THERE MAY BE INFINITELY MANY NEAR-COHERENCE CLASSES UNDER $\mathfrak{u} < \mathfrak{d}$

HEIKE MILDENBERGER

Abstract. We show that in the models of $\mathfrak{u} < \mathfrak{d}$ from [14] there are infinitely many near-coherence classes of ultrafilters, thus answering Banakh's and Blass' Question 30 of [3] negatively. By an unpublished result of Canjar, there are at least two classes in these models.

1. Introduction

Banakh and Blass [3] have shown that under $\mathfrak{u} \geq \mathfrak{d}$ there are $2^{\mathfrak{c}}$ near-coherence classes of ultrafilters and, in general, if there are infinitely many near-coherence classes, then there are $2^{\mathfrak{c}}$ many. They ask whether in the case of $\mathfrak{u} < \mathfrak{d}$ there are only finitely many near-coherence classes. We answer this question negatively.

There are models with exactly one class in [12, 13, 7]. It is likely that there is a model with exactly two classes, and for $n \in [3, \omega)$ it is open whether there exists a model with exactly n near-coherence classes of ultrafilters.

In the second section we recall some facts about the near-coherence relation. In the third section we investigate the only known type of models of $\mathfrak{u} < \mathfrak{d}$, whose number of near-coherence classes has not yet been known, namely the ones from [14]. We show that there are infinitely $2^{\mathfrak{c}}$ near-coherence classes in these models. We conclude with some remarks and open questions.

Our result is also a result on the number of composants of the Stone-Čech remainder $\mathbb{R}_+^* = \beta\mathbb{R}_+ \setminus \mathbb{R}_+$ of the closed half-line $\mathbb{R}_+$, as Mioduszewski [21, 22] has proved that the number of near-coherence classes of ultrafilters is the same as the number of composants of $\mathbb{R}_+^*$. Blass [5] contains a more streamlined proof and additional applications.

We use various results from the recent work by Banakh and Blass [3] and our and Blass' work [11] on the cofinalities of ultrapowers, and continue the analysis of possible cofinalities of reduced products. However, this is not pcf theory in the usual sense, as two of the first provisos for pcf theory, that the factors be pairwise different and each of them be strictly larger than the index set, are both not fulfilled.

In the remainder of this section we recall the relevant notions. We denote by ${}^{\omega}\omega$ the set of all functions from ω to ω and by $[\omega]^{\omega}$ the set of all infinite subsets of ω. We write X^c for $\omega \setminus X$. By a *filter* we mean a proper non-principal filter on ω. Let $\mathscr{F}$ be a filter on ω and let $f\colon \omega \to \omega$ be finite-to-one (that means that the preimage of each natural number is finite). Then also $f(\mathscr{F}) = \{X : f^{-1}(X) \in \mathscr{F}\}$ is a non-principal filter. Two filters $\mathscr{F}$

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and $\mathscr{G}$ are *nearly coherent*, iff there is some finite-to-one $f\colon \omega \to \omega$ such that $f(\mathscr{F}) \cup f(\mathscr{G})$ generates a filter, that means, for every $X \in f(\mathscr{F})$ for every $Y \in f(\mathscr{G})$ the set $X \cap Y$ is infinite. The restriction to finite-to-one monotone onto f gives the same equivalence relation by [4, Lemma 10]. In order to simplify the notation we henceforth mean by finite-to-one function finite-to-one monotone onto function. Often it is convenient to perceive such an f as a partition $\Pi = \{[f^-(i), f^-(i+1)) \ : \ i < \omega\}$ of ω into the intervals $[f^-(i), f^-(i+1))$, $i < \omega$, given by $f^-(0) = 0$, $f^-(i+1) = \max(f^{-1}\{i\}) + 1$. We call Π the partition given by f.

An ultrafilter is a maximal filter. A filter that is mapped by a finite-to-one function to an ultrafilter is called *almost ultra*.

We write $A \subseteq^* B$ iff $A \setminus B$ is finite. An ultrafilter $\mathscr{U}$ is called a *P_κ-point* if for every $\gamma < \kappa$, for every $A_i \in \mathscr{U}$, $i < \gamma$, there is some $A \in \mathscr{U}$ such that for all $i < \gamma$, $A \subseteq^* A_i$, such an A is called a *pseudo-intersection* of the A_i, $i < \gamma$. If we require only the existence of a pseudo-intersection and it need not be in $\mathscr{U}$, then we call it a *pseudo-P_κ-point.* A $P_{\aleph_1}$-point is called a P-point. An ultrafilter is called a *simple P_κ-point* if it is generated by a $\subseteq^*$-descending sequence $A_i \in \mathscr{U}$, $i < \kappa$. An ultrafilter is called a *Ramsey ultrafilter* if it is a P-point and for any partition of ω into finite intervals there is a set A in the ultrafilter that meets each interval in at most one point. An ultrafilter is called *rapid* if for any increasing sequence $\langle n_i \ : \ i < \omega\rangle$ of ω there is a set A in the ultrafilter that meets each interval $[0, n_i)$ in at most i points.

The space of non-principal ultrafilters has the topology given by the basic open sets $\{\mathscr{U} \ : \ A \in \mathscr{U}\}$, $A \in [\omega]^\omega$. These sets are clopen. For every filter $\mathscr{F}$ there is the closed set $[\mathscr{F}] = \{\mathscr{U} \ : \ \mathscr{F} \subseteq \mathscr{U}\}$. It follows that two filters $\mathscr{F}$ and $\mathscr{G}$ are nearly coherent iff there are nearly coherent ultrafilters $\mathscr{U} \in [\mathscr{F}]$ and $\mathscr{V} \in [\mathscr{G}]$. $S \subseteq {}^\omega\omega$ is a *test set* for $[\mathscr{F}]$ if for any $\mathscr{U}, \mathscr{V} \in [\mathscr{F}]$ the following holds: If $\mathscr{U}$ and $\mathscr{V}$ are nearly coherent, then there is some $f \in S$ such that $f(\mathscr{U}) = f(\mathscr{V})$.

Now we explain some cardinal characteristics: We consider the order of eventual domination: $f \leq^* g$ iff for all but finitely many n, $f(n) \leq g(n)$. This is the special case of $\mathscr{F}$ being the filter of all cofinite sets in the following notion: For a filter $\mathscr{F}$, we write $f \leq_{\mathscr{F}} g$ iff $\{n \ : \ f(n) \leq g(n)\} \in \mathscr{F}$. This is a quasi partial order. For an ultrafilter $\mathscr{U}$, $\leq_{\mathscr{U}}$ is a quasi linear order. After factoring by the equivalence relation of $f \equiv_{\mathscr{F}} g$ iff $\{n \ : \ f(n) = g(n)\} \in \mathscr{F}$, the order on the equivalence classes is well-defined and antisymmetric and thus “quasi” may be dropped.

