Order Types of Free Subsets

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July 6, 2000

Abstract

We give for ordinals α a lower bound for the least ordinal $\alpha(\beta)$ such that $\mathrm{Fr}^{\mathrm{ord}}_{\aleph_\xi}(\alpha(\beta), \beta)$ and show that given enough measurable cardinals there are forcing extensions where the given bounds are sharp.

1 Introduction

We consider the question whether wellordered structures have free subsets of given ordertype. Of course, the answer depends on assumptions beyond ZFC, because $\mathrm{Fr}^{\mathrm{ord}}_{\aleph_\xi}(\alpha(\beta), \beta)$ implies $\mathrm{Fr}_{\aleph_\xi}(\mathrm{card}(\alpha(\beta)), \mathrm{card}(\beta))$.

Notation: In order to simplify notation, we allow the function symbols in a type τ to be interpreted by partial functions.
For $U \subseteq A$, let $[U]^{\mathcal{A}}$ denote the substructure generated by U in $\mathcal{A}$. In contrast to this, let $\mathcal{A}{\restriction}U$ denote the structure with domain U and partial functions $f^{\mathcal{A}\restriction U}$ such that $f^{\mathcal{A}\restriction U}(\bar{u})$ is defined only if $\bar{u} \in U$ and $f^{\mathcal{A}\restriction U}(\bar{u}) \in U$. A type τ is called copious iff it is closed under Skolemfunctions. For a τ-structure $\mathcal{A}$, the length of $\mathcal{A}$, $\mathrm{lh}(\mathcal{A})$, is the cardinality $\mathrm{card}(\tau)$.

Definition 1.1 *$S \subseteq A$ is free in $\mathcal{A}$ iff for any $y \in S$ we have $y \notin [S \setminus \{y\}]^{\mathcal{A}}$.*

Definition 1.2 *For a cardinal μ and ordinals α, β let $\mathrm{Fr}^{\mathrm{ord}}_{\mu}(\alpha, \beta)$ be the following property: for any $\tau \dot\cup \{<\}$-structure $\mathcal{A} = (A, <^A, \ldots)$ of length $\leq \mu$ and $(A, <^A) \cong (\alpha, \in)$ there is a free subset S of $\mathcal{A}$ of order type β, i.e. $(S, <^A {\restriction} S^2) \cong (\beta, \in)$.*

Of course, for $\mathrm{card}(\alpha) \leq \mu$ and $\beta > 0$, $\mathrm{Fr}^{\mathrm{ord}}_\mu(\alpha, \beta)$ is not true.

Remarks: a) $S \subseteq U \subseteq A$. Let τ be copious. Then S is free in $\mathcal{A}$ iff S is free in $\mathcal{A}{\restriction}U$.
b) For μ infinite, $\mathrm{Fr}^{\mathrm{ord}}_\mu(\alpha', \beta)$ and $\alpha' < \alpha$ implies $\mathrm{Fr}^{\mathrm{ord}}_\mu(\alpha, \beta)$.

Definition 1.3 *Let μ, κ, λ be cardinals.* $\mathrm{Fr}_\mu(\kappa, \lambda)$ *iff for any τ-structure $\mathcal{A}$ of length $\leq \mu$ and* $\mathrm{card}(A) \geq \kappa$ *there is a free subset S of $\mathcal{A}$ with* $\mathrm{card}(S) \geq \lambda$.

Remarks: a) For μ infinite and λ a cardinal, $\mathrm{Fr}^{\mathrm{ord}}_\mu(\alpha, \lambda)$ implies $\mathrm{Fr}_\mu(\mathrm{card}(\alpha), \lambda)$ and $\mathrm{Fr}^{\mathrm{ord}}_\mu(\mathrm{card}(\alpha), \lambda)$.
b) For κ, λ cardinals $\mathrm{Fr}_\mu(\kappa, \lambda)$ implies $\mathrm{Fr}^{\mathrm{ord}}_\mu(\kappa, \lambda)$.

Hence $\min\{\alpha \mid \mathrm{Fr}^{\mathrm{ord}}_\mu(\alpha, \lambda)\} = \min\{\kappa \mid \mathrm{Fr}_\mu(\kappa, \lambda)\} = \min\{\mathrm{card}(\alpha) \mid \mathrm{Fr}^{\mathrm{ord}}_\mu(\alpha, \lambda)\}$ for cardinals λ.
For cardinals λ, [Ko1], [Ko2], [Ko3] and [Sh] give some information on the minimal κ with $\mathrm{Fr}_{\aleph_\xi}(\kappa, \lambda)$. In the following we consider ordinals β which are not cardinals.

