NEGATING CLUB AND STICK WITH STRATEGICALLY CLOSED FORCINGS

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Abstract. We continue the investigation of games played on forcing orders that might entail preservation properties. We investigate various strengthenings of properness and bounding for forcing orders that are defined by games in which the player aiming at completeness has a winning strategy. We show that some of these strategically closed forcings allow to negate club and stick, even under $\diamondsuit$ in the ground model.

1. Introduction

We define infinite two person games that are played on a forcing order. A forcing $\mathbb{P}$ is in a sense gentle if the player aiming at completeness has a winning strategy in a game played on $\mathbb{P}$. A weaker notion is given by the requirement that the opponent does not have a winning strategy. Examples are the properness game see, e.g., [6], the distributivity games and the bounding game (explained below). The game $\Game^1_{\cdot,1}(\mathbb{P})$ has appeared in [13], where we proved that "INC does not have a winning strategy in $\Game^1_{\cdot,1}(\mathbb{P}, M)$ for any countable elementary submodel $M \prec H(\chi)$ with $M \cap \omega_1$ in a stationary set $S \subseteq \omega_1$" guarantees that $\mathbb{P}$ preserves Souslin trees. Here we investigate forcings for which COM has a winning strategy in $\Game^1_{\cdot,1}(\mathbb{P}, M)$, and we consider versions of the game with fixed growth parameters f, $g\colon \omega \to \omega$, that $f(n)$ and $g(n)$ regulate the width and the depth of round n of the game. The hope was to find forcings over a ground model of diamond that preserve some weak versions of diamond. Examples are Hrušák's article on Baumgartner's unpublished proof of the Ostaszewski club in the product Sacks model [8] over a ground model of $\diamondsuit$, or of Guzmán's and Hrušák's proof of $\diamondsuit_{\mathfrak{d}}$ in an extension of a model of diamond by a strategically bounding forcing [7, Theorem 4.10]. Surprisingly, our Souslin tree preserving games negate this plan in a strong form: A winning strategy for the completeness player does not preclude destroying the Ostaszewski club $\clubsuit$ or the stick principle $\mathbf{!}$. These are our two main results.

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Theorem 3.1. *We assume CH and $2^{\aleph_1} = \aleph_2$. There is a countable support iteration $\mathbb{P} = \langle \mathbb{P}_\beta, \mathbb{Q}_\alpha : \alpha < \omega_2, \beta \leq \omega_2 \rangle$ such that $\mathbb{P}_0 = \{0\}$ and each $\mathbb{P}_\beta$ forces that* COM *has a winning strategy in $\Game^2_{f,g}(\dot{\mathbb{Q}}_\beta)$ for $f(n) \geq 2$ (and hence $\dot{\mathbb{Q}}_\beta$ is proper and ${}^\omega\omega$-bounding) and $\mathbb{P}$ forces that $\clubsuit$ fails.*

Theorem 3.8. *We assume* CH *and $2^{\aleph_1} = \aleph_2$. There is a ccountable support iteration $\mathbb{P} = \langle \mathbb{P}_\beta, \mathbb{Q}_\alpha : \alpha < \omega_2, \beta \leq \omega_2 \rangle$ such that $\mathbb{P}_0 = \{0\}$ and each $\mathbb{P}_\beta$ forces that* COM *has a winning strategy in $\Game^{2,+}_{f,g}(\underset{\sim}{\mathbb{Q}}_\beta)$ for* $\lim_n f(n) = \infty$ *and $\mathbb{P}$ forces that ¡ fails.*

We review the definitions of the mentioned guessing principles.

Definition 1.1. Let $S \subseteq \omega_1$ be a stationary set that contains only limit ordinals.

(1) (Stick) A ¡$_S$-witness is a set $A \subseteq \{u \subseteq \omega_1 : \text{otp}(u) = \omega \wedge \sup(u) \in S\}$ of size $\aleph_1$ such that for any $B \in [\omega_1]^{\aleph_1}$ there is a $u \in A$ such that $u \subseteq B$. If $S = \omega_1$, we drop the subscript here and also in the other principles.

(2) A $\clubsuit_S$-witness is a sequence $\bar{A} = \langle u_\alpha : \alpha \in S \rangle$ such that $u_\alpha \subseteq \alpha$, $\sup(u_\alpha) = \alpha$ for any $B \in [\omega_1]^{\aleph_1}$ there is some $\delta \in S$ such that $u_\delta \subseteq B$. We call $\clubsuit$ tiltan, Hebrew for clover (trefoil, trèfle).

(3) A $\diamondsuit_S$-witness or a diamond-sequence is a sequence $\bar{D} = \langle D_\alpha : \alpha \in S \rangle$ such that $D_\alpha \subseteq \alpha$, and for any $B \subseteq \omega_1$ there is some $\delta \in S$ such that $D_\delta = B \cap \delta$.

Remark 1.2. Suppose that A is a stick witness. Then also $\{u \in A : \text{otp}(u) = \omega\}$ is a stick witness. Suppose that $\langle u_\alpha : \alpha \in S \rangle$ is a $\clubsuit_S$-witness. Then for each $\alpha \in S$ we may chose a cofinal subset $v_\alpha \subseteq u_\alpha$ with $\text{otp}(v_\alpha) = \omega$ and get a $\clubsuit_S$-witness $\langle v_\alpha : \alpha \in S \rangle$. For stationary sets $S \subseteq T$, the principle $\clubsuit_S$ implies $\clubsuit_T$, and analogously for the ¡.

Lemma 1.3 ([17, Ch III, 7.2])**.** *If $\bar{A}$ is a $\clubsuit_S$-sequence then the following seemingly stronger guessing holds: For any $B \in [\omega_1]^{\aleph_1}$ there are stationarily many $\delta \in S$ such that $u_\delta \subseteq B$.*

Historic remarks. Jensen [11] introduced the diamond. Ostaszewski [14] introcuded the tiltan, which is also called Ostaszewski's club principle. Broverman [1] introduced the stick. CH implies the stick. Džamonja and Shelah [3] showed that ¡ (and indeed a slight weakening of $\clubsuit$) is strictly weaker than $\clubsuit$.

Burgess and Devlin proved that CH and $\clubsuit_S$ implies $\diamondsuit_S$. For a presentation you may look in [17, Chapter I]. Models of $2^{\aleph_0} \geq \aleph_2$ and $\clubsuit$ are given in [16, Section 5] and presented in [17, Chapter I, Section 5] via a collapse of $\aleph_1$. In [8], Hrušák shows that the countable support product of Sacks forcing gives a model of $\clubsuit_S$, and in [5] Fuchino, Shelah and Soukup introduce a technique called $*$-product of forcing orders and thus establish some models with large continuum and $\clubsuit_S$. Primavesi introduced [15] the following strong club principle. A super-$\clubsuit_S$-witness is a sequence $\bar{A} = \langle u_\alpha : \alpha \in S\rangle$ such that $u_\alpha \subseteq \alpha$, $\sup(u_\alpha) = \alpha$, and for any $B \in [\omega_1]^{\aleph_1}$ there is some $B' \in [B]^{\aleph_1}$ such that $\{\delta \in S : B' \cap \delta = u_\delta\}$ is stationary. Primavesi [15] proved

Theorem 1.4 (Primavesi [15])**.** *If for some stationary S there is a superclub sequence then there is a Souslin tree.*

Juhász' question whether $\clubsuit$ implies the existence of a Souslin tree, stays open.

The paper is organised as follows. In Section 2 we review some games played on forcing orders and state some connections between them. In Section 3 we prove the two main theorems. Section 4 contains some concluding remarks and open questions.

In forcing, we follow the Israeli tradition that stronger conditions are larger.

2. Infinite two person games on forcing orders

In this section, we recall and define games and put them into context. For background on infinite two person games, see, e.g., Kechris [12, Definition 8.10].

2.1. **Playing about distributivity, and strategic closure.** We consider distributivity games on forcing orders $\mathbb{P}$ in the style of Jech [9, 10] and many others.

COM (Black [10], II [4], Even [2])	$0_{\mathbb{P}}$		p_2	$\dots$	p_ω		$\dots$
INC (White, I , Odd)		p_1		$\dots$		$p_{\omega+1}$	$\dots$

The length of the game $G^{\mathsf{COM}}_\beta(\mathbb{P})$ is β for some $\beta \geq \omega$ and during a play the players generate a sequence $\langle p_\alpha : \alpha < \beta\rangle$. Hereby two players draw alternately, with COM playing at even stages, beginning with the weakest condition at stage 0 and at each stage, any player knows all the previous moves. The rules are $p_1 \leq p_2 \leq p_3 \leq p_4 \dots$ (in the Jerusalem notation). In the limit steps, COM moves. Player COM wins the game, if there are moves in any stage $\alpha < \beta$. Jech showed that any forcing $\mathbb{P}$ does not add new countable sequences (such a forcing is called countably distributive) iff INC does not have a winning strategy in $G^{\mathsf{COM}}_{\aleph_1}(\mathbb{P})$. An easy computation shows that a winning strategy for COM in $G^{\mathsf{COM}}_{\kappa+1}(\mathbb{P})$ allows COM to win any $G^{\mathsf{COM}}_\beta(\mathbb{P})$, $\beta \leq \kappa^+$.

