This is a contradiction. □

Theorem 5. *It is consistent with ZFC, relative to the consistency of ZF, to assume* $\max\{\mathfrak{b}, \mathfrak{s}\} < \mathfrak{f}$.

Proof. Let V be a model of *ZFC + CH*, and let $\kappa > \omega_1$ be a regular cardinal. Let P be a finite support iteration of Q (Definition 3.4) of length κ, and let G be P-generic over V. Then we have that $V[G] \models \mathfrak{b} = \mathfrak{s} = \omega_1$ and $V[G] \models \mathfrak{f} = \kappa$. The latter is clear by definition of Q. Since Q is Suslin *ccc*, $V \cap [\omega]^\omega$ is a splitting family in $V[G]$ (see [3] for definitions and proofs). By Corollary 3.6 and by Lemma 6.5.7 in [2], every well-ordered unbounded family in $V \cap \omega^\omega$ is unbounded in $V[G]$. Hence by the *CH* in V we conclude $V[G] \models \mathfrak{b} = \mathfrak{s} = \omega_1$. □

5. Finitely splitting

In [4], Kamburelis and Węglorz introduce a strengthening of splitting, called finitely splitting, and show that its norm $fs = \max(\mathfrak{b}, \mathfrak{s})$. We give a direct construction that shows that $\mathfrak{f} \geq fs$. Theorem 5 shows that there is no reverse construction.

The definition of fs is:

$$\begin{aligned} fs &= \min\{|\mathscr{A}| : \mathscr{A} \subseteq [\omega]^\omega \ \wedge\ \forall \text{ partitions } \mathscr{P} \text{ of an infinite subset of } \omega \\ &\quad \text{into finite sets} \\ &\quad \exists A \in \mathscr{A}\, (\exists^\infty P \in \mathscr{P}\ P \cap A = \emptyset \ \wedge\ \exists^\infty P \in \mathscr{P}\ A \supseteq P)\}. \end{aligned}$$

A family $\mathscr{A}$ as above is called a finitely splitting family.

Proposition 3. *Suppose $\mathscr{A}$ is a witness for the computation of $\mathfrak{f}_{1,1}$. Then from $\mathscr{A}$ we can construct a finitely splitting family of the same size.*

Proof. First we take again $\mathscr{A}' = \{e_Y[A] : Y, A \in \mathscr{A}\}$, as in the proof of Theorem 1 and in the proof of Lemma 1. Suppose we are given a partition as in the definition of fs, $\mathscr{P} = \{P_n : n \in \omega\}$. We take a partition of ω into intervals $\langle q_k : k \in \omega\rangle$ such that each $[q_k, q_{k+1})$ contains at least one P_n. According to the proofs of Theorem 1 or of Lemma 1, there are $A, Y \in \mathscr{A}$ and a strictly increasing sequence $\langle 2j_n : n \in \omega\rangle$, such that

$$\exists^\infty n\ (|[q_{2j_n+1}, q_{2j_{n+1}+1}) \cap e_Y[A]| = 1 \quad \text{and} \quad |[q_{2j_n}, q_{2j_n+2}) \cap e_Y[A]| = 1).$$

Now we take an increasing enumeration $\langle b_{Y,A}(n) : n \in \omega\rangle$ of $e_Y[A]$ for each $A, Y \in \mathscr{A}$, and define

$$B(Y, A) = \bigcup\{[b_{Y,A}(2n), b_{Y,A}(2n+1)) : n \in \omega\}.$$

The family $\{B(Y, A) : Y, A \in \mathscr{A}\}$ is a finitely splitting family. $\square$

6. Open questions

One can investigate whether the value of $\mathfrak{f}$ can be arranged more arbitrarily:

1. Can $\mathfrak{f}$ be singular?

2. Is $\max(\mathfrak{s}, \mathfrak{b}) < \mathfrak{f} < \min(\mathfrak{d}, \text{unif}(\mathbf{K}))$ consistent? Tomek Bartoszyński observed that one random real forces $\mathfrak{f} \leq \mathfrak{b}$, hence the combination of constructions leading to 5 and 4 does not give the desired result.

Nor does doing first 4, say with $\aleph_1$ and $\aleph_3$, and then 5, because of the Cohen reals coming with the finite support iteration of Q: adding one Cohen real makes $\text{unif}(\mathbf{K}) \leq \mathfrak{b}$ by Theorem 3.3.22 of [2].

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