A SIMPLE $P_{\aleph_1}$-POINT AND A SIMPLE $P_{\aleph_2}$-POINT

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Abstract. We answer the long-standing question whether it is consistent to have simple P-points of two different characters. For a filter $\mathcal{F}$ over ω Guzmán and Kalajdzievski introduced a parametrised version of Miller forcing called $\mathbb{PT}(\mathcal{F})$. By combining iterands of the type $\mathbb{PT}(\mathcal{F})$ with others we establish: It is consistent relative to ZFC that there is a simple $P_{\aleph_1}$-point and a simple $P_{\aleph_2}$-point. A main technical point is the use of properness and descriptive complexity in the limit steps of uncountable cofinality.

1. Introduction

The statement "There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points" is the first part of the title of an article by Andreas Blass and Saharon Shelah [7] from 1987. In that work a creature forcing with a norm and linear conditions is introduced and used for establishing a model in which any two non-principal ultrafilters are nearly coherent. A subforcing is supposed to give a simple $P_{\aleph_1}$-point and simple $P_{\aleph_2}$-point. The statement was considered as proved, however, in 2005 Alan Dow found a flaw in its proof, making the consistency of the existence of a simple $P_{\aleph_1}$-point and a simple $P_{\aleph_2}$-point again an open problem.

We will use a countable support iteration of a new forcing notion introduced by Guzmán and Kalajdzievski in [14] and apply it with particularly chosen parameters that are forced in intermediate steps. We further use an absoluteness argument in the steps of uncountable cofinality. Thus we establish that the existence of a simple $P_{\aleph_1}$-point and a simple $P_{\aleph_2}$-point is consistent relative to ZFC.

Our consistency result solves Nyikos' problems (1) and (4) of [21]. Our forcing also provides a new type of model of $\mathfrak{b} < \mathfrak{s}$. This constellation is still rare and it is established by a countable support construction with Shelah's creature forcing [23], Blass and Shelah's matrix construction in [8] — by flipping the matrix for $\mathfrak{u} < \mathfrak{d}$ we get a matrix forcing for $\mathfrak{b} < \mathfrak{u} = \mathfrak{s}$ see also [1] —, the $\mathfrak{b} < \mathfrak{s}$-matrix forcing by Brendle and Fischer [9], Dow's and Shelah's matrix model of a singular $\mathfrak{s}$ [12]. Splitting families in the ground model are destroyed by diagonalising an ultrafilter via Mathias forcing. The choice of a name of an ultrafilter such that no dominating real is added is an important technical step.

In particular, the even rarer inequality $\mathfrak{u} < \mathfrak{s}$ holds in our forcing extension, like in the countable support iteration of Blass–Shelah forcing [7] and Guzmán and Kalajdzievski's new forcing [14], parametrised by F_σ-generic ultrafilters.

We refer the reader to [4] for the definitions of the cardinal characteristics $\mathfrak{b}$, $\mathfrak{d}$, $\mathfrak{s}$ and $\mathfrak{u}$.

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In the remainder of this section we will recall the definitions that allow us to state a technical version of the main theorem in Theorem 1.10. We begin with some basic definitions concerning filters.

For a countable set A the set of finite/infinite subsets of A is denoted by $[A]^{<\omega}/[A]^{\omega}$.

Definition 1.1.

(1) For a set X, a *filter over* X is a non-empty subset of the power set $\mathcal{P}(X)$ that does not contain the empty set and that is closed under supersets and under finite intersections.

(2) By $\mathcal{F}r$ we denote the *Fréchet filter*, which is the filter of cofinite subsets of ω. Henceforth, by a filter we mean a filter over ω that contains the Fréchet filter. An *ultrafilter* is a maximal filter.

(3) A subset $\mathcal{B}$ of a filter $\mathcal{F}$ is called a *basis* of $\mathcal{F}$ if for every $F \in \mathcal{F}$ there is some $B \in \mathcal{B}$ such that $B \subseteq F$.

(4) For $\mathcal{E} \subseteq [\omega]^{\omega}$ such that for all $n \in \omega$ and $x_0, \dots, x_{n-1} \in \mathcal{E}$ we have $x_0 \cap \dots \cap x_{n-1} \in [\omega]^{\omega}$, we denote by filter$(\mathcal{E})$ the filter generated by $\mathcal{E} \cup \mathcal{F}r$, i.e.

$$\text{filter}(\mathcal{E}) = \{Y \subseteq \omega \mid \exists n \in \omega \exists x_0 \dots \exists x_{n-1} \in \mathcal{E}(Y \supseteq^* x_0 \cap \dots \cap x_{n-1})\}.$$

In order to include the case $n = 0$ we stipulate $\bigcap \emptyset = \omega$.

(5) The *character* of a filter $\mathcal{F}$ is the smallest size of a basis of $\mathcal{F}$ and denoted by $\chi(\mathcal{F})$. The ultrafilter number $\mathfrak{u}$ is the minimal character of a non-principal ultrafilter over ω.

(6) Let κ be an uncountable cardinal. An ultrafilter $\mathcal{U}$ over ω is called a *P_κ-point* if for any $\gamma < \kappa$, any $\subseteq^*$-descending sequence $\langle A_\beta \mid \beta < \gamma \rangle$ of elements of $\mathcal{U}$ has a pseudointersection $B \in \mathcal{U}$, that is, some B such that for $\beta < \gamma$, $B \subseteq^* A_\beta$. A $P_{\aleph_1}$-point is also just called a P-point.

(7) An ultrafilter $\mathcal{U}$ is called *simple* if there is an uncountable cardinal κ such that $\chi(\mathcal{U}) = \kappa$ and $\mathcal{U}$ is a P_κ-point.

If there is a simple P_κ-point, then κ is regular. A simple P_κ-point $\mathcal{U}$ has a basis $\mathcal{B}$ that consists of a strictly $\subseteq^*$-descending sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$.

Solomon [25] showed that any non-principal ultrafilter has character at least $\mathfrak{b}$. Nyikos [20], [21] showed: If $\mathcal{U}$ is a simple P_κ-point, then $\kappa = \mathfrak{b}$ or $\kappa = \mathfrak{d}$. In order to see this, assume that $\mathcal{U}$ is a simple P_κ-point and $\kappa > \mathfrak{b}$. We fix some $\leq^*$-unbounded family $\{f_\alpha \mid \alpha < \mathfrak{b}\}$ of strictly increasing functions. Then for every $A \in [\omega]^\omega$, $\{f_\alpha \restriction A \mid \alpha < \mathfrak{b}\}$ is unbounded in $\{f \mid f : A \to \omega\}$. For two functions f, g the relation $\leq_{\mathcal{U}}$ is defined by $f \leq_{\mathcal{U}} g$ if $\{n \mid f(n) \leq g(n)\} \in \mathcal{U}$. This relation is a linear order on the $=_{\mathcal{U}}$-equivalence classes, because $\mathcal{U}$ is an ultrafilter. Since $\mathcal{U}$ is a $P_{\mathfrak{b}^+}$-point, the family $\{f_\alpha \mid \alpha < \mathfrak{b}\}$ is $\leq_{\mathcal{U}}$-dominating. We let $\text{next}(n, X) = \min(X \setminus (n+1))$. For any basis $\mathcal{B}$ of $\mathcal{U}$, the family $\{f_\alpha(\text{next}(\cdot, X)) \mid X \in \mathcal{B}, \alpha < \mathfrak{b}\}$ is $\leq^*$-dominating. Thus we have that $\chi(\mathcal{U}) \geq \mathfrak{d}$. Since $\mathcal{U}$ is simple, there is a basis $\{B_\alpha \mid \alpha < \kappa\}$ of $\mathcal{U}$ such that for any $\alpha < \beta < \kappa$, we have $B_\beta \subseteq^* B_\alpha$ but $B_\alpha \not\subseteq^* B_\beta$. Then the family of increasing enumerations of the B_α, $\alpha < \kappa$, is strictly $\leq^*$-increasing. The cofinality of this enumeration is at most $\mathfrak{d}$ and since κ is regular, we have $\chi(\mathcal{U}) = \kappa \leq \mathfrak{d}$.

Definition 1.2. Let $\mathbb{P}$ be a notion of forcing. We say that $\mathbb{P}$ *preserves an ultrafilter* $\mathcal{W}$ *over* ω if

$$\mathbb{P} \Vdash \text{“}(\forall X \subseteq \omega)(\exists Y \in \mathcal{W})(Y \subseteq X \vee Y \subseteq \omega \smallsetminus X)\text{”},$$

and in the contrary case we say “$\mathbb{P}$ destroys $\mathcal{W}$”.

In the first case, we have in $\mathbf{V}^{\mathbb{P}}$,

$$\text{filter}^{\mathbf{V}^{\mathbb{P}}}(\mathcal{W}) = \{X \in [\omega]^\omega \cap \mathbf{V}^{\mathbb{P}} \mid (\exists Y \in \mathcal{W})(X \supseteq Y)\} \text{ is an ultrafilter.}$$

We identify $\mathcal{W}$ with the generated $\text{filter}^{\mathbf{V}^{\mathbb{P}}}(\mathcal{W})$ and we simply say "$\mathcal{W}$ is an ultrafilter in $\mathbf{V}^{\mathbb{P}}$." If $\mathcal{W}$ is a P-point in the ground model and $\mathbb{P}$ is proper and preserves $\mathcal{W}$, then $\mathcal{W}$ stays a P-point in the forcing extension by [7, Lemma 3.2].

The space 2^ω is endowed with the product topology of the discrete space $2 = \{0, 1\}$. Any subset F of ω is a point in 2^ω via its characteristic function χ_F. Collections $\mathcal{C}$ of subsets of ω are said to be of descriptive complexity Γ if the set $\{\chi_F \mid F \in \mathcal{C}\}$ is contained in Γ.

Definition 1.3.

(1) The partial order $\mathbb{F}_\sigma$ is the forcing with F_σ-filters over ω. Stronger filters are superfilters.

(2) If $\mathcal{F}$ is a filter, then $\mathbb{F}_\sigma(\mathcal{F})$ is the forcing with F_σ-filters that are compatible with $\mathcal{F}$, i.e. $\mathcal{G} \in \mathbb{F}_\sigma(\mathcal{F})$ iff $\mathcal{G}$ is an F_σ-filter and $\mathcal{G} \subseteq \mathcal{F}^+ = \{X \subseteq \omega \mid (\forall F \in \mathcal{F})(X \cap F \neq \emptyset)\}$.

Definition and Observation 1.4. [18, Lemma 6.1] Let G be an $\mathbb{F}_\sigma(\mathcal{F})$-generic filter. We let $\underset{\sim}{\mathcal{U}}$ be a $\mathbb{F}_\sigma(\mathcal{F})$-name for the union of G. By a density argument, the poset $\mathbb{F}_\sigma(\mathcal{F})$ forces that $\underset{\sim}{\mathcal{U}}$ is an ultrafilter that extends $\mathcal{F}$.

Remark 1.5. Since the forcing $\mathbb{F}_\sigma(\mathcal{F})$ is countably closed it does not add new reals and thus preserves any ultrafilter from the ground model.

The set of finite strictly increasing sequences of natural numbers is denoted by $\omega^{\uparrow<\omega}$. The length of $s \in \omega^{\uparrow<\omega}$ is its domain. For $s, t \in \omega^{<\omega}$, we say "*$t$ extends s*" or "*s is an initial segment of t*" and write $s \trianglelefteq t$ if $\text{dom}(s) \subseteq \text{dom}(t)$ and $s = t \restriction \text{dom}(s)$. The corresponding strict relation is denoted by $\vartriangleleft$.

Definition 1.6. A non-empty subset $p \subseteq \omega^{\uparrow<\omega}$ that is closed under initial segments is called a *tree*. The elements of a tree are called *nodes*. The range of a node t is denoted by $\text{rge}(t)$. A node s of a tree p is called a *splitting node of p* if s has more than one direct $\vartriangleleft$-successor in p and *ω-splitting node of p* if s has infinitely many direct $\vartriangleleft$-successors in p. The set of splitting nodes of p is denoted by $\text{spl}(p)$ while $\omega\text{-}\text{spl}(p)$ denotes the set of ω-splitting nodes of p.

Many well-known forcing notions such as Cohen-, Random-, Laver- and Mathias forcing have conditions that can be represented as particular kinds of trees. Stronger conditions are given by subtrees.

Miller forcing consists of all trees in which every node can be extended to a node which has infinitely many immediate successors.

In order to define a parametrised version of Miller forcing we will need some notions about blocks.

Definition 1.7. The elements of $\text{Fin} = [\omega]^{<\omega} \setminus \{\emptyset\}$ are called *blocks*. Let $\mathcal{F}$ be a filter over ω. We let

$$\begin{aligned}\mathcal{F}^{<\omega} &= \{X \subseteq \text{Fin} \mid (\exists A \in \mathcal{F})(X \supseteq [A]^{<\omega} \setminus \{\emptyset\})\},\\ (\mathcal{F}^{<\omega})^+ &= \{B \subseteq \text{Fin} \mid (\forall A \in \mathcal{F})([A]^{<\omega} \cap B \neq \emptyset)\}.\end{aligned}$$

The set $\mathcal{F}^{<\omega}$ is a filter over Fin and $(\mathcal{F}^{<\omega})^+$ is the corresponding coideal. The following forcing notion was introduced by Guzmán and Kalajdzievski [14] in order to prove that the ultrafilter number $\mathfrak{u}$ may be smaller than the almost disjointness number $\mathfrak{a}$ without using large cardinals.