A forcing $\mathbb{P}$ that does not add new sequences of ordinals of length $< \kappa$ is is called κ-distributive, or, caveat, elsewhere this might be called $(< \kappa)$-distributive. For longer games, it matters who plays first in limit steps. This is not the case for $\beta \leq \aleph_1$. The game $G^{\mathsf{COM}}_\beta(\mathbb{Q})$ is related to the distributivity of $\mathbb{Q}$ (and, equivalently, of $\mathrm{RO}(\mathbb{Q})$) as follows:

Theorem 2.1 (Foreman, [4, Theorem, page 718]). *Let κ be an infinite cardinal. Player* INC *does not have a winning strategy in* $G^{\mathsf{COM}}_{\kappa+1}(\mathbb{P})$ *iff the forcing* $\mathbb{P}$ *is* κ^+*-distributive.*

If COM even has a winning strategy in $G^{\mathsf{COM}}_{\beta}(\mathbb{P})$, then the forcing order $\mathbb{P}$ is called *weakly strategically* β*-closed*. If COM has a winning strategy even in the harder (for COM) game $G^{\mathsf{INC}}_{\beta}(\mathbb{P})$ where INC moves first in limits, then the forcing is called *strongly strategically* β*-closed*. For regular cardinals β, this notion is related to the preservation of stationary subsets of $[\chi]^{<\beta}$ for a regular cardinal $\chi \geq \beta$, for details see [2]. In the current work we touch only the case $\beta = \aleph_1$.

2.2. **The bounding game.** Games of length ω are used to describe features of names of ω-sequences.

The bounding game $\mathcal{BG}(\mathbb{P})$ looks as follows

COM		F_0		F_1		F_2		...
INC	p, D_0		D_1		D_2			...

The length of the game is ω. The rules are: $p \in \mathbb{P}$. Each D_n is an open dense subset of $\mathbb{P}$ above q_0 and F_n is a finite subset of D_n. Player COM wins the game $\mathcal{BG}(\mathbb{P})$, if there is some $q \geq p$ such that for any n, F_n is predense above q. In light of the distributivity games, one could see the length of the boundedness game as $\omega + 1$, if we count the witness q in the clause that describes who wins as a sort of a move. However, even if we talk about a winning strategy for COM, the witnessing condition q need not be computed by the strategy. A real is a function from ω to ω. A real $g \in \mathbf{V}[\mathbb{P}]$ is called unbounded if for any real $f \in \mathbf{V}$, $g \not\leq^* f$, i.e., there are infinitely many n with $f(n) < g(n)$.

Jech [10] and Zapletal [18, 3.10.7] showed:

Theorem 2.2. *Player* INC *does not have a winning strategy in the bounding game for* $\mathbb{P}$ *iff* $\mathbb{P}$ *does not add an unbounded real and is proper.*

Theorem 2.3 ([7, Theorem 6.5]). *If $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \beta < \delta, \alpha \leq \delta \rangle$ is a countable support iteration of forcings such that $\mathbb{P}_0 = \{0\}$ and for $\alpha < \delta$,*

$$\mathbb{P}_\alpha \Vdash \mathsf{COM} \text{ has a winning strategy in } \mathcal{BG}(\mathbb{Q}_\alpha),$$

then COM *also has a winning strategy in* $\mathcal{BG}(\mathbb{P}_\delta)$.

2.3. **Related games on boundedness.** Now we define a family of games that takes properties from both examples. The game $\Game^1_{\cdot,1}(\mathbb{Q})$ has appeared in [13] in the context of using that INC does not have a winning strategy for the preservation of Souslin trees. Here we require that COM has a winning strategy $\Game^1_{\cdot,1}(\mathbb{Q})$ and we consider some harder games for COM.

Definition 2.4. Let $\mathbb{Q}$ be an atomless forcing notion, and let $f, g \colon \omega \to \omega \setminus \{0\}$. We write $\Game^i_{f,1}(\mathbb{Q})$ if we mean that g is constantly 1. The index i gives the rules on the order of moves within the rounds.

(1) The game $\partial^1_{f,g}(\mathbb{Q})$.

(A) The game $\partial^1_{f,g}(\mathbb{Q})$ is played in ω moves between a player INC and a player COM. The moves can be grouped as follows: We write $\bar{p_n} = \langle p_{n,i} : i < f(n)g(n)\rangle$ and $\bar{q_n} = \langle q_{n,i} : i < f(n)g(n)\rangle$.

COM	$0_{\mathbb{Q}}$		$\bar{p_0}$		$\bar{p_1}$		$\dots$
INC		p		$\bar{q_0}$			$\dots$

Now, deviating from the grid, the grouping of the tuples $\bar{p_n}$, $\bar{q_n}$, is broken according to rules (1)(B^1) in version $\partial^1_{f,g}(\mathbb{Q})$ and to rules (2)(B^2) in version $\partial^2_{f,g}(\mathbb{Q})$ of the game. At the start of the game, Player INC chooses a condition $p \in \mathbb{Q}$.

(B^1) We start counting rounds after the double bar. In the n-th round a subgame of length $f(n) \cdot g(n)$ is played and it produces conditions $p_{n,\ell}$, $q_{n,\ell}$, for $\ell < f(n)g(n)$. The $p_{n,\ell}$, $q_{n,\ell}$ are played one by one alternately by COM and INC in the order $p_{n,0}$, $q_{n,0}$, $p_{n,1}$, $\dots$, $q_{n,f(n)g(n)-1}$. The additional rules are:

$(*)_1$ $p \le_{\mathbb{Q}} p_{n,\ell} \le_{\mathbb{Q}} q_{n,\ell}$ for any $\ell < f(n)g(n)$,

$(*)_2$ for $f(n) \le \ell < f(n)g(n)$, $p_{n,\ell} \ge_{\mathbb{Q}} q_{n,\ell-f(n)}$.

$(*)_3$ there is some onto function $s\colon f(n+1) \to f(n)$ such that for any $\ell < f(n+1)$, $p_{n+1,\ell} \ge_{\mathbb{Q}} q_{n,f(n)(g(n)-1)+s(\ell)}$.

(C) In the end, player COM wins if there is a condition $q \ge p$ such that

$$q \Vdash \forall n, \bigvee_{\ell<f(n)} q_{n,f(n)(g(n)-1)+\ell} \in \mathbf{G}.$$

Note that unboundedly many n would give the same result, by $(*)_i$, $i = 1, 2, 3$. Also

$$q \Vdash \forall k, \exists n \ge k, \bigvee_{\ell<f(n)} q_{n,f(n)(g(n)-1)+\ell} \in \mathbf{G}$$

describes the same criterion for winning.

(2) The game $\partial^2_{f,g}(\mathbb{Q})$.

(A) Is the same as in $\partial^1_{f,g}(\mathbb{Q})$, and also item (C) is the same.

(B^2) In the n-th move a subgame of $g(n)$ subrounds is played. In subround i, $0 \le i < g(n)$, Player COM chooses

$$\langle p_{n,f(n)\cdot i+\ell} : \ell < f(n)\rangle = \langle p_{n,i,\ell} : \ell < f(n)\rangle$$

at once and then Player INC chooses

$$\langle q_{n,f(n)\cdot i+\ell} : \ell < f(n)\rangle = \langle q_{n,i,\ell} : \ell < f(n)\rangle,$$

for any $0 \le i < g(n)$, $\ell < f(n)$,

$(*)_1$ $p \le_{\mathbb{Q}} p_{n,\ell} \le_{\mathbb{Q}} q_{n,\ell}$ for any $\ell < f(n)g(n)$,

$(*)_2$ for $f(n) \le \ell < f(n)g(n)$, $p_{n,\ell} \ge_{\mathbb{Q}} q_{n,\ell-f(n)}$, equivalently for $i < g(n)-1$, $\ell < f(n)$, $p_{n,i+1,\ell} \ge_{\mathbb{Q}} q_{n,i,\ell}$.

$(*)_3$ there is some onto function $s\colon f(n+1) \to f(n)$ such that for any $\ell < f(n+1)$, $p_{n+1,\ell} \ge_{\mathbb{Q}} q_{n,f(n)(g(n)-1)+s(\ell)}$.