2 The case $\lambda \cdot \beta$, $\mathrm{card}(\beta) \leq \lambda$

Regarding the lengths of the structures, we use the following fact from [Ko1]: If $\mathrm{Fr}_{\mu'}(\kappa, \lambda)$ and κ is minimal with this property then $\mathrm{Fr}_\mu(\kappa, \lambda)$ for any $\mu < \kappa$. Hence in case $\kappa > \lambda$ und κ minimal for $\mathrm{Fr}_{\mu'}(\kappa, \lambda)$ for some $\mu' < \kappa$, then for $\kappa > \mu \geq \lambda$ we have $\mathrm{Fr}_\mu(\kappa, \lambda)$ and the condition $\mathrm{card}(\beta) \leq \mu$ is no restriction.

Proposition 2.1 *Assume* $\mathrm{Fr}_\mu(\kappa, \lambda)$ *and* $\mathrm{card}(\beta) \leq \min(\lambda, \mu)$, $\lambda \geq \aleph_0$. *Then* $\mathrm{Fr}^{\mathrm{ord}}_\mu(\kappa \cdot \beta, \lambda \cdot \beta)$.

Proof: Let $\mathcal{A}$ be a $\tau \cup \{<\}$-structure of length $\leq \mu$ and $(A, <^A) \cong (\kappa \cdot \beta, \in)$. We expand $\mathcal{A}$ to a τ'-structure $\mathcal{A}'$ with $\mathrm{card}(\tau) \leq \mu$ by choosing for $\gamma < \beta$ interpretations

$$f^{\mathcal{A}'}_\gamma : \kappa \overset{bijective}{\to} [\kappa \cdot \gamma, \kappa \cdot (\gamma + 1)),$$

and adding Skolemfunctions as required. By $\mathrm{Fr}_\mu(\kappa, \lambda)$ there is a free subset S in $\mathcal{A}'{\restriction}\kappa$. We fix a function $h\colon \lambda \to \beta$ (independent of the interpretations of the functions in $\mathcal{A}'$) that is onto and

$$\forall \beta' \in \beta \; \mathrm{card}(h^{-1\prime\prime}\{\beta'\}) = \lambda.$$

Let $S = \{s_\varepsilon \mid \varepsilon \in \lambda\}$ with pairwise distinct s_ε. We set

$$\begin{aligned} \bar{s}_\varepsilon &= f^{\mathcal{A}'}_{h(\varepsilon)}(s_\varepsilon), \\ \bar{S} &= \{\bar{s}_\varepsilon \mid \varepsilon \in \lambda\}. \end{aligned}$$

$\bar{S}$ is free in $\mathcal{A}'$ and in $\mathcal{A}$:
By contradiction, assume that $h' \in \tau'$ be an m-ary function symbol and $\varepsilon_m \notin \{\varepsilon_i \mid i < m\}$ and $h'^{\mathcal{A}'}(\bar{s}_{\varepsilon_0}, \ldots \bar{s}_{\varepsilon_{m-1}}) = \bar{s}_{\varepsilon_m}$. Then

$$\begin{aligned} h'^{\mathcal{A}'}(f^{\mathcal{A}'}_{h(\varepsilon_0)}(s_{\varepsilon_0}), \ldots f^{\mathcal{A}'}_{h(\varepsilon_{m-1})}(s_{\varepsilon_{m-1}})) &= f^{\mathcal{A}'}_{h(\varepsilon_m)}(s_{\varepsilon_m}), \text{ and} \\ (f^{\mathcal{A}'}_{h(\varepsilon_m)})^{-1}(h'^{\mathcal{A}'}(f^{\mathcal{A}'}_{h(\varepsilon_0)}(s_{\varepsilon_0}), \ldots f^{\mathcal{A}'}_{h(\varepsilon_{m-1})}(s_{\varepsilon_{m-1}}))) &= s_{\varepsilon_m}. \end{aligned}$$

Since τ' is closed under Skolemfunctions, S is not free in $\mathcal{A}' \restriction \kappa$. Contradiction.
$\bar{S}$ has ordertype $\geq \lambda \cdot \beta$:
We show: For $\beta' < \beta$, $\operatorname{card}(\bar{S} \cap [\kappa \cdot \beta', \kappa \cdot (\beta' + 1))) = \lambda$. We have

$$\begin{aligned} \bar{s}_\varepsilon &\in [\kappa \cdot \beta', \kappa \cdot (\beta' + 1)) \text{ iff} \\ f^{\mathcal{A}'}_{h(\varepsilon)}(s_\varepsilon) &\in [\kappa \cdot \beta', \kappa \cdot (\beta' + 1)) \text{ iff} \\ h(\varepsilon) &= \beta'. \end{aligned}$$

and the latter is the case for λ many ε because of the choice of h. □

Now we can connect 2.1 with known results on consistency strengths and get for example:

Corollary 2.2 $\mathrm{Fr}^{\mathrm{ord}}_\omega(\omega_\omega \cdot \omega, \omega \cdot \omega)$ *is equiconsistent to the existence of a measurable cardinal.*

Proof: 2.1 shows that $\mathrm{Fr}^{\mathrm{ord}}_\omega(\omega_\omega \cdot \omega, \omega \cdot \omega)$ is not strictly stronger than $\mathrm{Fr}_\omega(\omega_\omega, \omega)$, which by [Ko1] is equiconsistent to a measurable. It is easy to see that $\mathrm{Fr}^{\mathrm{ord}}_\omega(\omega_\omega \cdot \omega, \omega \cdot \omega)$ is at least as strong as $\mathrm{Fr}_\omega(\omega_\omega, \omega)$.

Next we show under the additional assumptions that the bounds given in 2.1 are minimal.

Proposition 2.3 *Let* κ *be minimal with* $\mathrm{Fr}_\mu(\kappa, \lambda)$ *and assume* $\mathrm{cf}(\kappa) \leq \mu$ *(– hence* $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$ *–). Let* $\operatorname{card}(\beta) \leq \lambda$*. Then* $\kappa \cdot \beta$ *is minimal with* $\mathrm{Fr}^{\mathrm{ord}}_\mu(\kappa \cdot \beta, \lambda \cdot \beta)$.

Proof: We fix some examples that show $\neg\mathrm{Fr}_\mu(\kappa',\lambda)$ for $\kappa'<\kappa$, call them for further use $\mathcal{D}(\kappa',\lambda)$. Assume $\mathcal{D}(\kappa',\lambda)$ is a $\tau(\kappa')$-structure and the $\tau(\kappa')$ be pairwise disjoint. Let $\langle\kappa_i \mid i\in\mathrm{cf}(\kappa)\rangle$ be cofinal in κ. Expand κ by isomorphic copies of the $\mathcal{D}(\kappa_i,\lambda)$, $i<\mathrm{cf}(\kappa)$ and call this $\bigcup_{i<\mathrm{cf}(\kappa)}\tau(\kappa_i)$-structure $\tilde{\mathcal{D}}(\kappa,\lambda)$. We have: $\tilde{\mathcal{D}}(\kappa,\lambda)$ has a free subset of cardinality λ, any free subset of $\tilde{\mathcal{D}}(\kappa,\lambda)$ of cardinality λ is cofinal in κ, and there is no free subset of cardinality $>\lambda$.

Let $\gamma=\kappa\cdot\beta'+\gamma'<\kappa\cdot\beta$, $\gamma'<\kappa$. We expand for $\beta''<\beta'$ the interval $[\kappa\cdot\beta'',\kappa\cdot(\beta''+1))$ by the shift of $\tilde{\mathcal{D}}(\kappa,\lambda)$ onto this interval, and we expand the interval $[\kappa\cdot\beta',\kappa\cdot\beta'+\gamma'))$ by a copy of $\mathcal{D}(\mathrm{card}(\gamma'),\lambda)$. It is easy to see that any free subset in the resulting structure has ordertype less than $\lambda\cdot(\beta'+1)\le\lambda\cdot\beta$. $\square$

3 The case $\lambda\cdot\beta+\gamma$, $\mathrm{card}(\beta)\le\lambda$, $0<\gamma<\lambda$

This case is more complicated than the case $\gamma=0$. Assume $\mathrm{Fr}_\mu(\kappa,\lambda)$. For S free in $\mathcal{A}=(\kappa,\in,\ldots)$ we need some information on

$$\langle \mathrm{otp}(S\cap\delta)\mid\delta\in\kappa\rangle.$$

Example: If S is free in $\tilde{\mathcal{D}}(\aleph_\omega,\aleph_0)$ then $\mathrm{otp}(S\cap\aleph_n))\le n$.

Lemma 3.1 *Assume* $\mathrm{Fr}_\mu(\kappa,\lambda)$ *and* $\mathrm{cf}(\kappa)\le\mu$. *Then there is a function* $g\colon\lambda\to\kappa$ *such that for any structure on* κ *of length* $\le\mu$ *and for any* $\varepsilon\in\lambda$ *there is some* S *of cardinality* λ *free in* $\mathcal{A}$ *with* $\mathrm{otp}(S\cap g(\varepsilon))\ge\varepsilon$.