Definition 1.8. (See [14]) Let $\mathcal{F}$ be a filter over ω. The forcing $\mathbb{PT}(\mathcal{F})$ consists of conditions $p \subseteq \omega^{\uparrow<\omega}$ such that for each $s \in p$ there is $t \trianglerighteq s$, such that $t \in \omega\text{-}\operatorname{spl}(p)$ and

$$\begin{aligned}\operatorname{sucspl}_p(t) := \{&\operatorname{rge}(r) \setminus \operatorname{rge}(t) \mid r \text{ is a } \triangleleft\text{-minimal} \\ &\omega\text{-splitting node of } p \text{ strictly above } t\} \in (\mathcal{F}^{<\omega})^+.\end{aligned}$$

Such a t is called an $\mathcal{F}$-splitting node. We furthermore require of p that each ω-splitting node is an $\mathcal{F}$-splitting node[1] and there is a unique $\triangleleft$-minimal ω-splitting node called the *trunk of* p, $\operatorname{tr}(p)$. The set of $\mathcal{F}$-splitting nodes of p is denoted by $\mathcal{F}\text{-}\operatorname{spl}(p)$. A condition q is stronger than a condition p if $q \subseteq p$ and we write $q \leq p$.

Let G be $\mathbb{PT}(\mathcal{F})$-generic. Then we define the generic real

$$r_G = \bigcup\{\operatorname{tr}(p) \mid p \in G\}.$$

The $\mathbb{PT}(\mathcal{F})$-generic real r_G diagonalises $\mathcal{F}$ [14, Lemma 18], i.e., for any $F \in \mathcal{F}$ we have $\operatorname{rge}(r_G) \subseteq^* F$. In order to see this, we define the following manner of strengthening conditions.

Definition 1.9. For $p \in \mathbb{PT}(\mathcal{F})$ and $F \in \mathcal{F}$ we let $p \restriction\restriction F = q$ be the weakest strengthening of p such that

$$(\forall t \in \mathcal{F}\text{-}\operatorname{spl}(q))(\forall r \in \operatorname{sucspl}_q(t))(r \subseteq F).$$

Since $\mathcal{F}$ is a filter, we have $p \restriction\restriction F \in \mathbb{PT}(\mathcal{F})$. Since for every p, $p \restriction\restriction F \leq p$ and $p \restriction\restriction F \Vdash$ "$\underset{\sim}{r}_G \subseteq^* F$", the forcing $\mathbb{PT}(\mathcal{F})$ diagonalises $\mathcal{F}$.

Any node $t \in \omega^{\uparrow<\omega}$ can be mapped to $\operatorname{rge}(t) \in [\omega]^{<\omega}$, the range of t. Vice versa, any $r \in [\omega]^{<\omega}$ can be mapped to $\operatorname{en}(r) \in \omega^{\uparrow<\omega}$, the strictly increasing enumeration of r. Note that in contrast to Guzmán and Kalajdzievski, we do not identify $t \in p \subseteq \omega^{\uparrow<\omega}$ with its range. The function sending $t \in \omega^{\uparrow<\omega}$ to its range is an isomorphism witnessing

$$(\omega^{\uparrow<\omega}, \trianglelefteq) \cong ([\omega]^{<\omega}, \sqsubseteq),$$

where $\sqsubseteq$ denotes the end-extension relation on $\operatorname{Fin} \cup \{\emptyset\}$, i.e., $r \sqsubseteq s$ if $r \subseteq s$ and $\max(r) < \min(s \setminus r)$ or $r = \emptyset$ or $r = s$. Again, the strict relation corresponding to $\sqsubseteq$ is denoted by $\sqsubset$.

Having collected the necessary definitions we may now state the main theorem of this paper.

Theorem 1.10. *We assume* CH.

(**A**) *There are a countable support iteration* $\mathbb{P} = \langle \mathbb{P}_\gamma, \underset{\sim}{\mathbb{Q}}_\beta \mid \gamma \leq \aleph_2, \beta < \aleph_2\rangle$ *and a sequence of names* $\langle \underset{\sim}{\mathfrak{F}}_\gamma, \underset{\sim}{\mathfrak{U}}_\gamma, \underset{\sim}{r}_\beta \mid \gamma \leq \aleph_2, \beta < \aleph_2\rangle$ *such that:*
  1. (1) $\mathbb{P}_0 = \{0\}$, *and*
  2. (2) *For* $\beta < \aleph_2$ *we have the following:*
     - (i) $\mathbb{P}_\beta \Vdash \underset{\sim}{\mathfrak{F}}_\beta = \operatorname{filter}(\{\operatorname{rge}(\underset{\sim}{r}_\gamma) \mid \gamma < \beta\})$,
     - (ii) $\mathbb{P}_\beta * \mathbb{F}_\sigma(\underset{\sim}{\mathfrak{F}}_\beta) \Vdash \underset{\sim}{\mathfrak{U}}_\beta$ *is the* $\mathbb{F}_\sigma(\underset{\sim}{\mathfrak{F}}_\beta)$*-generic ultrafilter,*
     - (iii) $\mathbb{P}_\beta \Vdash \underset{\sim}{\mathbb{Q}}_\beta = \mathbb{F}_\sigma(\underset{\sim}{\mathfrak{F}}_\beta) * \mathbb{PT}(\underset{\sim}{\mathfrak{U}}_\beta)$ *and*

[1] There might be finitely splitting nodes. It is open whether the set of conditions without finitely splitting nodes is dense.

(iv) $\mathbb{P}_{\beta+1} \Vdash \underaccent{\sim}{r}_\beta$ *is the* $\mathbb{PT}(\underaccent{\sim}{\mathcal{U}}_\beta)$*-generic real.*

Statements (i) and (ii) also hold for $\beta = \aleph_2$*.*

(**B**) *Any* $\mathbb{P}$ *as in (***A***) is proper, does not collapse* $\aleph_2$*, and preserves any* P*-point from the ground model (and hence there is a simple* $P_{\aleph_1}$*-point) and forces that*

$$\text{filter}(\{\text{rge}(\underaccent{\sim}{r}_\gamma) \mid \gamma < \aleph_2\})$$

is a simple $P_{\aleph_2}$*-point and that* $2^\omega = \aleph_2$*.*

Remark 1.11. The existence of an iteration as in (**A**) follows from [24, Ch. III, Definition 3.1, Claim 3.1A, Theorem 3.1B]. Such an iteration is not as uniform as it may look at first sight. As we will see in Lemma 3.5, for $\beta \leq \aleph_2$ of uncountable cofinality we have

$$\mathbb{P}_\beta \Vdash \underaccent{\sim}{\mathcal{F}}_\beta \text{ is an ultrafilter.}$$

Hence for $\beta \leq \aleph_2$ of uncountable cofinality, in $\mathbf{V}^{\mathbb{P}_\beta}$, the forcing $\mathbb{F}_\sigma(\mathcal{F}_\beta)$ is equivalent to the one-point forcing $\{\underaccent{\sim}{\mathcal{F}}_\beta\}$ and we can let $\underaccent{\sim}{\mathcal{F}}_\beta = \underaccent{\sim}{\mathcal{U}}_\beta$.

Another important concept is the following.

Definition 1.12.

(1) A function $h\colon \omega \to \omega$ is called *finite-to-one* if for any n, the preimage of $\{n\}$, i.e. $h^{-1}[\{n\}]$, is finite (this includes the possibility of being empty).

(2) Two ultrafilters $\mathcal{F}$ and $\mathcal{U}$ over ω are called *nearly coherent* if there is a finite-to-one function h such that $h(\mathcal{F}) = h(\mathcal{U})$ where $h(\mathcal{U}) = \{X \subseteq \omega \mid h^{-1}[X] \in \mathcal{U}\}$.

By [3] the near-coherence relation is an equivalence relation on the set of ultrafilters, and its equivalence classes are called near-coherence classes.

Observation 1.13. *If* $\mathcal{U}$ *is a simple* $P_{\aleph_2}$*-point then* $\mathcal{U}$ *is not nearly coherent to any ultrafilter* $\mathcal{W}$ *with character* $\aleph_1$*.*

Proof. We let $\langle B_\alpha \mid \alpha < \aleph_2\rangle$ be an enumeration of a basis of $\mathcal{U}$ with $B_\alpha \supseteq^* B_\beta$ and $B_\alpha \not\subseteq^* B_\beta$ for $\alpha < \beta < \aleph_2$. We let $\{A_\alpha \mid \alpha < \aleph_1\}$ be a basis for $\mathcal{W}$. For a contradiction we assume that f is a finite-to-one function and $f(\mathcal{U}) = f(\mathcal{W})$. Then for every $\alpha < \aleph_2$ there is some $\beta_\alpha < \aleph_1$ such that $f[A_{\beta_\alpha}] \subseteq^* f[B_\alpha]$. Since $\aleph_2$ is regular, there is some $\beta < \aleph_1$ and there is an unbounded subset X of $\aleph_2$ such that for any $\alpha \in X$, $f[A_\beta] \subseteq^* f[B_\alpha]$. But then $\langle B_\alpha \mid \alpha < \aleph_2\rangle$ cannot be a descending basis of an ultrafilter. $\square$

Remark 1.14. Moreover, by [6], in the extension $\mathbf{V}^{\mathbb{P}_{\aleph_2}}$ we have $\mathfrak{s} = \aleph_2$ and that there are exactly two near coherence classes of ultrafilters. The reason for this is that the existence of a simple $P_{\aleph_1}$-point and a simple $P_{\aleph_2}$-point implies $\mathfrak{u} < \mathfrak{s}$. This further implies that there are at most two near coherence classes.

Remark 1.15. In [19] we construct another model with exactly two near coherence classes.

2. Canjar filters and parametrised Miller forcing

We are interested in filters over ω such that the associated forcing $\mathbb{PT}(\mathcal{F})$ does not add a dominating real. An example for such a filter is the following.

Definition 2.1. A filter $\mathcal{F}$ is called a *Canjar filter* if for any sequence $\langle X_n \mid n < \omega\rangle$ of elements of $(\mathcal{F}^{<\omega})^+$ there is a sequence $\langle Y_n \mid n < \omega\rangle$ such that $Y_n \in [X_n]^{<\omega}$ for all $n \in \omega$, and $\bigcup\{Y_n \mid n < \omega\} \in (\mathcal{F}^{<\omega})^+$.

In fact it is enough to consider decreasing sequences of elements in $(\mathcal{F}^{<\omega})^+$.

Remark 2.2 ([11, Claim 2.3]). A filter $\mathcal{F}$ is Canjar if and only if for any *decreasing* sequence $\langle X_n \mid n < \omega\rangle$ of elements of $(\mathcal{F}^{<\omega})^+$ there is a sequence $\langle Y_n \mid n < \omega\rangle$ such that $Y_n \in [X_n]^{<\omega}$ for all $n \in \omega$, and $\bigcup\{Y_n \mid n < \omega\} \in (\mathcal{F}^{<\omega})^+$.

Proof. The "only if" part is obvious, so assume we have any sequence $\langle X_n \mid n < \omega\rangle$ of elements in $(\mathcal{F}^{<\omega})^+$. Now define a decreasing sequence by

$$X'_n = \bigcup_{i\geq n} X_i.$$

If there is a sequence $\langle Y'_n \mid n < \omega\rangle$ such that $Y'_n \in [X'_n]^{<\omega}$ for all $n \in \omega$, and $\bigcup\{Y'_n \mid n < \omega\} \in (\mathcal{F}^{<\omega})^+$ we can set

$$Y_n = \bigcup_{i\leq n}(Y'_i \cap X_n).$$

Then for any $n \in \omega$ we have that $Y_n \in [X_n]^{<\omega}$ as a finite union of finite sets, each contained in X_n, and also $\bigcup\{Y_n \mid n < \omega\} = \bigcup\{Y'_n \mid n < \omega\} \in (\mathcal{F}^{<\omega})^+$. □

If $\mathcal{F}$ is a Canjar filter, then $\mathbb{PT}(\mathcal{F})$ does not add a dominating real by [14, Proposition 23]. It is open whether there are non-Canjar filters $\mathcal{F}$ such that $\mathbb{PT}(\mathcal{F})$ does not add dominating reals.

Definition 2.3. The following game $\mathcal{G}_{\mathrm{Canjar}}(\mathcal{F})$ is called *the Canjar game about* $\mathcal{F}$: Players I and II play alternately in ω many rounds.

I	X_0		X_1		X_2		$\dots$
II		Y_0		Y_1		Y_2	$\dots$

The rules are: $X_n \in (\mathcal{F}^{<\omega})^+$ and $Y_n \in [X_n]^{<\omega}$. Player II wins if

$$\bigcup\{Y_n \mid n \in \omega\} \in (\mathcal{F}^{<\omega})^+.$$

A filter is a Canjar filter if and only if Mathias forcing with second components in the filter does not add a dominating real [15, Theorem 5]. There are more equivalent formulations, see, e.g., [5, 11, 13]. The following one will be used further below.