(3) We write $\partial^i_{\cdot,g}(\mathbb{Q})$ or $\partial^i_{f,\cdot}(\mathbb{Q})$ or $\partial^i_{\cdot,\cdot}(\mathbb{Q})$, for the analogously defined games in which COM chooses $f(n)$ repectively $g(n)$ or both in the beginning of the n-th move. If $g = 1$ constantly and player COM chooses $f(n)$ in round n, then we write $\partial^i_{\cdot,1}(\mathbb{Q})$.

(4) If for any n, the conditions $p_{n,\ell}$, $\ell < f(n)$, are pairwise incompatible, then we can translate a play in $\partial^i_{f,g}(\mathbb{Q})$ into a tree of conditions, ordered according to strength, in a natural way. In detail, this goes as follows.

The conditions build up a tree of height $h(n) = \sum_{m \le n} g(m)$ and width $f(m)$ at level $h(m)$ for $m \le n$. For simplicity, we let $g(n) = 1$ for all n, and we define a tree W by recursion on levels W_n, $n < \omega$. Round n of the play defines the level W_{n+1} and the instruction on the tree order between W_{n+1} and W_n.

We let $W_0 = \{\emptyset\}$ and $q^{\text{tree}}_\emptyset = p^{\text{tree}}_\emptyset = p$. We let $W_1 = f(0)$ and let for $t \in W_1$, $p^{\text{tree}}_{\langle t\rangle} = p_{0,t}$, $q^{\text{tree}}_t = q_{0,t}$ and order $W_0 \cup W_1$ such that $t >_W \emptyset$ for $t \in W_1$.

Suppose that $p_{i,\ell}$ and $q_{i,\ell}$ were played in round $i \le n$, and we already defined p^{tree}_t, q^{tree}_t for $t \in \bigcup_{i \le n+1} W_i$ and the tree order $(\bigcup_{i\le n+1} W_i, <_W)$. Recall the onto function s from $(*)_3$. Now the $f(n+1)$ new elements $p_{n+1,\ell}$, $\ell < f(n+1)$, are placed in the new top level W_{n+2} by letting

(a) $W_{n+2} = \{t^\frown \ell : t \in W_{n+1}, \ell < f(n+1), p^{\text{tree}}_{t^\frown \ell} = p_{n+1,\ell} \ge q^{\text{tree}}_t = q_{n,s(\ell)}\}$,
(b) $t^\frown \ell >_W t$ for $t \in W_{n+1}$, $\ell < f(n+1)$ with $p^{\text{tree}}_{t^\frown\ell} \ge q^{\text{tree}}_t$,
(c) $q^{\text{tree}}_{t^\frown\ell} = q_{n+1,\ell}$, if $p^{\text{tree}}_{t^\frown\ell} = p_{n+1,\ell}$.

In applications to tree forcings $\mathbb{Q}$, each q^{tree}_t has a number of direct successors $p^{\text{tree}}_{t^\frown\ell}$, ℓ ranging over some successor branching size that is prescribed by the underlying tree forcing. Instead of f as a measure of width of the tree at level n we would take $f_{\text{succ}}\colon W \to \omega$ that describes how many direct successors each $t \in W$ and q^{tree}_t has. In Sacks forcing $\mathbb{S}$ it is easiest for COM to play in the game $\partial^2_{f,1}(\mathbb{S})$ where f_{succ} is constantly 2.

(5) If $M \prec (H(\chi), \in)$ is countable and $f, g, \mathbb{Q} \in M$ then we define $\partial^i_{f,g}(\mathbb{Q}, M)$ as above with the additional clause that all the moves are elements of $M \cap \mathbb{Q}$. The condition q in clause (C) need not be in M and also the winning criterion in clause (C) is formulated with $\mathbb{Q}$ (and not with the forcing $\mathbb{Q} \cap M$).

The games are related.

Observation 2.5. *(1) If* COM *has a winning strategy in* $\Game^i_{f_1,g_1}(\mathbb{Q})$ *and* $f_1 \leq f_2$, $g_1 \leq g_2$, *then* COM *has a winning strategy in* $\Game^i_{f_2,g_2}(\mathbb{Q})$, $\Game^i_{f_1,\cdot}(\mathbb{Q})$, $\Game^i_{\cdot,g_1}(\mathbb{Q})$ *and* $\Game^i_{\cdot,\cdot}(\mathbb{Q})$. *Here* i *is* 1, 2.

(2) For the same list of games we have: Player COM *can pretend that player* INC *has chosen* $q'_{n,\ell} \geq q_{n,\ell}$ *rather than* $q_{n,\ell}$ *and apply* COM*'s winning strategy to* $q'_{n,\ell}$. *This does not change* COM*'s winning.*

(3) If COM *has a winning strategy in* $\Game^2_{f,g}(\mathbb{Q})$, *then* COM *has a winning strategy in* $\Game^1_{f,g}(\mathbb{Q})$.

*(4) Items (1) to (3) also hold for "*INC *does not have a winning strategy" instead of "*COM *has a winning strategy" in both places.*

Now we look at forcings with trees of height ω as conditions.

Definition 2.6. A pair $(p, \leq_p)$ is a tree if $p \neq \emptyset$, $\leq_p$ is a partial order on p such that for each node $t \in p$ the set of $\leq_p$-predecessors is a finite linear order, and its order type is called the length of t.

A tree p is called perfect if each $s \in p$ there are $t_1, t_2 >_p s$ that are $\leq_p$-incomparable.

For a condition $p = (p, \leq_p)$ and a node $t \in p$, we let $p \restriction t = \{s \in p : t \leq_p s \vee s \leq_p t\}$. We write stem$(p)$ for the $\leq_q$-least node of p that has at least two immediate successors in p, if there is such a node. We use various underlying sets X. The set of finite sequences over X is denoted by $X^{<\omega}$. Typically a tree p is a subset of $X^{<\omega}$, and the tree order $\leq_p$ is the end extension order $\trianglelefteq$, where for $s, t \in X^{<\omega}$, $s \trianglelefteq t$ if $t \restriction \mathrm{dom}(s) = s$.

For testing our games, we work with Sacks forcing.

Definition 2.7. The Sacks forcing order $\mathbb{S}$ consists of perfect trees $p \subseteq 2^{<\omega}$. A condition q is stronger than a condition p if $q \subseteq p$. We write $q \geq p$ in Israeli notation.

Lemma 2.8. *We let* $\mathbb{Q}$ *be the Sacks forcing. Player* COM *has a winning strategy in* $\Game^2_{f,g}(\mathbb{Q})$ *for any* f *with* $\lim_n f(n) = \infty$ *and any* $g \geq 1$.

Proof. Here is a sketch with $f(n) = 2^n$. We let $f(-1) = g(-1) = 1$, $q_{-1,0} = p$. Suppose that $n \geq 1$ and that in round $n-1$ Player INC played the condition $q_{n-1,f(n-1)(g(n-1)-1)+i}$, $i < f(n-1)$.

The strategy will work for any $g(n) \geq 1$. In the first $f(n)$ moves of round n, player COM plays the restrictions

$$\begin{aligned}&\langle p_{n,i} : i < f(n)\rangle = \\ &\langle q_{n-1,f(n-1)(g(n-1)-1)+i} \restriction (\mathrm{stem}(q_{n-1,f(n-1)(g(n-1)-1)+i})^\frown \langle j\rangle) \\ &\quad : j \in 2, i < f(n-1)\rangle\end{aligned}$$

of $q_{n-1,f(n-1)(g(n-1)-1)+i}$, $i < f(n-1)$, according to their first splitting front. This tuple is $2 \cdot f(n-1) = f(n)$ long. In the higher subrounds (if

$g(n) \geq 2$) player COM just has to respect the rules, this is enough to ensure their winning. In the end we let q be the tree spanned in p by the nodes

$$\{\text{stem}(q_{n,f(n)\cdot(g(n)-1)+i}) \,:\, i < f(n), n \in \omega\}.$$

(Here we could equivalently write p's with the same indices, the result would be the same.) If $f(n)$ grows less fast, then COM has to keep track of freezing enough splitting nodes in former moves by INC. A slightly more complicated strategy exists. $\square$

Definition 2.9. Let $\mathcal{S} \subseteq [\alpha(*)]^\omega$ be stationary. $\mathbb{P}$ is $\mathcal{S}$-proper if for any countable $N \prec \mathcal{H}(\chi))$ such that $N \cap \alpha(*) \in \mathcal{S}$, for any $p \in \mathbb{P} \cap N$ there is $q \geq p$ that is $(N, \mathbb{P})$-generic. q is $(N, \mathbb{P})$-generic means: For any $D \in N$, if D is dense in $\mathbb{P}$ then $q \Vdash \underset{\sim}{\mathbf{G}} \cap D \neq \emptyset$.