A family $B \subseteq {}^\omega\omega$ is *unbounded* iff for every $g \in {}^\omega\omega$ there is some $f \in B$ such that $f \not\leq^* g$. The *bounding number* $\mathfrak{b}$ is the smallest cardinal of an unbounded family $B \subseteq {}^\omega\omega$. The bounding number of a filter $\mathscr{F}$, $\mathfrak{b}(\mathscr{F})$, is the smallest cardinal of a family B that is not bounded in $\leq_{\mathscr{F}}$, that is, for every g there is some $f \in B$ such that $f \not\leq_{\mathscr{F}} g$.

A family D is *dominating* iff for every $g \in {}^\omega\omega$ there is some $f \in D$ such that $g \leq^* f$. The dominating number $\mathfrak{d}$ is the smallest cardinal of a dominating family $D \subseteq {}^\omega\omega$. The dominating number of a filter $\mathscr{F}$, $\mathfrak{d}(\mathscr{F})$, is the smallest cardinal of a family D that is dominating in $\leq_{\mathscr{F}}$, that is for every g there is some $f \in D$ such that $g \leq_{\mathscr{F}} f$.

For an ultrafilter we have $\mathfrak{d}(\mathscr{U}) = \mathfrak{b}(\mathscr{U})$, and this cardinal is also called $\mathrm{cf}(\omega^\omega/\mathscr{U})$, the *cofinality of the ultrapower*. The cardinal $\mathfrak{mcf}$ is the minimal $\mathrm{cf}(\omega^\omega/\mathscr{U})$ for all non-principal ultrafilters $\mathscr{U}$.

A set $\mathscr{R} \subseteq [\omega]^\omega$ is called *unsplittable* or *refining* or *reaping* if for every infinite set X there is a set $R \in \mathscr{R}$ such that $R \subseteq X$ or $R \subseteq \omega \setminus X$ (we say R *reaps* X). The unsplitting or refining or reaping number $\mathfrak{r}$ is the smallest size of an unsplittable family. Balcar and Simon [2] showed that it also coincides with the smallest size of a pseudobase of an ultrafilter. A set $\mathscr{R}$ is a *pseudobase* of $\mathscr{F}$ if for every $F \in \mathscr{F}$ there is some $R \in \mathscr{R}$ such that $R \subseteq F$. $\pi\chi(\mathscr{F})$ denotes the smallest size of a pseudobase for $\mathscr{F}$. A set $\mathscr{B}$ is called a *base* of $\mathscr{F}$ iff $\mathscr{F} = \{X : (\exists B \in \mathscr{B})(X \supseteq B)\}$. A set $\mathscr{B}$ is called a *filter base* if it is closed under finite intersections and if it does not contain the empty set. The smallest size of a base of $\mathscr{F}$ is called $\chi(\mathscr{F})$, the character of $\mathscr{F}$. Nyikos [23] showed that $\pi\chi(\mathscr{F}) \cdot \mathfrak{d}(\mathscr{F}) \geq \mathfrak{d}$. Published proofs are in [4, Theorem 16] and [20, 3.1].

The ultrafilter characteristic $\mathfrak{u}$ is the minimal $\chi(\mathscr{U})$ for a non-principal ultrafilter $\mathscr{U}$. Solomon [24] showed $\mathfrak{b} \leq \mathfrak{u}$. The inequality $\mathfrak{r} \leq \mathfrak{u}$ is obvious.

A subset $\mathscr{G}$ of $[\omega]^\omega$ is called *groupwise dense* if $(\forall X \in \mathscr{G})(\forall Y \subseteq^* X)(Y \in \mathscr{G})$ and for every partition of ω into finite intervals $\{[\pi_i, \pi_{i+1}) : i < \omega\}$ there is an infinite set A such that $\bigcup\{[\pi_i, \pi_{i+1}) : i \in A\} \in \mathscr{G}$. The *groupwise density number*, $\mathfrak{g}$, is the smallest number of groupwise dense families with empty intersection. The *groupwise density number for filters*, $\mathfrak{g}_f$, is the smallest number of groupwise dense ideals with empty intersection. The inequality $\mathfrak{g} \leq \mathfrak{g}_f$ is obvious. Brendle [15] showed the relative consistency of $\mathfrak{g} < \mathfrak{g}_f = \mathfrak{b} = \mathfrak{c} = \aleph_2$. It is not known how to keep $\mathfrak{b}$ small let alone keep $\mathfrak{u}$ small in such a construction. The inequality $\mathfrak{b} > \mathfrak{g}_f$ is consistent: Blass' proof of $\mathfrak{g} \leq \mathrm{cf}(\mathfrak{c})$ [9, Theorem 8.6, Corollary 8.7] also shows $\mathfrak{g}_f \leq \mathrm{cf}(\mathfrak{c})$, and there is a model of $\mathrm{cf}(\mathfrak{c}) < \mathfrak{b}$. For working with groupwise dense families the *next-functions* are very useful: For an infinite subset X of ω let $\nu_X(n) = \min(X \cap [n, \infty))$.

The *filter dichotomy principle*, FD, says, that for every filter there is a finite-to-one function g such that $g(\mathscr{F})$ is either the filter of cofinite sets (also called the *Fréchet filter*) or an ultrafilter.

The *principle of near coherence of filters*, NCF, says, that any two filters are nearly coherent. Blass and Laflamme [10] showed that $\mathfrak{u} < \mathfrak{g}$ implies FD, and that FD implies NCF. All reversibilities are long-standing open questions. Some types of models of $\mathfrak{u} < \mathfrak{g}$ are known: an iteration of length $\aleph_2$ with countable support of Blass-Shelah forcing over a ground model of CH [12] gives $\aleph_1 = \mathfrak{u} < \mathfrak{s} = \mathfrak{g} = \mathfrak{d} = \mathfrak{c} = \aleph_2$ and an iteration of length $\aleph_2$ with countable support of Miller forcing over a ground model of CH [13] gives $\aleph_1 = \mathfrak{u} = \mathfrak{s} < \mathfrak{g} = \mathfrak{d} = \mathfrak{c} = \aleph_2$. Also a countable support iteration of Matet forcing [7] gives $\aleph_1 = \mathfrak{u} < \mathfrak{g} = \mathfrak{d} = \mathfrak{c} = \aleph_2$. Other tree forcings that preserve P-points can be interwoven into the iteration and, as long as at stationarily many steps a real is added to all groupwise dense families in the intermediate model, the outcome is $\mathfrak{u} < \mathfrak{g}$.