Proposition 2.4 ([11])**.** *A filter $\mathcal{F}$ is a Canjar filter if and only if player I does not have winning strategy in $\mathcal{G}_{\mathrm{Canjar}}(\mathcal{F})$.*

Lemma 2.5. *If a filter $\mathcal{F}$ is not Canjar, then player I has a winning strategy σ in the Canjar game $\mathcal{G}_{\mathrm{Canjar}}(\mathcal{F})$ such that for each $n \geq 1$*

$$\sigma(Y_0, \dots, Y_{n-1}) \supseteq \sigma(Y_0, \dots, Y_n) \supseteq \bigcup_{0\leq i\leq n} Y_i.$$

Proof. If $\mathcal{F}$ is not Canjar, then by Remark 2.2 there exists a decreasing sequence $\langle X_n \mid n < \omega \rangle$ of elements of $(\mathcal{F}^{<\omega})^+$ such that for any sequence $\langle Y_n \mid n < \omega \rangle$ with $Y_n \in [X_n]^{<\omega}$ for $n \in \omega$ we have $\bigcup \{Y_n \mid n < \omega\} \notin (\mathcal{F}^{<\omega})^+$. Obviously playing X_n in the n-th move is already a winning strategy for player I in the Canjar game. We now define for

$$\sigma(Y_0, \dots, Y_n) = X_{n+1} \cup \bigcup_{0 \leq i \leq n} Y_i$$

to obtain a winning strategy as claimed. □

Guzmán and Kalajdzievski introduced a family of strengthenings of Canjarity. For stating these strengthenings we first recall subrelations of the partial order on $\mathbb{PT}(\mathcal{F})$ that are related to finite subtrees.

Definition 2.6. Let $T \subseteq p$ be a finite tree. We write $q \leq_T p$ if $q \leq p$, $T \subseteq q$, and $T \cap \mathcal{F}\text{-}\operatorname{spl}(q) = T \cap \mathcal{F}\text{-}\operatorname{spl}(p)$.

Definition 2.7. Let $d\colon \omega^{\uparrow<\omega} \to \omega$ be a bijection such that $s \trianglelefteq t \to d(s) \leq d(t)$. For a filter $\mathcal{F}$, a condition $p \in \mathbb{PT}(\mathcal{F})$, and $n \in \omega$ we define

$$T(p,n) = \{t \in p \mid \exists s(t \trianglelefteq s \wedge s \in \mathcal{F}\text{-}\operatorname{spl}(p) \wedge d^{-1}(s) \leq n)\}.$$

We will use the following game which is closely related to the Canjar game.

Definition 2.8. For a filter $\mathcal{F}$ we consider the game $\mathcal{H}(\mathcal{F})$

I	p_0		p_1		p_2		$\dots$
II		n_0		n_1		n_2	$\dots$

with the following rules: For $i < \omega$

(1) $p_i \in \mathbb{PT}(\mathcal{F})$,
(2) $n_0 \in \omega$, $n_{i+1} > n_i$, and
(3) $p_{i+1} \leq_{T_i} p_i$ for $T_i = T(p_i, n_i)$.

Player II wins the game $\mathcal{H}(\mathcal{F})$ if $\bigcup\{T(p_i, n_i) \mid i < \omega\} \in \mathbb{PT}(\mathcal{F})$.

This game is equivalent to the Canjar game in the sense that for any filter $\mathcal{F}$ player I has a winning strategy in $\mathcal{G}_{\text{Canjar}}(\mathcal{F})$ if and only if player I has winning strategy in $\mathcal{H}(\mathcal{F})$. For our purposes we are only concerned with the "only if" part of this equivalence, for the other direction see [14, Proposition 27].

Lemma 2.9. *If a filter $\mathcal{F}$ is not Canjar, then player I has a winning strategy in the game $\mathcal{H}(\mathcal{F})$.*

Proof. By Proposition 2.4 and Lemma 2.5 player I has a winning strategy σ in the Canjar game $\mathcal{G}_{\text{Canjar}}(\mathcal{F})$ such that for each $n \geq 1$ we have

$$\sigma(Y_0, \dots, Y_{n-1}) \supseteq \sigma(Y_0, \dots, Y_n) \supseteq \bigcup_{0 \leq i \leq n} Y_i.$$

Now I plays in the game $\mathcal{H}(\mathcal{F})$ as follows. In the first move I plays $\sigma(\emptyset) = X_0$ in the Canjar game and chooses p_0 such that for any $s \in \mathcal{F}\text{-}\operatorname{spl}(p_0)$,

$$\operatorname{sucspl}_{p_0}(s) = \{y \in X_0 \mid \max(s) < \min(y)\}.$$

Now II plays some $n_0 \in \omega$ in the game $\mathcal{H}(\mathcal{F})$. Player I interprets n_0 in the Canjar game as

$$\begin{aligned} Y_0^s &= \{\mathrm{rge}(t) \setminus \mathrm{rge}(s) \mid t \in T(p_0, n_0)\} \cap \mathrm{sucspl}_{p_0}(s), \\ Y_0 &= \bigcup\{Y_0^s \mid s \in \mathcal{F}\text{-}\mathrm{spl}(p_0)\}. \end{aligned}$$

This is a finite subset of X_0. For $i \geq 1$, once n_{i-1} is played, it is interpreted as Y_{i-1} in the same fashion. Player I plays X_i according to σ and translates this to some p_i[2] such that $p_i \leq_{T(p_{i-1}, n_{i-1})} p_{i-1}$ and for each $s \in \mathcal{F}\text{-}\mathrm{spl}(p_i)$,

$$\mathrm{sucspl}_{p_i}(s) = \{y \in X_i \mid \max(s) < \min(y)\}.$$

Note that for the existence of p_i we use

$$X_{i-1} \supseteq X_i = \sigma(Y_0, \dots, Y_{i-1}) \supseteq \bigcup_{j<i} Y_j.$$

Player I wins the Canjar game with the strategy σ and hence $\bigcup\{Y_i \mid i < \omega\} \notin (\mathcal{F}^{<\omega})^+$. Now the strategy above is a winning strategy for player I in $\mathcal{H}(\mathcal{F})$: We show that $T = \bigcup\{T(p_i, n_i) \mid i < \omega\}$ fails to be a condition. If T has no ω-splitting node, then $T \notin \mathbb{PT}(\mathcal{F})$. If T has an ω-splitting node s, then

$$\mathrm{sucspl}_T(s) \subseteq \bigcup\{Y_i \mid i < \omega\} \notin (\mathcal{F}^{<\omega})^+.$$

□

We recall the P-point game.

Definition 2.10. Let $\mathcal{W}$ be an ultrafilter. The P-point game for $\mathcal{W}$, short $\mathcal{G}_{P\text{-point}}(\mathcal{W})$, is the following game:

I	X_0		X_1		X_2		$\dots$
II		Y_0		Y_1		Y_2	$\dots$

such that

(1) $X_i \in \mathcal{W}$,
(2) $Y_i \in [X_i]^{<\omega}$.

In the end player II wins if $\bigcup\{Y_i \mid i < \omega\} \in \mathcal{W}$.

Lemma 2.11 (Galvin and Shelah, see, e.g., [2, Theorem 4.4.4])**.** *Let $\mathcal{W}$ be an ultrafilter. $\mathcal{W}$ is a P-point if and only if player I does not have a winning strategy in $\mathcal{G}_{P\text{-point}}(\mathcal{W})$.*

Now we can give a version of Canjarity in relation to a P-point.

Definition 2.12 ([14, Def. 42])**.** Let $\mathcal{W}$ be an ultrafilter. Let d be a fixed bijection as in Definition 2.7. A filter $\mathcal{F}$ is called a $\mathcal{W}$-*Canjar filter* if player I does not have a winning strategy in the following game $\mathcal{H}(\mathcal{W}, \mathcal{F})$. The game is played in ω rounds and in round $2i$ player I plays Y_i and player II plays z_i, then in round $2i+1$ player I plays p_i and player II answers with n_i.

I	Y_0		p_0		Y_1		$\dots$
II		z_0		n_0		z_1	$\dots$

The rules are for $i < \omega$,

[2] Actually, p_i is determined by the requirements.

(1) $Y_i \in \mathcal{W}$,
(2) $z_i \in [Y_i]^{<\omega}$,
(3) $p_i \in \mathbb{PT}(\mathcal{F})$,
(4) $n_0 \in \omega$, $n_{i+1} > n_i$,
(5) $p_{i+1} \leq_{T_i} p_i$ for $T_i = T(p_i, n_i)$.

Player II wins if

$$\bigcup\{T(p_i, n_i) \mid i < \omega\} \in \mathbb{PT}(\mathcal{F}) \text{ and } \bigcup\{z_i \mid i < \omega\} \in \mathcal{W}.$$

The game $\mathcal{H}(\mathcal{W}, \mathcal{F})$ is a combination of the P-point game about $\mathcal{W}$ and the variation $\mathcal{H}(\mathcal{F})$ (from Definition 2.8) of the Canjar game $\mathcal{G}_{\text{Canjar}}(\mathcal{F})$ in which player I needs to only win one of the two partial games.

Lemma 2.13. *Let $\mathcal{W}$ be an ultrafilter. If a filter $\mathcal{F}$ is $\mathcal{W}$-Canjar then $\mathcal{W}$ is a P-point and $\mathcal{F}$ is Canjar.*

Proof. The moves $Y_0, z_0, Y_1, z_1, \ldots$ in $\mathcal{H}(\mathcal{W}, \mathcal{F})$ follow the rules of the P-point game $\mathcal{G}_{P\text{-point}}(\mathcal{W})$. If $\mathcal{W}$ is not a P-point, then player I has a winning strategy in this subgame and thus in $\mathcal{H}(\mathcal{W}, \mathcal{F})$.

Similarly, the moves $p_0, n_0, p_1, n_1, \ldots$ follow the rules of the game $\mathcal{H}(\mathcal{F})$ and if $\mathcal{F}$ is not a Canjar filter, player I has a winning strategy in this subgame by Lemma 2.9 and thus in $\mathcal{H}(\mathcal{W}, \mathcal{F})$.

□

Let $\mathcal{U}$ be a Canjar ultrafilter. Then $\mathcal{U}$ is a P-point. The easiest way to see this is to use that the Mathias forcing with $\mathcal{U}$ does not add a dominating real [10]. However, $\mathcal{U}$ is not $\mathcal{U}$-Canjar by part (b) of Lemma 2.15. Thus in general the reverse implication of the statement in Lemma 2.13 does not hold.

For properness and the preservation of $\aleph_1$, we use Axiom A, see, e.g., [16, Definition 31.10].

Definition 2.14. Let $\mathcal{F}$ be a filter and $p, q \in \mathbb{PT}(\mathcal{F})$.

(1) By induction on n we define $\mathcal{F}\text{-}\operatorname{spl}_n(p) \subseteq p$.
    $\mathcal{F}\text{-}\operatorname{spl}_0(p) = \{\operatorname{tr}(p)\}$.
    $\mathcal{F}\text{-}\operatorname{spl}_{n+1}(p)$ is the set of shortest nodes $s \in \mathcal{F}\text{-}\operatorname{spl}(p)$ that are not in $\mathcal{F}\text{-}\operatorname{spl}_n(p)$ but have a $\lhd$-predecessor in $\mathcal{F}\text{-}\operatorname{spl}_n(p)$.
(2) For $n \in \omega$ we write $q \leq_n p$ if $q \leq p$ and $\mathcal{F}\text{-}\operatorname{spl}_n(q) = \mathcal{F}\text{-}\operatorname{spl}_n(p)$. Note that $q \leq_n p$ implies $q \leq_k p$ for all $k < n$.

The next Lemma collects three technical properties of the forcing notions $\mathbb{PT}(\mathcal{F})$ and the notion of $\mathcal{W}$-Canjarity that we will use and that have been proved by Guzmán and Kalajdzievski.

Lemma 2.15. *Let $\mathcal{W}$ be a P-point.*

(a) [14, Lemma 18 and Proposition 20] *Let $\mathcal{F}$ be a filter. The forcing $\mathbb{PT}(\mathcal{F})$ with the suborders $\leq_n$, $n < \omega$, has Axiom A and diagonalises $\mathcal{F}$.*
(b) [14, Proposition 43] *Let $\mathcal{F}$ be a $\mathcal{W}$-Canjar filter. Then $\mathbb{PT}(\mathcal{F})$ preserves $\mathcal{W}$.*
(c) [14, Proposition 55] *The generic filter of the forcing $\mathbb{F}_\sigma$ is a $\mathcal{W}$-Canjar ultrafilter.*

In the remainder of this section we give a proof. The proof that $\mathbb{PT}(\mathcal{F})$ diagonalises the filter $\mathcal{F}$ was given after Definition 1.9.

Definition 2.16. Let p be a tree and let $s \in p$. We write $p_s = \{t \in p \mid t \trianglelefteq s \vee s \trianglelefteq t\}$.

For a set $X \subseteq \mathrm{Fin}$ we let minimal(X) be the set of elements of s of X such that there is no $r \in X$ with $r \sqsubset s$.