If $\mathbb{P}$ is $[\alpha(*)]^\omega$-proper for any uncountable regular $\alpha(*)$, then we say that $\mathbb{P}$ is proper.

For work on guessing principles on the space $[\omega_1]^{\aleph_1}$, we will use $\alpha(*) = \omega_1$. Then $\mathcal{S}$ can be replaced by its $\subseteq$-cofinal subset $S = \mathcal{S} \cap \omega_1$.

Definition 2.10. A forcing $\mathbb{P}$ is ${}^\omega\omega$*-bounding* if for every $\mathbb{P}$-name $\underset{\sim}{f}$ for a function from ω to ω and for any p, there are $g \in {}^\omega\omega \cap \mathbf{V}$ and $q \geq p$, $q \in \mathbb{P}$, such that $q \Vdash \forall n, \underset{\sim}{f}(n) \leq g(n)$.

In order to derive properness and bounding we work with the games $\Game^i(Q, M)$ from Definition 2.4(4) for countable elementary submodels $M \prec H(\chi)$, for some regular $\chi > 2^{|\mathbb{Q}|}$.

Lemma 2.11. *If Player* COM *has a winning strategy* **str** *in in* $\Game^i_{f,g}(\mathbb{Q})$ *and* $M \prec (H(\chi), \in)$ *is countable and* $f, g, \mathbb{Q}, \mathbf{str}, \mathbb{Q} \in M$, *then* **str** *is also a winning strategy for Player* COM *in* $\Game^i_{f,g}(\mathbb{Q}, M)$.

Proof. For $p, \underset{\sim}{\tau}, \mathbb{Q} \in M$ we have $M \models p \Vdash_{\mathbb{Q}} \varphi(\underset{\sim}{\tau})$ iff $V \models p \Vdash_{\mathbb{Q}} \varphi(\underset{\sim}{\tau})$. $\square$

Definition 2.12. We say "COM does not lose" for "INC does not have a winning strategy."

Definition 2.13. Let $S \subseteq \omega_1$ be stationary. We say "Player COM does not almost lose in $\Game^2_{f,g}(\mathbb{Q})$ for S" if for any countable $M \prec (H(\chi), \in)$ with $M \cap \omega_1 = \delta \in S$, $f, g, \mathbb{Q}, \in M$, **str** a strategy for Player INC not necessarily in M but $\mathbf{str}(\bar{p}) \in M$ for $\bar{p} \in M$, the Player COM does not lose against **str** in $\Game^2_{f,g}(\mathbb{Q}, M)$. Recall the definition winning in $\Game^2_{f,g}(\mathbb{Q}, M)$ in Definition 2.4(4): The condition q, and the full forcing is taken in the winning criterion

$$q \Vdash \forall n \in \omega, \bigvee_{\ell < f(n)} q_{n,f(n)(g(n)-1)+\ell} \in \underset{\sim}{\mathbf{G}}.$$

Corollary 2.14 (Of the previous Lemma)**.** *If* COM *has a winning strategy in* $\Game^2_{f,g}(\mathbb{Q})$, *then* COM *does not almost lose* $\Game^2_{f,g}(\mathbb{Q})$ *for* S.

"Does not almost lose for S" is maybe not a consequence of "does not lose". For countable $M \prec H(\theta)$ with $M \cap \omega_1 \in S$ Player INC "can look from outside M" for enumerating all dense sets in M or for knowing a cofinal ω-sequence in $\omega_1 \cap M$.

Lemma 2.15 (See [13, Lemma 1.7])**.** *Let $S \subseteq \omega_1$ be stationary. For any f and g and also for the versions in which COM chooses f and or g, we have: If COM does not almost lose $\eth^1_{f,g}(\mathbb{Q})$ for S, then $\mathbb{Q}$ is ω^ω-bounding and S-proper.*

Proof. Given a $\mathbb{Q}$-name $\underset{\sim}{f}$ of an increasing function from ω to ω and a condition $p \in \mathbb{Q}$, we take a countable model $(M, \in) \prec (H(\chi), \in)$ for some regular $\chi > (2^{|\mathbb{Q}|})^+$ such that $\underset{\sim}{f}$, p, f, g, $\mathbb{Q} \in M$ and such that $M \cap \omega_1 \in S$. We fix an enumeration $\langle D_n : n < \omega \rangle$ of all the open dense subsets of $\mathbb{Q}$ that are elements of M. After taking intersections of initial segments, we can assume $D_{n+1} \subseteq D_n$. Now COM and INC play $\eth^1_{f,g}(\mathbb{Q}, M)$ and INC starts with p. In round n, Player INC plays $\langle q_{n,i} : i < f(n) \rangle$ so strong that $q_{n,i} \in D_n \cap M$ and $q_{n,i}$ pins down $\underset{\sim}{f} \restriction (n+1)$, say to $g_{n,i}$. Player COM does not lose against this strategy of INC. Hence there is a play $\langle \bar{p_n}, \bar{q_n} : n < \omega \rangle$ according to Player INC's strategy of walking into the dense sets D_n and pinning down $\underset{\sim}{f} \restriction (n+1)$ in which Player COM wins . Let $q \geq p$ be a witnessing condition COM's winning as in Clause (C) of Definition 2.4(1),

$$q \Vdash \forall n \in \omega, \bigvee_{i < f(n)} q_{n,f(n)(g(n)-1)+i} \in \underset{\sim}{\mathbf{G}}.$$

Then q is $(M, \mathbb{Q}, p)$-generic. We let

$$g(n) = \max\{g_{n,i}(n) : i < f(n)\}.$$

Then,

$$q \Vdash (\forall n \in \omega) \underset{\sim}{f}(n) \leq g(n).$$

So $\mathbb{Q}$ is ${}^\omega\omega$-bounding. □

3. Negating club and stick with strategically closed interands

The following theorem gives information on countable support iterations $\mathbb{P} = \langle \mathbb{P}_\beta, \mathbb{Q}_\gamma : \gamma < \omega_1, \beta \leq \omega_2 \rangle$ with the $\aleph_2$-c.c. It shows that premises of the type "$\mathbb{P}_\beta$ forces that COM has a winning strategy in $\eth^2_{f,g}(\underset{\sim}{\mathbb{Q}_\beta})$", for $f(n) \geq 2$, are not sufficient to force $\clubsuit_S$ and not even for ❗.

Theorem 3.1. *We assume CH and $2^{\aleph_1} = \aleph_2$. There is a countable support iteration $\mathbb{P} = \langle \mathbb{P}_\beta, \mathbb{Q}_\alpha : \alpha < \omega_2, \beta \leq \omega_2 \rangle$ such that $\mathbb{P}_0 = \{0\}$ and each $\mathbb{P}_\beta$ forces that COM has a winning strategy in $\eth^2_{f,g}(\underset{\sim}{\mathbb{Q}_\beta})$ for $f(n) \geq 2$ (and hence $\underset{\sim}{\mathbb{Q}_\beta}$ is proper and ${}^\omega\omega$-bounding) and nevertheless $\mathbb{P}$ forces that $\clubsuit_S$ fails.*

Proof. The proof consists of two definitions and Lemma 3.4 and Lemma 3.5. We begin with a definition.

Definition 3.2. Let $S \subseteq \omega_1$ be stationary and let $\bar{a} = \langle a_\beta : \beta \in S\rangle$ be such that each $\beta \in S$ is a limit ordinal and that a_β is of order type ω and a cofinal subset of β. Then the sequence $\bar{a}$ is called a $\clubsuit_S$-candidate or tiltan candidate.

We plan that $\mathbb{P}$ is a countable support iteration of proper iterands of size $\aleph_1$.

A club-sequence $\bar{a}$ in $V^{\mathbb{P}}$ would have a $\mathbb{P}_\alpha$-name for some $\alpha < \omega_2$, and then the next iterand could be the anti-tiltan forcing $\mathbb{A}^{\bar{a}}$ of Lemma 3.5 below. We say "we work in $\mathbf{V}[\mathbb{Q}]$" if we argue in any generic extension. A $\mathbb{Q}$-name $\underaccent{\tilde}{x}$ is then written in $\mathbf{V}[\mathbb{Q}]$ as x, as a shortcut for picking a $\mathbb{P}$-generic filter $\mathbf{G}$ over $\mathbf{V}$ and work with for $\mathbf{V}[\mathbf{G}_\mathbb{Q}]$ and $x = \mathrm{val}(\underaccent{\tilde}{x}, \mathbf{G}_\mathbb{Q})$.