By [11], if $\mathfrak{r} < \mathfrak{s}$ then there are at most two classes. If $\mathfrak{s} \leq \mathfrak{r}$ and not NCF, then among the ultrafilters $\mathscr{U}$ with $\mathrm{cf}(\omega^\omega/\mathscr{U}) \leq \mathfrak{r}$ there is no pseudo-$P_{\mathfrak{r}^+}$-point, by [11]. By [20], NCF is equivalent to $\mathfrak{mcf} > \mathfrak{r}$, and by [8] together with [20], FD is equivalent to $\mathfrak{g}_f > \mathfrak{r}$. By [11, Theorems 3 and 16], $\mathfrak{mcf} \geq \mathfrak{g}_f$ and even $\mathfrak{b}(\mathscr{F}) \geq \mathfrak{g}_f$ for any non-feeble filter $\mathscr{F}$, which follows from the fact that the intersection of the groupwise dense ideals $\mathscr{G}_{f,\mathscr{F}} = \{Z \in [\omega]^\omega : \nu_Z >_{\mathscr{F}} f\}$ over a $\leq_{\mathscr{F}}$-unbounded family of f's is empty.

2. Test sets and many classes

In this section we review and extend some known results on the near-coherence relation.

If $f \colon \omega \to \omega$ is finite-to-one and $g \colon \omega \to \omega$ then we define $g^f(n) = \max\{g(k) : f(k) = n\}$.

Lemma 2.1. *(Probably first in the unpublished* [23]*) Let $\mathscr{F}$ be a filter. If $f \colon \omega \to \omega$ is finite-to-one then $\mathfrak{d}(\omega^\omega/\mathscr{F}) = \mathfrak{d}(\omega^\omega/f(\mathscr{F}))$ and $\mathfrak{b}(\omega^\omega/\mathscr{F}) = \mathfrak{b}(\omega^\omega/f(\mathscr{F}))$. More precisely, if g_α, $\alpha < \kappa$, are dominating/unbounded in $\omega^\omega/\mathscr{F}$, then g^f_α, $\alpha < \kappa$, are dominating/unbounded in $\omega^\omega/f(\mathscr{F})$. And vice versa, if g_α, $\alpha < \kappa$, are dominating/unbounded in $\omega^\omega/f(\mathscr{F})$ and every g_α is an increasing function, then $g_\alpha \circ f$, $\alpha < \kappa$, are dominating/unbounded in $\omega^\omega/\mathscr{F}$.*

The proof of our main theorem uses two recent results by Banakh and Blass. The first proposition is valid for filters instead of ultrafilters in the second place.

Proposition 2.2. *If a filter $\mathscr{F}$ and a filter $\mathscr{G}$ are not nearly coherent, then $\mathfrak{d}(\mathscr{F}) \leq \chi(\mathscr{G})$. Even more holds: Let $\mathscr{B}$ be a pseudobasis of $\mathscr{G}$ of size $\pi\chi(\mathscr{G})$. Then $\{\nu_X : X \in \mathscr{B}\}$ is a dominating family in $\leq_{\mathscr{F}}$ for all such $\mathscr{F}$.*

Proof. As in [3, Proposition 19].

Proposition 2.3. [3, Proposition 21] *Every filter $\mathscr{F}$ has a test set of size $\mathfrak{d}(\mathscr{F})$.* □

Suppose that $\mathfrak{u} < \mathfrak{d}$ and suppose not NCF. Then there are two non-nearly-coherent ultrafilters, call them $\mathscr{U}_P$ and $\mathscr{U}$. We assume that $\mathscr{U}_P$ is a witness for $\mathfrak{u}$. By Ketonen's [19] result, $\mathscr{U}_P$ is a P-point. Since $\mathrm{cf}(\omega^\omega/\mathscr{U}_P) = \mathfrak{d} > \mathfrak{u} \geq \mathfrak{r}$ and since any two ultrafilters $\mathscr{V}$ with $\mathrm{cf}(\omega^\omega/\mathscr{V}) > \mathfrak{r}$ are nearly coherent by [11, Theorem 12], we have that $\mathrm{cf}(\omega^\omega/\mathscr{U}) \leq \mathfrak{r} \leq \mathfrak{u}$. (By Aubrey's result [1] we have $\mathfrak{u} = \mathfrak{r}$ under $\mathfrak{r} < \mathfrak{d}$, so a fortiori under $\mathfrak{u} < \mathfrak{d}$. We will make use of this in Section 3.) By Lemma 2.1 we have that for all finite-to-one f, $\mathrm{cf}(\omega^\omega/f(\mathscr{U})) \leq \mathfrak{u} < \mathfrak{d}$ and hence that $f(\mathscr{U})$ is not generated by fewer than $\mathfrak{d}$ sets. This also follows from [4, Corollary 13].

Lemma 2.4. *If $\mathfrak{u} < \mathfrak{d}$, then $\mathfrak{d}$ is regular.*

Proof. We take an ultrafilter $\mathscr{U}_P$ with a base of size $\mathfrak{u}$. By the inequality $\mathrm{cf}(\omega^\omega/\mathscr{U}_P) \cdot \pi\chi(\mathscr{U}_P) \geq \mathfrak{d}$ we have $\mathrm{cf}(\omega^\omega/\mathscr{U}_P) \geq \mathfrak{d}$. Since always $\mathfrak{d}(\mathscr{U}_P) \leq \mathfrak{d}$, we have $\mathrm{cf}(\omega^\omega/\mathscr{U}_P) = \mathfrak{d}$. The cofinality of course is a regular cardinal. □

So, Canjar's elegant construction in ZFC of an ultrafilter $\mathscr{U}$ with $\mathrm{cf}({}^\omega\omega/\mathscr{U}) = \mathrm{cf}(\mathfrak{d})$ [16] does not help us under $\mathfrak{u} < \mathfrak{d}$ in finding a new cofinality and hence a new near-coherence class, as $\mathrm{cf}(\mathfrak{d}) = \mathfrak{d}$. Already the P-point $\mathscr{U}_P$ with $\mathfrak{d} = \mathrm{cf}({}^\omega\omega/\mathscr{U}_P)$ witnesses Canjar's existence theorem and any other ultrafilter $\mathscr{V}$ with $\mathfrak{r} < \mathfrak{d} = \mathrm{cf}({}^\omega\omega/\mathscr{V})$ is nearly coherent to it by [11, Theorem 12].