Lemma 2.17 ([14, Proposition 20])**.** *Let $\mathcal{F}$ be a filter, $p \in \mathbb{PT}(\mathcal{F})$, $s \in \mathcal{F}\text{-}\mathrm{spl}(p)$ and let D be a dense subset of $\mathbb{PT}(\mathcal{F})$. Let $\chi > 2^{\omega_1}$ be a regular cardinal and $M \prec H(\chi)$ be a countable elementary submodel with $\mathcal{F}$, p, $D \in M$. Then there is a $q \leq_0 p_s$, $q \in M$, such that q forces "$\underset{\sim}{G} \cap M \cap D \neq \emptyset$".*

Proof. We define

$$X = E(D, p, s) := \mathrm{minimal}\{\mathrm{rge}(r) \setminus \mathrm{rge}(s) \mid s \triangleleft r \wedge \exists q \leq p_s(\mathrm{tr}(q) = r \wedge q \in D)\}.$$

Then $X \in (\mathcal{F}^{<\omega})^+$, since otherwise we could find some $F \in \mathcal{F}$ such that no t in X is a subset of F. Thus the set D would not be dense below $q = p_s \upharpoonright\upharpoonright F$. For every $t \in X$ we take $q(t) \leq_0 p_{s^\frown \mathrm{en}(t)}$ such that $q(t) \in D$. We let $q = \bigcup\{q(t) \mid t \in X\}$.

We show $q \in \mathbb{PT}(\mathcal{F})$. For this we have to show that any infinitely splitting node $u \in q$ is $\mathcal{F}$-splitting in q. We let

$$\hat{X} = \{r \in p \mid r \triangleright s \wedge \mathrm{rge}(r) \setminus \mathrm{rge}(s) \in X\}.$$

We fix an infinitely splitting node $u \in q$. Note that $u \in p_s$.

First case: If there is some $r \in \hat{X}$, $u \trianglerighteq r$, then for the t with $s^\frown \mathrm{en}(t) = r$ we have $t \in X$, $u \in q(t)$ and u is $\mathcal{F}$-splitting in $q(t)$, hence $\mathcal{F}$-splitting in q.

Second case: The node u does not extend any node in $\hat{X}$. Since D is dense, by the forcing theorem, for any $q' \leq p$ there is some $r \in \hat{X} \cap q'$. So for any $w \in \mathrm{sucspl}_p(u)$ we pick the u' such that $w = \mathrm{rge}(u') \setminus \mathrm{rge}(u)$ and for u' there is $r \in \hat{X}$, $u' \trianglelefteq r$. Thus $u' \in q$ and $w \in \mathrm{sucspl}_q(u)$. Taking all w together yields $\mathrm{sucspl}_q(u) = \mathrm{sucspl}_p(u) \in (\mathcal{F}^{<\omega})^+$.

Now if $w \in \mathrm{sucspl}_q(s)$, the set $\{q(t) \mid t \in X \wedge s^\frown \mathrm{en}(t) \trianglerighteq s^\frown \mathrm{en}(w)\} \in M$ is a predense set above $q_{s^\frown \mathrm{en}(w)}$ and the latter forces $M \cap D \cap \underset{\sim}{G} \neq \emptyset$. □

Axiom A is immediate. We proved above (after Definition 1.9) that $\mathbb{PT}(\mathcal{F})$ diagonalises $\mathcal{F}$. Thus Lemma 2.15 part (a) is proved.

Now we turn to part (b).

For this we first give a shorter proof to the following.

Lemma 2.18 (Special case of [14, Lemma 11])**.** *Let $\mathcal{F}$ be a Canjar filter. For every sequence $\langle X_n \mid n < \omega\rangle$ of elements $X_n \in (\mathcal{F}^{<\omega})^+$ there is some $f \in {}^\omega\omega$ such that for $n \in \omega$ and*

$$Y_n = \{s \in X_n \mid s \subseteq [f(n), f(n+1))\}$$

we have $\bigcup\{Y_n \mid n < \omega\} \in (\mathcal{F}^{<\omega})^+$.

Proof. We let $f(0) = 0$. I and II play the Canjar game $\mathcal{G}_{\mathrm{Canjar}}(\mathcal{F})$ as follows

I	X'_0		X'_1		X'_2		$\dots$
II		Y'_0		Y'_1		Y'_2	$\dots$

such that

(1) $X'_0 = X_0$, $f(0) = 0$, and for $n \geq 1$ we let

$$f(n) = \max\{\max(s) + 1 \mid s \in Y'_{n-1}\},$$
$$X'_n = \{t \in X_n \mid \min(t) \geq f(n)\}.$$

Since $\mathcal{F}$ is non-principal, $X'_n \in (\mathcal{F}^{<\omega})^+$.

(2) Now player II plays $Y'_n \in [X'_n]^{<\omega}$.

In the end we take a play $\langle Y'_n \mid n < \omega\rangle$ such that II wins and obtain f. We observe that $Y'_n \subseteq Y_n$. □

Lemma 2.19 (A rest of Ramseyness, [14, Lemma 38])**.** *Let $\mathcal{F}$ be Canjar and $p \in \mathbb{PT}(\mathcal{F})$ and $c\colon \mathcal{F}\text{-}\mathrm{spl}(p) \to 2$. Then there is a $q \leq p$ such that $\mathcal{F}\text{-}\mathrm{spl}(q)$ is c-monochromatic.*

Proof. Assume there is no $q \leq p$ such that $\mathcal{F}\text{-}\mathrm{spl}(q)$ is 0-monochromatic, i.e., $1 \in c[\mathcal{F}\text{-}\mathrm{spl}(q)]$ for every $q \leq p$. Then we need to find a $q \leq p$ such that $\mathcal{F}\text{-}\mathrm{spl}(q)$ is 1-monochromatic. For $s \in \mathcal{F}\text{-}\mathrm{spl}(p)$ we define

$$X(s) = \{\mathrm{rge}(t) \setminus \mathrm{rge}(s) \mid t \in \mathcal{F}\text{-}\mathrm{spl}(p),\ s \vartriangleleft t,\ c(t) = 1\}.$$

We fix s. In order to show $X(s) \in (\mathcal{F}^{<\omega})^+$, for contradiction we assume that there exists an $A \in \mathcal{F}$ such that $r \not\subseteq A$ for any $r \in X(s)$. Then the condition $q = p_s \restriction\restriction A$ fulfils $c(t) = 0$ for every $t \in \mathcal{F}\text{-}\mathrm{spl}(q) \setminus \{s\}$. In other words, for any $t \in \mathrm{sucspl}_q(s)$, the set $\mathcal{F}\text{-}\mathrm{spl}(q_t)$ is 0-monochromatic, contrary to our assumption. Hence $X(s) \in (\mathcal{F}^{<\omega})^+$ is proved. Now we unfix s.

By Lemma 2.18 we can pick for each $s \in \mathcal{F}\text{-}\mathrm{spl}(p)$ some $Y(s) \subseteq X(s)$ such that $Y(s) \in (\mathcal{F}^{<\omega})^+$ and for every $n \in \omega$ the set $\{t \in Y(s) \mid t \cap n \neq \emptyset\}$ is finite. We will now inductively construct a fusion sequence

$$p \geq q_0 \geq_0 q_1 \geq_1 q_2 \geq_2 \dots$$

such that for every $n \in \omega$ and $s \in \mathcal{F}\text{-}\mathrm{spl}_n(q_n)$ we have $c(s) = 1$. In the end $q = \bigcap_{n\in\omega} q_n \leq p$ will be the desired condition.

Let $s_0 \in \mathcal{F}\text{-}\mathrm{spl}(p)$ be such that $c(s_0) = 1$ and set $q_0 = p_{s_0}$. If q_n is already constructed we set

$$q_{n+1} = \bigcup_{s\in\mathcal{F}\text{-}\mathrm{spl}_n(q_n)} \bigcup_{t\in Y(s)} p_{s^\frown \mathrm{en}(t)}.$$

Note that for $s \in \mathcal{F}\text{-}\mathrm{spl}_n(q_n)$ and $t \in Y(s)$ we have $p_{s^\frown\mathrm{en}(t)} = (q_n)_{s^\frown\mathrm{en}(t)}$.

For $s \in \mathcal{F}\text{-}\mathrm{spl}_n(q_n)$, we show that $\mathrm{sucspl}_{q_{n+1}}(s) \subseteq Y(s)$.

If $r \in \mathcal{F}\text{-}\mathrm{spl}(q_{n+1})$ is such that $\mathrm{rge}(r) \setminus \mathrm{rge}(s) \in \mathrm{sucspl}_{q_{n+1}}(s)$, by definition of q_{n+1} we have $r \in p_{s^\frown\mathrm{en}(t)}$ for some $t \in Y(s)$ of minimal length. Since r is an $\mathcal{F}$-splitting successor of s in q_{n+1} and since $\{t \in Y(s) \mid t \cap n \neq \emptyset\}$ is finite for each $n \in \omega$, we must have that $r = s^\frown\mathrm{en}(t)$ and $\mathrm{rge}(r) \setminus \mathrm{rge}(s) = t \in Y(s)$. □

Lemma 2.20 ([14, Lemma 40])**.** *Let $\mathcal{F}$ be a Canjar filter, $\underset{\sim}{B}$ be a $\mathbb{PT}(\mathcal{F})$-name and $p \in \mathbb{PT}(\mathcal{F})$ with $\mathrm{tr}(p) = s$ and $p \Vdash$"$\underset{\sim}{B}$ is an infinite subset of ω". Then there is a $q \leq_0 p = p_s$, $B_s \in [\omega]^\omega$, and there are finite sets $\langle F_n \mid n < \omega\rangle$, such that*

(1) $F_n \in [\mathrm{Fin}]^{<\omega}$ *and* $(\forall n)(\forall r \in F_n)(\forall t \in F_{n+1})(\max(r) < \min(t))$.

(2) $\mathrm{sucspl}_q(s) = \bigcup_{n<\omega} F_n$.

(3) $(\forall m \geq n)(\forall t \in F_m)(q_{s^\frown\mathrm{en}(t)} \Vdash \underset{\sim}{B} \cap (n+1) = B_s \cap (n+1))$.

Proof. For $n \in \omega$ we let

$$D_n = \{q' \in \mathbb{PT}(\mathcal{F}) \mid q' \leq p \text{ and } q' \text{ determines } \underset{\sim}{B} \cap (n+1)\}. \tag{2.1}$$

The set D_n is dense below p and with the definition of E from the proof of Lemma 2.17 we have

$$X_n = E(D_n, p, s) \in (\mathcal{F}^{<\omega})^+.$$

For $m \geq n$ and $A \subseteq n+1$ we set

$$Y(A,m,n) = \{u \in X_m \mid (\exists q' \leq_0 p_{s^\frown \mathrm{en}(u)})(q' \in D_m \text{ and } q' \Vdash \underset{\sim}{B} \cap (n+1) = A)\}.$$

For $n \in \omega$ we let

$$\mathcal{A}_n = \{A \subseteq n+1 \mid (\forall m \geq n)(Y(A,m,n) \in (\mathcal{F}^{<\omega})^+)\}.$$

Since for each m, the X_m is positive and is divided into finitely many pieces by the choice of $\underset{\sim}{B} \cap (n+1)$, for each m, at least one of the pieces is positive. By Kőnig's Lemma, $\mathcal{A}_n \neq \emptyset$.

Players I and II play the Canjar game: By induction on n we choose X'_n, A_n, and F_n. We start with $A_{-1} = \emptyset$, $F_{-1} = \emptyset$. In step n we choose $X'_n \in (\mathcal{F}^{<\omega})^+$ and $A_n \in \mathcal{A}_n$ such that

$$\begin{aligned} A_n \supseteq & A_{n-1} \text{ and} \\ X'_n = & \{t \in X_n \mid (\forall r \in F_{n-1})(\max(r) < \min(t)) \wedge \\ & (\exists q')(q' \leq_0 p_{s^\frown \mathrm{en}(t)} \wedge q' \Vdash \underset{\sim}{B} \cap (n+1) = A_n)\}. \end{aligned}$$

If a finite union is positive, then one of the parts is positive. Hence we can choose X'_n as above in $(\mathcal{F}^{<\omega})^+$. Player II chooses $F_n \in [X'_n]^{<\omega}$. We take a play such that II wins, and hence $\bigcup\{F_n \mid n < \omega\} \in (\mathcal{F}^{<\omega})^+$. We let

$$B_s = \bigcup\{A_n \mid n \in \omega\}$$

and for $t \in F_n$ we choose $q(t)$ such that $q(t) \leq_0 p_{s^\frown \mathrm{en}(t)}$ and $q(t) \Vdash \underset{\sim}{B} \cap (n+1) = A_n$. We let

$$q = \bigcup\{q(t) \mid n < \omega, t \in F_n\}.$$

Then (3) holds. □

By a fusion argument we get the following slight strengthening.

Lemma 2.21. *Let $\mathcal{F}$ be a Canjar filter, $\underset{\sim}{B}$ be a $\mathbb{PT}(\mathcal{F})$-name and $p \in \mathbb{PT}(\mathcal{F})$ with $p \Vdash$ "$\underset{\sim}{B}$ is an infinite subset of ω". Then there is a $q \leq_0 p$, and for each $s \in \mathcal{F}\text{-}\mathrm{spl}(q)$ there are $B_s \in [\omega]^\omega$ and finite sets $\langle F^s_n \mid n < \omega\rangle$, such that*

(1) $F^s_n \in [\mathrm{Fin}]^{<\omega}$ *and* $(\forall n)(\forall r \in F^s_n)(\forall t \in F^s_{n+1})(\max(r) < \min(t))$.
(2) $\mathrm{sucspl}_q(s) = \bigcup_{n<\omega} F^s_n$.
(3) $(\forall m \geq n)(\forall t \in F^s_m)(q_{s^\frown \mathrm{en}(t)} \Vdash \underset{\sim}{B} \cap (n+1) = B_s \cap (n+1))$.