Using $2^{\aleph_1} = \aleph_2$, we fix a book-keeping $\langle \bar{a}_\beta : \beta < \omega_2\rangle$ that gives all $\mathbb{P}'$-names of tiltan candidates for any $\mathbb{P}'$ that is a countable support iteration of proper iterands of size $\aleph_1$ (of the iterands) of length $\aleph_2$, each of the names appearing cofinally often in ω_2. Each such name is actually a $\mathbb{P}'_\beta$-name for some $\beta < \omega_2$. Details about the book-keeping technique can be found, e.g., in [17, Chapter III].

A particular tiltan candidate $\bar{a}$ is destroyed by a forcing that provides for a cofinal subset of ω_1, usually a new set, that does not contain any member of $\bar{a}$ as a subset. We start with a ground model of CH. An iteration of length $\aleph_2$ which at stage α adds via $\mathbb{Q}_\alpha$ a counterexample to the $\mathbb{P}_\alpha$-tiltan candidate $\bar{a_\alpha}$ given by the book-keeping, forces the failure of $\clubsuit_S$. Any counterexample $E \in [\aleph_1]^{\aleph_1} \cap \mathbf{V}[\mathbb{P}_{\alpha+1}]$ persists of course upwards in $\mathbf{V}[\mathbb{P}_{\aleph_2}]$, as the statement $\forall \delta \in S$, $a_\delta \not\subseteq E$ is Δ_0. So the only task is, given a tiltan candidate $\bar{a}$ and CH, to find a proper forcing $\mathbb{A}^{\bar{a}}$ of size $\aleph_1$ that forces that $\bar{a}$ is not a witness to $\clubsuit_S$ and such that COM has a winning strategy in $\eth^2_{f,g}(\mathbb{A}^{\bar{a}})$.

Definition 3.3. Let S be stationary and let $\bar{a}$ be a $\clubsuit_S$-candidate.

(1) We define a notion of forcing $\mathbb{Q}^{\bar{a}}$ as follows:
Conditions in $\mathbb{Q}^{\bar{a}}$ are partial functions $p\colon \alpha(p) \to 2$ for some $\alpha(p) = \mathrm{dom}(p) \in \omega_1$ with the following property:
$$\forall \delta \in S \cap (\alpha(p)+1),\ a_\delta \not\subseteq p^{-1}[\{1\}].$$
Stronger conditions are end extensions.

(2) We define a $\mathbb{Q}^{\bar{a}}$-name for a subset of ω_1 as follows:
$$\underaccent{\tilde}{E} = \{\langle \check{\beta}, p\rangle : p(\beta) = 1\}.$$

Lemma 3.4. *Let $\bar{a}$ be a $\clubsuit_S$-candidate. Then $\mathbb{Q}^{\bar{a}}$ forces that $\underaccent{\tilde}{E}$ is uncountable and for any $\alpha \in S$, $a_\alpha \not\subseteq \underaccent{\tilde}{E}$.*

Proof. By density, $\mathbb{Q}^{\bar{a}}$ forces that $\underaccent{\tilde}{E}$ is an unbounded subset of ω_1. We show that $\underaccent{\tilde}{E}$ is not predicted by $\bar{a}$ in the sense of $\clubsuit_S$. Let $p \in \mathbb{Q}^{\bar{a}}$. By definition of $\mathbb{Q}^{\bar{a}}$, we have for any $q \geq p$,
$$\forall \delta \in (\mathrm{dom}(q)+1) \cap S, a_\delta \not\subseteq q^{-1}[\{1\}].$$

Hence

$$p \Vdash \forall \delta \in S, a_\delta \not\subseteq \underset{\sim}{E}.$$

$\square$

We modify $\mathbb{Q}^{\bar{a}}$ now.

Lemma 3.5. *Assume* CH *and let* $S \subseteq \omega_1$ *be a stationary set of limit ordinals and let* $\bar{a} = \langle a_\delta : \delta \in S \rangle$ *be a tiltan candidate. Then there is a proper forcing* $\mathbb{A}^{\bar{a}}$ *that does not add reals (and hence is countably distributive), is of cardinality* $\aleph_1$ *and forces that* $\bar{a}$ *is not a tiltan sequence and if* $f(n) \geq 2$*, then* COM *has a winning strategy in* $\Game^2_{f,g}(\mathbb{A}^{\bar{a}})$.

Proof. We let $\mathbb{A}^{\bar{a}} = \mathbb{A}$ be of the form $\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}_1}$ as follows. $\mathbb{Q}_0 = ({}^{\omega_1 >}2, \trianglelefteq)$ is the $\aleph_1$-Cohen forcing, also written as $\mathrm{Add}(\aleph_1, 1)$. We let $\langle \alpha^*_\eta : \eta \in {}^{\omega_1 >}2 \rangle$ be a sequence of countable ordinals with no repetition. For transparency, we require $\alpha^*_\eta > \mathrm{dom}(\eta)$. Here we use CH. We let $\underset{\sim}{\eta_0}$ be the $\mathbb{Q}_0$-generic function from ω_1 to 2 and we let

$$\underset{\sim}{A} = \{\alpha^*_{\underset{\sim}{\eta_0} \restriction \varepsilon} : \varepsilon < \omega_1\}.$$

The forcing order $\mathbb{Q}_0$ is $< \omega_1$-closed and of cardinality $\aleph_1$ and hence does not add any countable sequence of ordinals. It forces $\underset{\sim}{\eta_0} \in {}^{\omega_1}2$ and $\underset{\sim}{A} \in [\omega_1]^{\aleph_1}$.

In $\mathbf{V}[\mathbb{Q}_0]$, we let

$$\mathbb{Q}_1 = \{p \in \mathbb{Q}^{\bar{a}} : p^{-1}[\{1\}] \subseteq A\},$$

and let the $\mathbb{Q}_1$-generic be called $\underset{\sim}{\eta_1}$. Since $\mathbb{Q}_0$ does not add new ω-sequences, the definition of $\mathbb{Q}^{\bar{a}}$ evaluated in $\mathbf{V}[\mathbb{Q}_0]$ is the same as the one in $\mathbf{V}$.

Now $\mathbb{A} = \mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}_1}$, and

$$\mathbb{A}' = \{(p(0), p(1)) \in \mathbb{A} : \mathrm{dom}(p(0)) \geq \mathrm{dom}(p(1)), p(1) \in \mathbf{V}\}$$

is a dense subset of $\mathbb{A}$. Note that $(p(0), p(1)) \in \mathbb{A}$ says $p(1)^{-1}[\{1\}] \subseteq \{\alpha_\nu : \nu \trianglelefteq p(0)\}$.

The forcing $\mathbb{A}$ forces that $\underset{\sim}{\eta_1}^{-1}[\{1\}] \in [\omega_1]^{\aleph_1}$ is not guessed by $\bar{a}$, because for any $\delta \in S$ for any $p \in \mathbb{A}$ with $\delta \in \mathrm{dom}(p(1)) + 1$ we have $a_\delta \not\subseteq p(1)^{-1}[\{1\}]$, and hence, by the definition of $\mathbb{A}$, $p \Vdash_{\mathbb{A}} a_\delta \not\subseteq \underset{\sim}{\eta_1}^{-1}[\{1\}]$.

Now we arrive to the main point:
If $f(n) \geq 2$ and $g(n) \geq 1$ then player COM has a winning strategy in $\Game^2_{f,g}(\mathbb{A})$.

We describe a strategy for COM. Player INC chooses a condition p in the very beginning of the play. Player COM plays all their moves in $\mathbb{A}'$. We let $p = (p(0), p(1))$ and have $p(0) = \eta \in {}^{\omega_1 >}2$. We let for $i = 0, 1$ $p^{[i]} = (p(0) {}^\frown \langle i \rangle, p(1))$.

We let

$$\begin{aligned} A_i = &\{\alpha_\nu : \eta {}^\frown \langle i \rangle \trianglelefteq \nu\}, \\ A_* = &\{\alpha_\nu : \nu \trianglelefteq \eta\}. \end{aligned}$$

So the (A_*, A_0, A_1) are pairwise disjoint and $p^{[i]}$ forces that $\underset{\sim}{A} \subseteq A_* \cup A_i$. In addition $A_* \subseteq \sup(A_*) := \varepsilon_* < \omega_1$.