Now we think of an inductive construction of $\mathscr{U}$ that is not nearly coherent to $\mathscr{U}_P$.

Let B_β, $\beta < \mathfrak{u}$, be an enumeration of a basis for $\mathscr{U}_P$. Let $f_\alpha, \alpha < \mathfrak{d}$, be a test set for near coherence of size $\mathfrak{d}$ (as in [4]). We fix these enumerations for the rest of the proof.

For every α there is some $X \in f_\alpha(\mathscr{U})$ such that $X^c \in f_\alpha(\mathscr{U}_P)$. We write X^c for $\omega \setminus X$. So $X^c \supseteq f_\alpha(B_{\beta_\alpha})$, for some $\beta_\alpha < \mathfrak{u}$, and $X \subseteq (f_\alpha(B_{\beta_\alpha}))^c$. We have $f_\alpha^{-1}(X) \subseteq f_\alpha^{-1}(f_\alpha(B_{\beta_\alpha}))^c)$ and $f_\alpha^{-1}(X) \in \mathscr{U}$. Now we have that $\{f_\alpha^{-1}((f_\alpha(B_{\beta_\alpha}))^c) : \alpha \in \mathfrak{d}\} \subseteq \mathscr{U}$, and hence $\{f_\alpha^{-1}((f_\alpha(B_{\beta_\alpha}))^c) : \alpha \in \mathfrak{d}\}$ has the finite intersection property. We write $\bar{\beta} = \langle \beta_\alpha : \alpha < \mathfrak{d}\rangle$. We set

$$\mathscr{H}_{\bar{\beta}} = \text{ the filter generated by } \{f_\alpha^{-1}((f_\alpha(B_{\beta_\alpha}))^c) : \alpha \in \mathfrak{d}\}.$$

Every filter that is not nearly coherent to $\mathscr{U}_P$ is a superset of some $\mathscr{H}_{\bar{\beta}}$. Now we let $\bar{\beta}$ range over those elements of ${}^{\mathfrak{d}}\mathfrak{u}$ such that $\mathscr{H}_{\bar{\beta}}$ is a filter. We do not know a second description of which $\bar{\beta}$ meet this condition. However, as the proof will automatically speak only about the ones who meet it, this does not harm. Every filter $\mathscr{H}_{\bar{\beta}}$ is not nearly coherent to $\mathscr{U}_P$.

By Proposition 2.2, we have that all filters with fewer than $\mathfrak{d}$ generators are nearly coherent to the P-point $\mathscr{U}_P$, because $\mathrm{cf}(\omega^\omega/\mathscr{U}_P) = \mathfrak{d}$. Hence there are at least $\mathfrak{d}$ different sets in the filter base $\{f_\alpha^{-1}((f_\alpha(B_{\beta_\alpha}))^c) : \alpha \in \mathfrak{d}\}$. Of course $\mathscr{H}_{\bar{\beta}}$ is not feeble, as every feeble filter is nearly coherent to any other filter.

We let $t(\mathscr{H})$ denote the minimal size of a test set for $[\mathscr{H}]$. By Proposition 2.3, $t(\mathscr{H}) \leq \mathfrak{d}(\mathscr{H})$.

Lemma 2.5. *If all extensions of $\mathscr{H}$ by fewer than $t(\mathscr{H})$ sets are not almost ultra, then we can construct by induction on $\alpha < t(\mathscr{H})$ infinitely many pairwise non-nearly-coherent ultrafilters in $[\mathscr{H}]$.*

Proof. By induction on $n < \omega$ we build $\mathscr{U}_n$. For $\mathscr{U}_0$ we take any ultrafilter above $\mathscr{H}$. Suppose $\mathscr{U}_i$, $i \leq n$, have already been built.

We build $\mathscr{V} \in [\mathscr{H}]$ by induction on $\alpha < t(\mathscr{H})$: By the definition of $t(\mathscr{H})$, there is a test set $\{f_\alpha : \alpha < t(\mathscr{H})\}$. By induction on α we define $\mathscr{V}_\alpha$, $A_{\alpha,i}$ for $i \leq n$ such that

(1) $\mathscr{V}_0 = \mathscr{H}$.

(2) For limit λ, $\mathscr{V}_\lambda = \bigcup\{\mathscr{V}_\alpha : \alpha < \lambda\}$.

(3) $\mathscr{V}_{\alpha+1}$ is generated by $\mathscr{V}_\alpha \cup \{A_{\alpha,i} : i \leq n\}$.

(4) For $i \leq n$: $f_\alpha(A_{\alpha,i}) \notin f_\alpha(\mathscr{U}_i)$.

(5) $\mathscr{V} = \bigcup\{\mathscr{V}_\alpha : \alpha < t(\mathscr{H})\}$. Now we let $\mathscr{U}_{n+1} \supseteq \mathscr{V}$ be an ultrafilter.

If this construction is accomplished, then $\mathscr{U}_i$ and $\mathscr{V}$, and hence also $\mathscr{U}_i$ and $\mathscr{U}_{n+1}$, are not nearly coherent.

Only the successor step requires some work: Let $\mathscr{V}_\alpha$ be already chosen. We first choose $A_{\alpha,i}$, $i \leq n$, such that $\langle \mathscr{V}_\alpha \cup \{A_{\alpha,i} \,:\, i \leq n\}\rangle$ has the finite intersection property and that $f_\alpha(A_{\alpha,i}) \notin f_\alpha(\mathscr{U}_i)$. Here the angled brackets $\langle\cdot\rangle$ denote the generated filter. The choice is carried out exactly as in Blass' proof of Theorem 14 in [4]. For completeness' sake, we insert his argument here:

It suffices to find a $B_i \in f_\alpha(\mathscr{U}_i)$ such that $f_\alpha^{-1}(B_i) \notin \langle \mathscr{V}_\alpha \cup \{A_{\alpha,j} \,:\, j < i\}\rangle$, for we can then set $A_{\alpha,i} = \omega \setminus f_\alpha^{-1}(B_i)$. Indeed, since $\langle \mathscr{V}_\alpha \cup \{A_{\alpha,j} \,:\, j < i\}\rangle$ is a filter not containing $f_\alpha^{-1}(B_i)$, it contains no subsets of $f_\alpha^{-1}(B_i)$, i.e., no sets disjoint from $A_{\alpha,i}$. Furthermore $f_\alpha(A_{\alpha,i})$ is disjoint from B_i, hence not in $f_\alpha(\mathscr{U}_i)$.