Here is the final step of the proof of part (b) of Lemma 2.15, where we literally follow the original proof.

Proof. Given p such that $p \Vdash \underset{\sim}{B} \in [\omega]^\omega$ we fix q, B_s, and F^s_n, $n < \omega$, $s \in \mathcal{F}\text{-}\mathrm{spl}(q)$, as in Lemma 2.21. Moreover by the Ramsey Lemma 2.19 we may strengthen q once (even lengthen the trunk) such that — for the strengthening, which we call q again — $(\forall t \in \mathcal{F}\text{-}\mathrm{spl}(q))(B_t \in \mathcal{W})$ or $(\forall t \in \mathcal{F}\text{-}\mathrm{spl}(q))(B^c_t \in \mathcal{W})$. We assume the first. Note that also the strengthened q has the analogon of property (3) of Lemma 2.21.

Let $s_0 = \mathrm{tr}(q)$.

We define a strategy for player I in $\mathcal{H}(\mathcal{W}, \mathcal{F})$ as follows

(i) Player I starts with $W_0 = B_{s_0}$

(ii) Assume player II plays $z_0 \in [W_0]^{<\omega}$. Letting $l_0 = \max(z_0)$, player I plays $p_0 = \bigcup\{q_{s_0 \frown \mathrm{en}(t)} \mid t \in F_i^{s_0}, i > l_0\}$. Note $p_0 \Vdash z_0 \subseteq \underset{\sim}{B}$.

(iii) Assume player II plays n_0. Now player I sets $T_0 = T(p_0, n_0)$ and $W_1 = \bigcap\{B_s \setminus (l_0 + 1) \mid s \in \mathcal{F}\text{-}\mathrm{spl}(p_0) \cap T_0\}$.

(iv) Suppose that the sequence $\langle W_0, z_0, p_0, n_0, \dots, W_m, z_m, p_m\rangle$ has been played such that for $i < m$, $T_i = T(p_i, n_i)$ and $p_{i+1} \leq_{T_i} p_i$ and for $i \leq m$, $p_i \Vdash z_i \subseteq \underset{\sim}{B}$. Now player II chooses n_m and player I lets $T_m = T(p_m, n_m)$, $l_m = \max(z_m)$, and

$$W_{m+1} = \bigcap\{B_s \cap [l_m + 1, \omega) \mid s \in \mathcal{F}\text{-}\mathrm{spl}(p_m) \cap T_m\}.$$

Assume that player II chooses $z_{m+1} \in [W_{m+1}]^{<\omega}$. Player I lets $l_{m+1} = \max(z_{m+1})$ and chooses for $s \in T_m \cap \mathcal{F}\text{-}\mathrm{spl}(p_m)$ a condition $p'_{s,m} \leq_0 (p_m)_s$ as follows: If $s \in T_m \cap \mathcal{F}\text{-}\mathrm{spl}(p_m)$ has a $\lhd$-larger element in T_m, then $p'_{s,m} \cap T_m = \{r \mid r \unlhd s\}$, and otherwise $p'_{s,m} = (p_m)_s$. Now player II plays

$$p_{m+1} = \bigcup\{(p'_{s,m})_{(s \frown \mathrm{en}(r))} \mid s \in T_m \cap \mathcal{F}\text{-}\mathrm{spl}(p_m) \wedge \\ j > l_{m+1} \wedge r \in F_j^s \wedge s^\frown \mathrm{en}(r) \in p_m\}.$$

By construction, $p_{m+1} \leq_{T(p_m, n_m)} p_m$ and $p_{m+1} \Vdash z_{m+1} \subseteq \underset{\sim}{B}$.

Since $\mathcal{F}$ is $\mathcal{W}$-Canjar, we know that σ is not a winning strategy. Suppose that player I played according to σ and player II won. Then we know that $U = \bigcup\{z_i \mid i < \omega\} \in \mathcal{W}$ and $q = \bigcup_{i<\omega} T_i \in \mathbb{PT}(\mathcal{F})$. By construction of q, we have $q \Vdash U \subseteq \underset{\sim}{B}$, and we are done. $\square$

So now part(b) is proved.

We turn to **part (c)** of Lemma 2.15. First we need some topology.

Definition 2.22. Let $X \subseteq \mathrm{Fin}$. We let

$$C(X) = \{A \subseteq \omega \mid (\forall s \in X)(s \cap A \neq \emptyset)\}.$$

Note that the set $C(X)$ is closed in the compact space 2^ω and hence compact.

Lemma 2.23 ([14, Lemma 47])**.** *Let $\mathcal{G}$ be a filter, $\mathcal{F} \in \mathbb{F}_\sigma(\mathcal{G})$ and $X \subseteq \mathrm{Fin}$. Then $\mathcal{F} \Vdash_{\mathbb{F}_\sigma(\mathcal{G})} X \in (\underset{\sim}{\mathcal{U}}(\mathcal{G})^{<\omega})^+$ if and only if $C(X) \subseteq \mathrm{filter}(\mathcal{F} \cup \mathcal{G})$.*

Proof. For the forward implication, let $H \notin \mathrm{filter}(\mathcal{F} \cup \mathcal{G})$. Then H^c is $\mathrm{filter}(\mathcal{F} \cup \mathcal{G})$-positive and $\mathcal{F} \cup \{H^c\} \leq \mathcal{F}$ is a condition in $\mathbb{F}_\sigma(\mathcal{G})$. Then there is some s in X with $s \subseteq H^c$. Thus $H \notin C(X)$.

For the reverse implication, suppose $C(X) \subseteq \mathrm{filter}(\mathcal{F} \cup \mathcal{G})$. Then for any A in $C(X)$ we have that the condition $\mathcal{F}$ forces "$A^c \notin \underset{\sim}{\mathcal{U}}(\mathcal{G})$". Hence $\mathcal{F}$ forces "if $D \in \underset{\sim}{\mathcal{U}}(\mathcal{G})$ then $D^c \notin C(X)$, i.e., there is $s \in X$ with $s \subseteq D$". $\square$

Lemma 2.24 ([14, Lemma 54])**.** *Let $\mathcal{F}$ be a filter and let $\mathcal{D} \subseteq \mathcal{F}$ be compact and let $X_1, \dots, X_n \in [\mathrm{Fin}]^\omega$ be such that $C(X_i) \subseteq \mathcal{F}$. Then there are $Y_i \in [X_i]^{<\omega}$, $1 \leq i \leq n$, such that for every $F \in \mathcal{D}$ and for every $1 \leq i \leq n$ and every $A_i^1, \dots, A_i^n \in C(Y_i)$ we have $F \cap \bigcap_{1 \leq i,j \leq n} A_i^j \neq \emptyset$.*

Proof. We consider the compact space $Z = (\prod_{i=1}^n \mathcal{P}(\omega)^n) \times \mathcal{D}$. For $\ell \in \omega$, we define the closed subspace $K(\ell)$ as follows:

$$
\begin{aligned}
K(\ell) = \{ (\langle \langle A_i^1, \dots, A_i^n \rangle \mid 1 \le i \le n \rangle, F) \in Z \mid \\
\bigwedge_{1 \le i,j \le n} A_i^j \in C(X_i \cap \mathcal{P}(\ell)) \wedge \bigcap_{1 \le i,j \le n} A_i^j \cap F = \emptyset \}.
\end{aligned}
\tag{2.2}
$$

Since $C(X_1), \dots, C(X_n), \mathcal{D} \subseteq \mathcal{F}$ we conclude that $\bigcap_{\ell \in \omega} K(\ell) = \emptyset$. Since Z is compact, there is some ℓ such that $K(\ell) = \emptyset$. Let $Y_i = X_i \cap \mathcal{P}(\ell)$. □

Now we perform the final step in Lemma 2.15 part(c):

Assume for a contradiction that $\mathcal{F}$ is a F_σ-filter and that $\mathcal{F}$ forces in $\mathbb{F}_\sigma$ that σ is a winning strategy for player I in $\mathcal{H}(\mathcal{W}, \underset{\sim}{\mathcal{U}})$. Since $\mathbb{F}_\sigma$ is a σ-closed forcing and since σ is a real, we can assume that σ is in the ground model. Let $\mathcal{F} = \bigcup_{i<\omega} \mathcal{C}_i$ for an increasing sequence $\mathcal{C}_i$ of compact sets. We show that there is a winning strategy for player I in $\mathcal{G}_{P\text{-point}}(\mathcal{W})$, and thus will have reached a contradiction.

We have: If $\mathcal{F} \Vdash p \in \mathbb{PT}(\underset{\sim}{\mathcal{U}})$ then $\mathcal{F} \Vdash (\forall s \in \omega\text{-}\operatorname{spl}(p))(\operatorname{sucspl}_p(s) \in (\underset{\sim}{\mathcal{U}}^{<\omega})^+)$. The latter means, according to Lemma 2.23, $C(\operatorname{sucspl}_p(s)) \subseteq \mathcal{F}$.

By Lemma 2.24 we have: For every $\mathcal{X} = \langle X_1, \dots, X_n \rangle$ such for every $1 \le i \le n$, $C(X_i) \subseteq \mathcal{F}$, and every $k \in \omega$ there is a function

$$F_{(\mathcal{X},k)} \colon \mathcal{X} \to [\mathrm{Fin}]^{<\omega}$$

with the following properties

(1) $Y_i = F_{(\mathcal{X},k)}(X_i) \in [X_i]^{<\omega}$ for every $1 \le i \le n$.
(2) For every $B \in \mathcal{C}_k$ and for every $A_i^1, \dots, A_i^n \in C(Y_i)$, $1 \le i \le n$, we have $B \cap \bigcap_{1 \le i,j \le n} A_i^j \ne \emptyset$.

We fix such a function.

Now we define a strategy π for player I in the P-point game $\mathcal{G}_{P\text{-point}}(\mathcal{W})$ as follows:

(i) $W_0 = \sigma(\emptyset) =: \pi(\emptyset)$.
(ii) Assume that player II plays $z_0 \in [W_0]^{<\omega}$ as response in $\mathcal{H}(\mathcal{W}, \underset{\sim}{\mathcal{U}})$. We let $p_0 = \sigma(\langle W_0, z_0 \rangle)$ and $s_0 = \operatorname{tr}(p_0)$. We take $n_0 > d^{-1}(s_0)$ so large such that

$$(\forall t \in F_{(\operatorname{sucspl}_{p_0}(s_0),0)}(\operatorname{sucspl}_{p_0}(s_0))(d^{-1}(s_0{}^\frown \operatorname{en}(t)) < n_0).$$

Now the strategy π in the game $\mathcal{G}_{P\text{-point}}(\mathcal{W})$ says: Player I will play $W_1 = \sigma(\langle W_0, z_0, p_0, n_0 \rangle)$.

(iii) In general assume that $\langle W_0, z_0, W_1, z_1, \dots, W_m \rangle$ has been played in the game $\mathcal{G}_{P\text{-point}}(\mathcal{W})$ and that in the same stage player I has constructed on the side a sequence

$$\langle W_0, z_0, p_0, n_0, W_1, z_1, p_1, \dots, n_{m-1}, W_m \rangle$$

in the game $\mathcal{H}(\mathcal{W}, \underset{\sim}{\mathcal{U}})$ following σ such that for each $i < m$ the integer n_i has the following property: Letting

$$\mathcal{X}_i = \{\operatorname{sucspl}_{p_i}(u) \mid u \in T(p_i, n_{i-1}) \cap \mathcal{F}\text{-}\operatorname{spl}(p_i)\}$$

we have taken n_i so large such that

$$
\begin{aligned}
&(\forall u \in T(p_i, n_{i-1}) \cap \mathcal{F}\text{-}\operatorname{spl}(p_i)) \\
&(\forall t \in F_{(\mathcal{X}_i,i)}(\operatorname{sucspl}_{p_i}(u)))\,(d^{-1}(u^\frown \operatorname{en}(t)) < n_i).
\end{aligned}
\tag{2.3}
$$

and player I has played $W_{i+1} = \sigma(\langle W_0, \dots, W_i, z_i, p_i, n_i\rangle) = \pi(z_0, \dots, z_i)$.