During a play $\langle p_{n,\ell}, q_{n,\ell} : n < \omega, \ell < f(n)g(n)\rangle$ with $p_{n,\ell} = (p_{n,\ell}(0), p_{n,\ell}(1))$ $q_{n,\ell} = (q_{n,\ell}(0), q_{n,\ell}(1))$, player COM plays $p_{n,\ell} \geq q_{n-1,f(n-1)(g(n-1)-1)+\ell}$ such that $\varepsilon_n(i) := \mathrm{dom}(p_{n,\ell}(i))$ does not depend on ℓ only on n and that

$$\varepsilon_n(1) \geq \sup\{\alpha_\nu + 1 : \nu \trianglelefteq q_{m,\ell}(0), m < n, \ell \leq f(m)g(m)\} + \varepsilon_*$$

and such that for any $m < n$ and $i = 0, 1$ and $p^{[i]} \leq_{\mathbb{A}} p_{n,i}$. So here we use that $f(n) \geq 2$. Moreover COM plays so that $p_{n,\ell} \geq_{\mathbb{A}} q_{m,f(m)\cdot(g(m)-1)+\ell}$ for $\ell < \min(f(n), f(m))$. Since for $m < n$, $i = 0, 1$, $p^{[i]} \leq p_{m,i} \leq q_{m,f(m)(g(m)-1)+i}$, these requirements respect the rules of the game $\Game^2_{f,g}(\mathbb{A})$ in Definition 2.4(2)(B).

Now we show: For some $\ell < 2$,

$$q_\ell = (\bigcup_n p_{n,\ell}(0), \bigcup_n p_{n,\ell}(1)) \in \mathbb{A}.$$

We let $\delta_{\max} = \bigcup\{\varepsilon_n(0) : n < \omega\}$. By the rules for the $p_{n,\ell}$ and the definition of $\mathbb{A}'$ we have $\varepsilon_* < \dots < \varepsilon_n(1) \leq \varepsilon_n(0) < \varepsilon_{n+1}(1) \dots$. Hence we have $\delta_{\max} = \bigcup\{\varepsilon_n(1) : n < \omega\} > \varepsilon_*$.

We have $q_\ell \in \mathbb{A}$ if and only if $\forall \delta \in S \cap (\delta_{\max} + 1)$, $a_\delta \not\subseteq q_\ell(1)^{-1}[\{1\}]$. Since the moves of the play are all in $\mathbb{A}$, only the limit $\delta = \delta_{\max}$ is in question. First case: $\delta_{\max} \notin S$. Then there is no requirement, and for $\ell = 0, 1$, we have $q_\ell \in \mathbb{A}$. Second case $\delta_{\max} \in S$. Since $A_0 \cap A_1 = \emptyset$, for some $\ell \in 2$, $q_\ell(0) \Vdash_{\mathbb{Q}_0} q_\ell(1)^{-1}[\{1\}] \subseteq \underset{\sim}{A}$ and $a_\delta \not\subseteq A_* \cup A_i$. So, $q_\ell \in \mathbb{A}$. □

Thus Theorem 3.1 is proved. □

We show that $\mathbb{Q}^{\bar{a}}$ alone does not necessarily do the job required in Theorem 3.1.

Remark 3.6. The forcing $\mathbb{Q}^{\bar{a}}$ is proper and does not add new reals.

Proof. Suppose that $M \prec H(\chi)$ is countable. Let $\langle D_m : m < \omega\rangle$ enumerate all open dense sets of $\mathbb{Q}^{\bar{a}}$ that are in M and let $\delta = \omega_1 \cap M = \sup\langle \alpha_n : n < \omega\rangle$. We assume that $\delta \in S$, the other case is easier. Then a_δ is cofinal in δ.

By induction on $n \in \omega$ we choose $p_n \in D_n$, $p_n \geq p$, $p_{n+1} \geq p_n$ such that $\exists \varepsilon \in \mathrm{dom}(p_n) \cap a_\delta \cap [\alpha_n, \delta), p_n(\varepsilon) = 0$. In the end $q = \bigcup_n p_n$ is as desired.

This shows also that COM does not lose the distributivity game $G_{\omega+1}(\mathbb{Q}^{\bar{a}})$ from Subsection 2.1 and hence the forcing does not add new ω-sequences (which for proper forcings is not stronger than saying does not add new reals). □

The forcing $\mathbb{Q}^{\bar{a}}$ is, in general, not good enough to serve as an iterand in Theorem 3.1.

Proposition 3.7. *Under $\diamondsuit_S$, there is a tiltan candidate $\bar{a}$ such that* COM *does not have a winning strategy in $\Game^1_{1,1}(\mathbb{Q}^{\bar{a}})$.*

Proof. Let $\langle D_\alpha : \alpha \in S\rangle$ be a diamond sequence. In addition we fix a continuous increasing sequence $\langle M_\alpha : \alpha \in \omega_1\rangle$ of elementary submodels of $H(\theta)$ such that $\bigcup M_\alpha \supseteq H(\omega_1)$ (this uses CH). So $M_\alpha \cap \omega_1 = \alpha$. For any $\alpha \in S$, if D_α codes M_α, s_α, and $\langle a_\beta : \beta < \alpha\rangle \in M_\alpha$ and $M \models s_\alpha$ is a strategy for

COM in $\mathbb{Q}^{\langle a_\beta\,:\,\beta<\alpha\rangle}$, then we want to define a_α such that s_α is not a winning strategy. This is done as follows: We fix an increasing cofinal sequence α_n converging to α. In detail, player INC starts with the empty condition, and then COM plays $s_\alpha(\emptyset)$. Then INC plays $q_0 = s_\alpha(\emptyset) \cup \{(\gamma_0, 1)\}$ for some $\gamma_0 \geq \alpha_0, \max(\mathrm{dom}(s_\alpha(\emptyset)) + 1)$. Then COM plays $p_1 = s_\alpha(q_0)$ and INC answers with $q_0 = s_\alpha(q_0) \cup \{(\gamma_1, 1)\}$ for some $\gamma_1 \geq \alpha_1, \max(\mathrm{dom}(s_\alpha(q_0))+1)$. In the end we let $a_\alpha = \{\gamma_n \,:\, n < \omega\}$ and see that INC wins. In the end, there is no winning strategy for COM in $\Game^1_{1,1}(\mathbb{Q}^{\bar{a}})$, since any such strategy would be guessed as some M_α, s_α, and then be defeated by INC. □

Now we end our digression and take up the main thread again.

Now, with some more technique, we show that also ¡ can be negated by strategically closed iterands.

Theorem 3.8. *We assume* CH *and* $2^{\aleph_1} = \aleph_2$. *There is a countable support iteration* $\mathbb{P} = \langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\alpha \,:\, \alpha < \omega_2, \beta \leq \omega_2\rangle$ *such that* $\mathbb{P}_0 = \{0\}$ *and each* $\mathbb{P}_\beta$ *forces that* COM *has a winning strategy in* $\Game^2_{f,g}(\underset{\sim}{\mathbb{Q}}_\beta)$ *for* $\lim_n f(n) = \infty$ *(and hence* $\underset{\sim}{\mathbb{Q}}_\beta$ *is proper and* ${}^\omega\omega$*-bounding) and nevertheless* $\mathbb{P}$ *forces that* ¡ *fails.*

Proof. The set-up is as in Theorem 3.1. For negating the stick starting from CH in the ground model, we do not need a book-keeping. It suffices to show that at stage $\mathbf{V}[\mathbb{P}_\alpha]$ there is an iterand $\mathbb{Q}_\alpha$ that forces that the ground model of this stage, i.e., $[\aleph_1]^{\aleph_0} \cap \mathbf{V}[\mathbb{P}_\alpha]$, is not a ¡-witness. (This persists of course upwards.) As $[\aleph_1]^{\aleph_0} \cap \mathbf{V}[\mathbb{P}_\alpha]$ would be the maximal stick witness, any stick witness at stage $\mathbf{V}[\mathbb{P}_\alpha]$ thus is ruled out in stage $\mathbf{V}[\mathbb{P}_{\alpha+1}]$. Now we focus on a single iterand. This time it has three sub-iterands.

Lemma 3.9. *Assume* CH. *There is a proper* ${}^\omega\omega$*-bounding forcing* $\mathbb{B}$ *of cardinality* $\aleph_1$ *and forces that* $[\aleph_1]^{\aleph_0}$ *is not a stick witness. Moreover* $\mathbb{B}$ *has the form* $\mathbb{Q}_0 * (\underset{\sim}{\mathbb{Q}_1} * \underset{\sim}{\mathbb{Q}_2})$*, where* $\mathbb{Q}_0 = \langle \underset{\sim}{\mathbb{S}_\beta} \,:\, \beta < \omega_1\rangle$ *is a countable support iteration of iterands in which* COM *has a winning strategy in* $\Game^2_{f,g}(\mathbb{S}_\beta)$ *if* $\lim f(n) = \infty$ *and* f *is increasing, and in* $\mathbf{V}[\mathbb{Q}_0]$, COM *has a winning strategy in* $\Game^2_{f,g}(\mathbb{Q}_1 * \underset{\sim}{\mathbb{Q}_2})$ *for any* f, g.