We suppose that no B_i of the desired sort exists and derive a contradiction. The supposition means that each $B_i \in f_\alpha(\mathscr{U}_i)$ also belongs to $f_\alpha(\langle \mathscr{V}_\alpha \cup \{A_{\alpha,j} \,:\, j < i\}\rangle)$. But by our inductive hypotheses $f_\alpha(\langle \mathscr{V}_\alpha \cup \{A_{\alpha,j} \,:\, j < i\}\rangle)$ is generated by $\mathscr{H}$ together with fewer than $t(\mathscr{H})$ sets. This contradicts the premise that no extension of $\mathscr{H}$ by fewer than $t(\mathscr{H})$ sets is almost ultra. $\square$

3. The models from [14]

The models constructed in [14] are so far the only models of $\mathfrak{s} \leq \mathfrak{r} = \mathfrak{u} < \mathfrak{d}$ (as it turns out in Proposition 3.1) and not NCF, so their investigation is worthwhile. In unpublished work Canjar showed that NCF does not hold in these models, see a remark following Question 30 in [3].

Here is a sketchy description of these models: Let V be a ground model of CH. Let ν and δ be regular cardinals such that $\aleph_1 \leq \nu < \delta$. First δ Cohen reals are added, call them r_α, $\alpha < \delta$. Thereafter ν Mathias reals are added by Mathias forcings $Q(\mathscr{U}_\xi)$, $\xi < \nu$, in a finite support support iteration. We call the whole forcing $\mathbb{P}$. The ultrafilters $\mathscr{U}_\xi$ are carefully chosen (— at least P-points with no rapid ultrafilters below them in the Rudin-Keisler ordering by [17], but not Ramsey ultrafilters as in the original Mathias forcing —) such that the Cohen reals are not bounded by fewer than δ reals in $V^{\mathbb{P}}$ and such that the Mathias reals s_ξ, $\xi < \nu$, generate an ultrafilter in $V^{\mathbb{P}}$.

A forcing condition in $Q(\mathscr{U}_\xi)$ is a pair (a, A), such that a is a finite set of natural numbers and $A \in \mathscr{U}_\xi$ and $\max(a) < \min(A)$. A condition (b, B) extends (a, A) iff $b \supseteq a$, $b \setminus a \subseteq A$, and $B \subseteq A$. In order to understand our proof it suffices to know that the forcing relation $\Vdash$ of $Q(\mathscr{U}_\xi)$ over $V(\delta, \xi)$ yields $(a, A) \Vdash a \subseteq \underset{\sim}{s_\xi} \subseteq a \cup A$. (We use $\underset{\sim}{s_\xi}$ for a $Q(\mathscr{U}_\xi)$-name of s_ξ). However, the fine construction in [14] that proves the existence of these models is long and cannot be repeated here.

For $\alpha \leq \delta$ and $\xi \leq \nu$ we set $V(\alpha, \xi) = V[(r_\beta \,:\, \beta < \alpha)][(s_\eta \,:\, \eta < \xi)]$. The really sophisticated part is to show for $\alpha \leq \delta$ and $\xi \leq \nu$ that the part of the ultrafilter $\mathscr{U}_\xi$ in $V(\alpha + 1, \xi)$ can be chosen such that no real with a $Q(\mathscr{U}_\xi \cap V(\alpha, \xi))$-name is dominating r_α in the Mathias extension built with $\mathscr{U}_\xi \cap V(\alpha + 1, \xi)$ over $V(\alpha + 1, \xi)$ (nor in later extensions). Moreover, $\mathscr{U}_\xi \cap V(\alpha + 1, \xi)$ is chosen such that $Q(\mathscr{U}_\xi \cap V(\alpha, \xi))$ is a complete subforcing of $Q(\mathscr{U}_\xi \cap V(\alpha + 1, \xi)$. The forcing is arranged such that $s_\xi \subseteq^* s_\eta$ for $\eta < \xi$ and hence the generated ultrafilter is a simple P_ν-point.

First we show that in $V^{\mathbb{P}}$, $\mathfrak{s} \leq \mathfrak{r}$, so that the results on models of $\mathfrak{r} < \mathfrak{s}$ (see [11]) do not apply. By Aubrey's result [1], $\mathfrak{r} = \mathfrak{u}$ in $V^{\mathbb{P}}$. So it suffices to show $\mathfrak{s} \leq \mathfrak{u}$.

Proposition 3.1. *In $V^{\mathbb{P}}$, $\mathfrak{s} \leq \nu$.*

Proof. Let s_ξ, $\xi < \nu$, be as in the construction there. As in [14] we set

$$X_\xi = \{n \in \omega \, : \, |s_\xi \cap n| \text{ is even}\}.$$

We repeat here Blass' and Shelah's proof [14, page 262]. Let $Y \in V(\delta, \xi) \cap [\omega]^\omega$. Suppose p is a condition in $Q(\mathscr{U}_\xi)$ forcing "$Y \subseteq^* X_\xi$" or forcing "$Y \subseteq^* X_\xi^c$" (over $V(\delta, \xi)$). Then there are a stronger condition and a finite modification of Y (which we just call Y again) such that (a, A) forces "$Y \subseteq X_\xi$" or that (a, A) forces "$Y \subseteq X_\xi^c$". Let $\max(a) < m \in A$ and choose $y \in Y$, $y > m$. Then $(a, A \setminus y) \Vdash \underset{\sim}{s_\xi} \cap y = a$ and $(a, A \setminus y \cup \{m\}) \Vdash \underset{\sim}{s_\xi} \cap y = a \cup \{m\}$. Hence, in case of even $|a|$, $(a, A \setminus y) \Vdash y \in \underset{\sim}{X_\xi}$ and $(a, A \setminus y \cup \{m\}) \Vdash y \in \underset{\sim}{X_\xi}^c$, and in case of odd $|a|$ the conclusion is vice versa. This contradicts the assumption about (a, A).

Hence for all $Y \in V(\delta, \xi)$, $Y \cap X_\xi$ and $Y \cap X_\xi^c$ are both infinite. So X_ξ splits all reals in $V(\delta, \xi)$, and $\{X_\xi \, : \, \xi < \nu\}$ is a splitting family. □

The existence of a (pseudo-) P_ν-point implies by [11, 2.3] that $\mathfrak{s} \geq \nu$. So, in $V^{\mathbb{P}}$, $\mathfrak{s} = \nu$. Now we state the main result:

Theorem 3.2. *In the model $V^{\mathbb{P}}$ there are $2^{\mathfrak{c}}$ near-coherence classes of ultrafilters.*

Proof. We shall show that there is a filter $\mathscr{H}$ that is non-nearly-coherent to $\mathscr{U}_P$ such that $\mathscr{H}$ extended by fewer than $t(\mathscr{H})$ sets is not almost ultra.