Assume that player II plays z_m as a response to $\langle W_0, \dots, W_m\rangle$ in $\mathcal{G}_{P\text{-point}}(\mathcal{W})$. Player I uses z_m in the game $\mathcal{H}(\mathcal{W}, \underset{\sim}{\mathcal{U}})$ and plays

$$p_m = \sigma(\langle W_0, \dots, W_m, z_m\rangle).$$

Let $n_m > n_{m-1}$ be such that for

$$\mathcal{X}_m = \{\operatorname{sucspl}_{p_m}(u) \mid u \in T(p_m, n_{m-1}) \cap \mathcal{F}\text{-}\operatorname{spl}(p_m)\}$$

we have

$$(\forall u \in T(p_m, n_{m-1}) \cap \mathcal{F}\text{-}\operatorname{spl}(p_m)) \tag{2.4}$$
$$(\forall t \in F_{(\mathcal{X}_m, m)}(\operatorname{sucspl}_{p_m}(u)))\,(d^{-1}(u^\frown \operatorname{en}(t)) < n_m).$$

Player I plays

$$W_{m+1} = \sigma(\langle W_0, \dots, W_m, z_m, p_m, n_m\rangle) =: \pi(z_0, \dots, z_m).$$

Now suppose that the play is finished. We let $Z = \bigcup\{z_i \mid i < \omega\}$ and $q = \bigcup_{i<\omega} T(p_i, n_i)$. The properties of the function F and statements (2.3) and (2.4) ensure that

$$\mathcal{K} := \operatorname{filter}(\mathcal{F} \cup \bigcup\{C(\operatorname{sucspl}_q(s)) \mid s \in \mathcal{F}\text{-}\operatorname{spl}(q)\})$$

is a F_σ*-filter*. Then $\mathcal{K} \leq_{\mathbb{F}_\sigma} \mathcal{F}$ and by Lemma 2.23 we have

$$\mathcal{K} \Vdash q \in \mathbb{PT}(\underset{\sim}{\mathcal{U}}) \wedge \sigma \text{ is a winning strategy for player I in } \mathcal{H}(\mathcal{W}, \underset{\sim}{\mathcal{U}}).$$

Hence $\mathcal{K} \Vdash Z \notin \mathcal{W}$. Since the latter two sets are in the ground model, we have in $\mathbf{V}$, $Z \notin \mathcal{W}$. Thus π is a winning strategy for player I in $\mathcal{G}_{P\text{-point}}(\mathcal{W})$, contradiction. Now Lemma 2.15 is proved.

Besides Lemma 2.15 we will also use the following which follows from the backward implication in Lemma 2.23.

Lemma 2.25. *Let $\mathcal{G}$ be a filter. If $\mathcal{F}$ is an F_σ-filter, $C(X) \subseteq \mathcal{F}$ and $\mathcal{F} \cup \mathcal{G}$ generates a filter, then $\mathcal{F} \Vdash_{\mathbb{F}_\sigma(\mathcal{G})} X \in (\underset{\sim}{\mathcal{U}}(\mathcal{G})^{<\omega})^+$.*

3. Proof of statement (**B**) of the theorem

We turn to new work and recall Definition 1.2 and the discussion thereafter.

Definition 3.1. For a notion of forcing $\mathbb{P}$ we say $\mathcal{W}$ *is a P-point in* $\mathbf{V}^{\mathbb{P}}$ if $\mathcal{W}$ has a $\mathbb{P}$-name and $\mathbb{P} \Vdash$ "$\mathcal{W}$ generates a P-point".

Note that such a P-point $\mathcal{W}$ is still a P-point in $\mathbf{V}^{\mathbb{P}'}$ if $\mathbb{P}$ is a complete subforcing of $\mathbb{P}'$ and the quotient forcing $\mathbb{P}'/\mathbb{P}$ is proper and preserves $\mathcal{W}$. This will apply to numerous P-points in intermediate stages of our construction.

We state and prove our main lemma. From now on we will drop the tildes underneath the names, except in cases where we want to stress technical arguments.

Lemma 3.2. *We assume* CH *and fix a P-point $\mathcal{E}$ from the ground model. We let $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta \mid \alpha \leq \aleph_2, \beta < \aleph_2\rangle$ be a countable support iteration and $\langle \mathcal{F}_\alpha, \mathcal{U}_\alpha, r_\beta \mid \alpha \leq \aleph_2, \beta < \aleph_2\rangle$ a sequence of names with the following properties:*

(1) $\mathbb{P}_0 = \{0\}$, *and*
(2) *For $\beta < \aleph_2$ we have the following:*
  (i) $\mathbb{P}_\beta \Vdash \mathcal{F}_\beta = \operatorname{filter}(\{\operatorname{rge}(r_\alpha) \mid \alpha < \beta\})$,
  (ii) $\mathbb{P}_\beta * \mathbb{F}_\sigma(\mathcal{F}_\beta) \Vdash \mathcal{U}_\beta$ *is the $\mathbb{F}_\sigma(\mathcal{F}_\beta)$-generic ultrafilter,*
  (iii) $\mathbb{P}_\beta \Vdash \mathbb{Q}_\beta = \mathbb{F}_\sigma(\mathcal{F}_\beta) * \mathbb{PT}(\mathcal{U}_\beta)$, *and*

(iv) $\mathbb{P}_{\beta+1} \Vdash r_\beta$ *is the* $\mathbb{PT}(\mathcal{U}_\beta)$*-generic real.*
Statements (i) and (ii) also hold for $\beta = \aleph_2$.

Then the following statements hold:

(A) *For any ordinal* $\beta \leq \aleph_2$ *and for any* $\alpha < \beta$ *with* $\mathrm{cf}(\alpha) \leq \omega$*, the forcing* $\mathbb{P}_\beta$ *is proper and preserves any* P*-point in* $\mathbf{V}^{\mathbb{P}_\alpha}$*. In particular,* $\mathcal{E}$ *is preserved in every step of the iteration. For* $\alpha < \aleph_2$*, we have* $|\mathbb{P}_\alpha| \leq \aleph_1$.

(B) *For any* $\beta \leq \aleph_2$ *with* $\mathrm{cf}(\beta) \leq \omega$,

$$\mathbb{P}_\beta \Vdash \mathcal{F}_\beta \text{ is an } F_\sigma\text{-filter},$$

$$\mathbb{P}_\beta * \mathbb{F}_\sigma(\mathcal{F}_\beta) \Vdash \mathcal{U}_\beta \text{ is a } \mathcal{W}\text{-Canjar ultrafilter for any } P\text{-point } \mathcal{W} \text{ in } \mathbf{V}^{\mathbb{P}_\beta}.$$

In particular,

$$\mathbb{P}_\beta * \mathbb{F}_\sigma(\mathcal{F}_\beta) \Vdash \mathcal{U}_\beta \text{ is an } \mathcal{E}\text{-Canjar ultrafilter.}$$

(C) *For any* $\beta \leq \aleph_2$*, if* $\mathrm{cf}(\beta) \geq \aleph_1$*, then*

$$\mathbb{P}_\beta \Vdash \mathcal{F}_\beta = \mathcal{U}_\beta \text{ is a } \mathcal{W}\text{-Canjar ultrafilter}$$
$$\text{for any } P\text{-point } \mathcal{W} \text{ in } \mathbf{V}^{\mathbb{P}_\alpha} \text{ for any } \alpha < \beta \text{ with } \mathrm{cf}(\alpha) \leq \omega.$$

In particular,

$$\mathbb{P}_\beta \Vdash \mathcal{U}_\beta \text{ is an } \mathcal{E}\text{-Canjar ultrafilter.}$$

(D) $\forall \alpha < \beta < \aleph_2$, $\mathbb{P}_{\beta+1} \Vdash \mathrm{rge}(r_\beta) \subseteq^* \mathrm{rge}(r_\alpha)$.

We prove the conclusion (A) by induction on $\beta \leq \aleph_2$. In order to do this, we carry conclusions (B), (C), and (D) along this induction.

We outline the organisation of the tasks: Conclusion (A) collects the properties that we carry over any step β i.e., from $\mathbb{P}_\alpha$, $\alpha < \beta$, to $\mathbb{P}_\beta$. In successor steps, we use

$$\mathbb{P}_\beta \Vdash \mathbb{Q}_\beta = \mathbb{F}_\sigma(\mathcal{F}_\beta) * \mathbb{PT}(\mathcal{U}_\beta).$$

We have two kinds of successor steps: Steps from $\mathbb{P}_\beta$ to $\mathbb{P}_{\beta+1}$ for $\mathrm{cf}(\beta) \leq \omega$ and steps from $\mathbb{P}_\beta$ to $\mathbb{P}_{\beta+1}$ for $\mathrm{cf}(\beta) = \aleph_1$. The two types differ strongly:

If $\mathrm{cf}(\beta) \leq \omega$, then $\mathcal{F}_\beta$ is countably generated. Now conclusion (B) is easy to see. Part (c) of Lemma 2.15 says: The $\mathbb{P}_\beta * \mathbb{F}_\sigma(\mathcal{F}_\beta)$-generic ultrafilter $\underset{\sim}{\mathcal{U}}_\beta$ is $\mathcal{W}$-Canjar for each P-point that is known in $\mathbb{P}_\beta$, that means by induction hypothesis any P-point in $\mathbb{P}_\alpha$ for a $\alpha \leq \beta$ with $\mathrm{cf}(\alpha) \leq \omega$. By induction hypothesis of (A) such a P-point $\mathcal{W}$ in $\mathbf{V}^{\mathbb{P}_\alpha}$ still is a P-point in $\mathbf{V}^{\mathbb{P}_\beta}$. Now conclusion (A) for $\mathbb{P}_{\beta+1}$ follows immediately by part (b) of Lemma 2.15. Conclusion (C) is vacuous, and Conclusion (D) for r_β follows from $\mathcal{F}_\beta \subseteq \mathcal{U}_\beta$.

If $\mathrm{cf}(\beta) \geq \aleph_1$, then by known facts on countable support iteration of proper iterands $\mathcal{F}_\beta = \mathcal{U}_\beta$ is a P-point. Now we face the only novel task: show that $\mathcal{U}_\beta$ is $\mathcal{W}$-Canjar for each P-point $\mathcal{W}$ that is known in $\mathbb{P}_\alpha$ for any $\alpha < \beta$ with $\mathrm{cf}(\alpha) \leq \omega$. For this we will use all four clauses (A) to (D). Any such $\mathcal{W}$ still generates a P-point in $\mathbb{P}_\beta$ by induction hypothesis. The ultrafilter $\mathcal{U}_\alpha$ for $\alpha < \beta$, $\mathrm{cf}(\alpha) \leq \omega$ is a P-point in the half-step $\mathbf{V}^{\mathbb{P}_\alpha * \mathbb{F}_\sigma(\mathcal{F}_\alpha)}$. The ultrafilter $\mathcal{U}_\alpha$ will be diagonalised by $\mathbb{P}_{\alpha+1}$ and does not generate a P-point in $\mathbb{P}_\beta$. The ultrafilter $\mathcal{U}_\alpha$ for $\alpha \leq \beta$, $\mathrm{cf}(\alpha) = \aleph_1$, is a P-point at stage $\mathbf{V}^{\mathbb{P}_\alpha}$. Again the ultrafilter $\mathcal{U}_\alpha$ will be diagonalised by $\mathbb{P}_{\alpha+1}$ and does not generate a P-point in $\mathbb{P}_\delta$ for $\delta \geq \alpha + 1$. Now, as above, Conclusion (A) for $\mathbb{P}_{\beta+1}$ is immediate by part (b) of Lemma 2.15, and conclusion (D) is derived quickly, as seen below. This ends the outline.

Now we carry out the induction.

For $\beta = 0$, Conclusion (B) follows from part (c) of Lemma 2.15 and the other conclusions are vacuously true. Note that filter$(\emptyset) = \mathcal{F}r$.

We suppose that $\beta \le \aleph_2$ is a **limit ordinal** and the lemma is proved for $\alpha < \beta$. For Conclusion (A) we cite:

Theorem 3.3. [7, Theorem 4.1] *If $\mathcal{W}$ is a P-point in $\mathbf{V}$, β is a limit ordinal and $\mathbb{P}_\beta = \langle \mathbb{P}_\alpha \mid \alpha < \beta \rangle$ is the countable support limit and for $\alpha < \beta$, the forcing $\mathbb{P}_\alpha$ is proper and preserves $\mathcal{W}$, then $\mathbb{P}_\beta$ is proper and preserves $\mathcal{W}$.*

Also the statement on the size of the forcing order is well-known [24, Theorem III.4.1]. Conclusion (D) is easy in the limit step.

Now we are concerned with the **successor step**. Properness is preserved since each iterand is proper by countable closure of $\mathbb{F}_\sigma$ and by part a) of Lemma 2.15.

We consider how to carry Conclusion (D) from $\mathbb{P}_\beta$ to $\mathbb{P}_{\beta+1}$. For this, the cofinality of β does not matter. For $\alpha < \beta < \aleph_2$, we have $\mathbb{P}_\beta \Vdash \mathrm{rge}(r_\alpha) \in \mathcal{F}_\beta$ and hence $\mathbb{P}_\beta * \mathbb{F}_\sigma(\mathcal{F}_\beta) \Vdash \mathrm{rge}(r_\alpha) \in \mathcal{U}_\beta$. Since the $\mathbb{PT}(\mathcal{U}_\beta)$-generic real diagonalises $\mathcal{U}_\beta$ (see Definition 1.8 and the paragraph after), we have $\mathbb{P}_{\beta+1} \Vdash \mathrm{rge}(r_\beta) \subseteq^* \mathrm{rge}(r_\alpha)$ for $\beta > \alpha$.

We turn to **the successor step for** $\mathrm{cf}(\beta) \le \omega$. Conclusion (B) has to be proved.