Proof. The ground model is $\mathbf{V}$. The iterand $\mathbb{B}$ has the form $\mathbb{B} = \mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}_1} * \underset{\sim}{\mathbb{Q}_2}$.

First sub-iterand: We first add via $\mathbb{Q}_0 = \langle \underset{\sim}{\mathbb{S}_\beta} \,:\, \beta < \omega_1\rangle$ in a countable support iteration $\aleph_1$ Sacks reals, and call the sequence of generic reals $\bar{\varrho} = \langle \underset{\sim}{\varrho_\alpha} \,:\, \alpha < \omega_1\rangle$.

Second sub-iterand: $\aleph_1$-Cohen

In $\mathbf{V}[\mathbb{Q}_0]$ we choose $\mathbb{Q}_1$. We let $\mathbb{Q}_1 = ({}^{\omega_1 >}2, \trianglelefteq)$ as defined in $\mathbf{V}[\mathbb{Q}_0]$. The $\mathbb{Q}_1$-generic function from ω_1 to 2 is named $\underset{\sim}{\eta_1}$. This is read also as a $\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}_1}$-name. We let

$$\mathbb{Q}_0 \Vdash \text{“}\langle \underset{\sim}{\alpha^*_\eta} \,:\, \eta \in ({}^{\omega_1 >}2)^{\mathbf{V}[\mathbb{Q}_0]}\rangle \text{ be a sequence of countable ordinals}$$
$$\text{with no repetition, and for any } \eta,\ \alpha^*_\eta > \mathrm{dom}(\eta).\text{”}$$

We fix the $\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}_1}$-name

$$\underset{\sim}{A} = \{\alpha^*_{\underset{\sim}{\eta_1}\restriction\alpha} \,:\, \alpha < \omega_1\}. \tag{3.1}$$

Third sub-iterand: "A-sparse $\mathbf{V}$-sparse $\aleph_1$-Cohen."
In $\mathbf{V}[\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}_1}]$ we define $\underset{\sim}{\mathbb{Q}_2}$ by letting $p \in \mathbb{Q}_2$ if for some $\varepsilon < \omega_1$, $p \colon \varepsilon \to 2$ and $p^{-1}[\{1\}] \subseteq A$ and $p^{-1}[\{1\}]$ contains no infinite subset of ω_1 from $\mathbf{V}$. The forcing $\mathbb{Q}_2$ is ordered by end extension. This is of course a forcing order in $\mathbf{V}[\mathbb{Q}_0 * \mathbb{Q}_1]$ since we have predicates for $\mathbf{V}$, $\mathbf{V}[\mathbb{Q}_0]$ and $\mathbf{V}[\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}_1}]$ in the $\mathbb{Q}_0 * \mathbb{Q}_1 * \mathbb{Q}_2$-forcing language. (Only $\mathbf{V}$ is used here.) Let $\underset{\sim}{\eta_2}$ be the $\mathbb{Q}_2$-generic function from ω_1 to 2, and let $\underset{\sim}{B}$ be a $\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}_1} * \underset{\sim}{\mathbb{Q}_2}$-name for $\eta_2^{-1}[\{1\}]$.

We now prove that $\mathbb{B} := \mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}_1} * \underset{\sim}{\mathbb{Q}_2}$ is as promised. Our aim is to show that $\underset{\sim}{B}$ is a witness for:

$$\mathbb{B} \Vdash \forall u \in [\aleph_1]^{\aleph_0} \cap \mathbf{V}, u \not\subseteq \underset{\sim}{B}.$$

First we show

$$\mathbb{B} \Vdash \underset{\sim}{B} \in [\aleph_1]^{\aleph_1}.$$

For this it suffice to show that for a given $\varepsilon < \omega_1$, in $\mathbf{V}[\mathbb{Q}_0 * \mathbb{Q}_1]$, the set

$$D_\varepsilon = \{p \in \mathbb{Q}_2 \,:\, \exists \varepsilon' \geq \varepsilon, p(\varepsilon') = 1 \wedge \varepsilon' \in A\}$$

is dense. Given p, we go back to $V[\mathbb{Q}_0]$ and find there some $r \in \mathbf{G}_{\mathbb{Q}_1}$, that forces $p \in \underset{\sim}{\mathbb{Q}_2}$. We assume that $\mathrm{dom}(r) \geq \varepsilon$. Then we extend $r \in bQ_1$ to some $r' \geq_{\mathbb{Q}_1} r$, $r' \in \mathbf{G}_{\mathbb{Q}_1}$ that forces some $\varepsilon' = \alpha^*_{\eta_1\restriction\mathrm{dom}(r')} = \alpha^*_{r'} > \varepsilon$ into $\underset{\sim}{A}$. Then we can let $p' = p \cup \{(\varepsilon', 1)\}$. We show that $p' \in \underset{\sim}{\mathbb{Q}_2}$: If $p^{-1}[\{1\}]$ did not have an infinite subset from the ground model, then also $p^{-1}[\{1\}] \cup \{\varepsilon'\} = (p')^{-1}[\{1\}]$ does not contain an infinite subset from the ground model. So we know that it is forced that $\underset{\sim}{B}$ has size $\aleph_1$.

Next we show in $\mathbf{V}[\mathbb{Q}_0]$: For $\mathbb{A} = \mathbb{Q}_1 * \underset{\sim}{\mathbb{Q}_2}$, Player COM has a winning strategy in $\Game^2_{f,g}(\mathbb{A})$. (Now we write undertilde-names only for objects that are not in $\mathbf{V}[\mathbb{Q}_0]$.) Since the first component $\mathbb{Q}_1$ of $\mathbb{A}$ is countably closed, we have the dense suborder $\mathbb{A}'$ of $\mathbb{A}$ in which the $(p(1), p(2)) \in \mathbb{A}'$ have $p(1) \Vdash p(2) \in \mathbb{Q}_2 \cap \mathbf{V}[\mathbb{Q}_0]$. This entails that $p(2)(\alpha^*_\eta) = 1$ occurs only for $\eta \trianglelefteq p(1)$. We let $\varepsilon_i(p) = \mathrm{dom}(p(i))$ for $i = 1, 2$. So in $\mathbf{V}[\mathbb{Q}_0]$, $p = (p(0), p(1))$ is an actual object (as opposed to names) and the $\varepsilon_i(p)$ are countable ordinals.

Now we describe the game. Let W be a subtree of ${}^{\omega>}\omega$, that means W is closed under initial segments. We write $\langle\rangle$ for the empty sequence in ${}^{\omega>}\omega$ and in W. We pick W so that W has no maximal node and no isolated ω-branch and for $n \in \omega$, $\mid W \cap {}^n\omega \mid = f(n)$. Let $\chi > 2^{\aleph_1}$ and $<_\chi$ be a well-ordering of $H(\chi)$ hence $\mathscr{B} = (H(\chi), \in, <_\chi)$ has Skolem functions. We describe a strategy for COM in $\Game^2_{f,g}(\mathbb{A})$.

We now work with the tree representation of the moves, according to Definition 2.4(4). For simplicity, we work the a binary tree $W = 2^{<\omega}$ and In the n-move, the index set $\ell \in f(n) = 2^n$ is replaced by $\ell \in W \cap {}^n\omega =: W_{n+1}$.

$W_0 = \{\emptyset\}$. Elements in W_{n+1} stem from round n and determine η of length $n+1$. With more notation, any f with $\lim_n f(n) = \infty$ can be used to build a perfect tree W. A tree order and incomparability preserving embedding to the binary tree into such a more general W would serve to describe a strategy along the lines below.