Lemma 3.3. *If there is a pseudo-$P_\mathfrak{u}$-point $\mathscr{U}_P$ with $\chi(\mathscr{U}_P) = \mathfrak{u} < \mathfrak{d}$, then $\mathfrak{b} = \mathfrak{u}$.*

Proof. We assume $\mathfrak{b} < \mathfrak{u}$. Let $\mathscr{U}_P$ be a pseudo-$P_\mathfrak{u}$-point. Then by [11, 2.4] $\mathrm{cf}(\omega^\omega/\mathscr{U}_P) = \mathfrak{b} < \mathfrak{d}$. But now we have $\mathrm{cf}(\omega^\omega/\mathscr{U}_P) \cdot \chi(\mathscr{U}_P) = \mathfrak{u} < \mathfrak{d}$, a contradiction. □

We think of the Cohen reals as subsets of ω and let the Cohen reals r_α, $\alpha < \delta$, be their strictly increasing enumerations. We set

$$\begin{aligned} X_{\alpha,\xi} &= \{r_\alpha(n) \, : \, |s_\xi \cap n| \text{ is even}\}, \\ \mathscr{H}_\xi &= \{X_{\alpha,\xi} \, : \, \alpha < \delta\}, \\ \mathscr{H} &= \{X_{\alpha,\xi} \, : \, \alpha < \delta, \xi < \nu, \xi \text{ is an even ordinal}\}, \\ \mathscr{H}_{\text{full}} &= \{X_{\alpha,\xi} \, : \, \alpha < \delta, \xi < \nu\}. \end{aligned}$$

Lemma 3.4. *For every $\xi < \nu$, for every $Y \in V(\delta, \xi)$ for every $\alpha_i < \delta$, $i < k$, we have: If $Y \cap \bigcap_{i<k} \mathrm{range}(r_{\alpha_i})$ is infinite, then the set*

$$Y \cap \bigcap_{0 \leq i < k} X_{\alpha_i,\xi}$$

is infinite.

Proof: We force with $Q(\mathscr{U}_\xi)$ over $V(\delta,\xi)$. We suppose for a contradiction that

$$(a,A) \Vdash_{Q(\mathscr{U}_\xi)} Y \subseteq \left(\bigcap_{0\le i<k} X_{\underset{\sim}{\alpha_i},\xi}\right)^c.$$

As in the proof of Proposition 3.1 we replaced almost inclusion by sharp inclusion. W.l.o.g., let $|a|$ be even. Since $Y \cap \bigcap_{i<k} \mathrm{range}(r_{\alpha_i})$ is infinite, there is some $y \in Y$ and there are n_i, $0 \le i < k$, such that $y = r_{\alpha_i}(n_i)$ for $0 \le i < k$ and such that $n_i > \max(a)$ for $0 \le i < k$. We take $m > \max\{n_i : 0 \le i < k\}$. Then

$$(a, A \setminus m) \Vdash_{Q(\mathscr{U}_\xi)} y \in Y \cap \bigcap_{0\le i<k} X_{\underset{\sim}{\alpha_i},\xi},$$

contrary to our assumption. □

Lemma 3.5. *$\mathscr{H}_{\mathrm{full}}$ has the finite intersection property.*

Proof: We prove the following claim by induction on ξ: For every (α_i,ξ_i), $i<\ell$, such that $\xi_i < \xi$, for every α_i, $\ell \le i < k$, the intersection $\bigcap_{i<\ell} X_{\alpha_i,\xi_i} \cap \bigcap_{i<k} \mathrm{range}(r_{\alpha_i})$ is infinite.

For $\xi = 0$, the claim follows from the properties of Cohen forcing. For limit steps ξ the claim follows immediately from the induction hypothesis, since we consider only finitely many ξ_i at a time.

Now for the step from ξ to $\xi+1$: Let X_{α_i,ξ_i}, $i<\ell$, and α_i, $\ell \le i < k$, be given. Suppose that $\xi = \xi_{\ell'} = \dots = \xi_{\ell-1}$, and for $0 \le i < \ell'$, $\xi_i < \xi$. By induction hypothesis, $Y = \bigcap_{0\le i<\ell'} X_{\alpha_i,\xi_i} \cap \bigcap_{0\le i<k} \mathrm{range}(r_{\alpha_i})$ is infinite. Now we apply Lemma 3.4 with Y and $X_{\alpha_i,\xi}$, $\ell' \le i < \ell$, and $X_{\alpha_i,\xi}$, $\ell \le i < k$. Taking the supersets $\mathrm{range}(r_{\alpha_i})$ of $X_{\alpha_i,\xi}$ for $\ell \le i < k$ gives the induction claim verbatim. □

Now we show that $\mathscr{H}_0$ and $\mathscr{U}_P$ are not nearly coherent. This will then allow us to use Propositions 2.2 and 2.3 and Lemma 2.5 for
$t(\mathscr{H}_0) \le \mathfrak{d}(\mathscr{H}_0) \le \chi(\mathscr{U}_P) = \mathfrak{u} = \nu$. Since $\mathscr{U}_P$ and hence also $f(\mathscr{U}_P)$ for every finite-to-one f are ultrafilters, $\mathscr{H}_0$ and $\mathscr{U}_P$ are nearly coherent iff for some finite-to-one f, $f(\mathscr{H}_0) \subseteq f(\mathscr{U}_P)$. We show the negation of the latter:

Theorem 3.6. *For every finite-to-one f, $f(\mathscr{H}_0) \not\subseteq f(\mathscr{U}_P)$.*

Proof. Assume that $p \in \mathbb{P}$, $p \Vdash f(\mathscr{H}_0) \subseteq f(\mathscr{U}_P)$. Then $p \Vdash (\forall \alpha < \delta)(\exists \xi < \nu)$ $f(X_{\alpha,0}) \supseteq f(s_\xi)$. Since δ is regular and $\nu < \delta$, there is some $\xi < \nu$ such that $p \Vdash$ "for a cofinal set C in δ, for all $\alpha \in C$, $f(X_{\alpha,0}) \supseteq f(s_\xi)$". Let $p \in V(\delta',\xi')$ and $f \in V(\delta',\xi')$, and we assume $\xi' \ge \xi$.