Lemma 3.4. *Let $\mathcal{W} \in \mathbf{V}^{\mathbb{P}_\beta}$ be a P-point. For $\mathrm{cf}(\beta) \le \omega$, $\mathcal{F}_\beta$ is an F_σ-filter and $\mathbb{P}_\beta * \mathbb{F}_\sigma(\mathcal{F}_\beta)$ forces the following: $\mathcal{U}_\beta$ is $\mathcal{W}$-Canjar.*

Proof. Since $\mathrm{cf}(\beta) = 1$ or $\mathrm{cf}(\beta) = \omega$ we may choose a cofinal (not necessarily strictly increasing) sequence $\langle \alpha_n \mid n < \omega \rangle$ converging to β. By part (D) of the induction hypothesis, in $\mathbf{V}^{\mathbb{P}_\beta}$, the filter $\mathcal{F}_\beta$ is generated by $\{\mathrm{rge}(r_{\alpha_n}) \mid n < \omega\}$. Thus

$$\mathcal{F}_\beta = \{A \subseteq \omega \mid (\exists n)(A \supseteq^* \mathrm{rge}(r_{\alpha_n}))\}$$

is an F_σ-filter and $\mathbb{F}_\sigma(\mathcal{F}_\beta)$ is equivalent to $\mathbb{F}_\sigma$ below the condition $\mathcal{F}_\beta$. Thus the part (c) of Lemma 2.15 yields that $\mathcal{U}_\beta$ is a $\mathcal{W}$-Canjar ultrafilter. $\square$

Now we turn to the new instances of Conclusion (A) of Lemma 3.2 in this successor step. Let $\mathcal{W}$ be a P-point in $\mathbf{V}^{\mathbb{P}_\beta}$ (see Definition 3.1). In particular, $\mathcal{W}$ could be a P-point from the ground model. Since $\mathbb{F}_\sigma(\mathcal{F}_\beta)$ is σ-closed, we only have to consider whether the second half of the iterand preserves $\mathcal{W}$. The second component of the iterand $\mathbb{Q}_\beta = \mathbb{F}_\sigma(\mathcal{F}_\beta) * \mathbb{PT}(\mathcal{U}_\beta)$ is $\mathbb{PT}(\mathcal{U}_\beta)$, and by Lemma 3.4, $\mathcal{U}_\beta$ is a $\mathcal{W}$-Canjar ultrafilter. Hence by Lemma 2.15 part (b) any P-point $\mathcal{W}$ is preserved.

Now we consider the **successor step for** $\mathrm{cf}(\beta) = \aleph_1$, that is we prove conclusion (C) and new instances of (A).

For a forcing $\mathbb{Q}$ and a condition $q \in \mathbb{Q}$, we write $\mathbb{Q} \restriction q$ for the forcing $\{p \in \mathbb{Q} \mid p \le q\}$ with the order of $\mathbb{Q}$. "φ holds in $\mathbf{V}^{\mathbb{Q}\restriction q}$" is used as a synonym to $q \Vdash_{\mathbb{Q}} \varphi$.

Lemma 3.5. *We let $\beta \le \aleph_2$, $\mathrm{cf}(\beta) \ge \aleph_1$ and we fix some $\alpha < \beta$ of countable cofinality and a P-point $\mathcal{W}$ in $\mathbf{V}^{\mathbb{P}_\alpha}$. Then $\mathbb{P}_\beta$ forces that $\mathcal{F}_\beta$ is a $\mathcal{W}$-Canjar ultrafilter.*

Proof. Let $\beta \le \aleph_2$ be of uncountable cofinality. Any name for a subset of ω appears in some $\mathbf{V}^{\mathbb{P}_\delta}$, $\delta < \beta$, by [22, page 96 ff]. Therefore an easy density argument shows that $\mathbb{P}_\beta \Vdash \mathcal{F}_\beta = \mathcal{U}_\beta$ is ultra.

The only not so easy statement is: $\mathbb{P}_\beta$ forces that $\mathcal{U}_\beta$ is $\mathcal{W}$-Canjar.

We fix $p_0 \in \mathbb{P}_\alpha$ such that $p_0 \Vdash$ "$\mathcal{W}$ is a P-point". By Lemma 3.4 we know that for any δ with $\alpha \le \delta < \beta$ such that $\mathrm{cf}(\delta) \le \omega$, the name $\mathcal{U}_\delta$ is forced by $(\mathbb{P}_\delta * \mathbb{F}_\sigma(\mathcal{F}_\delta)) \restriction p_0$ to be a $\mathcal{W}$-Canjar ultrafilter.

Suppose for a contradiction that we have $p \in \mathbb{P}_\beta \restriction p_0$ and a $\mathbb{P}_\beta$-name σ such that

$$p \Vdash_{\mathbb{P}_\beta} \sigma \text{ is a winning strategy for I in } \mathcal{H}(\mathcal{W}, \underset{\sim}{\mathcal{U}}_\beta). \tag{3.1}$$

Note that σ is a real. There are some $q \le p$ and $\delta_0 \in [\alpha, \beta)$ with $\mathrm{supp}(q) \subseteq \delta_0$ and $\mathrm{cf}(\delta_0) \le \omega$, such that below q the name σ is equivalent to a $\mathbb{P}_{\delta_0}$-name.

Let D be a countable cofinal subset of δ_0. If δ_0 is a successor $\delta_0 = \eta + 1$, then $D = \{\eta\}$. Now in $\mathbf{V}^{\mathbb{P}_{\delta_0} \restriction q}$,

$$\mathrm{filter}^{\mathbf{V}^{\mathbb{P}_{\delta_0} \restriction q}}(\{r_\varepsilon \mid \varepsilon \in \delta_0\}) = \mathcal{F}_{\delta_0}.$$

By part (D) of the induction hypothesis, in $\mathbf{V}^{\mathbb{P}_{\delta_0} \restriction q}$, the filter $\mathcal{F}_{\delta_0}$ is generated by $\{r_\varepsilon \mid \varepsilon \in D\}$ and hence an F_σ-filter.

We work below q and we identify σ with its $\mathbb{P}_{\delta_0}$-name. From part (B) of the induction hypothesis we get

$$(q, \mathcal{F}_{\delta_0}) \Vdash_{\mathbb{P}_{\delta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\delta_0})} \sigma \text{ is not a winning strategy for I in } \mathcal{H}(\mathcal{W}, \underset{\sim}{\mathcal{U}}_{\delta_0}). \tag{3.2}$$

Claim: The condition $(q, \mathcal{F}_{\delta_0})$ forces also in the subforcing $\mathbb{P}_{\delta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\delta_0})$ that σ is a legal strategy for player I in $\mathcal{H}(\mathcal{W}, \underset{\sim}{\mathcal{U}}_{\delta_0})$.

This is seen as follows: By statement (3.1), for any $m \in \omega$ we have

$$(q, \mathcal{F}_{\delta_0}) \Vdash_{\mathbb{P}_\beta} \forall (z_0, n_0, \dots, n_{m-1}, z_m), \sigma(z_0, n_0, \dots, n_{m-1}, z_m) = p_m \in \mathbb{P}(\underset{\sim}{\mathcal{U}}_\beta). \tag{3.3}$$

And p_m has a $\mathbb{P}_{\delta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\delta_0})$-name, since σ has a $\mathbb{P}_{\delta_0}$-name. By induction bypothesis for statement (D), we have

$$(q, \mathcal{F}_{\delta_0}) \Vdash_{\mathbb{P}_\beta} \underset{\sim}{\mathcal{U}}_{\delta_0} \subseteq \underset{\sim}{\mathcal{U}}_\beta.$$

Hence we have

$$(q, \mathcal{F}_{\delta_0}) \Vdash_{\mathbb{P}_\beta} \mathbb{PT}(\underset{\sim}{\mathcal{U}}_\beta) \subseteq \mathbb{PT}(\underset{\sim}{\mathcal{U}}_{\delta_0}).$$

Hence Statement (3.3) implies

$$(q, \mathcal{F}_{\delta_0}) \Vdash_{\mathbb{P}_\beta} p_m \in \mathbb{P}(\underset{\sim}{\mathcal{U}}_{\delta_0}). \tag{3.4}$$

The condition p_m is a $\mathbb{P}_{\delta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\delta_0})$-name and the statement "$p_m \in \mathbb{PT}(\underset{\sim}{\mathcal{U}}_{\delta_0})$" is a statement in the $\mathbb{P}_{\delta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\delta_0})$-forcing language. We claim that this statement is absolute between $\mathbf{V}^{\mathbb{P}_\beta \restriction (q, \mathcal{F}_{\delta_0})}$ and $\mathbf{V}^{\mathbb{P}_{\delta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\delta_0}) \restriction (q, \mathcal{F}_{\delta_0})}$.

The statement "$p_m \in \mathbb{PT}(\underset{\sim}{\mathcal{U}}_{\delta_0})$" reads as

$$(\forall t \in \omega\text{-}\mathrm{spl}(p_m))\, (\mathrm{sucspl}_{p_m}(t) \in ((\underset{\sim}{\mathcal{U}}_{\delta_0})^{<\omega})^+).$$

We fix t, $X \in \mathbf{V}^{\mathbb{P}_{\delta_0}}$, and a condition $(r, \mathcal{G}) \in \mathbb{P}_{\delta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\delta_0})$ below $(q, \mathcal{F}_{\delta_0})$ such that

$$(r, \mathcal{G}) \Vdash_{\mathbb{P}_\beta} t \in \omega\text{-}\mathrm{spl}(p_m) \wedge X = \mathrm{sucspl}_{p_m}(t).$$

Since $p_m \in V^{\mathbb{P}_{\delta_0} \restriction q}$ and since the forced statement is absolute, by [17, Lemma VII, 7.13] between we the complete subforcing $\mathbb{P}_{\delta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\delta_0}) \restriction (r, \mathcal{G})$ and the original forcing $\mathbb{P}_\beta \restriction (r, \mathcal{G})$, we also have

$$(r, \mathcal{G}) \Vdash_{\mathbb{P}_{\delta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\delta_0})} t \in \omega\text{-}\mathrm{spl}(p_m) \wedge X = \mathrm{sucspl}_{p_m}(t). \tag{3.5}$$

Our aim is to show

$$(r, \mathcal{G}) \Vdash_{\mathbb{P}_{\delta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\delta_0})} X \in ((\underset{\sim}{\mathcal{U}}_{\delta_0})^{<\omega})^+.$$

We go into a generic extension $V[G_\beta]$ with $(r, \mathcal{G}) \in G_\beta$. That $X \in ((\mathcal{U}_{\delta_0})^{<\omega})^+$ holds in the big model $V[G_\beta]$ means that for all F in the filter generated by U_{δ_0} (which is not an ultrafilter in $V[G_\beta]$ but this does not matter anyway), there is $s \in X$ with $s \subseteq F$. Now take any F in the ultrafilter U_{δ_0} in the small model $V[G_\beta \restriction (P_{\delta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\delta_0}))]$. Then we can find the required s in the big model and thus also in the small model, so that X is also positive in the latter. Thus the Claim is proved.

By the Claim, we have the following improvement of statement (3.2):

$$\begin{aligned}(q, \mathcal{F}_{\delta_0}) \Vdash_{\mathbb{P}_{\delta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\delta_0})} \quad & \sigma \text{ is a strategy for I in } \mathcal{H}(\mathcal{W}, \underset{\sim}{\mathcal{U}}_{\delta_0}),\\ & \text{and } \sigma \text{ is not a winning strategy for I.}\end{aligned}$$

By the countable closure of $\mathbb{F}_\sigma(\mathcal{F}_{\delta_0})$, the forcing theorem yields a condition $r \leq_{\mathbb{P}_{\delta_0}} q$ and a $\mathbb{P}_{\delta_0}$-name for an F_σ-filter $\mathcal{G}$ such that

$$r \Vdash_{\mathbb{P}_{\delta_0}} \mathcal{G} \leq \mathcal{F}_{\delta_0},$$

and a $\mathbb{P}_{\delta_0}$-name for a sequence $\bar{s} = \langle Y_i, z_i, p_i, n_i \mid i < \omega \rangle$ such that $(r, \mathcal{G})$ forces that $\bar{s}$ is a winning play for player II, i.e.,

$$\begin{aligned}(r, \mathcal{G}) \Vdash_{\mathbb{P}_{\delta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\delta_0})} \quad & \bar{s} \text{ is played according to } \sigma \text{ and}\\ & q' = \bigcup_{i \in \omega} T(p_i, n_i) \in \mathbb{PT}(\underset{\sim}{\mathcal{U}}_{\delta_0}) \text{ and } \bigcup_{i<\omega} z_i \in \mathcal{W}.\end{aligned} \tag{3.6}$$

The statement $\bigcup \{z_i \mid i < \omega\} \in \mathcal{W}$ is absolute and by [17, Lemma VII, 7.13 (b)], applied to $\mathbb{P}_{\delta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\delta_0}) \restriction (r, \mathcal{G})$ as a complete suborder of $\mathbb{P}_\beta \restriction (r, \mathcal{G})$, we have

$$(r, \mathcal{G}) \Vdash_{\mathbb{P}_\beta} \bigcup_{i<\omega} z_i \in \mathcal{W}.$$

Now we show:

$$(r, \mathcal{G}) \Vdash_{\mathbb{P}_\beta} q' \in \mathbb{PT}(\underset{\sim}{\mathcal{U}}_\beta). \tag{3.7}$$

Then

$$(r, \mathcal{G}) \Vdash_{\mathbb{P}_\beta} \sigma \text{ is not a winning strategy for I in } \mathcal{H}(\mathcal{W}, \underset{\sim}{\mathcal{U}}_\beta),$$

which will be a contradiction to statement (3.1).