Proof: For $\varepsilon\in\lambda$ define

$$\begin{aligned} M(\varepsilon) &= \{\delta\in\kappa\mid\forall\mathcal{A}\text{ with support }\kappa\\ &\qquad \exists S\text{ of cardinality }\lambda\text{ free in }\mathcal{A},\mathrm{otp}(S\cap\delta)\ge\varepsilon\}.\end{aligned}$$

$M(\varepsilon)\neq\emptyset$: If $\delta\notin M(\varepsilon)$ then there is a structure $\mathcal{A}_\delta$ say of type τ_δ such that for any free subset S in $\mathcal{A}_\delta$ of cardinality λ we have $\mathrm{otp}(S\cap\delta)<\varepsilon$. We may assume that the τ_δ be pairwise disjoint. If $M(\varepsilon)=\emptyset$ then

$$\mathcal{A}=(\kappa,(f^{\mathcal{A}_\delta})_{f\in\tau_\delta,\delta\in\kappa})$$

would not have any free subset of cardinality λ. If we restrict ourselves to δ's that are elements of a cofinal subset in κ, that would give a counterexample to $\mathrm{Fr}_\mu(\kappa, \lambda)$. Now define $g(\varepsilon) = \min(M(\varepsilon))$. $\square$

We call a g as in lemma 3.1 an *upper bound.* The g given in the proof of 3.1 is the lowest upper bound in the partial ordering $\leq$ on κ^λ: $f \leq g$ iff $f(\varepsilon) \leq g(\varepsilon)$ for all ε.

The following proposition is a generalisation of 2.1.

Proposition 3.2 *Let g be a upper bound for* $\mathrm{Fr}_\mu(\kappa, \lambda)$. $\gamma < \lambda$ *and* $\mathrm{card}(\beta) \leq \min(\lambda, \mu)$. *Then* $\mathrm{Fr}_\mu^{\mathrm{ord}}(\kappa \cdot \beta + g(\gamma), \lambda \cdot \beta + \gamma)$.

Proof: Let $\mathcal{A}$ be a $\tau \cup \{<\}$-structure with $(A, <^A) \cong \kappa \cdot \beta + g(\gamma)$. Let $\tau' \supseteq \tau \cup \{f_\gamma \mid \gamma \in \beta \cup \{-1\}\}$ be copious. We expand $\mathcal{A}$ to a τ'-structure $\mathcal{A}'$ by choosing interpretations

$$\begin{array}{rcl} f_\gamma^{\mathcal{A}'}\colon [g(\gamma), \kappa) & \overset{bijective}{\rightarrow} & [\kappa \cdot \gamma, \kappa \cdot (\gamma + 1)), \\ f_{-1}^{\mathcal{A}'}\colon (g(\gamma), \in) & \cong & ([\kappa \cdot \beta, \kappa \cdot \beta + g(\gamma)), \in), \end{array}$$

and interpreting the compositions in the canonical way. Take an S free in $\mathcal{A}$ with $\mathrm{otp}(S \cap g(\gamma)) \geq \gamma$. $S \cap g(\gamma) \supseteq \{t_i \mid i < \gamma\}$ with strictly ascending t_i. $S \cap [g(\gamma), \kappa) \supseteq \{s_\varepsilon \mid \varepsilon \in \lambda\}$. Set for $i < \gamma$: $\bar{t}_i = f_{-1}^{\mathcal{A}'}(t_i)$. We take h as in the proof of 2.1 and define the $\bar{s}_\varepsilon$ as there. Then $\{\bar{s}_\varepsilon \mid \varepsilon \in \lambda\} \cup \{\bar{t}_i \mid i < \gamma\}$ is a free subset in $\mathcal{A}$ of ordertype $\lambda \cdot \beta + \gamma$. $\square$
A lower threshold is given by

Proposition 3.3 *Let g be the lowest upper bound for* $\mathrm{Fr}_\mu(\kappa, \lambda)$. $\gamma < \lambda$ *and* $\mathrm{card}(\beta) \leq \lambda$. *Let* $\beta' < \kappa \cdot \beta + g(\gamma)$ *and* $\mathrm{cf}(\kappa) \leq \mu$. *Then* $\neg\mathrm{Fr}_\mu^{\mathrm{ord}}(\beta', \lambda \cdot \beta + \gamma)$.

Proof: Regard the defintion of *lowest* upper bound and use the techniques of 2.3. $\square$

Remark: We may try to change the ordering of the quantifiers in the notion of an upper bound:

Definition 3.4 *Assume* $\mathrm{Fr}_\mu(\kappa, \lambda)$. *An* upper bound for $\mathrm{Fr}_\mu(\kappa, \lambda)$ in the strong sense *is a function* $g\colon \lambda \to \kappa$ *such that for any structure on* κ *of length* $\leq \mu$ *there is some* S *of cardinality* λ *free in* $\mathcal{A}$ *and such that for each* $\varepsilon \in \lambda$ *we have* $\mathrm{otp}(S \cap g(\varepsilon)) \geq \varepsilon$.