Let e_ξ be the increasing enumeration of $f(s_\xi)$. We let

$$g_\xi(n) = \max(f^{-1}(\{e_\xi(n)\})) + 1.$$

One of the main properties of the forcing construction is that the function $k \mapsto g_\xi(2k)$ does not dominate all r_α for $\alpha \in C$. So we take some $\alpha < \delta$ such that $(\exists^\infty k)(r_\alpha(k) > g_\xi(2k))$. Then there are infinitely many n such that

$$\mathrm{range}(r_\alpha) \cap [g_\xi(n-1), g_\xi(n)) = \emptyset,$$

as otherwise we would have $(\forall^\infty k)(r_\alpha(k) \leq^* g_\xi(2k))$. But then $f(s_\xi) = \{e_\xi(n) : n \in \omega\} \not\subseteq f(\mathrm{range}(r_\alpha))$, and hence $f(s_\xi) \not\subseteq f(X_{\alpha,0})$. Contradiction. $\square$

So we have not NCF. We do not know Canjar's proof. By Proposition 2.2, $\mathfrak{d}(\mathscr{H}_0) \leq \chi(\mathscr{U}_P) = \mathfrak{u} = \nu$. In Lemma 3.7 we will show that $\mathscr{H}_0$ and also $\mathscr{H}$ fulfil the premise of Lemma 2.5. Of course $\mathfrak{d}(\mathscr{H}) \leq \mathfrak{d}(\mathscr{H}_0)$. We will prove Lemma 3.7 even for $\mathscr{H}$ and not only for $\mathscr{H}_0$. The result, that $\mathscr{H}$ is so far from being an ultrafilter is not used in the proof of Theorem 3.2. We just found it interesting, to see that using s_ξ for cofinally many $\xi < \nu$ does not yet lead to an ultrafilter.

Lemma 3.7. *In $V^{\mathbb{P}}$, each extension of $\mathscr{H}$ by fewer than ν sets is not almost ultra.*

Proof. Let $\kappa < \nu$, and let Y_i, $i < \kappa$, be infinite sets such that $\mathscr{H} \cup \{Y_i : i < \kappa\}$ has the finite intersection property. Let f be finite-to-one. Let $Y_i, f \in V(\delta, \xi)$ for some $\xi < \nu$, and let $\Pi = \{[\pi_i, \pi_{i+1}) : i < \omega\}$ the partition given by f. Let ξ be odd, otherwise increase it by one. Suppose, for a contradiction, that p is a condition in $Q(\mathscr{U}_\xi)$ forcing "$f(\mathscr{H} \cup \{Y_i : i < \kappa\})$ measures $f(X_{0,\xi})$". Strengthening p we may assume that there is some $k < \omega$ and there are $j_0 \in \omega$, $\beta_i < \kappa$, $\eta_i < \nu$, $\alpha_i < \delta$, $i < k$, such that the η_i are even and that p forces

$$(\forall j \in [j_0, \omega))(([\pi_j, \pi_{j+1}) \cap \bigcap_{i<k} X_{\alpha_i,\eta_i} \cap \bigcap_{i<k} Y_{\beta_i}) \neq \emptyset \to [\pi_j, \pi_{j+1}) \cap X_{0,\xi} = \emptyset)$$

or that p forces

$$(\forall j \in [j_0, \omega))(([\pi_j, \pi_{j+1}) \cap \bigcap_{i<k} X_{\alpha_i,\eta_i} \cap \bigcap_{i<k} Y_{\beta_i}) \neq \emptyset \to [\pi_j, \pi_{j+1}) \cap X_{0,\xi} \neq \emptyset).$$

The first statement is excluded, because by Lemma 3.4,

$$(\bigcap_{i<k,\eta_i<\xi} X_{\alpha_i,\eta_i} \cap \bigcap_{i<k} Y_{\beta_i}) \cap X_{0,\xi} \cap \bigcap_{i<k,\eta_i>\xi} X_{\alpha_i,\eta_i}$$

is infinite, and hence meets infinitely many of the $[\pi_j, \pi_{j+1})$. So the possibility that $f(\bigcap_{i<k} X_{\alpha_i,\eta_i} \cap \bigcap_{i<k} Y_{\beta_i}) \subseteq (f(X_{0,\xi}))^c$ is excluded.

We show that the second possibility does not occur either. We assume that $\eta_i < \xi$ for $0 \leq i < \ell$ and $\eta_i > \xi$ for $\ell \leq i < k$. By our assumption on $\mathscr{H}$ and Y_i, $Y = \bigcap_{i<k,\eta_i<\xi} X_{\alpha_i,\eta_i} \cap \bigcap_{i<k} Y_{\beta_i} \cap \bigcap_{\ell \leq i<k} \mathrm{range}(r_{\alpha_i}) \in V(\delta, \xi)$ is infinite. Suppose (a, A) is a condition in $Q(\mathscr{U}_\xi)$ forcing "every block of Π meets $X_{0,\xi}$ if it meets Y"(over $V(\delta, \xi)$). W.l.o.g., let $|a|$ be odd. We take $j \geq j_0$ such that $Y \cap [\pi_j, \pi_{j+1}) \cap \bigcap_{\ell \leq i<k} \mathrm{range}(r_{\alpha_i}) \neq \emptyset$ and such that $r_0(n) < \pi_j$ for $n \in a$. Then take j' such that for all $j'' \geq j'$, $r_0(j'') \geq \pi_{j+1}$. Now let $m > j'$. Then

$$(a, A \setminus m) \Vdash_{Q(\mathscr{U}_\xi)} Y \cap [\pi_j, \pi_{j+1}) \neq \emptyset \wedge \underset{\sim}{X}_{0,\xi} \cap [\pi_j, \pi_{j+1}) = \emptyset.$$

This contradicts the assumption about (a, A).

Let $\eta_i > \xi$ for $\ell \leq i < k$. We know that $Z = \{\min(\bigcap_{i<k,\eta_i<\xi} X_{\alpha_i,\eta_i} \cap \bigcap_{i<k} Y_{\beta_i} \cap \bigcap_{\ell \leq i<k} \mathrm{range}(r_{\alpha_i}) \cap [\pi_j, \pi_{j+1})) : j \in \omega, [\pi_j, \pi_{j+1}) \cap X_{0,\xi} = \emptyset\} \in V(\delta, \xi + 1)$ is infinite. By Lemma 3.4, also $Z \cap \bigcap_{\ell \leq i<k} X_{\alpha_i,\eta_i}$ is infinite. So also the second possibility, that $f(\bigcap_{i<k} X_{\alpha_i,\eta_i} \cap \bigcap_{i<k} Y_{\beta_i}) \subseteq f(X_{0,\xi})$, is excluded. $\square_{3.7,3.2}$

Remark: We do not know whether $\mathscr{H}_{\text{full}}$ is almost ultra.