The strategy **str** for COM is as follows:

(a) All conditions are in $\mathbb{A}'$,

(b) in the n-th subgame in the i-th move for $i < g(n)$, the player COM chooses the $f(n)$-tuple $\langle p_{n,i,\eta} : \eta \in W_{n+1}\rangle$, INC chooses $\langle q_{n,i,\eta} : \eta \in W_{n+1}\rangle$,

(c) for any n, $\langle \mathrm{dom}(p_{n,0,\eta}(1)), \mathrm{dom}(p_{n,0,\eta}(2)) : \eta \in W_{n+1}\rangle = (\varepsilon_n(1), \varepsilon_n(2))$ is constant and $\varepsilon_n(1) \le \varepsilon_n(2)$ and $\varepsilon_n(1) = \beta_n(1) + 1$ is a successor ordinal,

(d) for any n, the conditions $\langle p_{n,0,\eta}(1) : \eta \in W_{n+1}\rangle$ are pairwise distinct and hence pairwise incompatible,

(e) for any n, $i < g(n)$, $\eta \in W_{n+1}$, $p_{n,i+1,\eta} \ge q_{n,i,\eta}$,

(f) for $\eta \in W_{n+1}$, $p_{n+1,0,\eta} \ge q_{n,g(n)-1,\eta\restriction n}$,

(g) for $\eta' \in W_{n+1}$,

$$\begin{aligned}\varepsilon_n(1) = \mathrm{dom}(p_{n,0,\eta'}(1)) > \max(&\mathrm{dom}(q_{n-1,g(n-1)-1,\eta'\restriction n}(2)),\\ &\sup\{\alpha^*_\nu : \nu \trianglelefteq q_{n-1,g(n-1)-1,\eta'\restriction n}(1)\}) + 1,\end{aligned}$$

(here we let $q_{-1,\cdot,\cdot} = p$) and for any $\eta' = \eta^\frown\langle j\rangle \in W_{n+1}$, we have

$$p_{n,0,\eta^\frown\langle j\rangle}(1)(\beta_n(1)) = j,$$

and there is some $\varepsilon_2(n) \ge \varepsilon > \mathrm{dom}(q_{n-1,g(n-1)-1,\eta\restriction n}(2)), \varepsilon_n(1)$ with

$$\begin{aligned}&p_{n,0,\eta^\frown\langle j\rangle}(2)(\varepsilon) = 1,\\ &\qquad\qquad \varepsilon = \alpha^*_{p_{n,0,\eta^\frown\langle j\rangle}(1)^\frown\eta'} \text{ for some } \eta'.\end{aligned}$$

So for this ε we have for some η'', $\varepsilon = \alpha^*_{\eta''}$ and $\eta''(\beta_n(1)) = j$.[1]

(h) let N_n be the Skolem hull of $\langle p_{m,i,\eta}, q_{m,i,\eta} : m < n, i < g(m), \eta \in W_{m+1}\rangle$ in $\mathscr{B}$, and let $\langle \underset{\sim}{\tau}_{n,j} : j < \omega\rangle$ list the $\mathbb{Q}_1 * \mathbb{Q}_2$-names of ordinals in N_n. We demand that $p_{n,0,\eta}$ decides $\underset{\sim}{\tau}_{m,j}$ for $m, j < n$.

In the end of a play, the tuple

$$\mathbf{x} = \langle p_{n,i,\eta}, q_{n,i,\eta}, N_n, \underset{\sim}{\tau}_{n,j} : n < \omega, i < g(n), \eta \in W_{n+1}, j < \omega\rangle$$

belongs to $H(\aleph_1)$ and hence for some $\zeta < \omega_1$ it belongs to $\mathbf{V}[\langle \varrho_\xi : \xi < \zeta\rangle]$, where $\langle \varrho_\xi : \xi < \omega_1\rangle$ is the tuple of $\mathbb{Q}_0$-generic reals, the Sacks reals.

For every branch b through the tree $W = \bigcup\{W_n : n < \omega\}$ we define

$$q_b^+ = (\bigcup_n p_{n,0,b\restriction n}(1), \bigcup p_{n,0,b\restriction n}(2)).$$

[1] This will be important in the proof that the strategy works.

If there is some branch b such that $q_b^+ \in \mathbb{Q}_1 * \underset{\sim}{\mathbb{Q}_2}$, then it is an upper bound of $\{p_{n,0,b\restriction n} : n < \omega\}$, so it is $(\bigcup_n N_n, \mathbb{Q}_1 * \underset{\sim}{\mathbb{Q}_2})$-generic and forces that $\mathbf{G}[\mathbb{Q}_1 * \underset{\sim}{\mathbb{Q}_2}] \cap \bigcup N_n$ is in $\mathbf{V}[\mathbb{Q}_0]$, the we proved that $\mathbb{Q}_1 * \underset{\sim}{\mathbb{Q}_2}$ is proper and does not add any new real and in addition that the strategy is a winning stategy for COM.

In order to find such a branch b, we proceed as follows. We let $\delta = \omega_1 \cap \bigcup N_n$. By the rules of the strategy, $\delta = \sup\{\varepsilon_n(1) : n < \omega\} = \sup\{\varepsilon_n(2) : n < \omega\}$. The only question is whether $q_b^+(2)$, which is a function from δ to 2, does not contain a $u \in [\aleph_1]^{\aleph_0} \cap \mathbf{V}$ as a subset of $(q_b^+(2))^{-1}[\{1\}]$. We can restrict our attention to u that are cofinal subsets of δ of order type ω, since any u not of this form is excluded from being a subset of $(q_c^+(2))^{-1}[\{1\}]$, for any branch $c \in [W]$, since the play is played in $\mathbb{A}$.

Now $\mathbf{x}$ belongs to $\mathbf{V}[\langle \varrho_\xi : \xi < \zeta\rangle]$ and we can interpret ϱ_ζ as a branch b of the perfect tree $(W, <_W)$. Interpreting goes e.g. as follows: In W_{n+1} we read off $b \restriction n+1$. Suppose we already chose $\eta = b \restriction n$ interpreting $\varrho \restriction n$ in $p_{n,0,\eta}$, we append (n,j) to η and say $b \restriction n+1 = \eta^\frown \langle j\rangle$ interprets $\varrho \restriction (n+1)(n) = j$ if according to rule (g) there is some ε with $p_{n,0,\eta^\frown j}(2)(\varepsilon) = 1$ of the form $\varepsilon = \alpha^*_{p_{n,g(n)-1,\eta^\frown\langle j\rangle}(1)^\frown \eta'}$ for some η', and $p_{n,0,\eta^\frown\langle j\rangle}(1)(\beta_n(1)) = j$.

Now we look back at $\langle \alpha^*_\eta : \eta \in {}^{\omega_1>}2\rangle$ and the name $\underset{\sim}{A}$ from Equation (3.1). According to the rule (d) of the strategy, the $\mathbb{Q}_1$-coordinates of the nodes $b \restriction n$ of W result in pairwise incompatible conditions about $\underset{\sim}{\eta_1} \in {}^\delta 2$ (recall, this is the name for the $\mathbb{Q}_1$-generic function, defined shortly before Equation (3.1)). By rules (c) to (g) of the game and the coding rule, for any $a \in [\delta]^{\aleph_0}$ if a is cofinal in δ and any $b \in [W]$: If $n < \omega$ and $a \subseteq q_{n,g(n)-1,b\restriction n}(2)^{-1}[\{1\}]$, any entry $(\varepsilon, 1)$ in $p_{n,0,b\restriction n+1}(2)$ of the form $\varepsilon = \alpha_{p_{n,0,b\restriction n+1}(1)^\frown \eta'}$ with $\varepsilon \in a$ determines $q_b^+(1) \restriction \{\beta_k(1) : k \leq n\} = b \restriction (n+1)$. Therefore in $\mathbf{V}[\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}_1} * \underset{\sim}{\mathbb{Q}_2}]$ from any $a \subseteq \delta = \sup(a)$ with $\mathrm{otp}(a) = \omega$ and $a \subseteq p_b(2)^{-1}[\{1\}]$ we can compute $b \in [W]$, and hence for $b = \varrho_\zeta$, we have $a \notin \mathbf{V}$. So the choice of $b = \varrho_\zeta$ not in $\mathbf{V}[\langle \varrho_\varepsilon : \varepsilon < \zeta\rangle]$ guarantees that no element of $[\omega_1]^{\aleph_0} \cap V$ is a subset of $(q_b(2))^{-1}[\{1\}]$ and hence COM has won. □

This ends of Theorem 3.8. □

4. Concluding remarks and questions

We do not know the behaviour of iterations with respect to the games.

Question 4.1. *Suppose that along a say countable support iteration $\mathbb{P} = \langle \mathbb{P}_\gamma, \mathbb{Q}_\gamma : \gamma < \alpha\rangle$, with $\mathbb{P}_\gamma \Vdash$COM has a winning strategy in $\Game^i_{f,g}(\mathbb{Q}_\gamma)$.*

Does COM have a winning strategy in $\Game^i_{f,g}(\mathbb{P}_\alpha)$?

Our anti guessing theorems use a winning witness q that is not predicted by $\bigwedge_{n\in\omega} \bigvee_{\ell<f(n)} q_{n,\ell}$. A weak form of prediction would preclude our technique, namely we could change the winning criterion (C) in Definition 2.4 to

($C^{\text{compatible}}$) In the end, player COM wins if there is a condition $q \geq p$ such that for all n,

$$\forall \ell < f(n), q \not\leq q_{n,f(n)\cdot(g(n)-1)+\ell} \text{ and } q \Vdash \bigvee_{\ell<f(n)} q_{n,f(n)\cdot(g(n)-1)+\ell} \in \underset{\sim}{\mathbf{G}}.$$

Question 4.2. *Do the analogues to Theorem 3.1 and Theorem 3.8 for this type of game hold?*

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