Despite of the results in sections 4 and 5, there is the

Open Problem 3.5 *Assume* $\mathrm{Fr}_\mu(\kappa, \lambda)$ *and* $\mathrm{cf}(\kappa) \leq \mu$. *Is there an upper bound for* $\mathrm{Fr}_\mu(\kappa, \lambda)$ *in the strong sense?*

4 Countable Ordinals

Forcing $\mathrm{Fr}_{\aleph_\xi}(\aleph_{\xi+\lambda}, \lambda)$ starting from λ many measurables greater than $\aleph_\xi$ in case $\lambda < \aleph_\lambda$ or one measurable $> \aleph_\xi$ in case $\lambda = \omega$ (cf. [Sh] and [Ko1]) gives the upper bounds in the strong sense $g\colon \lambda \to \aleph_\lambda$:

$$g(\omega \cdot i + n) = \aleph_{\xi+\omega\cdot i+2n}, n \in \omega, i \in \lambda.$$

In order to lower these upper bounds, we may adjoin finitely many small ordinals to the sets of indiscernibles arising in above mentioned forcing proofs (cf. 4.6). In the case of $\xi = 0$ and one measurable cardinal this filling up technique leads to the upper bounds in the strong sense $g_k =$
$\langle \aleph_1, \aleph_2, \aleph_3, \ldots \aleph_{2k}, \aleph_{2k+2}, \aleph_{2k+4}, \ldots \rangle$ for $k \in \omega$. If the measurable cardinal is supposed to be greater than $\aleph_\xi$, the technique given in this section will work also for structures of length $\aleph_\xi$.

Theorem 4.1 *Given a measurable cardinal there is a forcing extension where* $\mathrm{Fr}^{\mathrm{ord}}_{\aleph_0}(\aleph_\omega \cdot \beta + \aleph_k, \omega \cdot \beta + k)$ *holds for* $k \in \omega$.

Actually we will show:

Theorem 4.2 *Given a measurable cardinal there is a forcing extension where for each* $k \in \omega$ *we have* $g_k = \langle \aleph_1, \aleph_2, \aleph_3, \ldots \aleph_{2k}, \aleph_{2k+2}, \aleph_{2k+4}, \ldots \rangle$ *as an upper bound in the strong sense for* $\mathrm{Fr}_{\aleph_0}(\aleph_\omega, \aleph_0)$.

and then apply 3.2.

Proof:
The forcing is the same as Koepke gives in [Ko1]. A closer look at the combinatorial components shows that there is a kind of normality (for a precise definition see theorem 4.4) in the coherent sequence of Ramsey cardinals. We will go along the lines of [Ko1] and indicate the additional conditions.

Let κ be a measurable cardinal and U an normal ultrafilter on κ. Let

$$P = \{(a, X) \mid a \in [\kappa]^{<\omega}, X \in U, \max a < \min X\}$$

be the set of Prikry conditions for κ, U, with the usual order. Let G be P-generic over V, let $\langle \kappa_i \mid i \in \omega \rangle$ be the Prikry sequence induced by G.

Lemma 4.3 *In $V[G]$, the following principle holds: if $f\colon [\kappa]^{<\omega} \to \kappa$ is regressive, i.e. $f(x) < \min x$ for $x \in [\kappa]^{<\omega}$, then there are $m \in \omega$ and $\langle A_i \mid m \leq i < \omega \rangle$ such that*
(i) $A_i \subseteq \kappa_i$ is cofinal in κ_i, and
(ii) if $x, y \in [\kappa]^{<\omega}$, $x, y \subseteq \bigcup\{A_i | m \leq i < \omega\}$ and if $\mathrm{card}(x \cap A_i) = \mathrm{card}(y \cap A_i)$ for every i and if $z \in [\min(x \cup y)]^{<\omega}$ then $f(z \cup x) = f(z \cup y)$.