4. Questions and remarks

The models we investigated have $\mathfrak{b} = \mathfrak{u} < \mathfrak{d} = \mathfrak{c}$. Interesting related questions are

Question 4.1. *Is $\mathfrak{b} < \mathfrak{u} < \mathfrak{d}$ consistent relative to ZFC?*

Question 4.2. *Is $\mathfrak{u} < \mathfrak{d}$ with a singular $\mathfrak{u}$ consistent relative to ZFC?*

The following result is a kind of reverse to Lemma 3.3:

Proposition 4.3. *If $\mathfrak{b} = \mathfrak{u} < \mathfrak{d}$, then there is an ultrafilter that is generated by a $\subseteq^*$-descending sequence.*

Proof. We enumerate a base $\{B_\alpha : \alpha < \mathfrak{b}\}$ of an ultrafilter, call it $\mathscr{U}_P$, witnessing $\mathfrak{u} = \mathfrak{b}$. For each $\beta < \mathfrak{b}$ the filter generated by $\{B_\alpha : \alpha \leq \beta\}$ is feeble [9, 9.10], i.e., there is finite-to-one function f_β such that $f_\beta(B_\alpha) =^* \omega$ for all $\alpha \leq \beta$. We assume that the finite-to-one f_α is monotone and onto, and we define the intervals $f_\alpha^{-1}(\{n\}) = [g_\alpha(n), g_\alpha(n+1))$. Possibly increasing the g_α and still keeping the property $f_\alpha(B_\alpha) =^* \omega$ we arrange that $g_\alpha \leq^* g_\beta$ for $\alpha < \beta < \mathfrak{b}$. Moreover, we use the iterates of the g_α's: $\tilde{g_\alpha}(0) = 0$, $\tilde{g_\alpha}(n+1) = g_\alpha(\tilde{g_\alpha}(n))$. Then we increase these functions even more in order to have $\tilde{g_\alpha} \leq^* \tilde{g_\beta}$ for $\alpha < \beta < \mathfrak{b}$. Now, since $\mathrm{cf}({}^\omega\omega/\mathscr{U}_P) = \mathfrak{d} > \mathfrak{b}$, we take some $g \geq_{\mathscr{U}_P} \tilde{g_\alpha}$ for all $\alpha < \mathfrak{b}$. And we take π_i such that $\forall n \leq \pi_i$, $g(n) < \pi_{i+1}$.

Then we set $X_\alpha = \{n : g(n) \geq \tilde{g_\alpha}(n)\}$. Since the X_α are $\subseteq^*$ decreasing, there is one $\ell \in \{0,1\}$, such that for all $\alpha < \mathfrak{b}$

$$Y_{\alpha,\ell} = \bigcup_{i<\omega} [\pi_{2i+\ell}, \pi_{2i+\ell+1}) \cap X_\alpha \in \mathscr{U}_P.$$

W.l.o.g., we assume $\ell = 0$. Then we set $g^-(k) = i$ for $k \in [\pi_{2i}, \pi_{2i+2})$. The $Y_{\alpha,0}$ are $\subseteq^*$-descending as well and we show $g^-(Y_{\alpha,0}) \subseteq^* g^-(B_\alpha)$: For this we take n_0 such that above n_0, in every interval $[g_\alpha(n), g_\alpha(n+1))$ there is some element of B_α. Then for all but finitely many $y \in Y_{\alpha,0}$ we have for some i such that $\pi_{2i} \geq n_0$

$$\pi_{2i} < y \leq \pi_{2i+1} \text{ and } y \leq g(y) < \pi_{2i+2}.$$

Then $\pi_{2i+2} > g(y) \geq \tilde{g_\alpha}(y) \geq g_\alpha(\tilde{g_\alpha}(y-1))$. And $\tilde{g_\alpha}(y-1) \geq y$, as we may assume that $g_\alpha(n) \geq n+1$ for all n. So,

$$[g_\alpha(\tilde{g_\alpha}(y-1)-1), g_\alpha(\tilde{g_\alpha}(y-1))) \subseteq [\tilde{g_\alpha}(y-1), \tilde{g_\alpha}(y)) \subseteq [\pi_{2i}, \pi_{2i+2})$$

as well. So $[\pi_{2i}, \pi_{2i+2})$ contains an element of B_α. $\square$

Remark 4.4. The descendingness of the Mathias reals seems to be crucial in Blass' and Shelah's work [14]. We look at an initial segment of the Blass Shelah construction with δ Cohen reals and ξ nested Mathias reals and that ξ is a limit of uncountable cofinality. Then $\{s_\zeta : \zeta < \xi\}$ generates an ultrafilter $\mathscr{U}_\xi$ in $V(\delta, \xi)$. If we want to go on with forcing with $Q(\mathscr{U})$ for some ultrafilter $\mathscr{U}$ and not add a pseudointersection to $\mathscr{U}_\xi$, then it is forbidden to force with an ultrafilter $\mathscr{U}$ that is above $\mathscr{U}_\xi$ in the Rudin-Keisler ordering (see [6, 18]).

Suppose, trying to ensure this, we force with $Q(\mathscr{U})$ for an ultrafilter $\mathscr{U}$ that is at least not nearly coherent to $\mathscr{U}_\xi$. For P-points $\mathscr{U}$, being not nearly coherent to $\mathscr{U}_\xi$ ensures being not Rudin-Keisler above $\mathscr{U}_\xi$. Then Proposition 2.2 yields $\mathfrak{d}(\mathscr{U}) \leq \mathrm{cf}(\xi)$. Now [17, Proposition 5] applies and yields that the restrictions of the members of the family dominating in $\leq_{\mathscr{U}}$ to the generic of $Q(\mathscr{U})$ form a new dominating set of size $\mathrm{cf}(\xi)$. This thwarts the plan of building a model of $\mathfrak{b} < \mathfrak{u} < \mathfrak{d}$ by modifying Blass' and Shelah's construction from [14] only in the choice of the Mathias forcings.

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Heike Mildenberger, Universität Wien, Kurt Gödel Research Center for Mathematical Logic, Währinger Str. 25, 1090 Vienna, Austria

E-mail address: heike@logic.univie.ac.at