In order to prove statement (3.7), it is enough to show

$$(r, \mathcal{G}) \Vdash_{\mathbb{P}_\beta} (\forall t \in \omega\text{-}\operatorname{spl}(q'))(\operatorname{sucspl}_{q'}(t) \in ((\underset{\sim}{\mathcal{U}}_\beta)^{<\omega})^+).$$

We fix t, $X \in \mathbf{V}^{\mathbb{P}_{\delta_0}}$, and a condition $(r', \mathcal{G}') \leq (r, \mathcal{G})$ in $\mathbb{P}_{\delta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\delta_0})$ such that

$$(r', \mathcal{G}') \Vdash_{\mathbb{P}_{\delta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\delta_0})} t \in \omega\text{-}\operatorname{spl}(q') \wedge X = \operatorname{sucspl}_{q'}(t). \tag{3.8}$$

Since $q' \in V^{\mathbb{P}_{\delta_0} \restriction r}$ we can use the forcing $\mathbb{P}_{\delta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\delta_0})$ instead of $\mathbb{P}_\beta$, again by [17, Lemma VII, 7.13 (b)]. Since $r' \Vdash_{\mathbb{P}_{\delta_0}} \mathcal{G}' \supseteq \mathcal{G}$ and $r' \Vdash_{\mathbb{P}_{\delta_0}} \mathcal{G} \supseteq \mathcal{F}_{\delta_0}$, we have $r' \Vdash_{\mathbb{P}_{\delta_0}} \mathcal{G}' \supseteq \mathcal{F}_{\delta_0}$. Our aim is to show

$$(r', \mathcal{G}') \Vdash_{\mathbb{P}_\beta} X \in ((\underset{\sim}{\mathcal{U}}_\beta)^{<\omega})^+. \tag{3.9}$$

Statements (3.6) and (3.8) yield

$$(r', \mathcal{G}') \Vdash_{\mathbb{P}_{\delta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\delta_0})} X \in ((\underset{\sim}{\mathcal{U}}_{\delta_0})^{<\omega})^+.$$

By Lemma 2.23 in $\mathbf{V}^{\mathbb{P}_{\delta_0} \restriction r'}$, this means

$$C(X) \subseteq \mathcal{G}'. \tag{3.10}$$

The property (3.10) is a Π^1_2-relation of a real parameter with a $(\mathbb{P}_{\delta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\delta_0}) \restriction (r', \mathcal{G}')$-name and hence is absolute by Shoenfield's absoluteness theorem [16, Theorem 25.20]. By [17, Lemma VII, 7.13 (b)], the absolute property (3.10) also holds in $\mathbf{V}^{\mathbb{P}_\beta \restriction (r', \mathcal{G}')}$ since $\mathbb{P}_{\delta_0} \restriction (r', \mathcal{G}')$ is a complete suborder of the forcing $\mathbb{P}_\beta \restriction (r', \mathcal{G}')$.

Since, in $\mathbf{V}^{\mathbb{P}_{\delta_0+1}}$, $r_{\delta_0} \in \mathcal{F}_{\delta_0+1}$, r_{δ_0} diagonalises $\mathcal{G}' \subseteq \mathcal{U}_{\delta_0}$, we have that $\mathbb{P}_\beta \restriction r' \Vdash_{\mathbb{P}_\beta} \underset{\sim}{\mathcal{G}'} \subseteq \underset{\sim}{\mathcal{F}}_\beta$. By Lemma 2.25, applied in $\mathbf{V}^{\mathbb{P}_\beta \restriction r'}$ to the condition $\mathrm{filter}^{\mathbf{V}^{\mathbb{P}_\beta \restriction r'}}(\mathcal{G}')$ in the forcing $\mathbb{F}_\sigma(\mathcal{F}_\beta)$ (— the trivial forcing —) and statement (3.10), we have in $\mathbf{V}^{\mathbb{P}_\beta \restriction r'}$

$$\mathrm{filter}^{\mathbf{V}^{\mathbb{P}_\beta \restriction r'}}(\mathcal{G}') \Vdash_{\mathbb{F}_\sigma(\mathcal{F}_\beta)} X \in ((\underset{\sim}{\mathcal{U}}_\beta)^{<\omega})^+.$$

Back in the ground model we have statement (3.9). $\square$

Now we turn to the new instances of Conclusion (A) of Lemma 3.2 in the successor step of uncountable cofinality. Since $\mathcal{F}_\beta$ is an ultrafilter, the forcing $\mathbb{F}_\sigma(\mathcal{F}_\beta)$ is the forcing with one condition. The second component of the iterand $\mathbb{Q}_\beta = \mathbb{F}_\sigma(\mathcal{F}_\beta) * \mathbb{PT}(\mathcal{U}_\beta)$ is $\mathbb{PT}(\mathcal{U}_\beta)$. By Lemma 3.5, for any P-point $\mathcal{W}$ in $\mathbf{V}^{\mathbb{P}_\alpha}$, for any $\alpha < \beta$ with $\mathrm{cf}(\alpha) \leq \omega$, the ultrafilter $\mathcal{U}_\beta$ is a $\mathcal{W}$-Canjar ultrafilter. Hence by part (b) of Lemma 2.15, the forcing $\mathbb{P}_{\beta+1}$ preserves the P-point $\mathcal{W}$.

Thus we carried the statements (A) to (D) of Lemma 3.2 for $\beta \leq \aleph_2$, and Lemma 3.2 is proved. This concludes the proof of Theorem 1.10.

We recall the concept of near coherence from Definition 1.12. By [3, Page 585] we have: If two ultrafilters $\mathcal{U}$, $\mathcal{V}$ are nearly coherent, then there is a finite-to-one weakly increasing surjective function h such that $h(\mathcal{U}) = h(\mathcal{V})$. A function h is weakly increasing if $m < n$ implies $h(m) \leq h(n)$.

Observation 3.6. *With $\mathcal{U}_\beta$ and $\mathcal{W}$ according to the notation of Lemma 3.4, the generic ultrafilter $\mathcal{U}_\beta$ is not nearly coherent to any such $\mathcal{W}$.*

Proof. First proof: $\mathbb{PT}(\mathcal{U}_\beta)$ diagonalises $\mathcal{U}_\beta$ and preserves $\mathcal{W}$ by Lemma 2.15. If $f(\mathcal{U}_\beta) = f(\mathcal{W})$ for a finite-to-one f, then any diagonalisation D of $\mathcal{U}_\beta$ destroys $\mathcal{W}$.

The second proof is by hand: Let $\mathcal{W}$ be an ultrafilter in $\mathbf{V}^{\mathbb{P}_\beta}$. Let h be a finite-to-one surjective staircase function. It suffices to consider finite-to-one functions from $\mathbf{V}^{\mathbb{P}_\beta}$, since $\mathbb{F}_\sigma(\mathcal{F}_\beta)$ is ω-closed and thus does not add reals ([18]). Let h be such a finite-to-one function. We choose π_n such $h(k) = n$ for $k \in [\pi_n, \pi_{n+1})$. Let p be an F_σ-filter. Since p is Borel, $h(p) = \{X \subseteq \omega \mid h^{-1}[X] \in p\}$ is analytic and hence not an ultrafilter. So there is an infinite set A such that

$$\bigcup\{[\pi_n, \pi_{n+1}) \mid n \in A\}, \bigcup\{[\pi_n, \pi_{n+1}) \mid n \notin A\} \notin p.$$

We assume that $X = \bigcup\{[\pi_n, \pi_{n+1}) \mid n \in A\} \in \mathcal{W}$ and consider $Y = \bigcup\{[\pi_n, \pi_{n+1}) \mid n \notin A\}$. Then

$$q = \mathrm{filter}(p \cup \{Y\})$$

fulfils $q \in \mathbb{F}_\sigma$, $q \leq p$. Of course q forces $Y \in \mathcal{U}_\beta$ as well as $X = h^{-1}[A] \in \mathcal{W}$. Thus q forces that $\mathcal{U}_\beta$ is not nearly coherent via h to $\mathcal{W}$. $\square$

Question 3.7. *Regarding Lemma 2.15, given a P-point $\mathcal{W}$, we ask for which filters $\mathcal{G}$ does $\mathbb{F}_\sigma(\mathcal{G})$ force that the generic ultrafilter $\underset{\sim}{\mathcal{U}}(\mathcal{G})$ is a $\mathcal{W}$-Canjar ultrafilter.*

In our proof we have examples of such $\mathcal{W}$, $\mathcal{G}$: We have a countably generated $\mathcal{G}$ and then for any P-point $\mathcal{W}$ the forcing $\mathbb{F}_\sigma(\mathcal{G})$ forces that $\underset{\sim}{\mathcal{U}}(\mathcal{G})$ is Canjar by part (c) of Lemma 2.15. At a forcing stage $\mathbb{P}_\beta$ of uncountable cofinality the filter $\mathcal{F}_\beta$ is a $\mathcal{W}$-Canjar ultrafilter for any P-point $\mathcal{W}$ from a strictly earlier stage and $\mathbb{F}_\sigma(\mathcal{F}_\beta)$ is the trivial forcing. A very deep example of a non-Canjar filter $\mathcal{G}$ with the property above for any P-point $\mathcal{W}$ is given by the dual filter $\mathcal{I}(\mathcal{A})^*$ of a Laflamme MAD family in [14, Proposition 70].

We conclude this section by showing that there are no rapid ultrafilters in the models of our main theorem.

Definition 3.8 (The Rudin–Keisler order)**.** Let $\mathcal{U}$ and $\mathcal{W}$ be two ultrafilters over ω. We write $\mathcal{U} \leq_{\mathrm{RK}} \mathcal{W}$ and say $\mathcal{U}$ is a Rudin–Keisler predecessor of $\mathcal{W}$ if there is a function $f\colon \omega \to \omega$ such that $f(\mathcal{W}) = \mathcal{U}$. For the definition of $f(\mathcal{U})$, recall the second part of Definition 1.12.

Definition 3.9 (Rapid ultrafilters)**.** An ultrafilter $\mathcal{R}$ is called *rapid* if for any $f\colon \omega \to \omega$ there is some $X = \{x_0, x_1, \dots\} \in \mathcal{R}$ such that

$$(\forall n)(f(n) < x_n).$$

An ultrafilter is rapid iff the set of the enumerating functions $\{\mathrm{en}(X) \mid X \in \mathcal{R}\}$ is a $\leq^*$-dominating family.

Remark 3.10. According to Lemma 3.5 and Lemma 2.13 in the forcing models of our main theorem, the simple $P_{\aleph_2}$-point generated by $\{r_\alpha \mid \alpha < \aleph_2\}$ is a Canjar ultrafilter. In particular, it does not have any rapid Rudin–Keisler-predecessor by [10, Lemma 4].

Lemma 3.11. *Let $h\colon \omega \to \omega$ be a finite-to-one surjective staircase function and let $\mathcal{R}$ be an ultrafilter. Then $\mathcal{R}$ is rapid if and only if $h(\mathcal{R})$ is rapid.*

Proof. We assume that $\mathcal{R}$ is rapid and show that $h(\mathcal{R})$ is rapid. Let $f\colon \omega \to \omega$ be given. For $n \in \omega$, we let $g(n) = \max h^{-1}[\{n\}]$. Note that $h \circ g = \mathrm{id}$ and that for $y > g(n)$, $h(y) > n$. Since $\mathcal{R}$ is rapid, there is some $X = \{x_0, x_1, \dots\} \in \mathcal{R}$ such that

$$(\forall n)((g \circ f)(n) < x_n).$$

Applying h to each side yields

$$(\forall n)(f(n) < h(x_n)).$$

Since $\{h(x_n) \mid n \in \omega\} \in h(\mathcal{R})$, we are done.

Now we assume that $h(\mathcal{R})$ is rapid and show that $\mathcal{R}$ is rapid. Let $f\colon \omega \to \omega$ be given. Since $h(\mathcal{R})$ is rapid, there is some $X = \{h(x_0), h(x_1), \dots\} \in h(\mathcal{R})$ such that

$$(\forall n)(f(n) < h(x_n) \leq x_n).$$

□

Proposition 3.12. *There is no rapid ultrafilter in the forcing extension of our main theorem.*

Proof. Let $\mathbb{P}$ be the forcing from the main theorem. We work in $\mathbf{V}^{\mathbb{P}}$. By Remark 1.14 there are exactly two near-coherence classes of ultrafilters and they are represented by a simple $P_{\aleph_1}$-point $\mathcal{W}$ in $\mathbf{V}$ and the simple $P_{\aleph_2}$-point $\mathcal{U} = \mathcal{U}_{\aleph_2}$. Let $\mathcal{R}$ be an ultrafilter in $\mathbf{V}^{\mathbb{P}}$. We have two cases. First case: $\mathcal{R}$ is nearly coherent to $\mathcal{W}$, via a surjective staircase function h. Since at each stage β, $\mathbb{PT}(\mathcal{U}_\beta)$ diagonalises $\mathcal{U}_\beta$ and hence adds an unbounded real by [25], the dominating number is $\aleph_2$. However, any rapid ultrafilter has character at least the dominating number. So $h(\mathcal{W}) = h(\mathcal{R})$, having character $\aleph_1$, cannot be rapid. By Lemma 3.11, $\mathcal{R}$ is not rapid.

Second case: $\mathcal{R}$ is nearly coherent to the Canjar ultrafilter $\mathcal{U}$. We let h be a finite-to-one surjective staircase function with $h(\mathcal{R}) = h(\mathcal{U})$. Then $h(\mathcal{R})$ is Rudin–Keisler below $\mathcal{U}$. By [10, Lemma 10], the ultrafilter $h(\mathcal{R})$ is not rapid, and by Lemma 3.11, also the ultrafilter $\mathcal{R}$ is not rapid. $\square$

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