Proof: Similar to lemma 3.1 of [Ko1]. And as in [Ko1] the lemma yields:

Theorem 4.4 *In $V[G]$ there is an ascending sequence $\langle \lambda_i \mid i \in \omega \rangle$ cofinal in κ which forms a coherent sequence of normal Ramsey cardinals, i.e. for every regressive $f\colon [\kappa]^{<\omega} \to \kappa$ there are $\langle A_i \mid i < \omega \rangle$ such that:*
(i) $A_i \subseteq \kappa_i$ is cofinal in λ_i, and
(ii) if $x, y \in [\kappa]^{<\omega}$, $x, y \subseteq \bigcup\{A_i \mid i < \omega\}$ and if $\mathrm{card}(x \cap A_i) = \mathrm{card}(y \cap A_i)$ for every i and if $z \in [\min(x \cup y)]^{<\omega}$ then $f(z \cup x) = f(z \cup y)$. $\square$

Let $(**)$ be the following assertion: If $f\colon [\aleph_\omega]^{<\omega} \to 2$ then there is $\langle C_i \mid i \in \omega \rangle$ such that:
(i) C_i is a cofinal subset of $\aleph_{2i+2}$, and
(ii) if $i_0 < \ldots < i_{n-1} < \omega$ and $\alpha_0, \beta_0 \in C_{i_0}, \ldots \alpha_{n-1}, \beta_{n-1} \in C_{i_{n-1}}$ and $c \in [\aleph_{i_0}]^{<\omega}$ then

$$f(c \cup \{\alpha_0, \ldots \alpha_{n-1}\}) = f(c \cup \{\beta_0, \ldots \beta_{n-1}\}).$$

Fix a coherent sequence $\langle \kappa_i \mid i \in \omega \rangle$ of normal Ramsey cardinals with supremum κ. Let $(P, \leq)$ be the following set of conditions

$$P = \{\langle p_i \mid i \in \omega \rangle \mid p_0 \in \mathrm{Col}(\aleph_1, \kappa_0), p_i \in \mathrm{Col}(\kappa_{i-1}^+, \kappa_i) \text{ for } 1 \leq i < \omega\},$$

where $\mathrm{Col}(\sigma, \rho)$ are the Levy conditions for collapsing the inaccessible ρ to σ^+; $\langle q_i \mid i \in \omega \rangle \leq \langle p_i \mid i \in \omega \rangle$ iff $\forall i\, q_i \supseteq p_i$. Let G be P-generic over V. In $V[G]$: $\kappa_0 = \aleph_2$, $\kappa_1 = \aleph_4, \ldots, \kappa = \aleph_\omega$.

Theorem 4.5 $(**)$ *holds in* $V[G]$.

Proof: Similar to theorem 4.3 in [Ko1], but with a coherent sequence of normal Ramsey cardinals. $\square$

Theorem 4.6 *If* $(**)$ *holds then for each* $k \in \omega$ *the function* $g_k = \langle \aleph_1, \aleph_2, \aleph_3, \ldots \aleph_{2k}, \aleph_{2k+2}, \aleph_{2k+4} \ldots \rangle$ *is an upper bound in the strong sense for* $\mathrm{Fr}_{\aleph_0}(\aleph_\omega, \aleph_0)$.

Proof:
Fix a $k \in \omega$. Let $\mathcal{A}$ be a τ-structure of countable type and with support $\aleph_\omega$. Take an enumeration $\{\phi_n \mid n \in \omega\}$ of L_τ in which each ϕ appears infinitely often and the free variables of ϕ_n are contained in $\{v_0, \ldots, v_{n-1}\}$. We define $f_{\mathcal{A}}\colon [\aleph_\omega]^{<\omega} \to 2$ by

$$f_{\mathcal{A}}(a) = \begin{cases} 0, & \text{if } \mathcal{A} \models \phi_{\mathrm{lh}(\bar{a})}(\bar{a}), \bar{a} \text{ increasing,} \\ 1, & \text{else.} \end{cases}$$

We apply $(**)$ to $f_{\mathcal{A}}$ and get $\langle C_n \mid n \in \omega \rangle$ as in $(**)$. We show the following
Claim: There is an S free in $\mathcal{A}$ such that

$$\begin{aligned} S \cap [\aleph_i, \aleph_{i+1}) &\neq \emptyset \text{ for } 0 \leq i < 2k, \\ S \cap C_i &\neq \emptyset \text{ for } i \geq k. \end{aligned}$$

By induction on i for $0 \leq i < 2k$ we choose $A_{2k-j}(2k-i) \subseteq [\aleph_{2k-j-1}, \aleph_{2k-j})$ for $0 \leq j \leq i$ decreasing in i, and $C_n(2k-i) \subseteq C_n$ for $n \geq 2k$ decreasing in i such that:

$$\begin{aligned} \mathrm{card}(A_{2k-j}(2k-i)) &= \aleph_{2k-i-1} \text{ for } 0 \leq j < i, \\ \mathrm{card}(A_{2k-i}(2k-i)) &= \aleph_{2k-i}, \\ \mathrm{card}(C_n(2k-i)) &= \aleph_{2k-i-1}, \\ A_{2k-i}(2k-i) &\cap [\aleph_{2k-i-1} \cup \bigcup_{j<i} A_{2k-j}(2k-i) \cup \bigcup_{n \geq 2k} C_n(2k-i)]^{\mathcal{A}} = \emptyset. \end{aligned}$$

Then we take $s_{2k-i} \in A_{2k-i}(1)$ for $0 \leq i < 2k$, and $s_{n+1} \in C_n(1)$ for $n \geq 2k$. $S = \{s_n \mid 1 \leq n \in \omega\}$ is free in $\mathcal{A}$:
For $n > 2k$, $s_n \notin [S \backslash \{s_n\}]^{\mathcal{A}}$, because s_n is a member of C_{n-1} and $\mathrm{card}(C_{n-1}) \geq 2$ and $\langle C_i \mid i \in \omega \rangle$ is a sequence as in $(**)$ for $f_{\mathcal{A}}$.
For $n \leq 2k$, $s_n \notin [S \setminus \{s_n\}]^{\mathcal{A}}$ because of the choice of $A_n(n)$. $\square$

5 Uncountable ordinals

We heavily refer to [Sh], but we use some modification of the partition theorems therein. We indicate how this modification is derived. Assume that there are measurable cardinals $\langle \kappa_i \mid i \in \lambda \rangle$ in increasing order, and $\kappa_0 > \aleph_\xi$. Let $P = \mathrm{Col}(\aleph_{\xi+1}, \kappa_0) \times \prod_{i \in \lambda} \mathrm{Col}(\kappa_i^+, \kappa_{i+1})$, and G be P-generic over V.

Lemma 5.1 *In $V[G]$, the following holds:*
a) For $n \in \omega, i < \lambda$: $\kappa_{\omega \cdot i+n} = (\aleph_{\xi+\omega \cdot i+2n+2})^{V[G]}$, and
b) $\forall f\colon [\aleph_{\xi+\lambda}]^{<\omega} \to 2\, \exists \langle S_{\omega \cdot i+n} \mid n \in \omega, i \in \lambda \rangle$ such that:
$S_{\omega \cdot i+n} \subseteq \aleph_{\xi+\omega \cdot i+2n+2}$ and $\mathrm{card}(S_{\omega \cdot i+n}) = \aleph_{\xi+\omega \cdot i+2n+2}$, and
$\forall i \in \lambda, n \in \omega, p \in [\aleph_{\xi+\omega \cdot i+2n}]^{<\omega}, m \in \omega$ with $\omega \cdot i+n \leq i(0) < i(1) \ldots < i(m-1) \forall a_\ell, b_\ell \in S_{i(\ell)}\ f(p \cup \{a_0, \ldots, a_{m-1}\}) = f(p \cup \{b_0, \ldots, b_{m-1}\})$.

Proof: [Sh] theorems 3 and 4 prove: In $V[G]$, GCH is true, and
$(*)$ for $n \in \omega, i < \lambda$: $\kappa_{\omega \cdot i+n} = (\aleph_{\xi+\omega \cdot i+2n+2})^{V[G]}$, and
$\forall \langle f_{i,n}\colon [\aleph_{\xi+\lambda}]^{<\omega} \to \aleph_{\xi+\omega \cdot i+2n+1} \mid n \in \omega, i \in \lambda \rangle\, \exists \langle S_{\omega \cdot i+n} \mid n \in \omega, i \in \lambda \rangle$
such that: $S_{\omega \cdot i+n} \subseteq \aleph_{\xi+\omega \cdot i+2n+2}$ and $\mathrm{card}(S_{\omega \cdot i+n}) = \aleph_{\xi+\omega \cdot i+2n+2}$, and
$\forall i \in \lambda, n \in \omega, m \in \omega$ with $\omega \cdot i+n \leq i(0) < i(1) \ldots < i(m-1)$
$\forall a_\ell, b_\ell \in S_{i(\ell)}\ f_{i,n}(\{a_0, \ldots, a_{m-1}\}) = f_{i,n}(\{b_0, \ldots, b_{m-1}\})$.
Now, given $f\colon [\aleph_{\xi+\lambda}]^{<\omega} \to 2$, we set for $n \in \omega, i \in \lambda, a \in [\aleph_{\xi+\lambda}]^{<\omega}$:

$$f_{i,n}(a) = \langle f(p \cup a) \mid p \in [\aleph_{\xi+\omega \cdot i+2n}]^{<\omega} \rangle,$$

and using GCH, we regard the latter as an element of $\aleph_{\xi+\omega \cdot i+2n+1}$. An application of $(*)$ gives the desired partition property. $\Box$

In order to cope with the types τ of cardinality $\aleph_\xi$, we fix an enumeration $\{\phi_i \mid i \in \aleph_\xi\}$ of L_τ. We define $f_{\mathcal{A}}\colon [\aleph_{\xi+\lambda}]^{<\omega} \to 2$ by

$$f_{\mathcal{A}}(a) = \begin{cases} 0, & \text{if } \mathcal{A} \models \phi_{\min a}(\bar{a}), \bar{a} \text{ increasing}, \min a < \aleph_\xi, \\ 1, & \text{else.} \end{cases}$$

As in lemma 4.6 (now between $\aleph_{\xi+k}$ and $\aleph_\xi$), we get from lemma 5.1 in $V[G]$: $\mathrm{Fr}_{\aleph_\xi}(\aleph_{\xi+\lambda}, \lambda)$, and for all $k \in \omega$ we have g_k as an upper bound in the strong sense, where

$$\begin{aligned} g_k(j) &= \aleph_{\xi+j} \text{ for } 0 \leq j \leq k \\ g_k(k+n+1) &= \aleph_{\xi+k+2n+2} \text{ for } n \in \omega \\ g_k(\omega \cdot (i+1)+n) &= \aleph_{\xi+\omega \cdot (i+1)+2n} \text{ for } n \in \omega, i \in \lambda. \end{aligned}$$

From this we get introducing suitable bijections as in 2.1, the following

Theorem 5.2 *If* $\lambda \leq \aleph_\xi$*, we have* $\mathrm{Fr}^{\mathrm{ord}}_{\aleph_\xi}(\alpha, \beta)$ *in* $V[G]$ *for*

$$\beta = \lambda_{m-1} \cdot \beta_{m-1} + \ldots + \lambda_0 \cdot \beta_0 + k,$$

with $\mathrm{card}(\beta_i) \leq \lambda_i$ *and* $\aleph_0 \leq \lambda_0 < \lambda_1 \ldots < \lambda_{m-1} \leq \lambda$*,* λ_i *cardinals,* $0 < k < \omega$ *and*

$$\alpha = \aleph_{\xi+\lambda_{m-1}} \cdot \beta_{m-1} + \ldots + \aleph_{\xi+\lambda_0} \cdot \beta_0 + \aleph_{\xi+k}.$$

Proof: We work in $V[G]$. Given $\mathcal{A}$ of length $\aleph_\xi$ such that $(A, \in) \cong \alpha$ and α as in the proposition, we expand $\mathcal{A}$ by the following functions (where the noncommutative sums are to be taken in decreasing order):

$$f_{-1}\colon \aleph_{\xi+k} \overset{bijective}{\rightarrow} [\sum_{m>i\geq 0} \aleph_{\xi+\lambda_i} \cdot \beta_i, \sum_{m>i\geq 0} \aleph_{\xi+\lambda_i} \cdot \beta_i + \aleph_{\xi+k}),$$

and for $\gamma < \beta_j, j < m$ with $\lambda_{-1} = k$:

$$f_{j,\gamma}\colon [\aleph_{\xi+\lambda_{j-1}}, \aleph_{\xi+\lambda_j}) \overset{bijective}{\rightarrow} [\sum_{j>i\geq 0} \aleph_{\xi+\lambda_i} \cdot \beta_i + \aleph_{\lambda_j} \cdot \gamma, \sum_{j>i\geq 0} \aleph_{\xi+\lambda_i} \cdot \beta_i + \aleph_{\lambda_j} \cdot (\gamma+1)).$$

Let $\mathcal{A}'$ be a Skolemstructure belonging to this expansion. Using $\mathrm{Fr}_{\aleph_\xi}(\aleph_{\xi+\lambda}, \lambda)$ with an upper bound g_k as above for the structure $\mathcal{A}' \restriction \aleph_{\xi+\lambda}$, we get a free subset $S = \{s_i \mid i \in \lambda\}$ in $\mathcal{A}' \restriction \aleph_{\xi+\lambda}$ with $s_i \in g_k(i+1) \setminus g_k(i)$ for $i \in \lambda$.
We fix for $j < m$ surjections $h_j\colon \lambda_j \rightarrow \beta_j$ such that $\forall \alpha \in \beta_j : \mathrm{card}(h_j^{-1}{}''\{\alpha\}) = \lambda_j$. Then

$$\{f_{-1}(s_i) \mid i < k\} \cup \{f_{j,h_j(i)}(s_i) \mid i \in \lambda, s_i \in \mathrm{dom}(f_{j,0}), j < m\}$$

is free in $\mathcal{A}'$ and of order type β. $\square$

Putting together the known examples that show in ZFC that $\aleph_{\xi+\lambda_j}$ is the smallest cardinal κ at which $\mathrm{Fr}_{\aleph_\xi}(\kappa, \lambda_j)$ may be consistent with ZFC it is easy to see our given upper bounds being minimal.

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